R2live 公式推导

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目录

摘要：基于 R2live 论文，结合 Fastlio2, 对公式进行详细推导和总结，给社区贡献一份力量。

关键词：R2live, Fastlio2, 公式

1 介绍

工作的主要贡献如下：

- 紧耦合了相机，雷达和 IMU 数据，实验表明，我们的方法在各种具有挑战性的场景中足够稳健，例如剧烈运动、传感器故障，甚至在具有大量移动物体和小 LiDAR 视场的狭窄隧道环境中。
- 提出了高速率的卡尔曼滤波器和低速率的因子图优化
- 通过紧耦合不同类型传感器，实现了更高精度的测量

2 系统概述

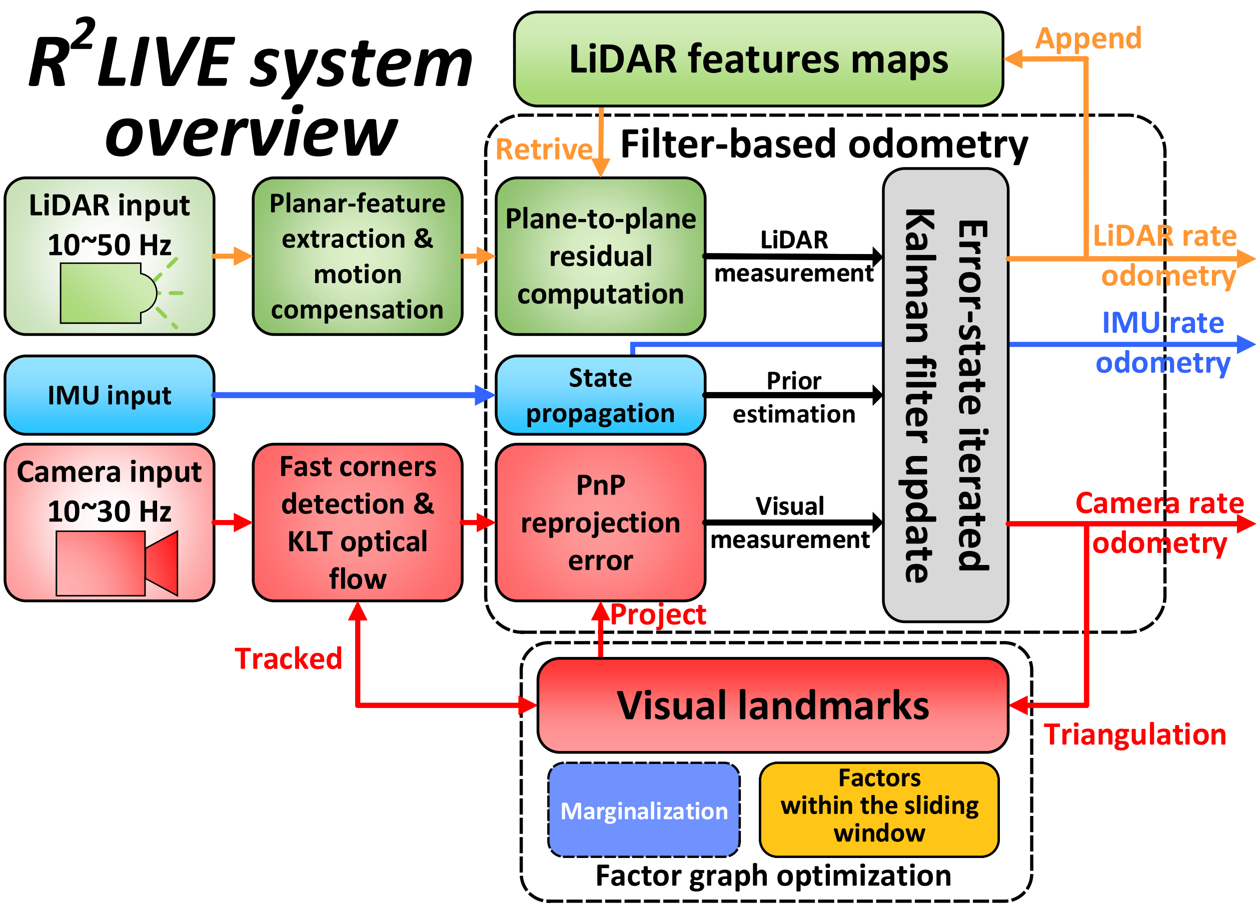

图 2-1 系统总览

从图中可以看到，分为 3 个线程执行：

1. 通过 IMU 的前向传播来传播机体状态，通过反向传播来完成点云去畸变。
2. 在两个雷达帧间有若干个 IMUPose, 通过每个 pose 的 R, t 来对该 IMUPose 时间间隔内的点云完成去畸变
3. 完成去去畸变后，对每个点 ${}^{G}p_i$ 在子地图上找到相临近的 5 个点构建平面，求该点到平面的残差，完成了雷达观测
4. 完成去去畸变后，对每个地图点 ${}^{G}p_i$, 构建 pnp 重投影残差，完成了视觉观测。
5. 将状态量，雷达观测，视觉观测送进 IESKF, 迭代更新状态量，构建最大后验估计来更新误差状态量，通过更新的误差状态量 x_k^{κ} 来计算优化的状态量 $\overline{x}_k$，最终用优化的状态量来将 lidar 坐标系下的点 ${}^{L}p_j$ 转换到世界坐标系下 ${}^{G}p_i$, 其转换公式为 ${}^{G}p_j = ^{G} T_I {}^{I}T_L {}^{L}p_i$

3 基于误差状态的卡尔曼滤波器

3.1 状态量及离散运动模型

1. 我们得到总状态量 x_k:

$$\mathrm{x}_i = \begin{bmatrix} {}^{G}\mathrm{R}_{I_i}^{T} & {}^{G}\mathrm{p}_{I_i}^{T} & {}^{I}\mathrm{R}_{C_i}^{T} & {}^{I}\mathrm{p}_{C_i}^{T} & {}^{G}\mathrm{v}_i^{T} & \mathrm{b}_{\mathrm{g}_i}^{T} & \mathrm{b}_{\mathrm{a}_i}^{T} \end{bmatrix}^{T} \tag{3-1}$$

2. 名义状态量 $\hat{x}_k$:

$$\hat{\mathrm{x}}_i = \begin{bmatrix} {}^{G}\hat{\mathrm{R}}_{I_i}^{T} & {}^{G}\hat{\mathrm{p}}_{I_i}^{T} & {}^{I}\hat{\mathrm{R}}_{C_i}^{T} & {}^{I}\hat{\mathrm{p}}_{C_i}^{T} & {}^{G}\hat{\mathrm{v}}_i^{T} & \hat{\mathrm{b}}_{\mathrm{g}_i}^{T} & \hat{\mathrm{b}}_{\mathrm{a}_i}^{T} \end{bmatrix}^{T} \tag{3-2}$$

3. 误差状态量 $\hat{\delta x}_k$:

$$\begin{aligned} &\delta\hat{\mathrm{x}}_i \triangleq \mathrm{x}_i \boxminus \hat{\mathrm{x}}_i \\ &= \begin{bmatrix} {}^{G}\delta\hat{\mathrm{r}}_{I_i}^{T} & {}^{G}\delta\hat{\mathrm{p}}_{I_i}^{T} & {}^{I}\delta\hat{\mathrm{r}}_{C_i}^{T} & {}^{I}\delta\hat{\mathrm{p}}_{C_i}^{T} & {}^{G}\delta\hat{\mathrm{v}}_i^{T} & \delta\hat{\mathrm{b}}_{\mathrm{g}_i}^{T} & \delta\hat{\mathrm{b}}_{\mathrm{a}_i}^{T} \end{bmatrix}^{T} \\ &\sim \mathcal{N}(0_{21\times 1}, \boldsymbol{\Sigma}_{\delta\hat{\mathrm{x}}_i}) \end{aligned} \tag{3-3}$$

${}^{G}\delta\hat{\mathrm{r}}_{I_i}^{T}$ 和 ${}^{I}\delta\hat{\mathrm{r}}_{C_i}^{T}$ 分别代表:

$${}^{G}\delta\hat{\mathrm{r}}_{I_i} = \mathrm{Log}({}^{G}\hat{\mathrm{R}}_{I_i}^{T}\,{}^{G}\mathrm{R}_{I_i}^{T}), \quad {}^{I}\delta\hat{\mathrm{r}}_{C_i} = \mathrm{Log}({}^{I}\hat{\mathrm{R}}_{C_i}^{T}\,{}^{I}\mathrm{R}_{C_i})$$

