ID	Problem Name	Problem Description
1	Multiples of 3 and 5	If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23. Find the sum of all the multiples of 3 or 5 below 1000.
2	Even Fibonacci numbers	Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 1 and 2, the first 10 terms will be: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.
3	Largest prime factor	The prime factors of 13195 are 5, 7, 13 and 29. What is the largest prime factor of the number 600851475143 ?
4	Largest palindrome product	A palindromic number reads the same both ways. The largest palindrome made from the product of two 2-digit numbers is 9009 = 91 × 99. Find the largest palindrome made from the product of two 3-digit numbers.
5	Smallest multiple	2520 is the smallest number that can be divided by each of the numbers from 1 to 10 without any remainder. What is the smallest positive number that is evenly divisible by all of the numbers from 1 to 20?

6	Sum square difference	The sum of the squares of the first ten natural numbers is, 12 + 22 + ... + 102 = 385 The square of the sum of the first ten natural numbers is, (1 + 2 + ... + 10)2 = 552 = 3025 Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is 3025 − 385 = 2640. Find the difference between the sum of the squares of the first one hundred natural numbers and the square of the sum.
7	10001st prime	By listing the first six prime numbers: 2, 3, 5, 7, 11, and 13, we can see that the 6th prime is 13. What is the 10 001st prime number?

8	Largest product in a series	The four adjacent digits in the 1000-digit number that have the greatest product are 9 × 9 × 8 × 9 = 5832. 73167176531330624919225119674426574742355349194934 96983520312774506326239578318016984801869478851843 85861560789112949495459501737958331952853208805511 12540698747158523863050715693290963295227443043557 66896648950445244523161731856403098711121722383113 62229893423380308135336276614282806444486645238749 30358907296290491560440772390713810515859307960866 70172427121883998797908792274921901699720888093776 65727333001053367881220235421809751254540594752243 52584907711670556013604839586446706324415722155397 53697817977846174064955149290862569321978468622482
9	Special Pythagorean triplet	A Pythagorean triplet is a set of three natural numbers, a < b < c, for which, a2 + b2 = c2 For example, 32 + 42 = 9 + 16 = 25 = 52. There exists exactly one Pythagorean triplet for which a + b + c = 1000. Find the product abc.
10	Summation of primes	The sum of the primes below 10 is 2 + 3 + 5 + 7 = 17. Find the sum of all the primes below two million.

11	Largest product in a grid	In the 20×20 grid below, four numbers along a diagonal line have been marked in red. 08 02 22 97 38 15 00 40 00 75 04 05 07 78 52 12 50 77 91 08 49 49 99 40 17 81 18 57 60 87 17 40 98 43 69 48 04 56 62 00 81 49 31 73 55 79 14 29 93 71 40 67 53 88 30 03 49 13 36 65 52 70 95 23 04 60 11 42 69 24 68 56 01 32 56 71 37 02 36 91 22 31 16 71 51 67 63 89 41 92 36 54 22 40 40 28 66 33 13 80 24 47 32 60 99 03 45 02 44 75 33 53 78 36 84 20 35 17 12 50 32 98 81 28 64 23 67 10 26 38 40 67 59 54 70 66 18 38 64 70 67 26 20 68 02 62 12 20 95 63 94 39 63 08 40 91 66 49 94 21 24 55 58 05 66 73 99 26 97 17 78 78 96 83 14 88 34 89 63 72 21 36 23 09 75 00 76 44 20 45 35 14 00 61 33 97 34 31 33 95 78 17 53 28 22 75 31 67 15 94 03 80 04 62 16 14 09 53 56 92

12	Highly divisible triangular number	The sequence of triangle numbers is generated by adding the natural numbers. So the 7th triangle number would be 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28. The first ten terms would be: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ... Let us list the factors of the first seven triangle numbers: 1: 1 3: 1,3 6: 1,2,3,6 10: 1,2,5,10 15: 1,3,5,15 21: 1,3,7,21 28: 1,2,4,7,14,28 We can see that 28 is the first triangle number to have over five divisors. What is the value of the first triangle number to have over five hundred divisors?

13	Large sum	Work out the first ten digits of the sum of the following one-hundred 50-digit numbers. 37107287533902102798797998220837590246510135740250 46376937677490009712648124896970078050417018260538 74324986199524741059474233309513058123726617309629 91942213363574161572522430563301811072406154908250 23067588207539346171171980310421047513778063246676 89261670696623633820136378418383684178734361726757 28112879812849979408065481931592621691275889832738 44274228917432520321923589422876796487670272189318 47451445736001306439091167216856844588711603153276 70386486105843025439939619828917593665686757934951 62176457141856560629502157223196586755079324193331

14	Longest Collatz sequence	The following iterative sequence is defined for the set of positive integers: n → n/2 (n is even) n → 3n + 1 (n is odd) Using the rule above and starting with 13, we generate the following sequence: 13 → 40 → 20 → 10 → 5 → 16 → 8 → 4 → 2 → 1 It can be seen that this sequence (starting at 13 and finishing at 1) contains 10 terms. Although it has not been proved yet (Collatz Problem), it is thought that all starting numbers finish at 1. Which starting number, under one million, produces the longest chain? NOTE: Once the chain starts the terms are allowed to go above one million.
15	Lattice paths	Starting in the top left corner of a 2×2 grid, and only being able to move to the right and down, there are exactly 6 routes to the bottom right corner. How many such routes are there through a 20×20 grid?
16	Power digit sum	215 = 32768 and the sum of its digits is 3 + 2 + 7 + 6 + 8 = 26. What is the sum of the digits of the number 21000?

17	Number letter counts	If the numbers 1 to 5 are written out in words: one, two, three, four, five, then there are 3 + 3 + 5 + 4 + 4 = 19 letters used in total. If all the numbers from 1 to 1000 (one thousand) inclusive were written out in words, how many letters would be used? NOTE: Do not count spaces or hyphens. For example, 342 (three hundred and forty-two) contains 23 letters and 115 (one hundred and fifteen) contains 20 letters. The use of "and" when writing out numbers is in compliance with British usage.

18	Maximum path sum I	By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23. 3 7 4 2 4 6 8 5 9 3 That is, 3 + 7 + 4 + 9 = 23. Find the maximum total from top to bottom of the triangle below: 75 95 64 17 47 82 18 35 87 10 20 04 82 47 65 19 01 23 75 03 34 88 02 77 73 07 63 67 99 65 04 28 06 16 70 92 41 41 26 56 83 40 80 70 33 41 48 72 33 47 32 37 16 94 29 53 71 44 65 25 43 91 52 97 51 14 70 11 33 28 77 73 17 78 39 68 17 57 91 71 52 38 17 14 91 43 58 50 27 29

19	Counting Sundays	You are given the following information, but you may prefer to do some research for yourself. 1 Jan 1900 was a Monday. Thirty days has September, April, June and November. All the rest have thirty-one, Saving February alone, Which has twenty-eight, rain or shine. And on leap years, twenty-nine. A leap year occurs on any year evenly divisible by 4, but not on a century unless it is divisible by 400. How many Sundays fell on the first of the month during the twentieth century (1 Jan 1901 to 31 Dec 2000)?
20	Factorial digit sum	$n!$ means $n \times (n - 1) \times \ldots \times 3 \times 2 \times 1$ For example, $10! = 10 \times 9 \times \ldots \times 3 \times 2 \times 1 = 3628800$, and the sum of the digits in the number $10!$ is $3 + 6 + 2 + 8 + 8 + 0 + 0 = 27$. Find the sum of the digits in the number $100!$
21	Amicable numbers	Let $d(n)$ be defined as the sum of proper divisors of n (numbers less than n which divide evenly into n). If $d(a) = b$ and $d(b) = a$, where $a \neq b$, then a and b are an amicable pair and each of a and b are called amicable numbers. For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore $d(220) = 284$. The proper divisors of 284 are 1, 2, 4, 71 and 142; so $d(284) = 220$. Evaluate the sum of all the amicable numbers under 10000.

22	Names scores	Using names.txt (right click and 'Save Link/Target As...'), a 46K text file containing over five-thousand first names, begin by sorting it into alphabetical order. Then working out the alphabetical value for each name, multiply this value by its alphabetical position in the list to obtain a name score. For example, when the list is sorted into alphabetical order, COLIN, which is worth 3 + 15 + 12 + 9 + 14 = 53, is the 938th name in the list. So, COLIN would obtain a score of 938 × 53 = 49714. What is the total of all the name scores in the file?

23	Non-abundant sums	A perfect number is a number for which the sum of its proper divisors is exactly equal to the number. For example, the sum of the proper divisors of 28 would be 1 + 2 + 4 + 7 + 14 = 28, which means that 28 is a perfect number. A number n is called deficient if the sum of its proper divisors is less than n and it is called abundant if this sum exceeds n. As 12 is the smallest abundant number, 1 + 2 + 3 + 4 + 6 = 16, the smallest number that can be written as the sum of two abundant numbers is 24. By mathematical analysis, it can be shown that all integers greater than 28123 can be written as the sum of two abundant numbers. However, this upper limit cannot be reduced any further by analysis even though it is known that the greatest number that cannot be expressed as the sum of two abundant numbers is less than this limit.
24	Lexicographic permutations	A permutation is an ordered arrangement of objects. For example, 3124 is one possible permutation of the digits 1, 2, 3 and 4. If all of the permutations are listed numerically or alphabetically, we call it lexicographic order. The lexicographic permutations of 0, 1 and 2 are: 012 021 102 120 201 210 What is the millionth lexicographic permutation of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9?

25	1000-digit Fibonacci number	The Fibonacci sequence is defined by the recurrence relation: Fn = Fn−1 + Fn−2, where F1 = 1 and F2 = 1. Hence the first 12 terms will be: F1 = 1 F2 = 1 F3 = 2 F4 = 3 F5 = 5 F6 = 8 F7 = 13 F8 = 21 F9 = 34 F10 = 55 F11 = 89 F12 = 144 The 12th term, F12, is the first term to contain three digits. What is the index of the first term in the Fibonacci sequence to contain 1000 digits?

26	Reciprocal cycles	A unit fraction contains 1 in the numerator. The decimal representation of the unit fractions with denominators 2 to 10 are given: 1/2 = 0.5 1/3 = 0.(3) 1/4 = 0.25 1/5 = 0.2 1/6 = 0.1(6) 1/7 = 0.(142857) 1/8 = 0.125 1/9 = 0.(1) 1/10 = 0.1 Where 0.1(6) means 0.166666..., and has a 1-digit recurring cycle. It can be seen that 1/7 has a 6-digit recurring cycle. Find the value of d < 1000 for which 1/d contains the longest recurring cycle in its decimal fraction part.

27	Quadratic primes	Euler discovered the remarkable quadratic formula: n 2 +n+41 n It turns out that the formula will produce 40 primes for the consecutive integer values 0≤n≤39 0 . However, when n=40, 40 2 +40+41=40(40+1)+41 n is divisible by 41, and certainly when n=41, 41 2 +41+41 n is clearly divisible by 41. The incredible formula
28	Number spiral diagonals	Starting with the number 1 and moving to the right in a clockwise direction a 5 by 5 spiral is formed as follows: 21 22 23 24 25 20 7 8 9 10 19 6 1 2 11 18 5 4 3 12 17 16 15 14 13 It can be verified that the sum of the numbers on the diagonals is 101. What is the sum of the numbers on the diagonals in a 1001 by 1001 spiral formed in the same way?

29	Distinct powers	Consider all integer combinations of ab for 2 ≤ a ≤ 5 and 2 ≤ b ≤ 5: 22=4, 23=8, 24=16, 25=32 32=9, 33=27, 34=81, 35=243 42=16, 43=64, 44=256, 45=1024 52=25, 53=125, 54=625, 55=3125 If they are then placed in numerical order, with any repeats removed, we get the following sequence of 15 distinct terms: 4, 8, 9, 16, 25, 27, 32, 64, 81, 125, 243, 256, 625, 1024, 3125 How many distinct terms are in the sequence generated by ab for 2 ≤ a ≤ 100 and 2 ≤ b ≤ 100?
30	Digit fifth powers	Surprisingly there are only three numbers that can be written as the sum of fourth powers of their digits: 1634 = 14 + 64 + 34 + 44 8208 = 84 + 24 + 04 + 84 9474 = 94 + 44 + 74 + 44 As 1 = 14 is not a sum it is not included. The sum of these numbers is 1634 + 8208 + 9474 = 19316. Find the sum of all the numbers that can be written as the sum of fifth powers of their digits.
31	Coin sums	In England the currency is made up of pound, £, and pence, p, and there are eight coins in general circulation: 1p, 2p, 5p, 10p, 20p, 50p, £1 (100p) and £2 (200p). It is possible to make £2 in the following way: 1×£1 + 1×50p + 2×20p + 1×5p + 1×2p + 3×1p How many different ways can £2 be made using any number of coins?

32	Pandigital products	We shall say that an n-digit number is pandigital if it makes use of all the digits 1 to n exactly once; for example, the 5-digit number, 15234, is 1 through 5 pandigital. The product 7254 is unusual, as the identity, 39 × 186 = 7254, containing multiplicand, multiplier, and product is 1 through 9 pandigital. Find the sum of all products whose multiplicand/multiplier/product identity can be written as a 1 through 9 pandigital. HINT: Some products can be obtained in more than one way so be sure to only include it once in your sum.
33	Digit cancelling fractions	The fraction 49/98 is a curious fraction, as an inexperienced mathematician in attempting to simplify it may incorrectly believe that 49/98 = 4/8, which is correct, is obtained by cancelling the 9s. We shall consider fractions like, 30/50 = 3/5, to be trivial examples. There are exactly four non-trivial examples of this type of fraction, less than one in value, and containing two digits in the numerator and denominator. If the product of these four fractions is given in its lowest common terms, find the value of the denominator.
34	Digit factorials	145 is a curious number, as 1! + 4! + 5! = 1 + 24 + 120 = 145. Find the sum of all numbers which are equal to the sum of the factorial of their digits. Note: as 1! = 1 and 2! = 2 are not sums they are not included.

35	Circular primes	The number, 197, is called a circular prime because all rotations of the digits: 197, 971, and 719, are themselves prime. There are thirteen such primes below 100: 2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, and 97. How many circular primes are there below one million?
36	Double-base palindromes	The decimal number, 585 = 10010010012 (binary), is palindromic in both bases. Find the sum of all numbers, less than one million, which are palindromic in base 10 and base 2. (Please note that the palindromic number, in either base, may not include leading zeros.)
37	Truncatable primes	The number 3797 has an interesting property. Being prime itself, it is possible to continuously remove digits from left to right, and remain prime at each stage: 3797, 797, 97, and 7. Similarly we can work from right to left: 3797, 379, 37, and 3. Find the sum of the only eleven primes that are both truncatable from left to right and right to left. NOTE: 2, 3, 5, and 7 are not considered to be truncatable primes.

