Module2_Groups_SpecialSubGraphs

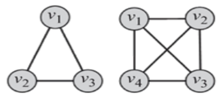
Reference Book:
Wasserman Stanley, and Katherine Faust. (2009), Social Network Analysis: Methods and Applications, Structural Analysis in the Social Sciences.

Special SubGraphs

Most common subgraph:

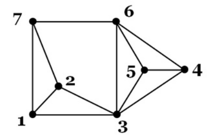
- **Clique**: a maximal complete subgraph in which all nodes inside the subgraph are adjacent to each other
  1. **The maximum clique**: **the clique** with the largest number of vertices
  2. **All maximal cliques**: cliques that are not subgraphs of a larger clique; i.e., cannot be further expanded

Examples:



Clique Example

- A clique in an undirected graph G = (V, E) is a subset of the vertex set C ⊆ V, such that for every two vertices in C, there exists an edge connecting the two.



- (3,4,5,6) is a clique in the above graph
- (1,2,7) is a clique
- (1,2,3) is a clique

Method to identify a Clique

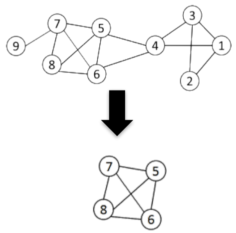
Systematic Pruning :

- When searching for cliques of size k or larger
  - If the clique is found, each node should have a degree equal to or more than $k-1$
- We can first prune all nodes (and edges connected to them) with degrees less than $k-1$
  - More nodes will have degrees less than $k-1$
  - Prune them recursively

Maximum Clique: Pruning

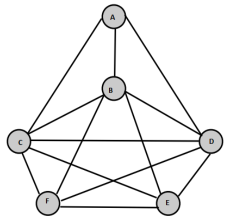
Example. To find a clique ≥ 4, remove all nodes with degree $\leq (4-1)-1=2$

- Remove nodes 2 and 9
- Remove nodes 1 and 3
- Remove node 4



Example

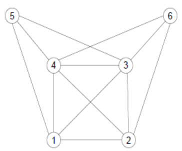
Find all Cliques



k-plex Example

Example

- 1-plex: {1,2,3,4}
  {1,3,4,5}
  {2,3,4,6}
- 2-plex: {1,2,3,4,5},
  {1,2,3,4,6}
- 3-plex: {1,2,3,4,5,6}

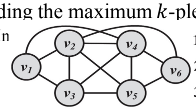


Relaxing Cliques

- ***k*-plex**: All nodes have a minimum degree that is not necessarily k - 1.
- For a set of vertices V, the structure is called a k-plex , if we have

$$d_v \geq |V| - k, \forall v \in V$$

- d_v is the degree of v in the induced subgraph
  - Number of nodes from V that are connected to v
- Clique of size k is a 1-plex
- As k gets larger in a k-plex, the structure gets increasingly relaxed
- Finding the maximum k-plex: **NP-hard**
  - In



1-plex : $\{v_2, v_3, v_4, v_5\}$
2-plex : $\{v_1, v_2, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5, v_6\}$
3-plex : $\{v_1, v_2, v_3, v_4, v_5, v_6\}$

k-plex Example

Example

- 1-plex: {1,2,3,4}
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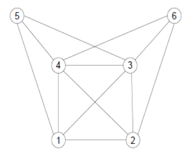
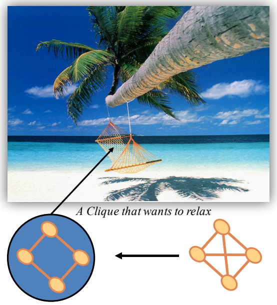


More Cliques Relaxing…

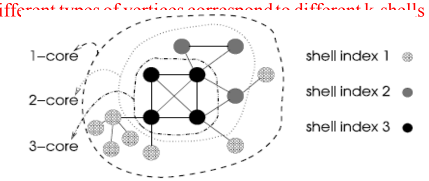
- ***k*-core:** a maximal connected subgraph in which all vertices have degree at least ***k***
- ***k*-shell:** nodes that are part of the k-core, but are not part of the $(k + 1)$ core



A Clique that wants to relax

k-core, k-shell - Example

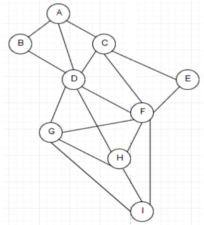
- k-core decomposition for a small graph:
  - Each closed line contains the set of vertices belonging to a given k-core
  - Different types of vertices correspond to different k-shells



- ***k*-core:** a maximal connected subgraph in which all vertices have degree at least ***k***
- ***k*-shell:** nodes that are part of the k-core, but are not part of the $(k + 1)$-core
- Reference: K-CORE DECOMPOSITION OF INTERNET GRAPHS: HIERARCHIES, SELF-SIMILARITY AND MEASUREMENT BIASES, J. I. ALVAREZ-HAMELIN, L. DALL'ASTA, A. BARRAT AND A. VESPIGNANI

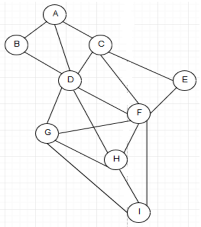
k- Core

Example : 2-core



k- Core

Example: k = ?

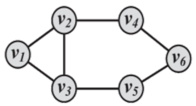


Special Subgraphs

1. ***k*-Clique**: a **maximal** subgraph in which the largest shortest path distance between nodes is less than or equal to k
   - Shortest path between any two nodes is always less than or equal to k.
   - Nodes on the shortest path need not be part of the subgraph
2. ***k*-Clan:** a k-clique where for all shortest paths within the subgraph the distance is equal or less than k.
   - k-cliques that have diameter less than or equal to k

   All k-clans are k-cliques but not vice versa
3. ***k*-Club**: follows the same definition as a k-clique
   - **Additional Constraint :** Nodes on the shortest paths should be part of the subgraph
   - **k-club is a maximal subgraph of diameter k**
   - No node can be added without increasing the diameter

     k-Clans = k-Cliques ∩ k-Clubs



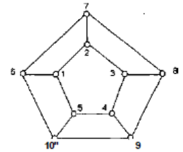
2-cliques :$\{v_1, v_2, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5, v_6\}$

2-clubs :$\{v_2, v_3, v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_5\}$

2-clans :$\{v_2, v_3, v_4, v_5, v_6\}$

Example

Identify 2-Cliques, 2-clubs and 2-clans from the given graph



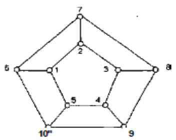
Example (cont'd)

2-Cliques: { 1,2,3,7}, { 2,3,4,8} , {3,4,5,9},{4,5,1,10}, { 5,1,6,2},

{1,2,7,6},{ 2,3,8,7} ,{ 3,4, 9, 8}, {4,5,10,9},{5,1,10,6},

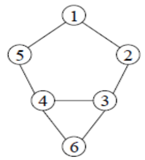
{6,7,1,10},{ 7,6,8,2},{ 8,3,7,9}, {9,4,8,10},{10,9,

{1,2,



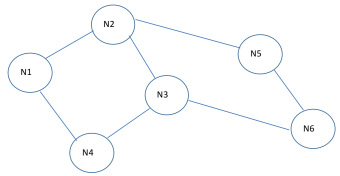
Example

Identify k-Clique, k-club and k-clan from the given graph



Example

Identify k-Clique, k-club and k-clan from the given graph



Example

Find all Cliques