4. 传感器输入 u_{m_i} 和 w_i

$$\mathrm{u}_i = \begin{bmatrix} \boldsymbol{\omega}_{m_i}^T & \mathrm{a}_{m_i}^T \end{bmatrix}^T, \quad \mathrm{w}_i = \begin{bmatrix} \mathrm{n}_{\mathrm{g}_i}^T & \mathrm{n}_{\mathrm{a}_i}^T & \mathrm{n}_{\mathrm{bg}_i}^T & \mathrm{n}_{\mathrm{ba}_i}^T \end{bmatrix}^T$$

$$\mathrm{f}(\mathrm{x}_i, \mathrm{u}_i, \mathrm{w}_i) = \begin{bmatrix} \boldsymbol{\omega}_{m_i} - \mathrm{b}_{\mathrm{g}_i} - \mathrm{n}_{\mathrm{g}_i} \\ {}^G\mathrm{v}_i \\ 0_{3\times 1} \\ 0_{3\times 1} \\ {}^G\mathrm{R}_{I_i}\left(\mathrm{a}_{m_i} - \mathrm{b}_{\mathrm{a}_i} - \mathrm{n}_{\mathrm{g}_i}\right) - {}^G\mathrm{g} \\ \mathrm{b}_{\mathrm{g}_i} \\ \mathrm{b}_{\mathrm{a}_i} \end{bmatrix}$$

5. 前向传播 $f(x_i, u_i, w_i)$:

$$\hat{\mathrm{x}}_{i+1} = \hat{\mathrm{x}}_i \boxplus \left(\Delta t \cdot \mathrm{f}(\hat{\mathrm{x}}_i, \mathrm{u}_i, 0)\right). \tag{3-4}$$

6. 几个重点公式要牢记:

关于 BCH **展开**

$$Exp(r+\delta r) = Exp(r + J_l \delta r) Exp(r) \qquad = Exp(r) Exp(J_r \delta r) \tag{3-5}$$

$$Log(\delta R R) = \phi + J_l^{-1}(\phi)\delta\phi \tag{3-6}$$

$$Log(R \delta R) = \phi + J_r^{-1}(\phi)\delta\phi \tag{3-7}$$

$$R = Exp\phi \tag{3-8}$$

$$\delta R = Exp(\delta\phi) \tag{3-9}$$

关于左右雅可比和罗德里格斯公式 左右雅可比的关系:

$$J_l(\theta a) = \frac{sin(\theta)}{\theta} I + (1 - \frac{sin(\theta)}{\theta}) a a^T + \frac{1 - cos(\theta)}{\theta} \hat{a} \tag{3-10}$$

$$J_l^{-1}(\theta a) = \frac{\theta}{2}\frac{cot(\theta)}{2} I + (1 - \frac{\theta}{2}\frac{cot(\theta)}{2}) a a^T - \frac{\theta}{2}\hat{a} \tag{3-11}$$

$$J_r(\theta a) = \frac{sin(\theta)}{\theta} I + (1 - \frac{sin(\theta)}{\theta}) a a^T - \frac{1 - cos(\theta)}{\theta} \hat{a} \tag{3-12}$$

$$J_r^{-1}(\theta a) = \frac{\theta}{2}\frac{cot(\theta)}{2} I + (1 - \frac{\theta}{2}\frac{cot(\theta)}{2}) a a^T + \frac{\theta}{2}\hat{a} \tag{3-13}$$

$$J_l(r)^T = J_l(-r) \tag{3-14}$$

$$J_r(r)^T = J_r(-r) \tag{3-15}$$

$$J_r(r) = J_l(-r) \tag{3-16}$$

$$J_r^{-1}(r) = J_l^{-1}(-r) \tag{3-17}$$

$$\tag{3-18}$$

罗德里格斯公式

$$R = Exp(\theta a) = cos(\theta)I + (1 - cos(\theta))aa^T + sin(\theta)\hat{a} \tag{3-19}$$

名义值和真实值

$$w_i = w_{m_i} - b_{w_i} - n_{w_i} \tag{3-20}$$

$$\hat{w}_i = w_{m_i} - \hat{b}_{w_i} \tag{3-21}$$

$$a_i = a_{m_i} - b_{a_i} - n_{a_i} \tag{3-22}$$

$$\hat{a}_i = a_{m_i} - b_{a_i} \tag{3-23}$$

$$w_i = \hat{w}_i - \delta b_{w_i} - n_{w_i} \tag{3-24}$$

$$a_i = \hat{a}_i - \delta b_{a_i} - n_{a_i} \tag{3-25}$$

3.2 状态传播

3.2.1 两种求导方式

此处参考王博的求导方法：考虑一个流形：$G = a \boxplus b \boxplus c \boxminus d$, 其中 $a, d \in \mathcal{R}, b, c \in SO3$, 那么有：

$$G = Log(d^{-1} a Exp(b) Exp(c))$$

其中，如果 $a, b, c, d \in \mathcal{R}$, 则：

$$G = a + b + c - d$$

, 那么得到：

$$\frac{\partial G}{\partial a} = \frac{\partial G}{\partial b} = \frac{\partial G}{\partial c} = -\frac{\partial G}{\partial d} = I_{3x3}$$

那么考虑 $a, d \in \mathcal{R}, b, c \in \mathcal{SO}3$, 那么如何求得 G 相对于 b, c 的导数？如下有两种方法：

方法 1-通过添加扰动的方法求解 给 G 添加 b,c 的扰动，得到：

$= Log(d^{-1}aExp(b+\delta b)Exp(c+\delta c))$

$= Log(d^{-1}aExp(b)Exp(J_r(b)\delta b))Exp(c)Exp(J_r(c)\delta c))$

这里用到了 BCH 展开

$= Log(d^{-1}aExp(b)Exp(c)Exp(c)^T Exp(J_r(b)\delta b)Exp(c)Exp(J_r(c)\delta c))$

添加 $Exp(c)Exp(c)^T$ 是为了让凑齐 $Log(d^{-1}aExp(b)Exp(c)$

$= Log(d^{-1}aExp(b)Exp(c)) + Log(Exp(c)^T Exp(J_r(b)\delta b)Exp(c)Exp(J_r(c)\delta c))$

直接展开 $Exp(J_r(c)\delta c)$

$= Log(d^{-1}aExp(b)Exp(c)) + Log(Exp(Exp(c)^T J_r(b)\delta b)Exp(J_r(c)\delta c))$

伴随的性质：$R^T Exp(a)R = Exp(R^T a)$

$= Log(d^{-1}aExp(b)Exp(c)) + Exp(c)^T J_r(b)\delta b + J_r(c)\delta c$

所以最终可以得到求导公式：

$$\frac{\partial G}{\partial b} = Exp(c)^T J_r(b)$$
$$\frac{\partial G}{\partial c} = J_r(c)$$

方法 2-**通过直接求导** a **的方法求解** 通过求导的方法也可以求得上述过程，看下图3-2:

误差状态量 $\delta\hat{x}_{i+1}$ 是真实状态 x_{i+1} 和名义状态量 $\hat{x}_{i+1}$ 的差：

$$\begin{aligned}
\delta\hat{\mathrm{x}}_{i+1} &= \mathrm{x}_{i+1} \boxminus \hat{\mathrm{x}}_{i+1} \\
&= (\mathrm{x}_i \boxplus (\Delta t\cdot \mathrm{f}(\mathrm{x}_i,\mathrm{u}_i,\mathrm{w}_i))) \boxminus (\hat{\mathrm{x}}_i \boxplus (\Delta t\cdot \mathrm{f}(\hat{\mathrm{x}}_i,\mathrm{u}_i,0))) \\
&\sim \mathcal{N}(0_{21\times 1}, \mathbf{\Sigma}_{\delta\hat{\mathrm{x}}_{i+1}})
\end{aligned} \tag{3-26}$$

其中：

$$\begin{aligned}
&\mathbf{\Sigma}_{\delta\hat{\mathrm{x}}_{i+1}} = \mathrm{F}_{\delta\hat{\mathrm{x}}}\mathbf{\Sigma}_{\delta\hat{\mathrm{x}}_i}\mathrm{F}_{\delta\hat{\mathrm{x}}}^T + \mathrm{F}_{\mathrm{w}}\mathrm{Q}\mathrm{F}_{\mathrm{w}}^T \\
&\mathrm{F}_{\delta\hat{\mathrm{x}}} = \left.\frac{\partial(\delta\hat{\mathrm{x}}_{i+1})}{\partial\delta\hat{\mathrm{x}}_i}\right|_{\delta\hat{\mathrm{x}}_i=0,\mathrm{w}_i=0}, \quad \mathrm{F}_{\mathrm{w}} = \left.\frac{\partial(\delta\hat{\mathrm{x}}_{i+1})}{\partial\mathrm{w}_i}\right|_{\delta\hat{\mathrm{x}}_i=0,\mathrm{w}_i=0}
\end{aligned} \tag{3-27}$$