38	Pandigital multiples	Take the number 192 and multiply it by each of 1, 2, and 3: 192 × 1 = 192 192 × 2 = 384 192 × 3 = 576 By concatenating each product we get the 1 to 9 pandigital, 192384576. We will call 192384576 the concatenated product of 192 and (1,2,3) The same can be achieved by starting with 9 and multiplying by 1, 2, 3, 4, and 5, giving the pandigital, 918273645, which is the concatenated product of 9 and (1,2,3,4,5). What is the largest 1 to 9 pandigital 9-digit number that can be formed as the concatenated product of an integer with (1,2, ... , n) where n > 1?
39	Integer right triangles	If p is the perimeter of a right angle triangle with integral length sides, {a,b,c}, there are exactly three solutions for p = 120. {20,48,52}, {24,45,51}, {30,40,50} For which value of p ≤ 1000, is the number of solutions maximised?
40	Champernowne's constant	An irrational decimal fraction is created by concatenating the positive integers: 0.123456789101112131415161718192021... It can be seen that the 12th digit of the fractional part is 1. If dn represents the nth digit of the fractional part, find the value of the following expression. d1 × d10 × d100 × d1000 × d10000 × d100000 × d1000000

41	Pandigital prime	We shall say that an n-digit number is pandigital if it makes use of all the digits 1 to n exactly once. For example, 2143 is a 4-digit pandigital and is also prime. What is the largest n-digit pandigital prime that exists?
42	Coded triangle numbers	The nth term of the sequence of triangle numbers is given by, tn = ½n(n+1); so the first ten triangle numbers are: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ... By converting each letter in a word to a number corresponding to its alphabetical position and adding these values we form a word value. For example, the word value for SKY is 19 + 11 + 25 = 55 = t10. If the word value is a triangle number then we shall call the word a triangle word. Using words.txt (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, how many are triangle words?

43	Sub-string divisibility	The number, 1406357289, is a 0 to 9 pandigital number because it is made up of each of the digits 0 to 9 in some order, but it also has a rather interesting sub-string divisibility property. Let d_1 be the 1st digit, d_2 be the 2nd digit, and so on. In this way, we note the following: $d_2d_3d_4=406$ is divisible by 2 $d_3d_4d_5=063$ is divisible by 3 $d_4d_5d_6=635$ is divisible by 5 $d_5d_6d_7=357$ is divisible by 7 $d_6d_7d_8=572$ is divisible by 11 $d_7d_8d_9=728$ is divisible by 13 $d_8d_9d_{10}=289$ is divisible by 17 Find the sum of all 0 to 9 pandigital numbers with this property.
44	Pentagon numbers	Pentagonal numbers are generated by the formula, $P_n=n(3n-1)/2$. The first ten pentagonal numbers are: 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, ... It can be seen that $P_4 + P_7 = 22 + 70 = 92 = P_8$. However, their difference, $70 - 22 = 48$, is not pentagonal. Find the pair of pentagonal numbers, P_j and P_k, for which their sum and difference are pentagonal and $D = \|P_k - P_j\|$ is minimised; what is the value of D?

45	Triangular, pentagonal, and hexagonal	Triangle, pentagonal, and hexagonal numbers are generated by the following formulae: Triangle Tn=n(n+1)/2 1, 3, 6, 10, 15, ... Pentagonal Pn=n(3n−1)/2 1, 5, 12, 22, 35, ... Hexagonal Hn=n(2n−1) 1, 6, 15, 28, 45, ... It can be verified that T285 = P165 = H143 = 40755. Find the next triangle number that is also pentagonal and hexagonal.
46	Goldbach's other conjecture	It was proposed by Christian Goldbach that every odd composite number can be written as the sum of a prime and twice a square. 9 = 7 + 2×12 15 = 7 + 2×22 21 = 3 + 2×32 25 = 7 + 2×32 27 = 19 + 2×22 33 = 31 + 2×12 It turns out that the conjecture was false. What is the smallest odd composite that cannot be written as the sum of a prime and twice a square?

47	Distinct primes factors	The first two consecutive numbers to have two distinct prime factors are: $14 = 2 \times 7$ $15 = 3 \times 5$ The first three consecutive numbers to have three distinct prime factors are: $644 = 2^2 \times 7 \times 23$ $645 = 3 \times 5 \times 43$ $646 = 2 \times 17 \times 19$. Find the first four consecutive integers to have four distinct prime factors each. What is the first of these numbers?
48	Self powers	The series, 11 + 22 + 33 + ... + 1010 = 10405071317. Find the last ten digits of the series, 11 + 22 + 33 + ... + 10001000.
49	Prime permutations	The arithmetic sequence, 1487, 4817, 8147, in which each of the terms increases by 3330, is unusual in two ways: (i) each of the three terms are prime, and, (ii) each of the 4-digit numbers are permutations of one another. There are no arithmetic sequences made up of three 1-, 2-, or 3-digit primes, exhibiting this property, but there is one other 4-digit increasing sequence. What 12-digit number do you form by concatenating the three terms in this sequence?

50	Consecutive prime sum	The prime 41, can be written as the sum of six consecutive primes: 41 = 2 + 3 + 5 + 7 + 11 + 13 This is the longest sum of consecutive primes that adds to a prime below one-hundred. The longest sum of consecutive primes below one-thousand that adds to a prime, contains 21 terms, and is equal to 953. Which prime, below one-million, can be written as the sum of the most consecutive primes?
51	Prime digit replacements	By replacing the 1st digit of the 2-digit number *3, it turns out that six of the nine possible values: 13, 23, 43, 53, 73, and 83, are all prime. By replacing the 3rd and 4th digits of 56**3 with the same digit, this 5-digit number is the first example having seven primes among the ten generated numbers, yielding the family: 56003, 56113, 56333, 56443, 56663, 56773, and 56993. Consequently 56003, being the first member of this family, is the smallest prime with this property. Find the smallest prime which, by replacing part of the number (not necessarily adjacent digits) with the same digit, is part of an eight prime value family.
52	Permuted multiples	It can be seen that the number, 125874, and its double, 251748, contain exactly the same digits, but in a different order. Find the smallest positive integer, x, such that 2x, 3x, 4x, 5x, and 6x, contain the same digits.

53	Combinatoric selections	There are exactly ten ways of selecting three from five, 12345: 123, 124, 125, 134, 135, 145, 234, 235, 245, and 345 In combinatorics, we use the notation, (5 3)=10 (. In general, (n r)= n! r!(n−r)! (, where r≤n r , n!=n×(n−1)×...×3×2×1 n

54	Poker hands	In the card game poker, a hand consists of five cards and are ranked, from lowest to highest, in the following way: High Card: Highest value card. One Pair: Two cards of the same value. Two Pairs: Two different pairs. Three of a Kind: Three cards of the same value. Straight: All cards are consecutive values. Flush: All cards of the same suit. Full House: Three of a kind and a pair. Four of a Kind: Four cards of the same value. Straight Flush: All cards are consecutive values of same suit. Royal Flush: Ten, Jack, Queen, King, Ace, in same suit. The cards are valued in the order: 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace. If two players have the same ranked hands then the rank made up of the highest value wins; for example, a pair

55	Lychrel numbers	If we take 47, reverse and add, 47 + 74 = 121, which is palindromic. Not all numbers produce palindromes so quickly. For example, 349 + 943 = 1292, 1292 + 2921 = 4213 4213 + 3124 = 7337 That is, 349 took three iterations to arrive at a palindrome. Although no one has proved it yet, it is thought that some numbers, like 196, never produce a palindrome. A number that never forms a palindrome through the reverse and add process is called a Lychrel number. Due to the theoretical nature of these numbers, and for the purpose of this problem, we shall assume that a number is Lychrel until proven otherwise. In addition you are given that for every number below ten-thousand, it will either (i) become a palindrome in less than fifty iterations, or, (ii) no one, with all the computing power that exists, has managed so far to map it to a
56	Powerful digit sum	A googol (10100) is a massive number: one followed by one-hundred zeros; 100100 is almost unimaginably large: one followed by two-hundred zeros. Despite their size, the sum of the digits in each number is only 1. Considering natural numbers of the form, ab, where a, b < 100, what is the maximum digital sum?

57	Square root convergents	It is possible to show that the square root of two can be expressed as an infinite continued fraction. 2 – √ =1+ 1 2+ 1 2+ 1 2+... 2 By expanding this for the first four iterations, we get: 1+ 1 2 = 3 2 =1.5 1

58	Spiral primes	Starting with 1 and spiralling anticlockwise in the following way, a square spiral with side length 7 is formed. 37 36 35 34 33 32 31 38 17 16 15 14 13 30 39 18 5 4 3 12 29 40 19 6 1 2 11 28 41 20 7 8 9 10 27 42 21 22 23 24 25 26 43 44 45 46 47 48 49 It is interesting to note that the odd squares lie along the bottom right diagonal, but what is more interesting is that 8 out of the 13 numbers lying along both diagonals are prime; that is, a ratio of 8/13 ≈ 62%. If one complete new layer is wrapped around the spiral above, a square spiral with side length 9 will be formed. If this process is continued, what is the side length of the square spiral for which the ratio of primes along both diagonals first falls below 10%?

59	XOR decryption	Each character on a computer is assigned a unique code and the preferred standard is ASCII (American Standard Code for Information Interchange). For example, uppercase A = 65, asterisk (*) = 42, and lowercase k = 107. A modern encryption method is to take a text file, convert the bytes to ASCII, then XOR each byte with a given value, taken from a secret key. The advantage with the XOR function is that using the same encryption key on the cipher text, restores the plain text; for example, 65 XOR 42 = 107, then 107 XOR 42 = 65. For unbreakable encryption, the key is the same length as the plain text message, and the key is made up of random bytes. The user would keep the encrypted message and the encryption key in different locations, and without both "halves", it is impossible to decrypt the message. Unfortunately, this method is
60	Prime pair sets	The primes 3, 7, 109, and 673, are quite remarkable. By taking any two primes and concatenating them in any order the result will always be prime. For example, taking 7 and 109, both 7109 and 1097 are prime. The sum of these four primes, 792, represents the lowest sum for a set of four primes with this property. Find the lowest sum for a set of five primes for which any two primes concatenate to produce another prime.

61	Cyclical figurate numbers	Triangle, square, pentagonal, hexagonal, heptagonal, and octagonal numbers are all figurate (polygonal) numbers and are generated by the following formulae: Triangle P3,n=n(n+1)/2 1, 3, 6, 10, 15, ... Square P4,n=n2 1, 4, 9, 16, 25, ... Pentagonal P5,n=n(3n−1)/2 1, 5, 12, 22, 35, ... Hexagonal P6,n=n(2n−1) 1, 6, 15, 28, 45, ... Heptagonal P7,n=n(5n−3)/2 1, 7, 18, 34, 55, ... Octagonal P8,n=n(3n−2) 1, 8, 21, 40, 65, ... The ordered set of three 4-digit numbers: 8128, 2882, 8281, has three interesting properties. The set is cyclic, in that the last two digits of each number is the first two digits of the next number (including the last number with the first). Each polygonal type: triangle (P3,127=8128), square (P4,91=8281),
62	Cubic permutations	The cube, 41063625 (3453), can be permuted to produce two other cubes: 56623104 (3843) and 66430125 (4053). In fact, 41063625 is the smallest cube which has exactly three permutations of its digits which are also cube. Find the smallest cube for which exactly five permutations of its digits are cube.

63	Powerful digit counts	The 5-digit number, 16807=75, is also a fifth power. Similarly, the 9-digit number, 134217728=89, is a ninth power. How many n-digit positive integers exist which are also an nth power?
64	Odd period square roots	All square roots are periodic when written as continued fractions and can be written in the form: N − − √ = a 0 + 1 a 1 + 1 a 2 + 1 a3+... N For example, let us consider 23 −

65	Convergents of e	The square root of 2 can be written as an infinite continued fraction. 2 − √ =1+ 1 2+ 1 2+ 1 2+ 1 2+... 2 The infinite continued fraction can be written, 2 − √ =[1;(2)] 2 , (2) (

66	Diophantine equation	Consider quadratic Diophantine equations of the form: x2 – Dy2 = 1 For example, when D=13, the minimal solution in x is 6492 – 13×1802 = 1. It can be assumed that there are no solutions in positive integers when D is square. By finding minimal solutions in x for D = {2, 3, 5, 6, 7}, we obtain the following: 32 – 2×22 = 1 22 – 3×12 = 1 92 – 5×42 = 1 52 – 6×22 = 1 82 – 7×32 = 1 Hence, by considering minimal solutions in x for D ≤ 7, the largest x is obtained when D=5. Find the value of D ≤ 1000 in minimal solutions of x for which the largest value of x is obtained.

67	Maximum path sum II	By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23. 3 7 4 2 4 6 8 5 9 3 That is, 3 + 7 + 4 + 9 = 23. Find the maximum total from top to bottom in triangle.txt (right click and 'Save Link/Target As...'), a 15K text file containing a triangle with one-hundred rows. NOTE: This is a much more difficult version of Problem 18. It is not possible to try every route to solve this problem, as there are 299 altogether! If you could check one trillion (1012) routes every second it would take over twenty billion years to check them all. There is an efficient algorithm to solve it. ;o)

68	Magic 5-gon ring	Consider the following "magic" 3-gon ring, filled with the numbers 1 to 6, and each line adding to nine. Working clockwise, and starting from the group of three with the numerically lowest external node (4,3,2 in this example), each solution can be described uniquely. For example, the above solution can be described by the set: 4,3,2; 6,2,1; 5,1,3. It is possible to complete the ring with four different totals: 9, 10, 11, and 12. There are eight solutions in total. Total Solution Set 9 4,2,3; 5,3,1; 6,1,2 9 4,3,2; 6,2,1; 5,1,3 10 2,3,5; 4,5,1; 6,1,3 10 2,5,3; 6,3,1; 4,1,5 11 1,4,6; 3,6,2; 5,2,4 11 1,6,4; 5,4,2; 3,2,6 12 1,5,6; 2,6,4; 3,4,5 12 1,6,5; 3,5,4; 2,4,6 By concatenating each group it is possible to form 9-digit strings; the

69	Totient maximum	Euler's Totient function, φ(n) [sometimes called the phi function], is used to determine the number of numbers less than n which are relatively prime to n. For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine, φ(9)=6. n Relatively Prime φ(n) n/φ(n) 2 1 1 2 3 1,2 2 1.5 4 1,3 2 2 5 1,2,3,4 4 1.25 6 1,5 2 3 7 1,2,3,4,5,6 6 1.1666... 8 1,3,5,7 4 2 9 1,2,4,5,7,8 6 1.5 10 1,3,7,9 4 2.5 It can be seen that n=6 produces a maximum n/φ(n) for n ≤ 10. Find the value of n ≤ 1,000,000 for which n/φ(n) is a maximum.
70	Totient permutation	Euler's Totient function, φ(n) [sometimes called the phi function], is used to determine the number of positive numbers less than or equal to n which are relatively prime to n. For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine, φ(9)=6. The number 1 is considered to be relatively prime to every positive number, so φ(1)=1. Interestingly, φ(87109)=79180, and it can be seen that 87109 is a permutation of 79180. Find the value of n, 1 < n < 107, for which φ(n) is a permutation of n and the ratio n/φ(n) produces a minimum.

