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## **Interdisciplinary Applied Mathematics**

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# An Invitation to 3-D Vision

## From Images to Geometric Models

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With 170 Illustrations



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*To my mother and my father (Y.M.)*

*To Giuseppe Torresin, Engineer (S.S.)*

*To my parents (J.K.)*

*To my mother (S.S.S.)*

# Preface

This book is intended to give students at the advanced undergraduate or introductory graduate level, and researchers in computer vision, robotics and computer graphics, a self-contained introduction to the geometry of three-dimensional (3-D) vision. This is the study of the reconstruction of 3-D models of objects from a collection of 2-D images. An essential prerequisite for this book is a course in linear algebra at the advanced undergraduate level. Background knowledge in rigid-body motion, estimation and optimization will certainly improve the reader's appreciation of the material but is not critical since the first few chapters and the appendices provide a review and summary of basic notions and results on these topics.

## *Our motivation*

Research monographs and books on geometric approaches to computer vision have been published recently in two batches: The first was in the mid 1990s with books on the geometry of two views, see e.g. [Faugeras, 1993, Kanatani, 1993b, Maybank, 1993, Weng et al., 1993b]. The second was more recent with books focusing on the geometry of multiple views, see e.g. [Hartley and Zisserman, 2000] and [Faugeras and Luong, 2001] as well as a more comprehensive book on computer vision [Forsyth and Ponce, 2002]. We felt that the time was ripe for synthesizing the material in a unified framework so as to provide a self-contained exposition of this subject, which can be used both for pedagogical purposes and by practitioners interested in this field. Although the approach we take in this book deviates from several other classical approaches, the techniques we use are mainly linear algebra and our book gives a comprehensive view of what is known

to date on the geometry of 3-D vision. It also develops homogeneous terminology on a solid analytical foundation to enable what should be a great deal of future research in this young field.

Apart from a self-contained treatment of geometry and algebra associated with computer vision, the book covers relevant aspects of the image formation process, basic image processing, and feature extraction techniques – essentially all that one needs to know in order to build a system that can automatically generate a 3-D model from a set of 2-D images.

### *Organization of the book*

This book is organized as follows: Following a brief introduction, Part I provides background material for the rest of the book. Two fundamental transformations in multiple-view geometry, namely, rigid-body motion and perspective projection, are introduced in Chapters 2 and 3, respectively. Feature extraction and correspondence are discussed in Chapter 4.

Chapters 5, 6, and 7, in Part II, cover the classic theory of two-view geometry based on the so-called epipolar constraint. Theory and algorithms are developed for both discrete and continuous motions, both general and planar scenes, both calibrated and uncalibrated camera models, and both single and multiple moving objects.

Although the epipolar constraint has been very successful in the two-view case, Part III shows that a more proper tool for studying the geometry of multiple views is the so-called *rank condition on the multiple-view matrix* (Chapter 8), which unifies all the constraints among multiple images that are known to date. The theory culminates in Chapter 9 with a unified theorem on a rank condition for arbitrarily mixed point, line, and plane features. It captures *all* possible constraints among multiple images of these geometric primitives, and serves as a key to both geometric analysis and algorithmic development. Chapter 10 uses the rank condition to reexamine and unify the study of single-view and multiple-view geometry given scene knowledge such as symmetry.

Based on the theory and conceptual algorithms developed in the early part of the book, Chapters 11 and 12, in Part IV, demonstrate practical reconstruction algorithms step-by-step, as well as discuss possible extensions of the theory covered in this book. An outline of the logical dependency among chapters is given in Figure 1.

### *Curriculum options*

Drafts of this book and the exercises in it have been used to teach a one-semester course at the University of California at Berkeley, the University of Illinois at Urbana-Champaign, Washington University in St. Louis, the George Mason University and the University of Pennsylvania, and a one-quarter course at the University of California at Los Angeles. There is apparently adequate material for two semesters or three quarters of lectures. Advanced topics suggested in Part IV or chosen by the instructor can be added to the second half of the sec-