3.2.2 由以上方法推导状态传播矩阵

将 (3-3) 和 (3-26) 结合，我们得到：

$$\begin{aligned}
\delta\hat{\mathrm{x}}_{i+1} &= \mathrm{x}_{i+1} \boxminus \hat{\mathrm{x}}_{i+1} \\
&= (\mathrm{x}_i \boxplus (\Delta t\cdot \mathrm{f}(\mathrm{x}_i,\mathrm{u}_i,\mathrm{w}_i))) \boxminus (\hat{\mathrm{x}}_i \boxplus (\Delta t\cdot \mathrm{f}(\hat{\mathrm{x}}_i,\mathrm{u}_i,0)))
\end{aligned}$$

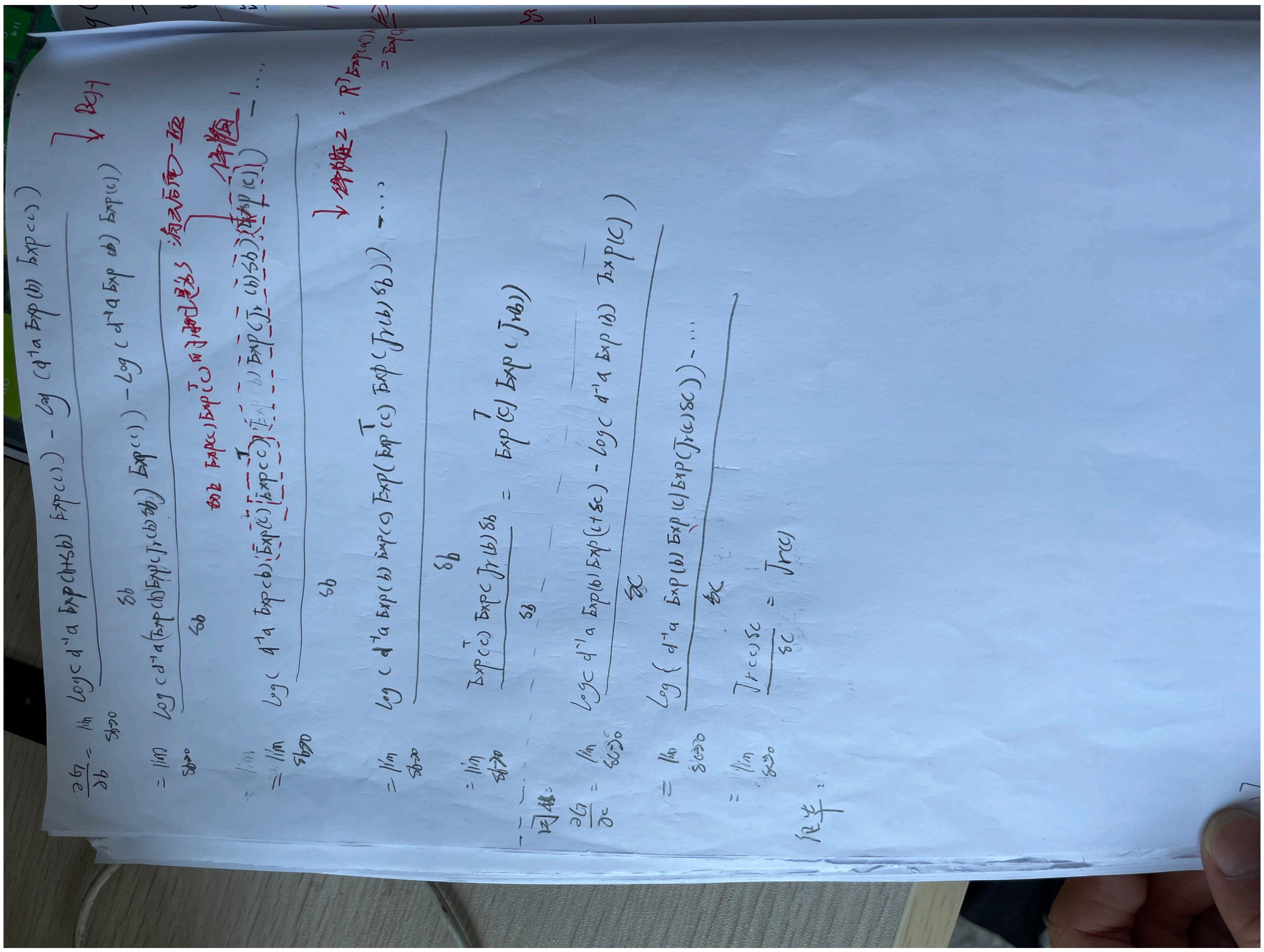

图 3-2 log 的推导

$$= \begin{bmatrix} \mathrm{Log}\left(\left({}^{G}\hat{\mathrm{R}}_{I_i}\mathrm{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)\right)^{T}\cdot\left({}^{G}\hat{\mathrm{R}}_{I_i}\mathrm{Exp}\left({}^{G}\delta\mathrm{r}_{I_i}\right)\mathrm{Exp}\left(\boldsymbol{\omega}_i\Delta t\right)\right)\right) \\ {}^{G}\mathrm{p}_{I_i} +^{G} v_{I_i}\delta t - ({}^{G}\hat{p}_{I_i} +^{G} v_i\delta t) \\ {}^{I}\delta\mathrm{r}_{C_i} \\ {}^{I}\delta\mathrm{p}_{C_i} \\ {}^{G}v_{I_i} =^{G} R_{I_i}a_i\Delta t -^{G} \hat{R}_{I_i}\hat{a}_i\Delta t \\ \mathrm{b}_{g_i} + \mathrm{n}_{\mathrm{bg}_i}\Delta t - \hat{b}_{g_i} \\ \mathrm{b}_{a_i} + \mathrm{n}_{\mathrm{ba}_i} - \hat{b}_{a_i} \end{bmatrix}$$

对于

1. ${}^{G}R_{I_i} =^{G} \hat{R}_{I_i}Exp({}^{G}\delta r_{I_i})$，为名义值和扰动值的差值
2. $\delta^{G}\hat{r}_{I_i} = Log({}^{G}\hat{R}_{I_i}{}^{T}\,{}^{G}R_{I_i})$
3. $\delta^{I}\hat{r}_{C_i} = Log({}^{I}\hat{R}_{C_i}{}^{T}\,{}^{I}R_{C_i})$

其中：

$$\hat{\boldsymbol{\omega}}_i = \boldsymbol{\omega}_{m_i} - \mathrm{b}_{\mathrm{g}_i},\ \boldsymbol{\omega}_i = \hat{\boldsymbol{\omega}}_i - \delta\mathrm{b}_{\mathrm{g}_i} - \mathrm{n}_{\mathrm{g}_i} \tag{3-28}$$

$$\hat{\mathrm{a}}_i = \mathrm{a}_{m_i} - \mathrm{b}_{\mathrm{a}_i},\ \mathrm{a}_i = \hat{\mathrm{a}}_i - \delta\mathrm{b}_{\mathrm{a}_i} - \mathrm{n}_{\mathrm{a}_i} \tag{3-29}$$