71	Ordered fractions	Consider the fraction, n/d, where n and d are positive integers. If n<d and HCF(n,d)=1, it is called a reduced proper fraction. If we list the set of reduced proper fractions for d ≤ 8 in ascending order of size, we get: 1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8 It can be seen that 2/5 is the fraction immediately to the left of 3/7. By listing the set of reduced proper fractions for d ≤ 1,000,000 in ascending order of size, find the numerator of the fraction immediately to the left of 3/7.
72	Counting fractions	Consider the fraction, n/d, where n and d are positive integers. If n<d and HCF(n,d)=1, it is called a reduced proper fraction. If we list the set of reduced proper fractions for d ≤ 8 in ascending order of size, we get: 1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8 It can be seen that there are 21 elements in this set. How many elements would be contained in the set of reduced proper fractions for d ≤ 1,000,000?

73	Counting fractions in a range	Consider the fraction, n/d, where n and d are positive integers. If n<d and HCF(n,d)=1, it is called a reduced proper fraction. If we list the set of reduced proper fractions for d ≤ 8 in ascending order of size, we get: 1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8 It can be seen that there are 3 fractions between 1/3 and 1/2. How many fractions lie between 1/3 and 1/2 in the sorted set of reduced proper fractions for d ≤ 12,000?
74	Digit factorial chains	The number 145 is well known for the property that the sum of the factorial of its digits is equal to 145: 1! + 4! + 5! = 1 + 24 + 120 = 145 Perhaps less well known is 169, in that it produces the longest chain of numbers that link back to 169; it turns out that there are only three such loops that exist: 169 → 363601 → 1454 → 169 871 → 45361 → 871 872 → 45362 → 872 It is not difficult to prove that EVERY starting number will eventually get stuck in a loop. For example, 69 → 363600 → 1454 → 169 → 363601 (→ 1454) 78 → 45360 → 871 → 45361 (→ 871) 540 → 145 (→ 145) Starting with 69 produces a chain of five non-repeating terms, but the longest non-repeating chain with a starting number below one million is sixty terms. How many chains, with a starting

75	Singular integer right triangles	It turns out that 12 cm is the smallest length of wire that can be bent to form an integer sided right angle triangle in exactly one way, but there are many more examples. 12 cm: (3,4,5) 24 cm: (6,8,10) 30 cm: (5,12,13) 36 cm: (9,12,15) 40 cm: (8,15,17) 48 cm: (12,16,20) In contrast, some lengths of wire, like 20 cm, cannot be bent to form an integer sided right angle triangle, and other lengths allow more than one solution to be found; for example, using 120 cm it is possible to form exactly three different integer sided right angle triangles. 120 cm: (30,40,50), (20,48,52), (24,45,51) Given that L is the length of the wire, for how many values of L ≤ 1,500,000 can exactly one integer sided right angle triangle be formed?
76	Counting summations	It is possible to write five as a sum in exactly six different ways: 4 + 1 3 + 2 3 + 1 + 1 2 + 2 + 1 2 + 1 + 1 + 1 1 + 1 + 1 + 1 + 1 How many different ways can one hundred be written as a sum of at least two positive integers?

77	Prime summations	It is possible to write ten as the sum of primes in exactly five different ways: 7 + 3 5 + 5 5 + 3 + 2 3 + 3 + 2 + 2 2 + 2 + 2 + 2 + 2 What is the first value which can be written as the sum of primes in over five thousand different ways?
78	Coin partitions	Let p(n) represent the number of different ways in which n coins can be separated into piles. For example, five coins can be separated into piles in exactly seven different ways, so p(5)=7. OOOOO OOOO O OOO OO OOO O O OO OO O OO O O O O O O O O Find the least value of n for which p(n) is divisible by one million.
79	Passcode derivation	A common security method used for online banking is to ask the user for three random characters from a passcode. For example, if the passcode was 531278, they may ask for the 2nd, 3rd, and 5th characters; the expected reply would be: 317. The text file, keylog.txt, contains fifty successful login attempts. Given that the three characters are always asked for in order, analyse the file so as to determine the shortest possible secret passcode of unknown length.

80	Square root digital expansion	It is well known that if the square root of a natural number is not an integer, then it is irrational. The decimal expansion of such square roots is infinite without any repeating pattern at all. The square root of two is 1.41421356237309504880..., and the digital sum of the first one hundred decimal digits is 475. For the first one hundred natural numbers, find the total of the digital sums of the first one hundred decimal digits for all the irrational square roots.
81	Path sum: two ways	In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by only moving to the right and down, is indicated in bold red and is equal to 2427. 131 201 630 537 805 673 96 803 699 732 234 342

82	Path sum: three ways	NOTE: This problem is a more challenging version of Problem 81. The minimal path sum in the 5 by 5 matrix below, by starting in any cell in the left column and finishing in any cell in the right column, and only moving up, down, and right, is indicated in red and bold; the sum is equal to 994. 131 201 630 537 805 673 96 803 699

83	Path sum: four ways	NOTE: This problem is a significantly more challenging version of Problem 81. In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by moving left, right, up, and down, is indicated in bold red and is equal to 2297. (| | | | | | 131 201 630 537 805 673 96 803 699

84	Monopoly odds	In the game, Monopoly, the standard board is set up in the following way: GO A1 CC1 A2 T1 R1 B1 CH1 B2 B3 JAIL H2 C1 T2 U1 H1 C2 CH3 C3 R4 R2 G3 D1 CC3 CC2 G2 D2 G1 D3 G2J F3 U2 F2 F1 R3 E3 E2 CH2 E1 FP A player starts on the GO square and adds the scores on two 6-sided dice to determine the number of squares they advance in a clockwise direction. Without any further rules we would expect to visit each square with equal probability: 2.5%. However, landing on G2J (Go To Jail), CC (community chest), and CH (chance) changes this distribution. In addition to G2J, and one card from
85	Counting rectangles	By counting carefully it can be seen that a rectangular grid measuring 3 by 2 contains eighteen rectangles: Although there exists no rectangular grid that contains exactly two million rectangles, find the area of the grid with the nearest solution.

86	Cuboid route	A spider, S, sits in one corner of a cuboid room, measuring 6 by 5 by 3, and a fly, F, sits in the opposite corner. By travelling on the surfaces of the room the shortest "straight line" distance from S to F is 10 and the path is shown on the diagram. However, there are up to three "shortest" path candidates for any given cuboid and the shortest route doesn't always have integer length. It can be shown that there are exactly 2060 distinct cuboids, ignoring rotations, with integer dimensions, up to a maximum size of M by M by M, for which the shortest route has integer length when M = 100. This is the least value of M for which the number of solutions first exceeds two thousand; the number of solutions when M = 99 is 1975. Find the least value of M such that the number of solutions first exceeds one million.
87	Prime power triples	The smallest number expressible as the sum of a prime square, prime cube, and prime fourth power is 28. In fact, there are exactly four numbers below fifty that can be expressed in such a way: 28 = 22 + 23 + 24 33 = 32 + 23 + 24 49 = 52 + 23 + 24 47 = 22 + 33 + 24 How many numbers below fifty million can be expressed as the sum of a prime square, prime cube, and prime fourth power?

88	Product-sum numbers	A natural number, N, that can be written as the sum and product of a given set of at least two natural numbers, {a1, a2, ... , ak} is called a product-sum number: N = a1 + a2 + ... + ak = a1 × a2 × ... × ak. For example, 6 = 1 + 2 + 3 = 1 × 2 × 3. For a given set of size, k, we shall call the smallest N with this property a minimal product-sum number. The minimal product-sum numbers for sets of size, k = 2, 3, 4, 5, and 6 are as follows. k=2: 4 = 2 × 2 = 2 + 2 k=3: 6 = 1 × 2 × 3 = 1 + 2 + 3 k=4: 8 = 1 × 1 × 2 × 4 = 1 + 1 + 2 + 4 k=5: 8 = 1 × 1 × 2 × 2 × 2 = 1 + 1 + 2 + 2 + 2 k=6: 12 = 1 × 1 × 1 × 1 × 2 × 6 = 1 + 1 + 1 + 1 + 2 + 6 Hence for 2≤k≤6, the sum of all the minimal product-sum numbers is 4+6+8+12 = 30; note that 8 is only counted once in the sum.

89	Roman numerals	For a number written in Roman numerals to be considered valid there are basic rules which must be followed. Even though the rules allow some numbers to be expressed in more than one way there is always a "best" way of writing a particular number. For example, it would appear that there are at least six ways of writing the number sixteen: IIIIIIIIIIIIIIII VIIIIIIIIIII VVIIIIII XIIIIII VVVI XVI However, according to the rules only XIIIIII and XVI are valid, and the last example is considered to be the most efficient, as it uses the least number of numerals. The 11K text file, roman.txt (right click and 'Save Link/Target As...'), contains one thousand numbers written in valid, but not necessarily minimal, Roman

90	Cube digit pairs	Each of the six faces on a cube has a different digit (0 to 9) written on it; the same is done to a second cube. By placing the two cubes side-by-side in different positions we can form a variety of 2-digit numbers. For example, the square number 64 could be formed: In fact, by carefully choosing the digits on both cubes it is possible to display all of the square numbers below one-hundred: 01, 04, 09, 16, 25, 36, 49, 64, and 81. For example, one way this can be achieved is by placing {0, 5, 6, 7, 8, 9} on one cube and {1, 2, 3, 4, 8, 9} on the other cube. However, for this problem we shall allow the 6 or 9 to be turned upside-down so that an arrangement like {0, 5, 6, 7, 8, 9} and {1, 2, 3, 4, 6, 7} allows for all nine square numbers to be displayed; otherwise it would be impossible to obtain 09.
91	Right triangles with integer coordinates	The points P (x1, y1) and Q (x2, y2) are plotted at integer co-ordinates and are joined to the origin, O(0,0), to form ΔOPQ. There are exactly fourteen triangles containing a right angle that can be formed when each co-ordinate lies between 0 and 2 inclusive; that is, 0 ≤ x1, y1, x2, y2 ≤ 2. Given that 0 ≤ x1, y1, x2, y2 ≤ 50, how many right triangles can be formed?

92	Square digit chains	A number chain is created by continuously adding the square of the digits in a number to form a new number until it has been seen before. For example, 44 → 32 → 13 → 10 → 1 → 1 85 → 89 → 145 → 42 → 20 → 4 → 16 → 37 → 58 → 89 Therefore any chain that arrives at 1 or 89 will become stuck in an endless loop. What is most amazing is that EVERY starting number will eventually arrive at 1 or 89. How many starting numbers below ten million will arrive at 89?
93	Arithmetic expressions	By using each of the digits from the set, {1, 2, 3, 4}, exactly once, and making use of the four arithmetic operations (+, −, *, /) and brackets/parentheses, it is possible to form different positive integer targets. For example, 8 = (4 * (1 + 3)) / 2 14 = 4 * (3 + 1 / 2) 19 = 4 * (2 + 3) − 1 36 = 3 * 4 * (2 + 1) Note that concatenations of the digits, like 12 + 34, are not allowed. Using the set, {1, 2, 3, 4}, it is possible to obtain thirty-one different target numbers of which 36 is the maximum, and each of the numbers 1 to 28 can be obtained before encountering the first non-expressible number. Find the set of four distinct digits, a < b < c < d, for which the longest set of consecutive positive integers, 1 to n, can be obtained, giving your answer as a string: abcd.

94	Almost equilateral triangles	It is easily proved that no equilateral triangle exists with integral length sides and integral area. However, the almost equilateral triangle 5-5-6 has an area of 12 square units. We shall define an almost equilateral triangle to be a triangle for which two sides are equal and the third differs by no more than one unit. Find the sum of the perimeters of all almost equilateral triangles with integral side lengths and area and whose perimeters do not exceed one billion (1,000,000,000).
95	Amicable chains	The proper divisors of a number are all the divisors excluding the number itself. For example, the proper divisors of 28 are 1, 2, 4, 7, and 14. As the sum of these divisors is equal to 28, we call it a perfect number. Interestingly the sum of the proper divisors of 220 is 284 and the sum of the proper divisors of 284 is 220, forming a chain of two numbers. For this reason, 220 and 284 are called an amicable pair. Perhaps less well known are longer chains. For example, starting with 12496, we form a chain of five numbers: 12496 → 14288 → 15472 → 14536 → 14264 (→ 12496 → ...) Since this chain returns to its starting point, it is called an amicable chain. Find the smallest member of the longest amicable chain with no element exceeding one million.

96	Su Doku	Su Doku (Japanese meaning number place) is the name given to a popular puzzle concept. Its origin is unclear, but credit must be attributed to Leonhard Euler who invented a similar, and much more difficult, puzzle idea called Latin Squares. The objective of Su Doku puzzles, however, is to replace the blanks (or zeros) in a 9 by 9 grid in such that each row, column, and 3 by 3 box contains each of the digits 1 to 9. Below is an example of a typical starting puzzle grid and its solution grid. 0 0 3 9 0 0 0 0 1 0 2 0 3 0 5 8 0 6 6 0 0 0 0 1 4 0 0 0 0 8 7 0 0 0 0 6 1 0 2 0 0 0
97	Large non-Mersenne prime	The first known prime found to exceed one million digits was discovered in 1999, and is a Mersenne prime of the form 26972593−1; it contains exactly 2,098,960 digits. Subsequently other Mersenne primes, of the form 2p−1, have been found which contain more digits. However, in 2004 there was found a massive non-Mersenne prime which contains 2,357,207 digits: 28433×27830457+1. Find the last ten digits of this prime number.

98	Anagramic squares	By replacing each of the letters in the word CARE with 1, 2, 9, and 6 respectively, we form a square number: 1296 = 362. What is remarkable is that, by using the same digital substitutions, the anagram, RACE, also forms a square number: 9216 = 962. We shall call CARE (and RACE) a square anagram word pair and specify further that leading zeroes are not permitted, neither may a different letter have the same digital value as another letter. Using words.txt (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, find all the square anagram word pairs (a palindromic word is NOT considered to be an anagram of itself). What is the largest square number formed by any member of such a pair? NOTE: All anagrams formed must be contained in the given text file.