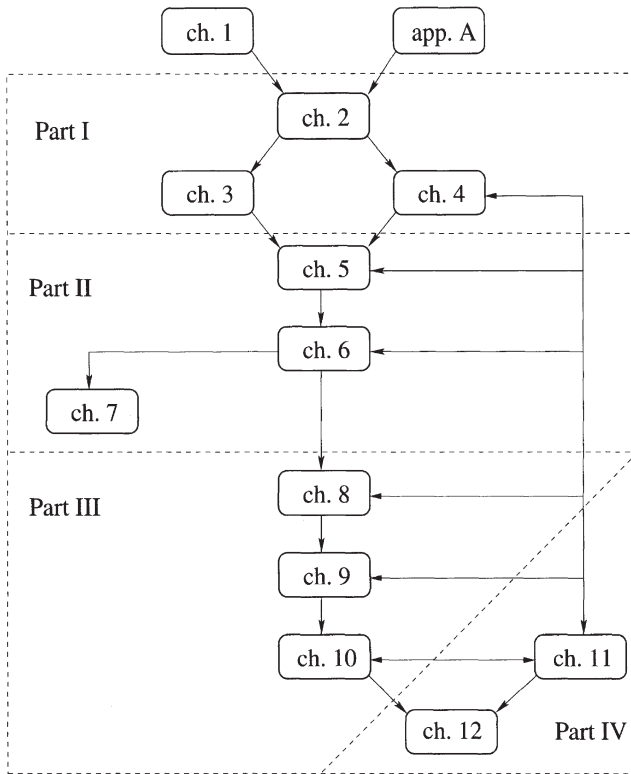


Figure 1. Organization of the book: logical dependency among parts and chapters.

ond semester if a two-semester course is offered. Below are some suggestions for course development based on this book:

1. *A one-semester course:* Appendix A, Chapters 1–6, and part of Chapters 8–10.
2. *A two-quarter course:* Chapters 1–6 for the first quarter, and Chapters 8–10, 12 for the second quarter.
3. *A two-semester course:* Appendix A and Chapters 1–6 for the first semester; Chapters 7–10 and the instructor's choice of some advanced topics from Chapter 12 for the second semester.
4. *A three-quarter sequence:* Chapters 1–6 for the first quarter, Chapters 7–10 for the second quarter, and the instructor's choice of advanced topics and projects from Chapters 11 and 12 for the third quarter.

Chapter 11 plays a special role in this book: Its purpose is to make it easy for the instructor to develop and assign experimental exercises or course projects along with other chapters being taught throughout the course. Relevant code is available



at <http://vision.ucla.edu/MASKS>, from which students may get hands-on experience with a minimum version of a working computer vision system. This chapter can also be used by practitioners who are interested in using the algorithms developed in this book, without necessarily delving into the details of the mathematical formulation. Finally, an additional purpose of this chapter is to summarize “the book in one chapter,” which can be used in the first lecture as an overview of what is to come.

Exercises are provided at the end of each chapter. They consist of mainly three types:

1. *drill exercises* that help students understand the theory covered in each chapter;
2. *advanced exercises* that guide students to creatively develop a solution to a specialized case that is related to but not necessarily covered by the general theorems in the book;
3. *programming exercises* that help students grasp the algorithms developed in each chapter.

Solutions to selected exercises are available, along with software for examples and algorithms, at <http://vision.ucla.edu/MASKS>.

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Stefano Soatto, Los Angeles, California  
Jana Košecká, Fairfax, Virginia  
Shankar Sastry, Berkeley, California  
Spring, 2003

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Some of the seeds of this work go back to the early nineties, when Ruggero Frezza, Pietro Perona, and Giorgio Picci started looking at the problem of structure from motion within the context of systems and controls. They taught a precursor of this material in a course at the University of Padova in 1991. Appendix B was inspired by the beautiful lectures of Giorgio Picci. We owe them our sincere gratitude for their vision and for their initial efforts that sparked our interest in the field.

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# Contents

<b>Preface</b>	<b>vii</b>
<b>Acknowledgments</b>	<b>xi</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Visual perception from 2-D images to 3-D models . . . . .	1
1.2 A mathematical approach . . . . .	8
1.3 A historical perspective . . . . .	9
<b>I Introductory Material</b>	<b>13</b>
<b>2 Representation of a Three-Dimensional Moving Scene</b>	<b>15</b>
2.1 Three-dimensional Euclidean space . . . . .	16
2.2 Rigid-body motion . . . . .	19
2.3 Rotational motion and its representations . . . . .	22
2.3.1 Orthogonal matrix representation of rotations . . . . .	22
2.3.2 Canonical exponential coordinates for rotations . . . . .	25
2.4 Rigid-body motion and its representations . . . . .	28
2.4.1 Homogeneous representation . . . . .	29
2.4.2 Canonical exponential coordinates for rigid-body motions . . . . .	31
2.5 Coordinate and velocity transformations . . . . .	34
2.6 Summary . . . . .	37