最终我们可以得到最终形式：

$$\begin{aligned} &\mathrm{Log}\left(\left({}^{G}\hat{\mathrm{R}}_{I_i}\mathrm{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)\right)^{T}\cdot\left({}^{G}\hat{\mathrm{R}}_{I_i}\mathrm{Exp}\left({}^{G}\delta\mathrm{r}_{I_i}\right)\mathrm{Exp}\left(\boldsymbol{\omega}_i\Delta t\right)\right)\right) \\ =&\mathrm{Log}\left(\mathrm{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)^{T}\cdot\left(\mathrm{Exp}\left({}^{G}\delta\mathrm{r}_{I_i}\right)\cdot\mathrm{Exp}\left(\boldsymbol{\omega}_i\Delta t\right)\right)\right) \\ \approx&\mathrm{Log}\left(\mathrm{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)^{T}\mathrm{Exp}\left({}^{G}\delta\mathrm{r}_{I_i}\right)\mathrm{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)\cdot\right. \\ &\qquad \mathrm{Exp}\left(-\mathrm{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)\left(\delta\mathrm{b}_{g_i} + \mathrm{n}_{\mathrm{g}_i}\right)\right)) \\ \approx&\mathrm{Exp}\left(\hat{\boldsymbol{\omega}}_i\Delta t\right)\cdot{}^{G}\delta\mathrm{r}_{I_i} - \mathrm{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)^{T}\delta\mathrm{b}_{\mathrm{g}_i} - \mathrm{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)^{T}\mathrm{n}_{\mathrm{g}_i} \\ &\left({}^{G}\mathrm{R}_{I_i}\mathrm{Exp}\left({}^{G}\delta\mathrm{r}_{I_i}\right)\right)\mathrm{a}_i\Delta t \\ \approx&\left({}^{G}\mathrm{R}_{I_i}\left(\mathrm{I} + [{}^{G}\delta\mathrm{r}_{I_i}]_{\times}\right)\right)\left(\hat{\mathrm{a}}_i - \delta\mathrm{b}_{\mathrm{a}_i} - \mathrm{n}_{\mathrm{a}_i}\right)\Delta t \\ \approx&{}^{G}\mathrm{R}_{I_i}\hat{\mathrm{a}}_i\Delta t -^{G}\mathrm{R}_{I_i}\delta\mathrm{b}_{\mathrm{a}_i}\Delta t -^{G}\mathrm{R}_{I_i}\mathrm{n}_{\mathrm{a}_i}\Delta t - {}^{G}\mathrm{R}_{I_i}\left[\hat{\mathrm{a}}_i\right]_{\times}{}^{G}\delta\mathrm{r}_{I_i} \end{aligned}$$

最终，我们可以的得到：$\mathrm{F}_{\delta\mathrm{x}}$ 和 F_{w} 如下：

$$\begin{aligned} &\mathrm{F}_{\delta\hat{\mathrm{x}}} = \left.\frac{\partial\left(\delta\hat{\mathrm{x}}_{i+1}\right)}{\partial\delta\hat{\mathrm{x}}_i}\right|_{\delta\hat{\mathrm{x}}_i=0,\mathrm{w}_i=0} \\ &= \begin{bmatrix} \mathrm{Exp}(-\hat{\boldsymbol{\omega}}_i\Delta t) & 0 & 0 & 0 & 0 & -\mathrm{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)^{T} & 0 \\ 0 & \mathrm{I} & 0 & 0 & \mathrm{I}\Delta t & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{I} & 0 & 0 & 0 \\ -{}^{G}\hat{\mathrm{R}}_{I_i}[\hat{\mathrm{a}}_i]_{\times}\Delta t & 0 & 0 & 0 & \mathrm{I} & 0 & -{}^{G}\hat{\mathrm{R}}_{I_i}\Delta t \\ 0 & 0 & 0 & 0 & 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
\mathrm{F_w} &= \left.\frac{\partial\left(\delta\hat{\mathrm{x}}_{i+1}\right)}{\partial \mathrm{w}_i}\right|_{\delta\hat{\mathrm{x}}_i=0,\mathrm{w}_i=0} \\
&= \begin{bmatrix} -\mathrm{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -{}^G\hat{\mathrm{R}}_{I_i}\Delta t & 0 & 0 \\ 0 & 0 & \mathrm{I}\Delta t & 0 \\ 0 & 0 & 0 & \mathrm{I}\Delta t \end{bmatrix}
\end{aligned}
$$

3.3 先验分布和迭代初始化

让传播和协方差传播在第 $(k+1)$-th 的雷达/相机测量中停止 (可以看到 Fig.3-3), 传播的状态量和协方差分别是 $\hat{\mathrm{x}}_{k+1}$ and $\boldsymbol{\Sigma}_{\delta\hat{\mathrm{x}}_{k+1}}$, 在状态 x_{k+1} 在融合 $(k+1)$-th 测量前施加了先验分布如下:

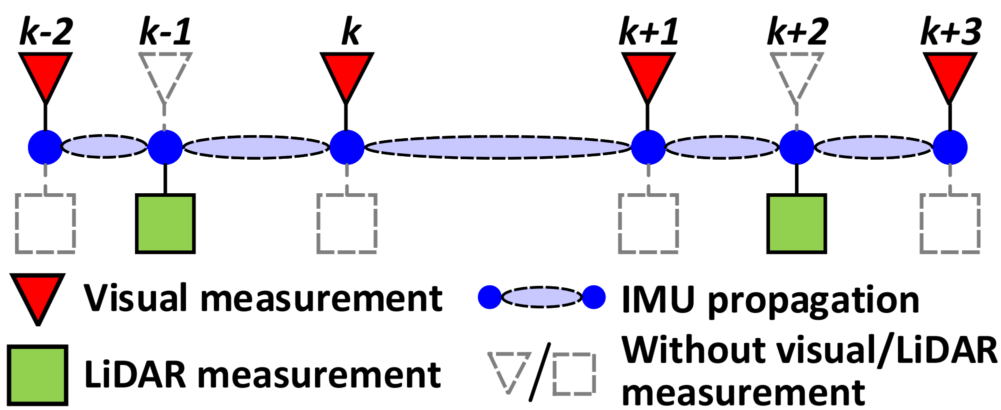

图 3-3 误差迭代卡尔曼滤波的更新

3.3.1 先验分布的公式

$$
\delta\hat{x}_{k+1} = \mathrm{x}_{k+1} \boxminus \hat{\mathrm{x}}_{k+1} \sim \mathcal{N}(0, \boldsymbol{\Sigma}_{\delta\hat{\mathrm{x}}_{k+1}}). \tag{3-30}
$$

3.3.2 迭代初始化

在 (3-30) 中的先验分布将会融合雷达或相机估计来构建 MAP 估计 (称之为 $\check{\mathrm{x}}_{k+1}$)。最大后验估计 MAP$\check{\mathrm{x}}_{k+1}$ 由先验状态 $\hat{\mathrm{x}}_{k+1}$ 初始化, 而且被不断迭代更新, 在每一次迭代中, 误差状态量 $\delta\check{\mathrm{x}}_{k+1}$, 真实状态量 x_{k+1} 和估计状态量 $\check{\mathrm{x}}_{k+1}$ 被定义为:

$$
\delta\check{\mathrm{x}}_{k+1} \triangleq \mathrm{x}_{k+1} \boxminus \check{\mathrm{x}}_{k+1}, \tag{3-31}
$$

它将会被 MAP(3-30) 和雷达/视觉观测中最小化, 因此, 由先验分布 (3-30) 表示的 x_{k+1} 应转换为 $\delta\check{\mathrm{x}}_{k+1}$ 的等效先验分布:

$$
\begin{aligned}
\delta x_{k+1} = \mathrm{x}_{k+1} \boxminus \hat{\mathrm{x}}_{k+1} &= (\check{\mathrm{x}}_{k+1} \boxplus \delta\check{\mathrm{x}}_{k+1}) \boxminus \hat{\mathrm{x}}_{k+1} \\
&\approx \check{\mathrm{x}}_{k+1} \boxminus \hat{\mathrm{x}}_{k+1} + \boldsymbol{\mathcal{H}}\delta\check{\mathrm{x}}_{k+1} \\
&\sim \mathcal{N}(0, \boldsymbol{\Sigma}_{\delta\hat{\mathrm{x}}_{k+1}}),
\end{aligned} \tag{3-32}
$$

其中

$$
\boldsymbol{\mathcal{H}} = \frac{(\check{\mathrm{x}}_{k+1} \boxplus \delta\check{\mathrm{x}}_{k+1}) \boxminus \hat{\mathrm{x}}_{k+1}}{\partial\delta\check{\mathrm{x}}_{k+1}}|_{\delta\check{\mathrm{x}}_{k+1}=0}
$$

在后面详细推导。实际上 (3-36) 对 $\delta\check{\mathrm{x}}_{k+1}$ 施加了先验分布：

$$
\delta\check{\mathrm{x}}_{k+1} \sim \mathcal{N}(-\boldsymbol{\mathcal{H}}^{-1}\left(\check{\mathrm{x}}_{k+1} \boxminus \hat{\mathrm{x}}_{k+1}\right), \boldsymbol{\mathcal{H}}^{-1}\boldsymbol{\Sigma}_{\delta\hat{\mathrm{x}}_{k+1}}\boldsymbol{\mathcal{H}}^{-T}) \tag{3-33}
$$