99	Largest exponential	Comparing two numbers written in index form like 211 and 37 is not difficult, as any calculator would confirm that 211 = 2048 < 37 = 2187. However, confirming that 632382518061 > 519432525806 would be much more difficult, as both numbers contain over three million digits. Using base_exp.txt (right click and 'Save Link/Target As...'), a 22K text file containing one thousand lines with a base/exponent pair on each line, determine which line number has the greatest numerical value. NOTE: The first two lines in the file represent the numbers in the example given above.
100	Arranged probability	If a box contains twenty-one coloured discs, composed of fifteen blue discs and six red discs, and two discs were taken at random, it can be seen that the probability of taking two blue discs, P(BB) = (15/21)×(14/20) = 1/2. The next such arrangement, for which there is exactly 50% chance of taking two blue discs at random, is a box containing eighty-five blue discs and thirty-five red discs. By finding the first arrangement to contain over 1012 = 1,000,000,000,000 discs in total, determine the number of blue discs that the box would contain.

101	Optimum polynomial	If we are presented with the first k terms of a sequence it is impossible to say with certainty the value of the next term, as there are infinitely many polynomial functions that can model the sequence. As an example, let us consider the sequence of cube numbers. This is defined by the generating function, un = n3: 1, 8, 27, 64, 125, 216, ... Suppose we were only given the first two terms of this sequence. Working on the principle that "simple is best" we should assume a linear relationship and predict the next term to be 15 (common difference 7). Even if we were presented with the first three terms, by the same principle of simplicity, a quadratic relationship should be assumed. We shall define OP(k, n) to be the nth term of the optimum polynomial generating function for the first k terms of a sequence. It should be clear that OP(k, n) will accurately generate

102	Triangle containment	Three distinct points are plotted at random on a Cartesian plane, for which -1000 ≤ x, y ≤ 1000, such that a triangle is formed. Consider the following two triangles: A(-340,495), B(-153,-910), C(835,-947) X(-175,41), Y(-421,-714), Z(574,-645) It can be verified that triangle ABC contains the origin, whereas triangle XYZ does not. Using triangles.txt (right click and 'Save Link/Target As...'), a 27K text file containing the co-ordinates of one thousand "random" triangles, find the number of triangles for which the interior contains the origin. NOTE: The first two examples in the file represent the triangles in the example given above.

103	Special subset sums: optimum	Let S(A) represent the sum of elements in set A of size n. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true: S(B) ≠ S(C); that is, sums of subsets cannot be equal. If B contains more elements than C then S(B) > S(C). If S(A) is minimised for a given n, we shall call it an optimum special sum set. The first five optimum special sum sets are given below. n = 1: {1} n = 2: {1, 2} n = 3: {2, 3, 4} n = 4: {3, 5, 6, 7} n = 5: {6, 9, 11, 12, 13} It seems that for a given optimum set, A = {a1, a2, ... , an}, the next optimum set is of the form B = {b, a1+b, a2+b, ... ,an+b}, where b is the "middle" element on the previous row. By applying this "rule" we would expect the optimum set for n = 6 to be

104	Pandigital Fibonacci ends	The Fibonacci sequence is defined by the recurrence relation: $F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1$ and $F_2 = 1$. It turns out that F_{541}, which contains 113 digits, is the first Fibonacci number for which the last nine digits are 1-9 pandigital (contain all the digits 1 to 9, but not necessarily in order). And F_{2749}, which contains 575 digits, is the first Fibonacci number for which the first nine digits are 1-9 pandigital. Given that F_k is the first Fibonacci number for which the first nine digits AND the last nine digits are 1-9 pandigital, find k.

105	Special subset sums: testing	Let S(A) represent the sum of elements in set A of size n. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true: S(B) ≠ S(C); that is, sums of subsets cannot be equal. If B contains more elements than C then S(B) > S(C). For example, {81, 88, 75, 42, 87, 84, 86, 65} is not a special sum set because 65 + 87 + 88 = 75 + 81 + 84, whereas {157, 150, 164, 119, 79, 159, 161, 139, 158} satisfies both rules for all possible subset pair combinations and S(A) = 1286. Using sets.txt (right click and "Save Link/Target As..."), a 4K text file with one-hundred sets containing seven to twelve elements (the two examples given above are the first two sets in the file), identify all the special sum sets, A1, A2, ..., Ak, and find the value of S(A1) + S(A2) + ... + S(Ak). NOTE: This problem is related to

106	Special subset sums: meta-testing	Let S(A) represent the sum of elements in set A of size n. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true: S(B) ≠ S(C); that is, sums of subsets cannot be equal. If B contains more elements than C then S(B) > S(C). For this problem we shall assume that a given set contains n strictly increasing elements and it already satisfies the second rule. Surprisingly, out of the 25 possible subset pairs that can be obtained from a set for which n = 4, only 1 of these pairs need to be tested for equality (first rule). Similarly, when n = 7, only 70 out of the 966 subset pairs need to be tested. For n = 12, how many of the 261625 subset pairs that can be obtained need to be tested for equality? NOTE: This problem is related to Problem 103 and Problem 105.

107	Minimal network	The following undirected network consists of seven vertices and twelve edges with a total weight of 243. The same network can be represented by the matrix below. A B C D E F G A - 16 12 21 - - - B 16 - - 17 20 - - C 12 - - 28 - 31 - D 21 17 28 - 18 19 23 E - 20 - 18 - - 11 F - - 31 19 - - 27 G - - - 23 11 27 - However, it is possible to optimise the network by removing some edges and still ensure that all points on the network remain connected. The network which achieves the maximum saving is shown below. It has a weight of 93, representing a saving of 243 − 93 = 150 from the original network. Using network.txt (right click and 'Save Link/Target As...'), a 6K text file

108	Diophantine reciprocals I	In the following equation x, y, and n are positive integers. 1 x + 1 y = 1 n 1x+1y=1n For n = 4 there are exactly three distinct solutions: 1 5 + 1 20 1 6 + 1 12 1 8

109	Darts	In the game of darts a player throws three darts at a target board which is split into twenty equal sized sections numbered one to twenty. The score of a dart is determined by the number of the region that the dart lands in. A dart landing outside the red/green outer ring scores zero. The black and cream regions inside this ring represent single scores. However, the red/green outer ring and middle ring score double and treble scores respectively. At the centre of the board are two concentric circles called the bull region, or bulls-eye. The outer bull is worth 25 points and the inner bull is a double, worth 50 points. There are many variations of rules but in the most popular game the players will begin with a score 301 or 501 and the first player to reduce their running total to zero is a winner. However, it is normal to play a "doubles out" system,

110	Diophantine reciprocals II	In the following equation x, y, and n are positive integers. 1 x + 1 y = 1 n It can be verified that when n = 1260 there are 113 distinct solutions and this is the least value of n for which the total number of distinct solutions exceeds one hundred. What is the least value of n for which the number of distinct solutions exceeds four million? NOTE: This problem is a much more difficult version of Problem 108 and as it is well beyond the limitations of a brute force approach it requires a clever implementation.

111	Primes with runs	Considering 4-digit primes containing repeated digits it is clear that they cannot all be the same: 1111 is divisible by 11, 2222 is divisible by 22, and so on. But there are nine 4-digit primes containing three ones: 1117, 1151, 1171, 1181, 1511, 1811, 2111, 4111, 8111 We shall say that M(n, d) represents the maximum number of repeated digits for an n-digit prime where d is the repeated digit, N(n, d) represents the number of such primes, and S(n, d) represents the sum of these primes. So M(4, 1) = 3 is the maximum number of repeated digits for a 4-digit prime where one is the repeated digit, there are N(4, 1) = 9 such primes, and the sum of these primes is S(4, 1) = 22275. It turns out that for d = 0, it is only possible to have M(4, 0) = 2 repeated digits, but there are N(4, 0) = 13 such cases. In the same way we obtain the following results for 4-digit primes.

112	Bouncy numbers	Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468. Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420. We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349. Clearly there cannot be any bouncy numbers below one-hundred, but just over half of the numbers below one-thousand (525) are bouncy. In fact, the least number for which the proportion of bouncy numbers first reaches 50% is 538. Surprisingly, bouncy numbers become more and more common and by the time we reach 21780 the proportion of bouncy numbers is equal to 90%. Find the least number for which the proportion of bouncy numbers is exactly 99%.

113	Non-bouncy numbers	Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468. Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420. We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349. As n increases, the proportion of bouncy numbers below n increases such that there are only 12951 numbers below one-million that are not bouncy and only 277032 non-bouncy numbers below 1010. How many numbers below a googol (10100) are not bouncy?
114	Counting block combinations I	A row measuring seven units in length has red blocks with a minimum length of three units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one grey square. There are exactly seventeen ways of doing this. How many ways can a row measuring fifty units in length be filled? NOTE: Although the example above does not lend itself to the possibility, in general it is permitted to mix block sizes. For example, on a row measuring eight units in length you could use red (3), grey (1), and red (4).

115	Counting block combinations II	NOTE: This is a more difficult version of Problem 114. A row measuring n units in length has red blocks with a minimum length of m units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square. Let the fill-count function, F(m, n), represent the number of ways that a row can be filled. For example, F(3, 29) = 673135 and F(3, 30) = 1089155. That is, for m = 3, it can be seen that n = 30 is the smallest value for which the fill-count function first exceeds one million. In the same way, for m = 10, it can be verified that F(10, 56) = 880711 and F(10, 57) = 1148904, so n = 57 is the least value for which the fill-count function first exceeds one million. For m = 50, find the least value of n for which the fill-count function first exceeds one million.

116	Red, green or blue tiles	A row of five grey square tiles is to have a number of its tiles replaced with coloured oblong tiles chosen from red (length two), green (length three), or blue (length four). If red tiles are chosen there are exactly seven ways this can be done. If green tiles are chosen there are three ways. And if blue tiles are chosen there are two ways. Assuming that colours cannot be mixed there are 7 + 3 + 2 = 12 ways of replacing the grey tiles in a row measuring five units in length. How many different ways can the grey tiles in a row measuring fifty units in length be replaced if colours cannot be mixed and at least one coloured tile must be used? NOTE: This is related to Problem 117.
117	Red, green, and blue tiles	Using a combination of grey square tiles and oblong tiles chosen from: red tiles (measuring two units), green tiles (measuring three units), and blue tiles (measuring four units), it is possible to tile a row measuring five units in length in exactly fifteen different ways. How many ways can a row measuring fifty units in length be tiled? NOTE: This is related to Problem 116.

118	Pandigital prime sets	Using all of the digits 1 through 9 and concatenating them freely to form decimal integers, different sets can be formed. Interestingly with the set {2,5,47,89,631}, all of the elements belonging to it are prime. How many distinct sets containing each of the digits one through nine exactly once contain only prime elements?
119	Digit power sum	The number 512 is interesting because it is equal to the sum of its digits raised to some power: 5 + 1 + 2 = 8, and 83 = 512. Another example of a number with this property is 614656 = 284. We shall define an to be the nth term of this sequence and insist that a number must contain at least two digits to have a sum. You are given that a2 = 512 and a10 = 614656. Find a30.
120	Square remainders	Let r be the remainder when (a−1)n + (a+1)n is divided by a2. For example, if a = 7 and n = 3, then r = 42: 63 + 83 = 728 ≡ 42 mod 49. And as n varies, so too will r, but for a = 7 it turns out that rmax = 42. For 3 ≤ a ≤ 1000, find ∑ rmax.

121	Disc game prize fund	A bag contains one red disc and one blue disc. In a game of chance a player takes a disc at random and its colour is noted. After each turn the disc is returned to the bag, an extra red disc is added, and another disc is taken at random. The player pays £1 to play and wins if they have taken more blue discs than red discs at the end of the game. If the game is played for four turns, the probability of a player winning is exactly 11/120, and so the maximum prize fund the banker should allocate for winning in this game would be £10 before they would expect to incur a loss. Note that any payout will be a whole number of pounds and also includes the original £1 paid to play the game, so in the example given the player actually wins £9. Find the maximum prize fund that should be allocated to a single game in which fifteen turns are played.

122	Efficient exponentiation	The most naive way of computing n15 requires fourteen multiplications: n × n × ... × n = n15 But using a "binary" method you can compute it in six multiplications: n × n = n2 n2 × n2 = n4 n4 × n4 = n8 n8 × n4 = n12 n12 × n2 = n14 n14 × n = n15 However it is yet possible to compute it in only five multiplications: n × n = n2 n2 × n = n3 n3 × n3 = n6 n6 × n6 = n12 n12 × n3 = n15 We shall define m(k) to be the minimum number of multiplications to compute nk; for example m(15) = 5. For 1 ≤ k ≤ 200, find ∑ m(k).
123	Prime square remainders	Let pn be the nth prime: 2, 3, 5, 7, 11, ..., and let r be the remainder when (pn−1)n + (pn+1)n is divided by pn2. For example, when n = 3, p3 = 5, and 43 + 63 = 280 ≡ 5 mod 25. The least value of n for which the remainder first exceeds 109 is 7037. Find the least value of n for which the remainder first exceeds 1010.

124	Ordered radicals	The radical of n, rad(n), is the product of the distinct prime factors of n. For example, 504 = 23 × 32 × 7, so rad(504) = 2 × 3 × 7 = 42. If we calculate rad(n) for 1 ≤ n ≤ 10, then sort them on rad(n), and sorting on n if the radical values are equal, we get: Unsorted Sorted n rad(n) n rad(n) k 1 1 1 1

125	Palindromic sums	The palindromic number 595 is interesting because it can be written as the sum of consecutive squares: 62 + 72 + 82 + 92 + 102 + 112 + 122. There are exactly eleven palindromes below one-thousand that can be written as consecutive square sums, and the sum of these palindromes is 4164. Note that 1 = 02 + 12 has not been included as this problem is concerned with the squares of positive integers. Find the sum of all the numbers less than 108 that are both palindromic and can be written as the sum of consecutive squares.

126	Cuboid layers	The minimum number of cubes to cover every visible face on a cuboid measuring 3 x 2 x 1 is twenty-two. If we then add a second layer to this solid it would require forty-six cubes to cover every visible face, the third layer would require seventy-eight cubes, and the fourth layer would require one-hundred and eighteen cubes to cover every visible face. However, the first layer on a cuboid measuring 5 x 1 x 1 also requires twenty-two cubes; similarly the first layer on cuboids measuring 5 x 3 x 1, 7 x 2 x 1, and 11 x 1 x 1 all contain forty-six cubes. We shall define C(n) to represent the number of cuboids that contain n cubes in one of its layers. So C(22) = 2, C(46) = 4, C(78) = 5, and C(118) = 8. It turns out that 154 is the least value of n for which C(n) = 10. Find the least value of n for which C(n) = 1000.

127	abc-hits	The radical of n, rad(n), is the product of distinct prime factors of n. For example, 504 = 23 × 32 × 7, so rad(504) = 2 × 3 × 7 = 42. We shall define the triplet of positive integers (a, b, c) to be an abc-hit if: GCD(a, b) = GCD(a, c) = GCD(b, c) = 1 a < b a + b = c rad(abc) < c For example, (5, 27, 32) is an abc-hit, because: GCD(5, 27) = GCD(5, 32) = GCD(27, 32) = 1 5 < 27 5 + 27 = 32 rad(4320) = 30 < 32 It turns out that abc-hits are quite rare and there are only thirty-one abc-hits for c < 1000, with ∑c = 12523. Find ∑c for c < 120000.