2.7	Exercises . . . . .	38
2.A	Quaternions and Euler angles for rotations . . . . .	40
<b>3</b>	<b>Image Formation</b>	<b>44</b>
3.1	Representation of images . . . . .	46
3.2	Lenses, light, and basic photometry . . . . .	47
3.2.1	Imaging through lenses . . . . .	48
3.2.2	Imaging through a pinhole . . . . .	49
3.3	A geometric model of image formation . . . . .	51
3.3.1	An ideal perspective camera . . . . .	52
3.3.2	Camera with intrinsic parameters . . . . .	53
3.3.3	Radial distortion . . . . .	58
3.3.4	Image, preimage, and coimage of points and lines . . . . .	59
3.4	Summary . . . . .	62
3.5	Exercises . . . . .	62
3.A	Basic photometry with light sources and surfaces . . . . .	65
3.B	Image formation in the language of projective geometry . . . . .	70
<b>4</b>	<b>Image Primitives and Correspondence</b>	<b>75</b>
4.1	Correspondence of geometric features . . . . .	76
4.1.1	From photometric features to geometric primitives . . . . .	77
4.1.2	Local vs. global image deformations . . . . .	78
4.2	Local deformation models . . . . .	80
4.2.1	Transformations of the image domain . . . . .	80
4.2.2	Transformations of the intensity value . . . . .	82
4.3	Matching point features . . . . .	82
4.3.1	Small baseline: feature tracking and optical flow . . . . .	84
4.3.2	Large baseline: affine model and normalized cross-correlation . . . . .	88
4.3.3	Point feature selection . . . . .	90
4.4	Tracking line features . . . . .	92
4.4.1	Edge features and edge detection . . . . .	93
4.4.2	Composition of edge elements: line fitting . . . . .	94
4.4.3	Tracking and matching line segments . . . . .	95
4.5	Summary . . . . .	96
4.6	Exercises . . . . .	97
4.A	Computing image gradients . . . . .	99
<b>II</b>	<b>Geometry of Two Views</b>	<b>107</b>
<b>5</b>	<b>Reconstruction from Two Calibrated Views</b>	<b>109</b>
5.1	Epipolar geometry . . . . .	110
5.1.1	The epipolar constraint and the essential matrix . . . . .	110
5.1.2	Elementary properties of the essential matrix . . . . .	113

5.2	Basic reconstruction algorithms . . . . .	117
5.2.1	The eight-point linear algorithm . . . . .	117
5.2.2	Euclidean constraints and structure reconstruction . .	124
5.2.3	Optimal pose and structure . . . . .	125
5.3	Planar scenes and homography . . . . .	131
5.3.1	Planar homography . . . . .	131
5.3.2	Estimating the planar homography matrix . . . . .	134
5.3.3	Decomposing the planar homography matrix . . . . .	136
5.3.4	Relationships between the homography and the essential matrix . . . . .	139
5.4	Continuous motion case . . . . .	142
5.4.1	Continuous epipolar constraint and the continuous essential matrix . . . . .	142
5.4.2	Properties of the continuous essential matrix . . . . .	144
5.4.3	The eight-point linear algorithm . . . . .	148
5.4.4	Euclidean constraints and structure reconstruction . .	152
5.4.5	Continuous homography for a planar scene . . . . .	154
5.4.6	Estimating the continuous homography matrix . . . . .	155
5.4.7	Decomposing the continuous homography matrix . .	157
5.5	Summary . . . . .	158
5.6	Exercises . . . . .	159
5.A	Optimization subject to the epipolar constraint . . . . .	165
<b>6</b>	<b>Reconstruction from Two Uncalibrated Views</b>	<b>171</b>
6.1	Uncalibrated camera or distorted space? . . . . .	174
6.2	Uncalibrated epipolar geometry . . . . .	177
6.2.1	The fundamental matrix . . . . .	177
6.2.2	Properties of the fundamental matrix . . . . .	178
6.3	Ambiguities and constraints in image formation . . . . .	181
6.3.1	Structure of the intrinsic parameter matrix . . . . .	182
6.3.2	Structure of the extrinsic parameters . . . . .	184
6.3.3	Structure of the projection matrix . . . . .	184
6.4	Stratified reconstruction . . . . .	185
6.4.1	Geometric stratification . . . . .	185
6.4.2	Projective reconstruction . . . . .	188
6.4.3	Affine reconstruction . . . . .	192
6.4.4	Euclidean reconstruction . . . . .	194
6.4.5	Direct stratification from multiple views (preview) . .	196
6.5	Calibration with scene knowledge . . . . .	198
6.5.1	Partial scene knowledge . . . . .	199
6.5.2	Calibration with a rig . . . . .	201
6.5.3	Calibration with a planar pattern . . . . .	202
6.6	Dinner with Kruppa . . . . .	204
6.7	Summary . . . . .	206
6.8	Exercises . . . . .	206