注意 2 个公式的区别第 1 个代表传播值，第 2 个代表迭代值，第 3 个代表迭代值和传播值的关系

- $$x_{i+1} = \hat{x}_{i+1} \boxplus \delta\hat{x}_{i+1} \tag{3-34}$$

- $$x_{i+1} = \check{x}_{i+1} \boxplus \delta\check{x}_{i+1} \tag{3-35}$$

- $$
\begin{aligned}
\delta x_{k+1} = \mathrm{x}_{k+1} \boxminus \hat{\mathrm{x}}_{k+1} &= (\check{\mathrm{x}}_{k+1} \boxplus \delta\check{\mathrm{x}}_{k+1}) \boxminus \hat{\mathrm{x}}_{k+1} \\
&\approx \check{\mathrm{x}}_{k+1} \boxminus \hat{\mathrm{x}}_{k+1} + \boldsymbol{\mathcal{H}}\delta\check{\mathrm{x}}_{k+1} \\
&\sim \mathcal{N}(0, \boldsymbol{\Sigma}_{\delta\hat{\mathrm{x}}_{k+1}}),
\end{aligned} \tag{3-36}
$$

下面来详细推导 $\mathcal{H}$:

$$
\begin{aligned}
\boldsymbol{\mathcal{H}} &= \frac{(\check{\mathrm{x}}_{k+1} \boxplus \delta\check{\mathrm{x}}_{k+1}) \boxminus \hat{\mathrm{x}}_{k+1}}{\partial\delta\check{\mathrm{x}}_{k+1}}|_{\delta\check{\mathrm{x}}_{k+1}=0} \\
&= \begin{bmatrix}
\mathrm{A} & 0 & 0 & 0 & 0_{3\times9} \\
0 & \mathrm{I} & 0 & 0 & 0_{3\times9} \\
0 & 0 & \mathrm{B} & 0 & 0_{3\times9} \\
0 & 0 & 0 & \mathrm{I} & 0_{3\times9} \\
0 & 0 & 0 & 0 & \mathrm{I}_{9\times9}
\end{bmatrix}
\end{aligned}
$$

A 和 B 分别为 $\mathrm{A} = \mathrm{J}_r^{-1}(\mathrm{Log}({}^G\hat{\mathrm{R}}_{I_{k+1}}{}^T\,{}^G\check{\mathrm{R}}_{I_{k+1}}))$ 和 $\mathrm{B} = \mathrm{J}_r^{-1}(\mathrm{Log}({}^I\hat{\mathrm{R}}_{C_{k+1}}{}^T\,{}^I\check{\mathrm{R}}_{C_{k+1}}))$. 省略下标，$\mathcal{H}$ 的详细

推导：

$$H = \check{\mathrm{x}}_{k+1} \boxplus \delta\check{\mathrm{x}}_{k+1} \boxminus \hat{x}_{k+1} \tag{3-37}$$

$$= \begin{bmatrix} Log({}^G\hat{R}_I^{T\,G}\check{R}_I Exp(\delta^G\check{r}_I)) \\ {}^G\check{p}_I + \delta\check{p}_I -{}^G \hat{p}_I \\ Log({}^I\hat{R}_I^{T\,I}\check{R}_C Exp(\delta^G\check{r}_C)) \\ {}^G\check{p}_I +{}^G \delta\check{p}_I -{}^G \hat{p}_I \\ {}^G\check{v}_I +{}^G \delta\check{v}_I -{}^G \hat{v}_I \\ \check{b}_g + \delta\check{b}_g - \hat{b}_g \\ \check{b}_a + \delta\check{b}_a - \hat{b}_a \end{bmatrix} \tag{3-38}$$

下面参考我的推导过程3-4：

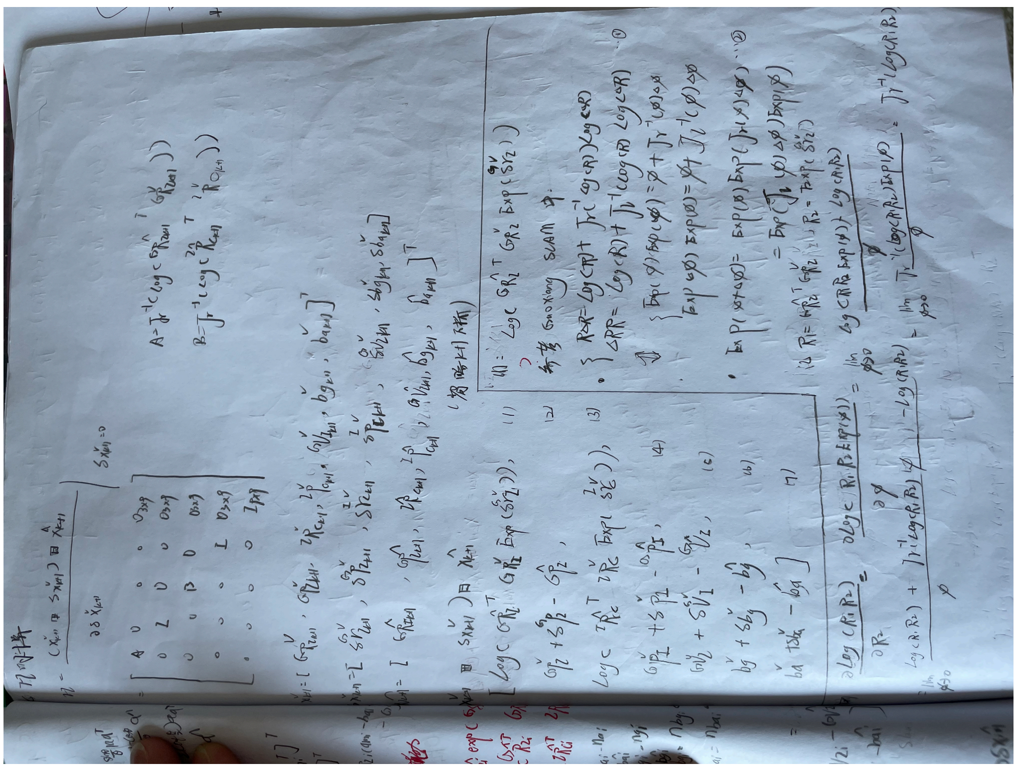

图 3-4 H 矩阵的推导

3.4 雷达残差推导

如果第 $(k+1)$ 是雷达帧，那么我们原始点集合 $\{{}^Gp_i\}$ 中找到相对于 ${}^Gp_{L_j}$ 的相邻 5 个点，然后在两个雷达时间间隔内对所有的雷达点 ${}^Lp_j$ 进行运动补偿，最后计算 $\{{}^Gp_i\}$ 到邻近平面的点面残差

$\check{\mathrm{x}}_{k+1}$ 是当前帧 x_{k+1} 的估计，我们可以用 ${}^L\mathrm{p}_j$ 从雷达坐标系转换为世界坐标系：${}^G\mathrm{p}_j = {}^G\check{\mathrm{R}}_{I_{k+1}}({}^I\mathrm{R}_L{}^L\mathrm{p}_j + {}^I\mathrm{p}_L) + {}^G\check{\mathrm{p}}_{I_{k+1}}$. 这是点面残差的定义：

$$\mathrm{r}_l(\check{\mathrm{x}}_{k+1}, {}^L\mathrm{p}_j) = \mathrm{u}_j^T\left({}^G\mathrm{p}_j - \mathrm{q}_j\right) \tag{3-39}$$

n_j 是测量的噪声 ${}^L\mathrm{p}_j$, 我们可以给通过补偿噪声 ${}^L\mathrm{p}_j$ 来得到真实的状态 ${}^L\mathrm{p}_j^{\mathrm{gt}}$:

$$ {}^L\mathrm{p}_j = {}^L\mathrm{p}_j^{\mathrm{gt}} + \mathrm{n}_j, \mathrm{n}_j \sim \mathcal{N}(0, \mathbf{\Sigma}_{\mathrm{n}_j}). \tag{3-40}$$

这个真实的点在拟合平面上的残差应该为 0:

$$0 = \mathrm{r}_l(\mathrm{x}_{k+1}, {}^L\mathrm{p}_j^{\mathrm{gt}}) = \mathrm{r}_l(\check{\mathrm{x}}_{k+1}, {}^L\mathrm{p}_j) + \mathrm{H}_j^l \delta\check{\mathrm{x}}_{k+1} + \boldsymbol{\alpha}_j, \tag{3-41}$$