128	Hexagonal tile differences	A hexagonal tile with number 1 is surrounded by a ring of six hexagonal tiles, starting at "12 o'clock" and numbering the tiles 2 to 7 in an anti-clockwise direction. New rings are added in the same fashion, with the next rings being numbered 8 to 19, 20 to 37, 38 to 61, and so on. The diagram below shows the first three rings. By finding the difference between tile n and each of its six neighbours we shall define PD(n) to be the number of those differences which are prime. For example, working clockwise around tile 8 the differences are 12, 29, 11, 6, 1, and 13. So PD(8) = 3. In the same way, the differences around tile 17 are 1, 17, 16, 1, 11, and 10, hence PD(17) = 2. It can be shown that the maximum value of PD(n) is 3. If all of the tiles for which PD(n) = 3 are listed in ascending order to form a sequence, the 10th tile would be 271.
129	Repunit divisibility	A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111. Given that n is a positive integer and GCD(n, 10) = 1, it can be shown that there always exists a value, k, for which R(k) is divisible by n, and let A(n) be the least such value of k; for example, A(7) = 6 and A(41) = 5. The least value of n for which A(n) first exceeds ten is 17. Find the least value of n for which A(n) first exceeds one-million.

130	Composites with prime repunit property	A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111. Given that n is a positive integer and GCD(n, 10) = 1, it can be shown that there always exists a value, k, for which R(k) is divisible by n, and let A(n) be the least such value of k; for example, A(7) = 6 and A(41) = 5. You are given that for all primes, p > 5, that p − 1 is divisible by A(p). For example, when p = 41, A(41) = 5, and 40 is divisible by 5. However, there are rare composite values for which this is also true; the first five examples being 91, 259, 451, 481, and 703. Find the sum of the first twenty-five composite values of n for which GCD(n, 10) = 1 and n − 1 is divisible by A(n).
131	Prime cube partnership	There are some prime values, p, for which there exists a positive integer, n, such that the expression n3 + n2p is a perfect cube. For example, when p = 19, 83 + 82×19 = 123. What is perhaps most surprising is that for each prime with this property the value of n is unique, and there are only four such primes below one-hundred. How many primes below one million have this remarkable property?

132	Large repunit factors	A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k. For example, R(10) = 1111111111 = 11×41×271×9091, and the sum of these prime factors is 9414. Find the sum of the first forty prime factors of R(109).
133	Repunit nonfactors	A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111. Let us consider repunits of the form R(10n). Although R(10), R(100), or R(1000) are not divisible by 17, R(10000) is divisible by 17. Yet there is no value of n for which R(10n) will divide by 19. In fact, it is remarkable that 11, 17, 41, and 73 are the only four primes below one-hundred that can be a factor of R(10n). Find the sum of all the primes below one-hundred thousand that will never be a factor of R(10n).
134	Prime pair connection	Consider the consecutive primes p1 = 19 and p2 = 23. It can be verified that 1219 is the smallest number such that the last digits are formed by p1 whilst also being divisible by p2. In fact, with the exception of p1 = 3 and p2 = 5, for every pair of consecutive primes, p2 > p1, there exist values of n for which the last digits are formed by p1 and n is divisible by p2. Let S be the smallest of these values of n. Find ∑ S for every pair of consecutive primes with 5 ≤ p1 ≤ 1000000.

135	Same differences	Given the positive integers, x, y, and z, are consecutive terms of an arithmetic progression, the least value of the positive integer, n, for which the equation, x2 − y2 − z2 = n, has exactly two solutions is n = 27: 342 − 272 − 202 = 122 − 92 − 62 = 27 It turns out that n = 1155 is the least value which has exactly ten solutions. How many values of n less than one million have exactly ten distinct solutions?
136	Singleton difference	The positive integers, x, y, and z, are consecutive terms of an arithmetic progression. Given that n is a positive integer, the equation, x2 − y2 − z2 = n, has exactly one solution when n = 20: 132 − 102 − 72 = 20 In fact there are twenty-five values of n below one hundred for which the equation has a unique solution. How many values of n less than fifty million have exactly one solution?

137	Fibonacci golden nuggets	Consider the infinite polynomial series A F (x)=x F 1 + x 2 F 2 + x 3 F 3 +... A , where F k F is the k k

138	Special isosceles triangles	Consider the isosceles triangle with base length, b=16 b , and legs, L=17 L . By using the Pythagorean theorem it can be seen that the height of the triangle, h= 17 2 − 8 2 − − − − − − − √

139	Pythagorean tiles	Let (a, b, c) represent the three sides of a right angle triangle with integral length sides. It is possible to place four such triangles together to form a square with length c. For example, (3, 4, 5) triangles can be placed together to form a 5 by 5 square with a 1 by 1 hole in the middle and it can be seen that the 5 by 5 square can be tiled with twenty-five 1 by 1 squares. However, if (5, 12, 13) triangles were used then the hole would measure 7 by 7 and these could not be used to tile the 13 by 13 square. Given that the perimeter of the right triangle is less than one-hundred million, how many Pythagorean triangles would allow such a tiling to take place?

140	Modified Fibonacci golden nuggets	Consider the infinite polynomial series A G (x)=x G 1 + x 2 G 2 + x 3 G 3 +⋯ A , where G k G is the k k

141	ing progressive numbers, n, which are al	A positive integer, n, is divided by d and the quotient and remainder are q and r respectively. In addition d, q, and r are consecutive positive integer terms in a geometric sequence, but not necessarily in that order. For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that 4, 6, 9 are consecutive terms in a geometric sequence (common ratio 3/2). We will call such numbers, n, progressive. Some progressive numbers, such as 9 and 10404 = 1022, happen to also be perfect squares. The sum of all progressive perfect squares below one hundred thousand is 124657. Find the sum of all progressive perfect squares below one trillion (1012).
142	Perfect Square Collection	Find the smallest x + y + z with integers x > y > z > 0 such that x + y, x − y, x + z, x − z, y + z, y − z are all perfect squares.

143	vestigating the Torricelli point of a triang	Let ABC be a triangle with all interior angles being less than 120 degrees. Let X be any point inside the triangle and let XA = p, XC = q, and XB = r. Fermat challenged Torricelli to find the position of X such that p + q + r was minimised. Torricelli was able to prove that if equilateral triangles AOB, BNC and AMC are constructed on each side of triangle ABC, the circumscribed circles of AOB, BNC, and AMC will intersect at a single point, T, inside the triangle. Moreover he proved that T, called the Torricelli/Fermat point, minimises p + q + r. Even more remarkable, it can be shown that when the sum is minimised, AN = BM = CO = p + q + r and that AN, BM and CO also intersect at T. If the sum is minimised and a, b, c, p, q and r are all positive integers we shall call triangle ABC a Torricelli triangle. For example, a = 399, b = 455, c = 511 is an example of a Torricelli triangle,

144	stigating multiple reflections of a laser b	In laser physics, a "white cell" is a mirror system that acts as a delay line for the laser beam. The beam enters the cell, bounces around on the mirrors, and eventually works its way back out. The specific white cell we will be considering is an ellipse with the equation 4x2 + y2 = 100 The section corresponding to −0.01 ≤ x ≤ +0.01 at the top is missing, allowing the light to enter and exit through the hole. The light beam in this problem starts at the point (0.0,10.1) just outside the white cell, and the beam first impacts the mirror at (1.4,-9.6). Each time the laser beam hits the surface of the ellipse, it follows the usual law of reflection "angle of incidence equals angle of reflection." That is, both the incident and reflected beams make the same angle with the normal line at the point of incidence. In the figure on the left, the red line
145	y reversible numbers are there below or	Some positive integers n have the property that the sum [n + reverse(n)] consists entirely of odd (decimal) digits. For instance, 36 + 63 = 99 and 409 + 904 = 1313. We will call such numbers reversible; so 36, 63, 409, and 904 are reversible. Leading zeroes are not allowed in either n or reverse(n). There are 120 reversible numbers below one-thousand. How many reversible numbers are there below one-billion (109)?

146	Investigating a Prime Pattern	The smallest positive integer n for which the numbers n2+1, n2+3, n2+7, n2+9, n2+13, and n2+27 are consecutive primes is 10. The sum of all such integers n below one-million is 1242490. What is the sum of all such integers n below 150 million?
147	Rectangles in cross-hatched grids	In a 3x2 cross-hatched grid, a total of 37 different rectangles could be situated within that grid as indicated in the sketch. There are 5 grids smaller than 3x2, vertical and horizontal dimensions being important, i.e. 1x1, 2x1, 3x1, 1x2 and 2x2. If each of them is cross-hatched, the following number of different rectangles could be situated within those smaller grids: 1x1 1 2x1 4 3x1 8 1x2 4 2x2 18 Adding those to the 37 of the 3x2 grid, a total of 72 different rectangles could be situated within 3x2 and smaller grids. How many different rectangles could be situated within 47x43 and smaller grids?

148	Exploring Pascal's triangle	We can easily verify that none of the entries in the first seven rows of Pascal's triangle are divisible by 7: 1 1 1 1 2 1 1 3 3 1 1 4 6 4 1 1 5 10 10 5 1 1 6 15 20 15 6 1 However, if we check the first one hundred rows, we will find that only 2361 of the 5050 entries are not divisible by 7. Find the number of entries which are not divisible by 7 in the first one billion (109) rows of Pascal's triangle.

149	arching for a maximum-sum subsequen	Looking at the table below, it is easy to verify that the maximum possible sum of adjacent numbers in any direction (horizontal, vertical, diagonal or anti-diagonal) is 16 (= 8 + 7 + 1). −2 5 3 2 9 −6 5 1 3 2 7 3 −1 8 −4 8 Now, let us repeat the search, but on a much larger scale: First, generate four million pseudo-random numbers using a specific form of what is known as a "Lagged Fibonacci Generator": For 1 ≤ k ≤ 55, sk = [100003 − 200003k + 300007k3] (modulo 1000000) − 500000. For 56 ≤ k ≤ 4000000, sk = [sk−24 + sk−55 + 1000000] (modulo 1000000) − 500000. Thus, s10 = −393027 and s100 = 86613. The terms of s are then arranged in a 2000×2000 table, using the first 2000

150	iangular array for a sub-triangle having r	In a triangular array of positive and negative integers, we wish to find a sub-triangle such that the sum of the numbers it contains is the smallest possible. In the example below, it can be easily verified that the marked triangle satisfies this condition having a sum of −42. We wish to make such a triangular array with one thousand rows, so we generate 500500 pseudo-random numbers sk in the range ±219, using a type of random number generator (known as a Linear Congruential Generator) as follows: t := 0 for k = 1 up to k = 500500: t := (615949*t + 797807) modulo 220 sk := t−219 Thus: s1 = 273519, s2 = −153582, s3 = 450905 etc Our triangular array is then formed using the pseudo-random numbers

151	eets of standard sizes: an expected-value	A printing shop runs 16 batches (jobs) every week and each batch requires a sheet of special colour-proofing paper of size A5. Every Monday morning, the foreman opens a new envelope, containing a large sheet of the special paper with size A1. He proceeds to cut it in half, thus getting two sheets of size A2. Then he cuts one of them in half to get two sheets of size A3 and so on until he obtains the A5-size sheet needed for the first batch of the week. All the unused sheets are placed back in the envelope. At the beginning of each subsequent batch, he takes from the envelope one sheet of paper at random. If it is of size A5, he uses it. If it is larger, he repeats the 'cut-in-half' procedure until he has what he needs and any remaining sheets are always placed back in the envelope. Excluding the first and last batch of the

152	Writing 1/2 as a sum of inverse squares	There are several ways to write the number 1/2 as a sum of inverse squares using distinct integers. For instance, the numbers {2,3,4,5,7,12,15,20,28,35} can be used: 1 2 = 1 2 2 + 1 3 2 + 1 4 2 + 1 5 2 + 1

153	Investigating Gaussian Integers	As we all know the equation x2=-1 has no solutions for real x. If we however introduce the imaginary number i this equation has two solutions: x=i and x=-i. If we go a step further the equation (x-3)2=-4 has two complex solutions: x=3+2i and x=3-2i. x=3+2i and x=3-2i are called each others' complex conjugate. Numbers of the form a+bi are called complex numbers. In general a+bi and a−bi are each other's complex conjugate. A Gaussian Integer is a complex number a+bi such that both a and b are integers. The regular integers are also Gaussian integers (with b=0). To distinguish them from Gaussian integers with b ≠ 0 we call such integers "rational integers." A Gaussian integer is called a divisor of a rational integer n if the result is also a Gaussian integer.

154	Exploring Pascal's pyramid	A triangular pyramid is constructed using spherical balls so that each ball rests on exactly three balls of the next lower level. Then, we calculate the number of paths leading from the apex to each position: A path starts at the apex and progresses downwards to any of the three spheres directly below the current position. Consequently, the number of paths to reach a certain position is the sum of the numbers immediately above it (depending on the position, there are up to three numbers above it). The result is Pascal's pyramid and the numbers at each level n are the coefficients of the trinomial expansion (x + y + z)n. How many coefficients in the expansion of (x + y + z)200000 are multiples of 1012?

155	Counting Capacitor Circuits	An electric circuit uses exclusively identical capacitors of the same value C. The capacitors can be connected in series or in parallel to form sub-units, which can then be connected in series or in parallel with other capacitors or other sub-units to form larger sub-units, and so on up to a final circuit. Using this simple procedure and up to n identical capacitors, we can make circuits having a range of different total capacitances. For example, using up to n=3 capacitors of 60 F each, we can obtain the following 7 distinct total capacitance values: If we denote by D(n) the number of distinct total capacitance values we can obtain when using up to n equal-valued capacitors and the simple procedure described above, we have: D(1)=1, D(2)=3, D(3)=7 ... Find D(18). Reminder : When connecting capacitors C1, C2 etc in parallel, the

156	Counting Digits	Starting from zero the natural numbers are written down in base 10 like this: 0 1 2 3 4 5 6 7 8 9 10 11 12.... Consider the digit d=1. After we write down each number n, we will update the number of ones that have occurred and call this number f(n,1). The first values for f(n,1), then, are as follows: n f(n,1) 0 0 1 1 2 1 3 1 4 1 5 1 6 1 7 1 8 1 9 1 10 2 11 4 12 5 Note that f(n,1) never equals 3. So the first two solutions of the equation f(n,1)=n are n=0 and n=1.