6.A	From images to fundamental matrices . . . . .	211
6.B	Properties of Kruppa's equations . . . . .	215
6.B.1	Linearly independent Kruppa's equations under special motions . . . . .	217
6.B.2	Cheirality constraints . . . . .	223
<b>7</b>	<b>Estimation of Multiple Motions from Two Views</b>	<b>228</b>
7.1	Multibody epipolar constraint and the fundamental matrix . .	229
7.2	A rank condition for the number of motions . . . . .	234
7.3	Geometric properties of the multibody fundamental matrix .	237
7.4	Multibody motion estimation and segmentation . . . . .	242
7.4.1	Estimation of epipolar lines and epipoles . . . . .	243
7.4.2	Recovery of individual fundamental matrices . . . . .	247
7.4.3	3-D motion segmentation . . . . .	248
7.5	Multibody structure from motion . . . . .	249
7.6	Summary . . . . .	252
7.7	Exercises . . . . .	253
7.A	Homogeneous polynomial factorization . . . . .	256
<b>III</b>	<b>Geometry of Multiple Views</b>	<b>261</b>
<b>8</b>	<b>Multiple-View Geometry of Points and Lines</b>	<b>263</b>
8.1	Basic notation for the (pre)image and coimage of points and lines . . . . .	264
8.2	Preliminary rank conditions of multiple images . . . . .	267
8.2.1	Point features . . . . .	267
8.2.2	Line features . . . . .	270
8.3	Geometry of point features . . . . .	273
8.3.1	The multiple-view matrix of a point and its rank . . .	273
8.3.2	Geometric interpretation of the rank condition . . . .	276
8.3.3	Multiple-view factorization of point features . . . . .	278
8.4	Geometry of line features . . . . .	283
8.4.1	The multiple-view matrix of a line and its rank . . . .	283
8.4.2	Geometric interpretation of the rank condition . . . .	284
8.4.3	Trilinear relationships among points and lines . . . .	288
8.5	Uncalibrated factorization and stratification . . . . .	289
8.5.1	Equivalent multiple-view matrices . . . . .	290
8.5.2	Rank-based uncalibrated factorization . . . . .	291
8.5.3	Direct stratification by the absolute quadric constraint	292
8.6	Summary . . . . .	294
8.7	Exercises . . . . .	295
8.A	Proof for the properties of bilinear and trilinear constraints .	305
<b>9</b>	<b>Extension to General Incidence Relations</b>	<b>310</b>



9.1	Incidence relations among points, lines, and planes . . . . .	310
9.1.1	Incidence relations in 3-D space . . . . .	310
9.1.2	Incidence relations in 2-D images . . . . .	312
9.2	Rank conditions for incidence relations . . . . .	313
9.2.1	Intersection of a family of lines . . . . .	313
9.2.2	Restriction to a plane . . . . .	316
9.3	Universal rank conditions on the multiple-view matrix . . . . .	320
9.4	Summary . . . . .	324
9.5	Exercises . . . . .	326
9.A	Incidence relations and rank conditions . . . . .	330
9.B	Beyond constraints among four views . . . . .	331
9.C	Examples of geometric interpretation of the rank conditions . . . . .	333
9.C.1	Case 2: $0 \leq \text{rank}(M) \leq 1$ . . . . .	333
9.C.2	Case 1: $1 \leq \text{rank}(M) \leq 2$ . . . . .	335
<b>10</b>	<b>Geometry and Reconstruction from Symmetry</b>	<b>338</b>
10.1	Symmetry and multiple-view geometry . . . . .	338
10.1.1	Equivalent views of symmetric structures . . . . .	339
10.1.2	Symmetric structure and symmetry group . . . . .	341
10.1.3	Symmetric multiple-view matrix and rank condition . . . . .	344
10.1.4	Homography group for a planar symmetric structure . . . . .	346
10.2	Symmetry-based 3-D reconstruction . . . . .	348
10.2.1	Canonical pose recovery for symmetric structure . . . . .	349
10.2.2	Pose ambiguity from three types of symmetry . . . . .	350
10.2.3	Structure reconstruction based on symmetry . . . . .	357
10.3	Camera calibration from symmetry . . . . .	364
10.3.1	Calibration from translational symmetry . . . . .	365
10.3.2	Calibration from reflective symmetry . . . . .	365
10.3.3	Calibration from rotational symmetry . . . . .	366
10.4	Summary . . . . .	367
10.5	Exercises . . . . .	368
<b>IV</b>	<b>Applications</b>	<b>373</b>
<b>11</b>	<b>Step-by-Step Building of a 3-D Model from Images</b>	<b>375</b>
11.1	Feature selection . . . . .	378
11.2	Feature correspondence . . . . .	380
11.2.1	Feature tracking . . . . .	380
11.2.2	Robust matching across wide baselines . . . . .	385
11.3	Projective reconstruction . . . . .	391
11.3.1	Two-view initialization . . . . .	391
11.3.2	Multiple-view reconstruction . . . . .	394
11.3.3	Gradient descent nonlinear refinement (“bundle adjustment”) . . . . .	397