它包括 $\delta\check{\mathrm{x}}_{k+1}$ 的先验分布，在 (3-41)。在 (3-41), x_{k+1} 由它的误差 $\delta\check{\mathrm{x}}_{k+1}$ 所定义，$\boldsymbol{\alpha}_j \sim \mathcal{N}(0, \mathbf{\Sigma}_{\boldsymbol{\alpha}_j})$:

$$\begin{aligned}
\mathrm{H}_j^l =& \frac{\partial \mathrm{r}_l(\check{\mathrm{x}}_{k+1} \boxplus \delta\check{\mathrm{x}}_{k+1}, {}^L\mathrm{p}_j)}{\partial \delta\check{\mathrm{x}}_{k+1}}\Big|_{\delta\check{\mathrm{x}}_{k+1}=0} \\
\mathbf{\Sigma}_{\boldsymbol{\alpha}_j} =& \mathrm{F}_{\mathrm{p}_j} \mathbf{\Sigma}_{\mathrm{n}_j} \mathrm{F}_{\mathrm{p}_j}^T \\
\mathrm{F}_{\mathrm{p}_j} =& \left(\frac{\partial \mathrm{r}_l(\check{\mathrm{x}}_{k+1}, {}^L\mathrm{p}_j)}{\partial {}^L\mathrm{p}_j} \right) = {}^G\check{\mathrm{R}}_{I_{k+1}} {}^I\mathrm{R}_L
\end{aligned} \tag{3-42}$$

H_j^l 的结果如下:

$$\mathrm{H}_j^l = \mathrm{u}_j^T \begin{bmatrix} -{}^G\check{\mathrm{R}}_{I_{k+1}} [\mathrm{P_a}]_\times & \mathrm{I}_{3\times3} & 0_{3\times15} \end{bmatrix}$$

其中 $\mathrm{P_a} = {}^I\mathrm{R}_L {}^L\mathrm{p}_j + {}^I\mathrm{p}_L$.

下面是对 H_j^l 的详细推导

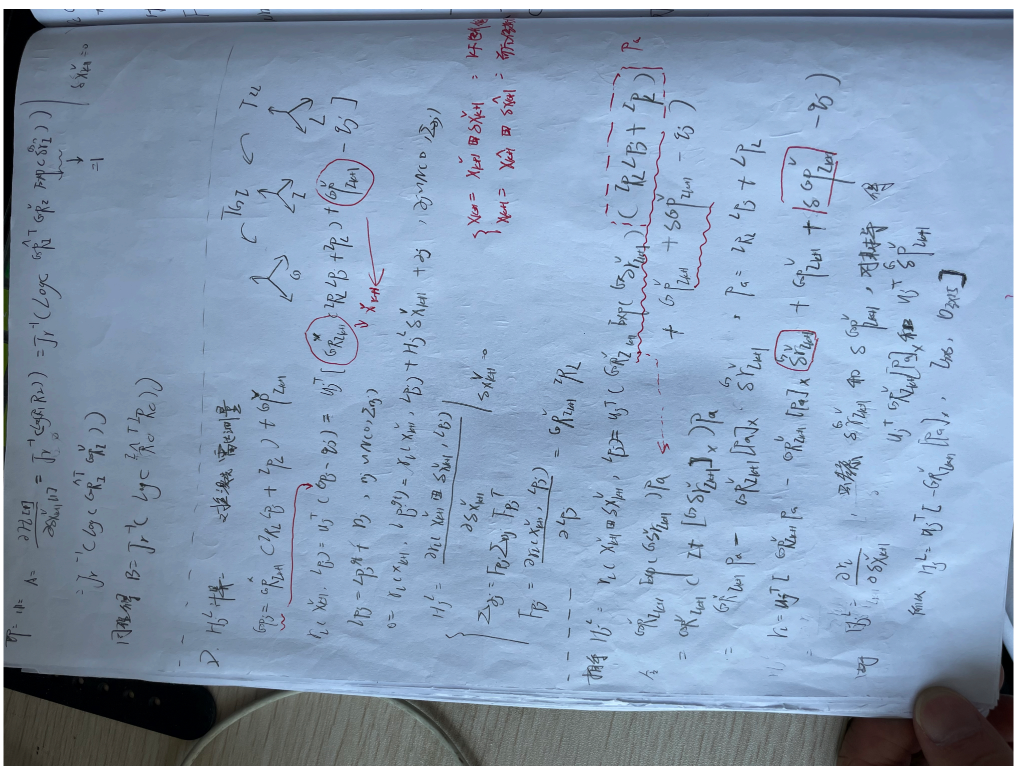

图 3-5 雷达残差推导

3.5 相机残差推导

如果第 k+1 帧是相机帧，我们从去畸变的图像 $\mathcal{C}_{k+1}$ 中获取 FAST 角点，在 $\mathcal{C}_{k+1}$ 利用 KLT 光流追踪方法追踪特征点，如在 $\mathcal{C}_{k+1}$ 的特这个特征点丢失或者没有被追踪，我们用新的估计位姿来三角化新的特征点在三维空间中。视觉 landmark 和特征点的重投影误差在第 k+1 帧中被用来更新当前的估计状态 $\check{\mathrm{x}}_{k+1}$。对于一个特征点 ${}^{C}\mathrm{p}_s = \begin{bmatrix} u_s & v_s \end{bmatrix}^T \in \mathcal{C}_{k+1}$ 其中 s 是特征点的索引, 他在 3D 空间中对应的 landmark 是 ${}^{G}\mathrm{P}_s$, 视觉观测残差 ${}^{C}\mathrm{p}_s$ 是:

$$
\begin{aligned}
{}^{C}\mathrm{P}_s &= \left({}^{G}\check{\mathrm{R}}_{I_{k+1}} {}^{I}\check{\mathrm{R}}_{C_{k+1}}\right)^T {}^{G}\mathrm{P}_s \\
&\qquad - \left({}^{I}\check{\mathrm{R}}_{C_{k+1}}\right)^T {}^{G}\check{\mathrm{p}}_{I_{k+1}} - {}^{I}\check{\mathrm{p}}_{C_{k+1}} \\
\mathrm{r}_c\left(\check{\mathrm{x}}_{k+1}, {}^{C}\mathrm{p}_s, {}^{G}\mathrm{P}_s\right) &= {}^{C}\mathrm{p}_s - \boldsymbol{\pi}({}^{C}\mathrm{P}_s)
\end{aligned}
\tag{3-43}
$$

$\boldsymbol{\pi}(\cdot)$ 是针孔投影模型. 现在考虑观测噪声，我们得到:

$$
{}^{G}\mathrm{P}_s = {}^{G}\mathrm{P}_s^{\mathrm{gt}} + \mathrm{n}_{\mathrm{P}_s},\ \mathrm{n}_{\mathrm{P}_s} \sim \mathcal{N}(0, \boldsymbol{\Sigma}_{\mathrm{n}_{\mathrm{P}_s}}) \tag{3-44}
$$

$$
{}^{C}\mathrm{p}_s = {}^{C}\mathrm{p}_s^{\mathrm{gt}} + \mathrm{n}_{\mathrm{p}_s},\ \mathrm{n}_{\mathrm{p}_s} \sim \mathcal{N}(0, \boldsymbol{\Sigma}_{\mathrm{n}_{\mathrm{p}_s}}) \tag{3-45}
$$

其中 ${}^{G}\mathrm{P}_s^{\mathrm{gt}}$ 和 ${}^{C}\mathrm{p}_s^{\mathrm{gt}}$ 是 ${}^{G}\mathrm{P}_s$ 和 ${}^{C}\mathrm{p}_s$ 的真实值, 与此同时,我们得到了视觉残差的一阶泰勒展开 $\mathrm{r}_c(\mathrm{x}_{k+1}, {}^{C}\mathrm{p}_s^{\mathrm{gt}})$ as:

$$
\begin{aligned}
0 &= \mathrm{r}_c(\mathrm{x}_{k+1}, {}^{C}\mathrm{p}_s^{\mathrm{gt}}, {}^{G}\mathrm{P}_s^{\mathrm{gt}}) \\
&\approx \mathrm{r}_c\left(\check{\mathrm{x}}_{k+1}, {}^{C}\mathrm{p}_s, {}^{G}\mathrm{P}_s\right) + \mathrm{H}_s^c \delta\check{\mathrm{x}}_{k+1} + \boldsymbol{\beta}_s,
\end{aligned}
\tag{3-46}
$$

它包括 $\delta\check{\mathrm{x}}_{k+1}$ 的先验分布. 在 (3-46), $\boldsymbol{\beta}_s \sim \mathcal{N}(0, \boldsymbol{\Sigma}_{\boldsymbol{\beta}_s})$:

$$
\begin{aligned}
\mathrm{H}_s^c &= \frac{\partial \mathrm{r}_c(\check{\mathrm{x}}_{k+1} \boxplus \delta\check{\mathrm{x}}_{k+1}, {}^{C}\mathrm{p}_s, {}^{G}\mathrm{P}_s)}{\partial \delta\check{\mathrm{x}}_{k+1}}|_{\delta\check{\mathrm{x}}_{k+1}=0} \\
\boldsymbol{\Sigma}_{\boldsymbol{\beta}_s} &= \boldsymbol{\Sigma}_{\mathrm{n}_{\mathrm{p}_s}} + \mathrm{F}_{\mathrm{P}_s} \boldsymbol{\Sigma}_{\mathrm{P}_s} \mathrm{F}_{\mathrm{P}_s}{}^T \\
\mathrm{F}_{\mathrm{P}_s} &= \frac{\partial \mathrm{r}_c(\check{\mathrm{x}}_{k+1}, {}^{C}\mathrm{p}_s, {}^{G}\mathrm{P}_s)}{\partial {}^{G}\mathrm{P}_s}
\end{aligned}
\tag{3-47}
$$

相机残差推导即是计算特征点的重投影误差，以下是我的推导过程3-6：

3.6 迭代更新部分

这里直接给我的推导部分，请看3-9

4 实验部分

todo...