157	ng the diophantine equation 1/a+1/b= p	Consider the diophantine equation 1/a+1/b= p/10n with a, b, p, n positive integers and a ≤ b. For n=1 this equation has 20 solutions that are listed below: 1/1+1/1=20/10 1/1+1/2=15/10 1/1+1/5=12/10 1/1+1/10=11/10 1/2+1/2=10/10 1/2+1/5=7/10 1/2+1/10=6/10 1/3+1/6=5/10 1/3+1/15=4/10 1/4+1/4=5/10 1/4+1/20=3/10 1/5+1/5=4/10 1/5+1/10=3/10 1/6+1/30=2/10 1/10+1/10=2/10 1/11+1/110=1/10 1/12+1/60=1/10 1/14+1/35=1/10 1/15+1/30=1/10 1/20+1/20=1/10 How many solutions has this equation for 1 ≤ n ≤ 9?

158	ly one character comes lexicographically	Taking three different letters from the 26 letters of the alphabet, character strings of length three can be formed. Examples are 'abc', 'hat' and 'zyx'. When we study these three examples we see that for 'abc' two characters come lexicographically after its neighbour to the left. For 'hat' there is exactly one character that comes lexicographically after its neighbour to the left. For 'zyx' there are zero characters that come lexicographically after its neighbour to the left. In all there are 10400 strings of length 3 for which exactly one character comes lexicographically after its neighbour to the left. We now consider strings of $n \le 26$ different characters from the alphabet. For every n, p(n) is the number of strings of length n for which exactly one character comes lexicographically after its neighbour to the left. What is the maximum value of p(n)?

159	Digital root sums of factorisations	A composite number can be factored many different ways. For instance, not including multiplication by one, 24 can be factored in 7 distinct ways: 24 = 2x2x2x3 24 = 2x3x4 24 = 2x2x6 24 = 4x6 24 = 3x8 24 = 2x12 24 = 24 Recall that the digital root of a number, in base 10, is found by adding together the digits of that number, and repeating that process until a number is arrived at that is less than 10. Thus the digital root of 467 is 8. We shall call a Digital Root Sum (DRS) the sum of the digital roots of the individual factors of our number. The chart below demonstrates all of the DRS values for 24. Factorisation Digital Root Sum 2x2x2x3 9
160	Factorial trailing digits	For any N, let f(N) be the last five digits before the trailing zeroes in N!. For example, 9! = 362880 so f(9)=36288 10! = 3628800 so f(10)=36288 20! = 2432902008176640000 so f(20)=17664 Find f(1,000,000,000,000)

161	Triominoes	A triomino is a shape consisting of three squares joined via the edges. There are two basic forms: If all possible orientations are taken into account there are six: Any n by m grid for which nxm is divisible by 3 can be tiled with triominoes. If we consider tilings that can be obtained by reflection or rotation from another tiling as different there are 41 ways a 2 by 9 grid can be tiled with triominoes: In how many ways can a 9 by 12 grid be tiled in this way by triominoes?
162	Hexadecimal numbers	In the hexadecimal number system numbers are represented using 16 different digits: 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F The hexadecimal number AF when written in the decimal number system equals 10x16+15=175. In the 3-digit hexadecimal numbers 10A, 1A0, A10, and A01 the digits 0,1 and A are all present. Like numbers written in base ten we write hexadecimal numbers without leading zeroes. How many hexadecimal numbers containing at most sixteen hexadecimal digits exist with all of the digits 0,1, and A present at least once? Give your answer as a hexadecimal number. (A,B,C,D,E and F in upper case, without any leading or trailing code that marks the number as hexadecimal and without leading zeroes , e.g. 1A3F and not: 1a3f and not 0x1a3f and not $1A3F and not #1A3F and not

163	Cross-hatched triangles	Consider an equilateral triangle in which straight lines are drawn from each vertex to the middle of the opposite side, such as in the size 1 triangle in the sketch below. Sixteen triangles of either different shape or size or orientation or location can now be observed in that triangle. Using size 1 triangles as building blocks, larger triangles can be formed, such as the size 2 triangle in the above sketch. One-hundred and four triangles of either different shape or size or orientation or location can now be observed in that size 2 triangle. It can be observed that the size 2 triangle contains 4 size 1 triangle building blocks. A size 3 triangle would contain 9 size 1 triangle building blocks and a size n triangle would thus contain n2 size 1 triangle building blocks. If we denote T(n) as the number of triangles present in a triangle of size n, then
164	three consecutive digits have a sum gre	How many 20 digit numbers n (without any leading zero) exist such that no three consecutive digits of n have a sum greater than 9?

165	Intersections	A segment is uniquely defined by its two endpoints. By considering two line segments in plane geometry there are three possibilities: the segments have zero points, one point, or infinitely many points in common. Moreover when two segments have exactly one point in common it might be the case that that common point is an endpoint of either one of the segments or of both. If a common point of two segments is not an endpoint of either of the segments it is an interior point of both segments. We will call a common point T of two segments L1 and L2 a true intersection point of L1 and L2 if T is the only common point of L1 and L2 and T is an interior point of both segments. Consider the three segments L1, L2, and L3: L1: (27, 44) to (12, 32) L2: (46, 53) to (17, 62)
166	Criss Cross	A 4x4 grid is filled with digits d, $0 \le d \le 9$. It can be seen that in the grid 6 3 3 0 5 0 4 3 0 7 1 4 1 2 4 5 the sum of each row and each column has the value 12. Moreover the sum of each diagonal is also 12. In how many ways can you fill a 4x4 grid with the digits d, $0 \le d \le 9$ so that each row, each column, and both diagonals have the same sum?

167	Investigating Ulam sequences	For two positive integers a and b, the Ulam sequence U(a,b) is defined by U(a,b)1 = a, U(a,b)2 = b and for k > 2, U(a,b)k is the smallest integer greater than U(a,b)(k-1) which can be written in exactly one way as the sum of two distinct previous members of U(a,b). For example, the sequence U(1,2) begins with 1, 2, 3 = 1 + 2, 4 = 1 + 3, 6 = 2 + 4, 8 = 2 + 6, 11 = 3 + 8; 5 does not belong to it because 5 = 1 + 4 = 2 + 3 has two representations as the sum of two previous members, likewise 7 = 1 + 6 = 3 + 4. Find ∑ U(2,2n+1)k for 2 ≤ n ≤10, where k = 1011.
168	Number Rotations	Consider the number 142857. We can right-rotate this number by moving the last digit (7) to the front of it, giving us 714285. It can be verified that 714285=5×142857. This demonstrates an unusual property of 142857: it is a divisor of its right-rotation. Find the last 5 digits of the sum of all integers n, 10 < n < 10100, that have this property.

169	different ways a number can be expresse	Define f(0)=1 and f(n) to be the number of different ways n can be expressed as a sum of integer powers of 2 using each power no more than twice. For example, f(10)=5 since there are five different ways to express 10: 1 + 1 + 8 1 + 1 + 4 + 4 1 + 1 + 2 + 2 + 4 2 + 4 + 4 2 + 8 What is f(1025)?
170	9 pandigital that can be formed by con	Take the number 6 and multiply it by each of 1273 and 9854: 6 × 1273 = 7638 6 × 9854 = 59124 By concatenating these products we get the 1 to 9 pandigital 763859124. We will call 763859124 the "concatenated product of 6 and (1273,9854)". Notice too, that the concatenation of the input numbers, 612739854, is also 1 to 9 pandigital. The same can be done for 0 to 9 pandigital numbers. What is the largest 0 to 9 pandigital 10-digit concatenated product of an integer with two or more other integers, such that the concatenation of the input numbers is also a 0 to 9 pandigital 10-digit number?

171	for which the sum of the squares of the	For a positive integer n, let f(n) be the sum of the squares of the digits (in base 10) of n, e.g. f(3) = 32 = 9, f(25) = 22 + 52 = 4 + 25 = 29, f(442) = 42 + 42 + 22 = 16 + 16 + 4 = 36 Find the last nine digits of the sum of all n, 0 < n < 1020, such that f(n) is a perfect square.
172	estigating numbers with few repeated di	How many 18-digit numbers n (without leading zeros) are there such that no digit occurs more than three times in n?
173	tiles how many different "hollow" square	We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry. For example, using exactly thirty-two square tiles we can form two different square laminae: With one-hundred tiles, and not necessarily using all of the tiles at one time, it is possible to form forty-one different square laminae. Using up to one million tiles how many different square laminae can be formed?

174	/" square laminae that can form one, two	We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry. Given eight tiles it is possible to form a lamina in only one way: 3x3 square with a 1x1 hole in the middle. However, using thirty-two tiles it is possible to form two distinct laminae. If t represents the number of tiles used, we shall say that t = 8 is type L(1) and t = 32 is type L(2). Let N(n) be the number of t ≤ 1000000 such that t is type L(n); for example, N(15) = 832. What is ∑ N(n) for 1 ≤ n ≤ 10?

175	r of different ways a number can be exp	Define f(0)=1 and f(n) to be the number of ways to write n as a sum of powers of 2 where no power occurs more than twice. For example, f(10)=5 since there are five different ways to express 10: 10 = 8+2 = 8+1+1 = 4+4+2 = 4+2+2+1+1 = 4+4+1+1 It can be shown that for every fraction p/q (p>0, q>0) there exists at least one integer n such that f(n)/f(n-1)=p/q. For instance, the smallest n for which f(n)/f(n-1)=13/17 is 241. The binary expansion of 241 is 11110001. Reading this binary number from the most significant bit to the least significant bit there are 4 one's, 3 zeroes and 1 one. We shall call the string 4,3,1 the Shortened Binary Expansion of 241.
176	ght-angled triangles that share a cathet	The four right-angled triangles with sides (9,12,15), (12,16,20), (5,12,13) and (12,35,37) all have one of the shorter sides (catheti) equal to 12. It can be shown that no other integer sided right-angled triangle exists with one of the catheti equal to 12. Find the smallest integer that can be the length of a cathetus of exactly 47547 different integer sided right-angled triangles.

177	Integer angled Quadrilaterals	Let ABCD be a convex quadrilateral, with diagonals AC and BD. At each vertex the diagonal makes an angle with each of the two sides, creating eight corner angles. For example, at vertex A, the two angles are CAD, CAB. We call such a quadrilateral for which all eight corner angles have integer values when measured in degrees an "integer angled quadrilateral". An example of an integer angled quadrilateral is a square, where all eight corner angles are 45°. Another example is given by DAC = 20°, BAC = 60°, ABD = 50°, CBD = 30°, BCA = 40°, DCA = 30°, CDB = 80°, ADB = 50°. What is the total number of non-similar integer angled quadrilaterals? Note: In your calculations you may assume that a calculated angle is integral if it is within a tolerance of 10-9 of an integer value.
178	Step Numbers	Consider the number 45656. It can be seen that each pair of consecutive digits of 45656 has a difference of one. A number for which every pair of consecutive digits has a difference of one is called a step number. A pandigital number contains every decimal digit from 0 to 9 at least once. How many pandigital step numbers less than 1040 are there?

179	Consecutive positive divisors	Find the number of integers 1 < n < 107, for which n and n + 1 have the same number of positive divisors. For example, 14 has the positive divisors 1, 2, 7, 14 while 15 has 1, 3, 5, 15.
180	tional zeros of a function of three variab	For any integer n, consider the three functions f1,n(x,y,z) = xn+1 + yn+1 − zn+1 f2,n(x,y,z) = (xy + yz + zx)*(xn-1 + yn-1 − zn-1) f3,n(x,y,z) = xyz*(xn-2 + yn-2 − zn-2) and their combination fn(x,y,z) = f1,n(x,y,z) + f2,n(x,y,z) − f3,n(x,y,z) We call (x,y,z) a golden triple of order k if x, y, and z are all rational numbers of the form a / b with 0 < a < b ≤ k and there is (at least) one integer n, so that fn(x,y,z) = 0. Let s(x,y,z) = x + y + z. Let t = u / v be the sum of all distinct s(x,y,z) for all golden triples (x,y,z) of order 35. All the s(x,y,z) and t must be in reduced form. Find u + v.
181	many ways objects of two different colo	Having three black objects B and one white object W they can be grouped in 7 ways like this: (BBBW) (B,BBW) (B,B,BW) (B,B,B,W) (B,BB,W) (BBB,W) (BB,BW) In how many ways can sixty black objects B and forty white objects W be thus grouped?

182	RSA encryption	The RSA encryption is based on the following procedure: Generate two distinct primes p and q. Compute n=pq and φ=(p-1)(q-1). Find an integer e, 1<e<φ, such that gcd(e,φ)=1. A message in this system is a number in the interval [0,n-1]. A text to be encrypted is then somehow converted to messages (numbers in the interval [0,n-1]). To encrypt the text, for each message, m, c=me mod n is calculated. To decrypt the text, the following procedure is needed: calculate d such that ed=1 mod φ, then for each encrypted message, c, calculate m=cd mod n. There exist values of e and m such that me mod n=m. We call messages m for which me mod n=m unconcealed messages. An issue when choosing e is that there should not be too many unconcealed messages.

183	Maximum product of parts	Let N be a positive integer and let N be split into k equal parts, r = N/k, so that N = r + r + ... + r. Let P be the product of these parts, P = r × r × ... × r = rk. For example, if 11 is split into five equal parts, 11 = 2.2 + 2.2 + 2.2 + 2.2 + 2.2, then P = 2.25 = 51.53632. Let M(N) = Pmax for a given value of N. It turns out that the maximum for N = 11 is found by splitting eleven into four equal parts which leads to Pmax = (11/4)4; that is, M(11) = 14641/256 = 57.19140625, which is a terminating decimal. However, for N = 8 the maximum is achieved by splitting it into three equal parts, so M(8) = 512/27, which is a non-terminating decimal. Let D(N) = N if M(N) is a non-terminating decimal and D(N) = -N if M(N) is a terminating decimal. For example, ∑ D(N) for 5 ≤ N ≤ 100 is 2438.

184	Triangles containing the origin	Consider the set Ir of points (x,y) with integer co-ordinates in the interior of the circle with radius r, centered at the origin, i.e. x2 + y2 < r2. For a radius of 2, I2 contains the nine points (0,0), (1,0), (1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1) and (1,-1). There are eight triangles having all three vertices in I2 which contain the origin in the interior. Two of them are shown below, the others are obtained from these by rotation. For a radius of 3, there are 360 triangles containing the origin in the interior and having all vertices in I3 and for I5 the number is 10600. How many triangles are there containing the origin in the interior and having all three vertices in I105?