11.4	Upgrade from projective to Euclidean reconstruction . . . . .	398
11.4.1	Stratification with the absolute quadric constraint . . .	399
11.4.2	Gradient descent nonlinear refinement (“Euclidean bundle adjustment”) . . . . .	402
11.5	Visualization . . . . .	403
11.5.1	Epipolar rectification . . . . .	404
11.5.2	Dense matching . . . . .	407
11.5.3	Texture mapping . . . . .	409
11.6	Additional techniques for image-based modeling . . . . .	409
<b>12</b>	<b>Visual Feedback</b>	<b>412</b>
12.1	Structure and motion estimation as a filtering problem . . . .	414
12.1.1	Observability . . . . .	415
12.1.2	Realization . . . . .	418
12.1.3	Implementation issues . . . . .	420
12.1.4	Complete algorithm . . . . .	422
12.2	Application to virtual insertion in live video . . . . .	426
12.3	Visual feedback for autonomous car driving . . . . .	427
12.3.1	System setup and implementation . . . . .	428
12.3.2	Vision system design . . . . .	429
12.3.3	System test results . . . . .	431
12.4	Visual feedback for autonomous helicopter landing . . . . .	432
12.4.1	System setup and implementation . . . . .	433
12.4.2	Vision system design . . . . .	434
12.4.3	System performance and evaluation . . . . .	436
<b>V</b>	<b>Appendices</b>	<b>439</b>
<b>A</b>	<b>Basic Facts from Linear Algebra</b>	<b>441</b>
A.1	Basic notions associated with a linear space . . . . .	442
A.1.1	Linear independence and change of basis . . . . .	442
A.1.2	Inner product and orthogonality . . . . .	444
A.1.3	Kronecker product and stack of matrices . . . . .	445
A.2	Linear transformations and matrix groups . . . . .	446
A.3	Gram-Schmidt and the QR decomposition . . . . .	449
A.4	Range, null space (kernel), rank and eigenvectors of a matrix	451
A.5	Symmetric matrices and skew-symmetric matrices . . . . .	454
A.6	Lyapunov map and Lyapunov equation . . . . .	456
A.7	The singular value decomposition (SVD) . . . . .	457
A.7.1	Algebraic derivation . . . . .	457
A.7.2	Geometric interpretation . . . . .	459
A.7.3	Some properties of the SVD . . . . .	459
<b>B</b>	<b>Least-Variance Estimation and Filtering</b>	<b>462</b>

B.1	Least-variance estimators of random vectors . . . . .	463
B.1.1	Projections onto the range of a random vector . . . . .	464
B.1.2	Solution for the linear (scalar) estimator . . . . .	464
B.1.3	Affine least-variance estimator . . . . .	465
B.1.4	Properties and interpretations of the least-variance estimator . . . . .	466
B.2	The Kalman-Bucy filter . . . . .	468
B.2.1	Linear Gaussian dynamical models . . . . .	468
B.2.2	A little intuition . . . . .	469
B.2.3	Observability . . . . .	471
B.2.4	Derivation of the Kalman filter . . . . .	472
B.3	The extended Kalman filter . . . . .	476
<b>C</b>	<b>Basic Facts from Nonlinear Optimization</b>	<b>479</b>
C.1	Unconstrained optimization: gradient-based methods . . . . .	480
C.1.1	Optimality conditions . . . . .	481
C.1.2	Algorithms . . . . .	482
C.2	Constrained optimization: Lagrange multiplier method . . . . .	484
C.2.1	Optimality conditions . . . . .	485
C.2.2	Algorithms . . . . .	485
	<b>References</b>	<b>487</b>
	<b>Glossary of Notation</b>	<b>509</b>
	<b>Index</b>	<b>513</b>