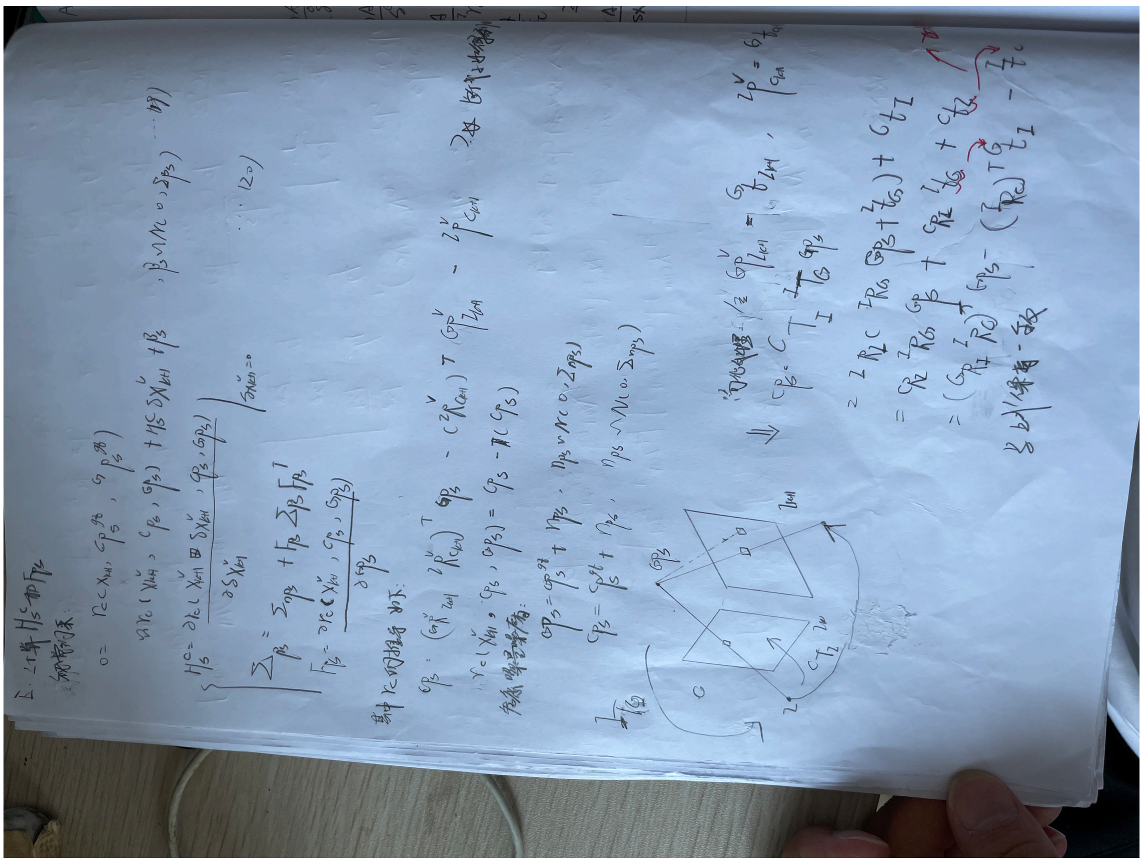

图 3-6 视觉残差 1

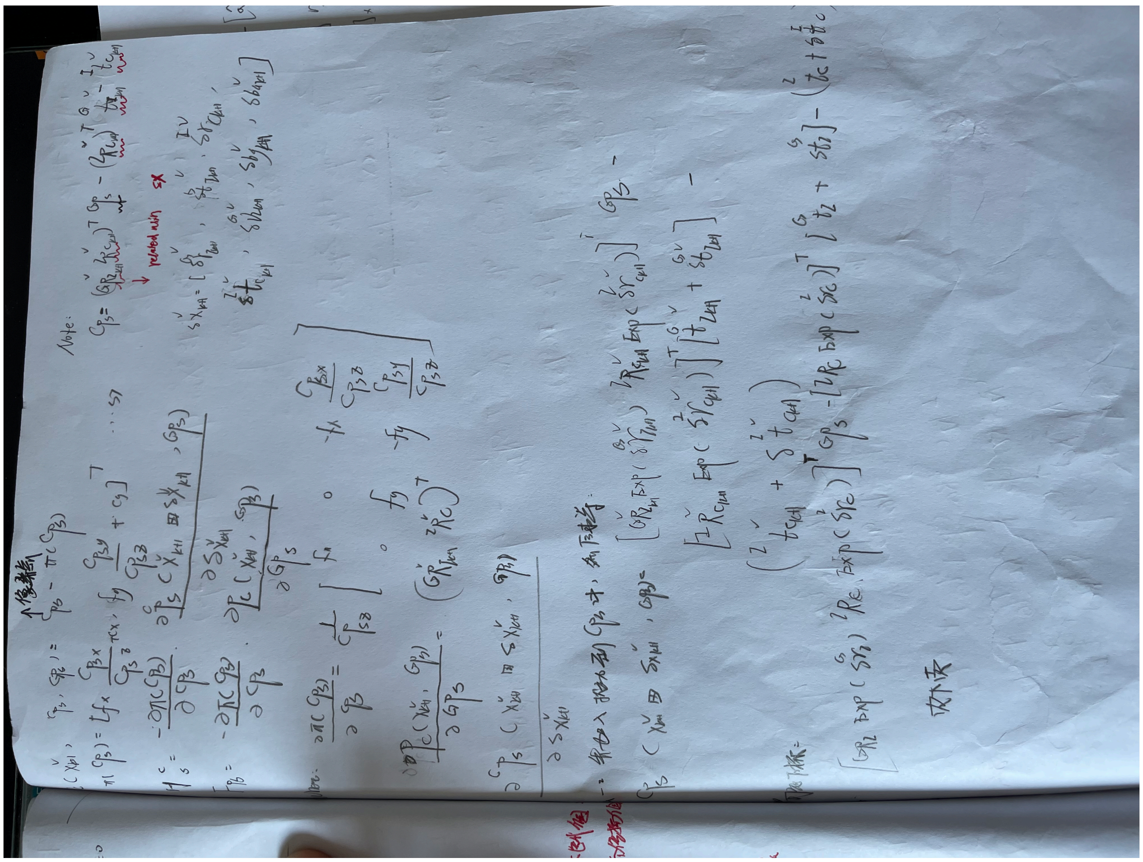

图 3-7 视觉残差 2

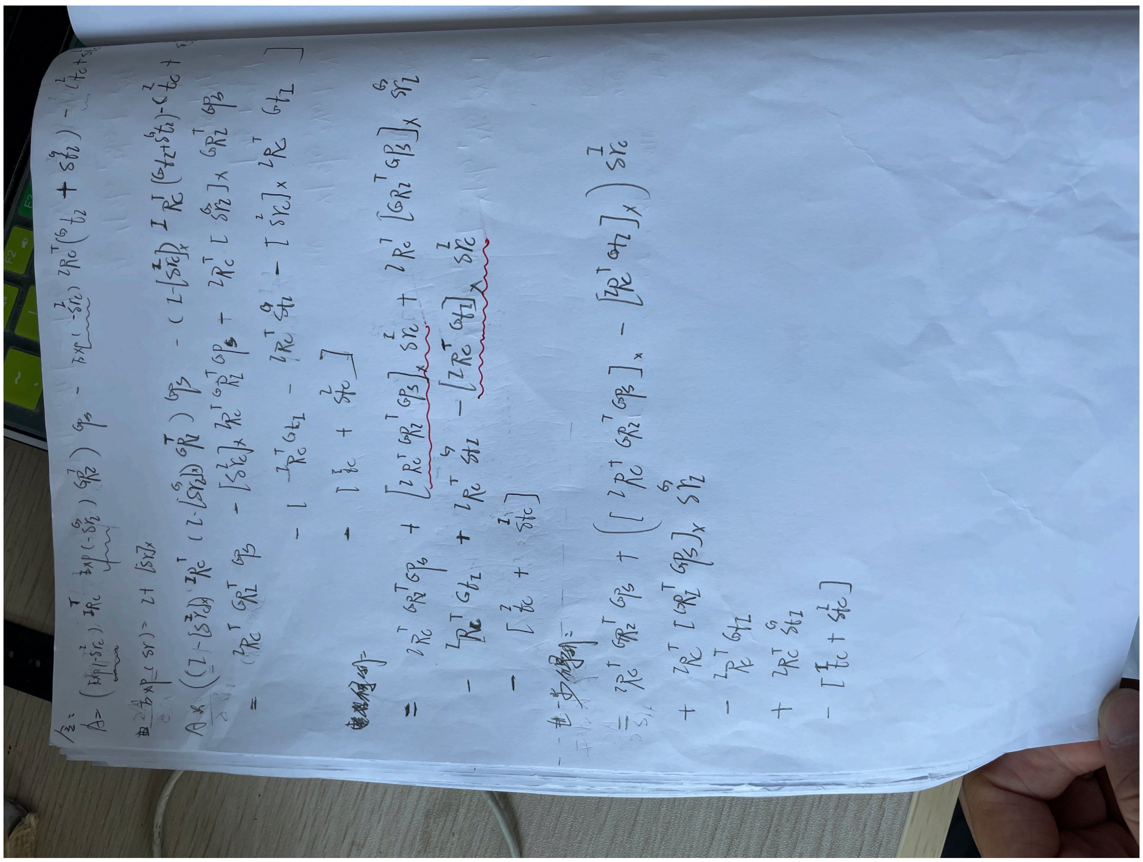

图 3-8 视觉残差 3

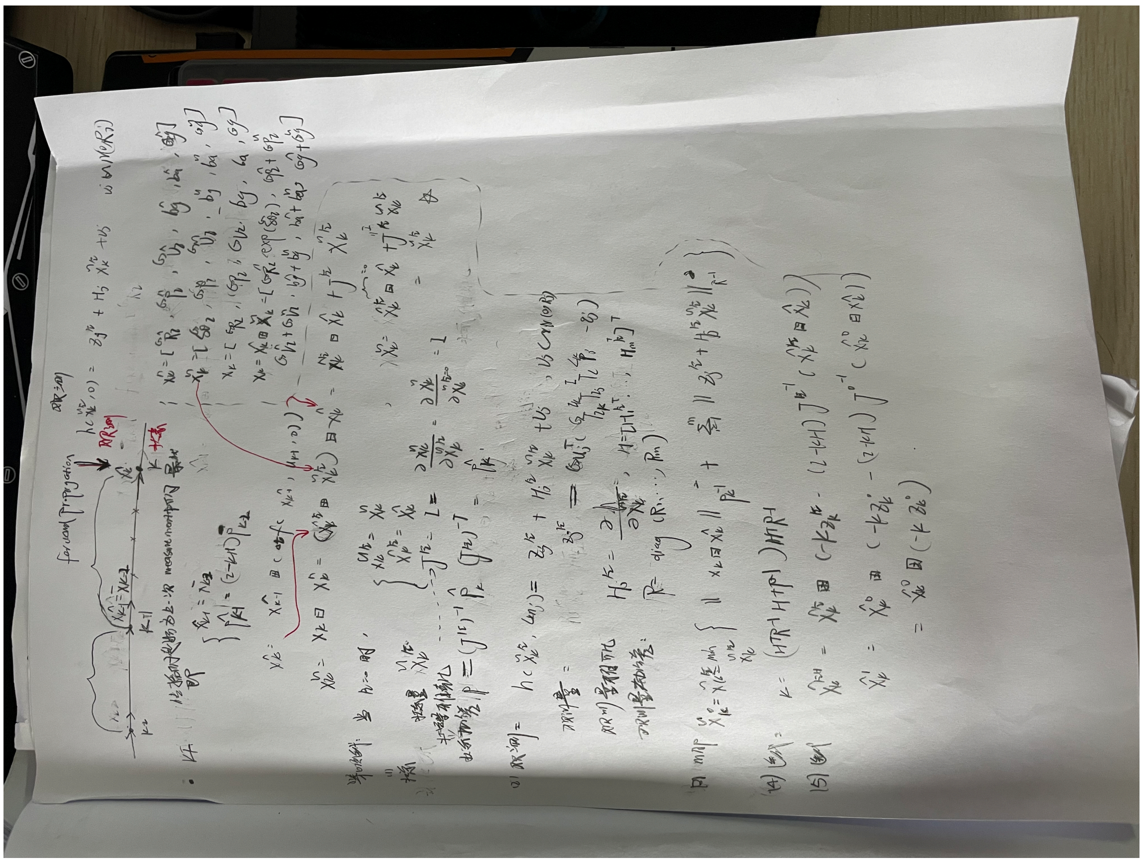