185	Number Mind	The game Number Mind is a variant of the well known game Master Mind. Instead of coloured pegs, you have to guess a secret sequence of digits. After each guess you're only told in how many places you've guessed the correct digit. So, if the sequence was 1234 and you guessed 2036, you'd be told that you have one correct digit; however, you would NOT be told that you also have another digit in the wrong place. For instance, given the following guesses for a 5-digit secret sequence, 90342 ;2 correct 70794 ;0 correct 39458 ;2 correct 34109 ;1 correct 51545 ;2 correct 12531 ;1 correct The correct sequence 39542 is unique. Based on the following guesses, 5616185650518293 ;2 correct 3847439647293047 ;1 correct 5855462940810587 ;3 correct

186	Connectedness of a network	Here are the records from a busy telephone system with one million users: RecNr Caller Called 1 200007 100053 2 600183 500439 3 600863 701497 ... ... The telephone number of the caller and the called number in record n are Caller(n) = S2n-1 and Called(n) = S2n where S1,2,3,... come from the "Lagged Fibonacci Generator": For 1 ≤ k ≤ 55, Sk = [100003 - 200003k + 300007k3] (modulo 1000000) For 56 ≤ k, Sk = [Sk-24 + Sk-55] (modulo 1000000) If Caller(n) = Called(n) then the user is assumed to have misdialled and the call fails; otherwise the call is successful. From the start of the records, we say that any pair of users X and Y are friends if X calls Y or vice-versa.
187	Semiprimes	A composite is a number containing at least two prime factors. For example, 15 = 3 × 5; 9 = 3 × 3; 12 = 2 × 2 × 3. There are ten composites below thirty containing precisely two, not necessarily distinct, prime factors: 4, 6, 9, 10, 14, 15, 21, 22, 25, 26. How many composite integers, n < 108, have precisely two, not necessarily distinct, prime factors?

188	The hyperexponentiation of a number	The hyperexponentiation or tetration of a number a by a positive integer b, denoted by a↑↑b or ba, is recursively defined by: a↑↑1 = a, a↑↑(k+1) = a(a↑↑k). Thus we have e.g. 3↑↑2 = 33 = 27, hence 3↑↑3 = 327 = 7625597484987 and 3↑↑4 is roughly 103.6383346400240996*10^12. Find the last 8 digits of 1777↑↑1855.
189	Tri-colouring a triangular grid	Consider the following configuration of 64 triangles: We wish to colour the interior of each triangle with one of three colours: red, green or blue, so that no two neighbouring triangles have the same colour. Such a colouring shall be called valid. Here, two triangles are said to be neighbouring if they share an edge. Note: if they only share a vertex, then they are not neighbours. For example, here is a valid colouring of the above grid: A colouring C' which is obtained from a colouring C by rotation or reflection is considered distinct from C unless the two are identical. How many distinct valid colourings are there for the above configuration?

190	Maximising a weighted product	Let Sm = (x1, x2, ... , xm) be the m-tuple of positive real numbers with x1 + x2 + ... + xm = m for which Pm = x1 * x22 * ... * xmm is maximised. For example, it can be verified that [P10] = 4112 ([ ] is the integer part function). Find Σ[Pm] for 2 ≤ m ≤ 15.
191	Prize Strings	A particular school offers cash rewards to children with good attendance and punctuality. If they are absent for three consecutive days or late on more than one occasion then they forfeit their prize. During an n-day period a trinary string is formed for each child consisting of L's (late), O's (on time), and A's (absent). Although there are eighty-one trinary strings for a 4-day period that can be formed, exactly forty-three strings would lead to a prize: OOOO OOOA OOOL OOAO OOAA OOAL OOLO OOLA OAOO OAOA OAOL OAAO OAAL OALO OALA OLOO OLOA OLAO OLAA AOOO AOOA AOOL AOAO AOAA AOAL AOLO AOLA AAOO AAOA AAOL AALO AALA ALOO ALOA ALAO ALAA LOOO LOOA LOAO LOAA LAOO LAOA LAAO How many "prize" strings exist over a 30-day period?

192	Best Approximations	Let x x be a real number. A best approximation to x x for the denominator bound d d is a rational number r s r in reduced form, with s≤d s , such that any rational number which is closer to x x than r s r
193	Squarefree Numbers	A positive integer n is called squarefree, if no square of a prime divides n, thus 1, 2, 3, 5, 6, 7, 10, 11 are squarefree, but not 4, 8, 9, 12. How many squarefree numbers are there below 250?

194	Coloured Configurations	Consider graphs built with the units A: and B: , where the units are glued along the vertical edges as in the graph . A configuration of type (a,b,c) is a graph thus built of a units A and b units B, where the graph's vertices are coloured using up to c colours, so that no two adjacent vertices have the same colour. The compound graph above is an example of a configuration of type (2,2,6), in fact of type (2,2,c) for all c ≥ 4. Let N(a,b,c) be the number of configurations of type (a,b,c). For example, N(1,0,3) = 24, N(0,2,4) = 92928 and N(2,2,3) = 20736. Find the last 8 digits of N(25,75,1984).
195	l circles of triangles with one angle of 60	Let's call an integer sided triangle with exactly one angle of 60 degrees a 60-degree triangle. Let r be the radius of the inscribed circle of such a 60-degree triangle. There are 1234 60-degree triangles for which r ≤ 100. Let T(n) be the number of 60-degree triangles for which r ≤ n, so T(100) = 1234, T(1000) = 22767, and T(10000) = 359912. Find T(1053779).

196	Prime triplets	Build a triangle from all positive integers in the following way: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 . . . Each positive integer has up to eight neighbours in the triangle. A set of three primes is called a prime triplet if one of the three primes has the other two as neighbours in the triangle. For example, in the second row, the prime numbers 2 and 3 are elements of some prime triplet. If row 8 is considered, it contains two primes which are elements of some
197	ng the behaviour of a recursively definec	Given is the function f(x) = ⌊230.403243784-x2⌋ × 10-9 (⌊ ⌋ is the floor-function), the sequence un is defined by u0 = -1 and un+1 = f(un). Find un + un+1 for n = 1012. Give your answer with 9 digits after the decimal point.

198	Ambiguous Numbers	A best approximation to a real number x x for the denominator bound d d is a rational number r s r (in reduced form) with s≤d s , so that any rational number p q p which is closer to x x than r s r has

199	Iterative Circle Packing	Three circles of equal radius are placed inside a larger circle such that each pair of circles is tangent to one another and the inner circles do not overlap. There are four uncovered "gaps" which are to be filled iteratively with more tangent circles. At each iteration, a maximally sized circle is placed in each gap, which creates more gaps for the next iteration. After 3 iterations (pictured), there are 108 gaps and the fraction of the area which is not covered by circles is 0.06790342, rounded to eight decimal places. What fraction of the area is not covered by circles after 10 iterations? Give your answer rounded to eight decimal places using the format x.xxxxxxxx .
200	ne-proof sqube containing the contiguo	We shall define a sqube to be a number of the form, p2q3, where p and q are distinct primes. For example, 200 = 5223 or 120072949 = 232613. The first five squbes are 72, 108, 200, 392, and 500. Interestingly, 200 is also the first number for which you cannot change any single digit to make a prime; we shall call such numbers, prime-proof. The next prime-proof sqube which contains the contiguous sub-string "200" is 1992008. Find the 200th prime-proof sqube containing the contiguous sub-string "200".

201	Subsets with a unique sum	For any set A of numbers, let sum(A) be the sum of the elements of A. Consider the set B = {1,3,6,8,10,11}. There are 20 subsets of B containing three elements, and their sums are: sum({1,3,6}) = 10, sum({1,3,8}) = 12, sum({1,3,10}) = 14, sum({1,3,11}) = 15, sum({1,6,8}) = 15, sum({1,6,10}) = 17, sum({1,6,11}) = 18, sum({1,8,10}) = 19, sum({1,8,11}) = 20, sum({1,10,11}) = 22, sum({3,6,8}) = 17, sum({3,6,10}) = 19, sum({3,6,11}) = 20, sum({3,8,10}) = 21, sum({3,8,11}) = 22, sum({3,10,11}) = 24, sum({6,8,10}) = 24, sum({6,8,11}) = 25, sum({6,10,11}) = 27, sum({8,10,11}) = 29.

202	Laserbeam	Three mirrors are arranged in the shape of an equilateral triangle, with their reflective surfaces pointing inwards. There is an infinitesimal gap at each vertex of the triangle through which a laser beam may pass. Label the vertices A, B and C. There are 2 ways in which a laser beam may enter vertex C, bounce off 11 surfaces, then exit through the same vertex: one way is shown below; the other is the reverse of that. There are 80840 ways in which a laser beam may enter vertex C, bounce off 1000001 surfaces, then exit through the same vertex. In how many ways can a laser beam enter at vertex C, bounce off 12017639147 surfaces, then exit through the same vertex?

203	Squarefree Binomial Coefficients	The binomial coefficients (n k) (can be arranged in triangular form, Pascal's triangle, like this: 1 1 1 1 2 1 1 3 3 1 1 4 6 4 1 1 5 10 10 5 1 1 6 15 20 15 6 1 1 7 21 35 35 21 7 1 .... It can be seen that the first eight rows of Pascal's triangle contain twelve distinct numbers: 1, 2, 3, 4, 5, 6, 7, 10, 15, 20, 21 and 35. A positive integer n is called squarefree if no square of a prime divides n. Of the twelve distinct numbers in the first eight rows of Pascal's triangle, all

204	Generalised Hamming Numbers	A Hamming number is a positive number which has no prime factor larger than 5. So the first few Hamming numbers are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15. There are 1105 Hamming numbers not exceeding 108. We will call a positive number a generalised Hamming number of type n, if it has no prime factor larger than n. Hence the Hamming numbers are the generalised Hamming numbers of type 5. How many generalised Hamming numbers of type 100 are there which don't exceed 109?
205	Dice Game	Peter has nine four-sided (pyramidal) dice, each with faces numbered 1, 2, 3, 4. Colin has six six-sided (cubic) dice, each with faces numbered 1, 2, 3, 4, 5, 6. Peter and Colin roll their dice and compare totals: the highest total wins. The result is a draw if the totals are equal. What is the probability that Pyramidal Pete beats Cubic Colin? Give your answer rounded to seven decimal places in the form 0.abcdefg
206	Concealed Square	Find the unique positive integer whose square has the form 1_2_3_4_5_6_7_8_9_0, where each "_" is a single digit.

207	Integer partition equations	For some positive integers k, there exists an integer partition of the form $4^t = 2^t + k$, where 4^t, 2^t, and k are all positive integers and t is a real number. The first two such partitions are $4^1 = 2^1 + 2$ and $4^{1.5849625...} = 2^{1.5849625...} + 6$. Partitions where t is also an integer are called perfect. For any m ≥ 1 let P(m) be the proportion of such partitions that are perfect with k ≤ m. Thus P(6) = 1/2. In the following table are listed some values of P(m) P(5) = 1/1 P(10) = 1/2 P(15) = 2/3 P(20) = 1/2 P(25) = 1/2 P(30) = 2/5 ... P(180) = 1/4 P(185) = 3/13
208	Robot Walks	A robot moves in a series of one-fifth circular arcs (72°), with a free choice of a clockwise or an anticlockwise arc for each step, but no turning on the spot. One of 70932 possible closed paths of 25 arcs starting northward is Given that the robot starts facing North, how many journeys of 70 arcs in length can it take that return it, after the final arc, to its starting position? (Any arc may be traversed multiple times.)

209	Circular Logic	A k-input binary truth table is a map from k input bits (binary digits, 0 [false] or 1 [true]) to 1 output bit. For example, the 2-input binary truth tables for the logical AND and XOR functions are: x y x AND y 0 0 0 0 1 0 1 0 0 1 1 1 x y x XOR y 0 0 0 0 1 1 1 0 1 1 1 0 How many 6-input binary truth tables, τ, satisfy the formula τ(a, b, c, d, e, f) AND τ(b, c, d, e, f, a XOR (b AND c)) = 0 for all 6-bit inputs (a, b, c, d, e, f)?
210	Obtuse Angled Triangles	Consider the set S(r) of points (x,y) with integer coordinates satisfying $\|x\| + \|y\| \le r$. Let O be the point (0,0) and C the point (r/4,r/4). Let N(r) be the number of points B in S(r), so that the triangle OBC has an obtuse angle, i.e. the largest angle α satisfies 90°<α<180°. So, for example, N(4)=24 and N(8)=100. What is N(1,000,000,000)?

211	Divisor Square Sum	For a positive integer n, let σ2(n) be the sum of the squares of its divisors. For example, σ2(10) = 1 + 4 + 25 + 100 = 130. Find the sum of all n, 0 < n < 64,000,000 such that σ2(n) is a perfect square.
212	Combined Volume of Cuboids	An axis-aligned cuboid, specified by parameters { (x0,y0,z0), (dx,dy,dz) }, consists of all points (X,Y,Z) such that x0 ≤ X ≤ x0+dx, y0 ≤ Y ≤ y0+dy and z0 ≤ Z ≤ z0+dz. The volume of the cuboid is the product, dx × dy × dz. The combined volume of a collection of cuboids is the volume of their union and will be less than the sum of the individual volumes if any cuboids overlap. Let C1,...,C50000 be a collection of 50000 axis-aligned cuboids such that Cn has parameters x0 = S6n-5 modulo 10000 y0 = S6n-4 modulo 10000 z0 = S6n-3 modulo 10000 dx = 1 + (S6n-2 modulo 399) dy = 1 + (S6n-1 modulo 399) dz = 1 + (S6n modulo 399) where S1,...,S300000 come from the "Lagged Fibonacci Generator": For 1 ≤ k ≤ 55, Sk = [100003 - 200003k + 300007k3] (modulo 1000000)

213	Flea Circus	A 30×30 grid of squares contains 900 fleas, initially one flea per square. When a bell is rung, each flea jumps to an adjacent square at random (usually 4 possibilities, except for fleas on the edge of the grid or at the corners). What is the expected number of unoccupied squares after 50 rings of the bell? Give your answer rounded to six decimal places.
214	Totient Chains	Let φ be Euler's totient function, i.e. for a natural number n, φ(n) is the number of k, 1 ≤ k ≤ n, for which gcd(k,n) = 1. By iterating φ, each positive integer generates a decreasing chain of numbers ending in 1. E.g. if we start with 5 the sequence 5,4,2,1 is generated. Here is a listing of all chains with length 4: 5,4,2,1 7,6,2,1 8,4,2,1 9,6,2,1 10,4,2,1 12,4,2,1 14,6,2,1 18,6,2,1 Only two of these chains start with a prime, their sum is 12. What is the sum of all primes less than 40000000 which generate a chain of length 25?