图 3-9 迭代更新 1

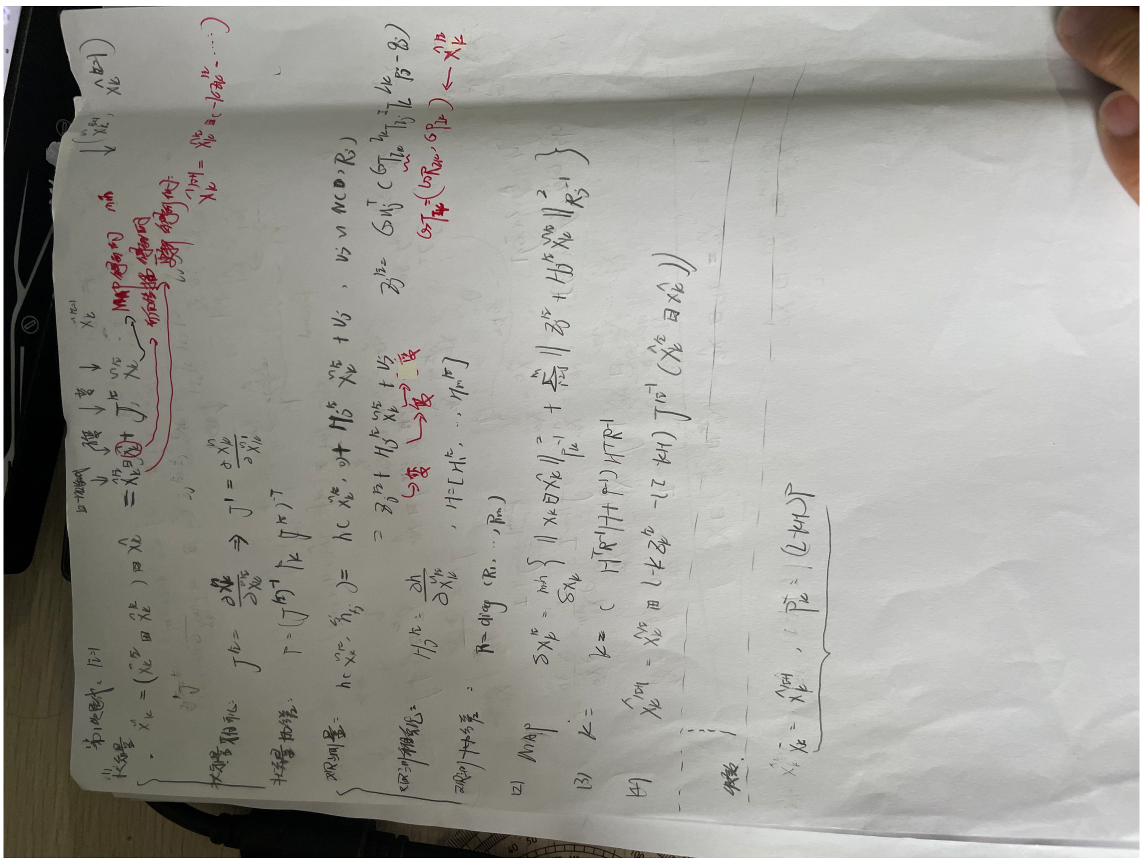

图 3-10 迭代更新 2

5 致谢

这个篇推导的目的是为了加深对公式的理解，总结一些个人理解，其次致敬徐杰，王泽霖博士，他们是我滤波 SLAM 的入门，我也希望能向前辈们一样优秀，为社区做贡献！

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