215	Crack-free Walls	Consider the problem of building a wall out of 2×1 and 3×1 bricks (horizontal×vertical dimensions) such that, for extra strength, the gaps between horizontally-adjacent bricks never line up in consecutive layers, i.e. never form a "running crack". For example, the following 9×3 wall is not acceptable due to the running crack shown in red: There are eight ways of forming a crack-free 9×3 wall, written W(9,3) = 8. Calculate W(32,10).
216	ating the primality of numbers of the for	Consider numbers t(n) of the form t(n) = 2n2-1 with n > 1. The first such numbers are 7, 17, 31, 49, 71, 97, 127 and 161. It turns out that only 49 = 7*7 and 161 = 7*23 are not prime. For n ≤ 10000 there are 2202 numbers t(n) that are prime. How many numbers t(n) are prime for n ≤ 50,000,000 ?
217	Balanced Numbers	A positive integer with k (decimal) digits is called balanced if its first ⌈k/2⌉ digits sum to the same value as its last ⌈k/2⌉ digits, where ⌈x⌉, pronounced ceiling of x, is the smallest integer ≥ x, thus ⌈π⌉ = 4 and ⌈5⌉ = 5. So, for example, all palindromes are balanced, as is 13722. Let T(n) be the sum of all balanced numbers less than 10n. Thus: T(1) = 45, T(2) = 540 and T(5) = 334795890. Find T(47) mod 315

218	Perfect right-angled triangles	Consider the right angled triangle with sides a=7, b=24 and c=25. The area of this triangle is 84, which is divisible by the perfect numbers 6 and 28. Moreover it is a primitive right angled triangle as gcd(a,b)=1 and gcd(b,c)=1. Also c is a perfect square. We will call a right angled triangle perfect if -it is a primitive right angled triangle -its hypotenuse is a perfect square We will call a right angled triangle super-perfect if -it is a perfect right angled triangle and -its area is a multiple of the perfect numbers 6 and 28. How many perfect right-angled triangles with c≤1016 exist that are not super-perfect?

219	Skew-cost coding	Let A and B be bit strings (sequences of 0's and 1's). If A is equal to the leftmost length(A) bits of B, then A is said to be a prefix of B. For example, 00110 is a prefix of 001101001, but not of 00111 or 100110. A prefix-free code of size n is a collection of n distinct bit strings such that no string is a prefix of any other. For example, this is a prefix-free code of size 6: 0000, 0001, 001, 01, 10, 11 Now suppose that it costs one penny to transmit a '0' bit, but four pence to transmit a '1'. Then the total cost of the prefix-free code shown above is 35 pence, which happens to be the cheapest possible for the skewed pricing scheme in question. In short, we write Cost(6) = 35. What is Cost(109) ?

220	Heighway Dragon	Let D0 be the two-letter string "Fa". For n≥1, derive Dn from Dn-1 by the string-rewriting rules: "a" → "aRbFR" "b" → "LFaLb" Thus, D0 = "Fa", D1 = "FaRbFR", D2 = "FaRbFRRLFaLbFR", and so on. These strings can be interpreted as instructions to a computer graphics program, with "F" meaning "draw forward one unit", "L" meaning "turn left 90 degrees", "R" meaning "turn right 90 degrees", and "a" and "b" being ignored. The initial position of the computer cursor is (0,0), pointing up towards (0,1). Then Dn is an exotic drawing known as the Heighway Dragon of order n. For example, D10 is shown below; counting each "F" as one step, the highlighted spot at (18,16) is the position reached after 500 steps. What is the position of the cursor after 1012 steps in D50 ? Give your answer in the form x,y with

221	Alexandrian Integers	We shall call a positive integer A an "Alexandrian integer", if there exist integers p, q, r such that: A = p · q · r and 1 A = 1 p + 1 q + 1 r For example, 630 is an Alexandrian integer (p = 5, q = −7, r = −18). In fact, 630 is the 6th Alexandrian integer, the first 6 Alexandrian integers being: 6, 42, 120, 156, 420 and 630. Find the 150000th Alexandrian integer.
222	Sphere Packing	What is the length of the shortest pipe, of internal radius 50mm, that can fully contain 21 balls of radii 30mm, 31mm, ..., 50mm? Give your answer in micrometres (10-6 m) rounded to the nearest integer.
223	Almost right-angled triangles I	Let us call an integer sided triangle with sides a ≤ b ≤ c barely acute if the sides satisfy a2 + b2 = c2 + 1. How many barely acute triangles are there with perimeter ≤ 25,000,000?

224	Almost right-angled triangles II	Let us call an integer sided triangle with sides a ≤ b ≤ c barely obtuse if the sides satisfy a2 + b2 = c2 - 1. How many barely obtuse triangles are there with perimeter ≤ 75,000,000?
225	Tribonacci non-divisors	The sequence 1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201 ... is defined by T1 = T2 = T3 = 1 and Tn = Tn-1 + Tn-2 + Tn-3. It can be shown that 27 does not divide any terms of this sequence. In fact, 27 is the first odd number with this property. Find the 124th odd number that does not divide any terms of the above sequence.

226	A Scoop of Blancmange	The blancmange curve is the set of points (x,y) (such that 0≤x≤1 0 and y= ∑ n=0 ∞ s(2 n x) 2 n y , where s(x) s is the distance from x x

227	The Chase	"The Chase" is a game played with two dice and an even number of players. The players sit around a table; the game begins with two opposite players having one die each. On each turn, the two players with a die roll it. If a player rolls a 1, he passes the die to his neighbour on the left; if he rolls a 6, he passes the die to his neighbour on the right; otherwise, he keeps the die for the next turn. The game ends when one player has both dice after they have been rolled and passed; that player has then lost. In a game with 100 players, what is the expected number of turns the game lasts? Give your answer rounded to ten significant digits.
228	Minkowski Sums	Let Sn be the regular n-sided polygon – or shape – whose vertices vk (k = 1,2,…,n) have coordinates: xk = cos(2k-1/n ×180°) yk = sin(2k-1/n ×180°) Each Sn is to be interpreted as a filled shape consisting of all points on the perimeter and in the interior. The Minkowski sum, S+T, of two shapes S and T is the result of adding every point in S to every point in T, where point addition is performed coordinate-wise: (u, v) + (x, y) = (u+x, v+y). For example, the sum of S3 and S4 is the six-sided shape shown in pink below: How many sides does S1864 + S1865 + … + S1909 have?

229	Four Representations using Squares	Consider the number 3600. It is very special, because 3600 = 482 + 362 3600 = 202 + 2×402 3600 = 302 + 3×302 3600 = 452 + 7×152 Similarly, we find that 88201 = 992 + 2802 = 2872 + 2×542 = 2832 + 3×522 = 1972 + 7×842. In 1747, Euler proved which numbers are representable as a sum of two squares. We are interested in the numbers n which admit representations of all of the following four types: n = a12 + b12 n = a22 + 2 b22 n = a32 + 3 b32

230	Fibonacci Words	For any two strings of digits, A and B, we define FA,B to be the sequence (A,B,AB,BAB,ABBAB,...) in which each term is the concatenation of the previous two. Further, we define DA,B(n) to be the nth digit in the first term of FA,B that contains at least n digits. Example: Let A=1415926535, B=8979323846. We wish to find DA,B(35), say. The first few terms of FA,B are: 1415926535 8979323846 14159265358979323846 897932384614159265358979323846 14159265358979323846897932384614159265358979323846 Then DA,B(35) is the 35th digit in the fifth term, which is 9. Now we use for A the first 100 digits of π behind the decimal point: 14159265358979323846264338327950288419716939937510 58209749445923078164062862089986

231	prime factorisation of binomial coefficie	The binomial coefficient (10 3)=120 (. 120= 2 3 ×3×5=2×2×2×3×5 120 , and 2+2+2+3+5=14 2 . So the sum of the terms in the prime factorisation of (10 3) (is 14

232	The Race	Two players share an unbiased coin and take it in turns to play "The Race". On Player 1's turn, he tosses the coin once: if it comes up Heads, he scores one point; if it comes up Tails, he scores nothing. On Player 2's turn, she chooses a positive integer T and tosses the coin T times: if it comes up all Heads, she scores 2T-1 points; otherwise, she scores nothing. Player 1 goes first. The winner is the first to 100 or more points. On each turn Player 2 selects the number, T, of coin tosses that maximises the probability of her winning. What is the probability that Player 2 wins? Give your answer rounded to eight decimal places in the form 0.abcdefgh .
233	Lattice points on a circle	Let f(N) be the number of points with integer coordinates that are on a circle passing through (0,0), (N,0),(0,N), and (N,N). It can be shown that f(10000) = 36. What is the sum of all positive integers N ≤ 1011 such that f(N) = 420 ?

234	Semidivisible numbers	For an integer n ≥ 4, we define the lower prime square root of n, denoted by lps(n), as the largest prime ≤ √n and the upper prime square root of n, ups(n), as the smallest prime ≥ √n. So, for example, lps(4) = 2 = ups(4), lps(1000) = 31, ups(1000) = 37. Let us call an integer n ≥ 4 semidivisible, if one of lps(n) and ups(n) divides n, but not both. The sum of the semidivisible numbers not exceeding 15 is 30, the numbers are 8, 10 and 12. 15 is not semidivisible because it is a multiple of both lps(15) = 3 and ups(15) = 5. As a further example, the sum of the 92 semidivisible numbers up to 1000 is 34825. What is the sum of all semidivisible numbers not exceeding 999966663333 ?
235	An Arithmetic Geometric sequence	Given is the arithmetic-geometric sequence u(k) = (900-3k)rk-1. Let s(n) = Σk=1...nu(k). Find the value of r for which s(5000) = -600,000,000,000. Give your answer rounded to 12 places behind the decimal point.

236	Luxury Hampers	Suppliers 'A' and 'B' provided the following numbers of products for the luxury hamper market: Product 'A' 'B' Beluga Caviar 5248 640 Christmas Cake 1312 1888 Gammon Joint 2624 3776 Vintage Port 5760 3776 Champagne Truffles 3936 5664 Although the suppliers try very hard to ship their goods in perfect condition, there is inevitably some spoilage - i.e. products gone bad. The suppliers compare their performance using two types of statistic: The five per-product spoilage rates for each supplier are equal to the number of products gone bad divided by the number of products supplied, for each of the five products in turn. The overall spoilage rate for each supplier is equal to the total number of products gone bad divided by the total number of products provided by that
237	Tours on a 4 x n playing board	Let T(n) be the number of tours over a 4 × n playing board such that: The tour starts in the top left corner. The tour consists of moves that are up, down, left, or right one square. The tour visits each square exactly once. The tour ends in the bottom left corner. The diagram shows one tour over a 4 × 10 board: T(10) is 2329. What is T(1012) modulo 108?

238	Infinite string tour	Create a sequence of numbers using the "Blum Blum Shub" pseudo-random number generator: s0 = 14025256 sn+1 = sn2 mod 20300713 Concatenate these numbers s0s1s2… to create a string w of infinite length. Then, w = 14025256741014958470038053646… For a positive integer k, if no substring of w exists with a sum of digits equal to k, p(k) is defined to be zero. If at least one substring of w exists with a sum of digits equal to k, we define p(k) = z, where z is the starting position of the earliest such substring. For instance: The substrings 1, 14, 1402, … with respective sums of digits equal to 1, 5, 7, … start at position 1, hence p(1) = p(5) = p(7) = … = 1. The substrings 4, 402, 4025, … with respective sums of digits equal to
239	Twenty-two Foolish Primes	A set of disks numbered 1 through 100 are placed in a line in random order. What is the probability that we have a partial derangement such that exactly 22 prime number discs are found away from their natural positions? (Any number of non-prime disks may also be found in or out of their natural positions.) Give your answer rounded to 12 places behind the decimal point in the form 0.abcdefghijkl.

240	Top Dice	There are 1111 ways in which five 6-sided dice (sides numbered 1 to 6) can be rolled so that the top three sum to 15. Some examples are: D1,D2,D3,D4,D5 = 4,3,6,3,5 D1,D2,D3,D4,D5 = 4,3,3,5,6 D1,D2,D3,D4,D5 = 3,3,3,6,6 D1,D2,D3,D4,D5 = 6,6,3,3,3 In how many ways can twenty 12-sided dice (sides numbered 1 to 12) be rolled so that the top ten sum to 70?
241	Perfection Quotients	For a positive integer n, let σ(n) be the sum of all divisors of n, so e.g. σ(6) = 1 + 2 + 3 + 6 = 12. A perfect number, as you probably know, is a number with σ(n) = 2n. Let us define the perfection quotient of a positive integer as p(n) = σ(n) n . Find the sum of all positive integers n ≤ 1018 for which p(n) has the form k + 1⁄2, where k is an integer.

242	Odd Triplets	Given the set {1,2,...,n}, we define f(n,k) as the number of its k-element subsets with an odd sum of elements. For example, f(5,3) = 4, since the set {1,2,3,4,5} has four 3-element subsets having an odd sum of elements, i.e.: {1,2,4}, {1,3,5}, {2,3,4} and {2,4,5}. When all three values n, k and f(n,k) are odd, we say that they make an odd-triplet [n,k,f(n,k)]. There are exactly five odd-triplets with n ≤ 10, namely: [1,1,f(1,1) = 1], [5,1,f(5,1) = 3], [5,5,f(5,5) = 1], [9,1,f(9,1) = 5] and [9,9,f(9,9) = 1]. How many odd-triplets are there with n ≤ 1012 ?
243	Resilience	A positive fraction whose numerator is less than its denominator is called a proper fraction. For any denominator, d, there will be d−1 proper fractions; for example, with d = 12: 1/12 , 2/12 , 3/12 , 4/12 , 5/12 , 6/12 , 7/12 , 8/12 , 9/12 , 10/12 , 11/12 . We shall call a fraction that cannot be cancelled down a resilient fraction. Furthermore we shall define the resilience of a denominator, R(d), to be the ratio of its proper fractions that are resilient; for example, R(12) = 4/11 . In fact, d = 12 is the smallest denominator having a resilience R(d) < 4/10 . Find the smallest denominator d, having a resilience R(d) < 15499/94744 .

244	Sliders	You probably know the game Fifteen Puzzle. Here, instead of numbered tiles, we have seven red tiles and eight blue tiles. A move is denoted by the uppercase initial of the direction (Left, Right, Up, Down) in which the tile is slid, e.g. starting from configuration (S), by the sequence LULUR we reach the configuration (E): (S) , (E) For each path, its checksum is calculated by (pseudocode): checksum = 0 checksum = (checksum × 243 + m1) mod 100 000 007 checksum = (checksum × 243 + m2) mod 100 000 007 … checksum = (checksum × 243 + mn) mod 100 000 007 where mk is the ASCII value of the kth letter in the move sequence and the ASCII values for the moves are: L 76

245	Coresilience	We shall call a fraction that cannot be cancelled down a resilient fraction. Furthermore we shall define the resilience of a denominator, R(d), to be the ratio of its proper fractions that are resilient; for example, R(12) = 4⁄11. The resilience of a number d > 1 is then φ(d) d − 1 , where φ is Euler's totient function. We further define the coresilience of a number n > 1 as C(n) = n − φ(n) n − 1 . The coresilience of a prime p is C(p) = 1 p − 1 . Find the sum of all composite integers 1 < n ≤ 2×1011, for which C(n) is a unit fraction.

246	Tangents to an ellipse	A definition for an ellipse is: Given a circle c with centre M and radius r and a point G such that d(G,M)<r, the locus of the points that are equidistant from c and G form an ellipse. The construction of the points of the ellipse is shown below. Given are the points M(-2000,1500) and G(8000,1500). Given is also the circle c with centre M and radius 15000. The locus of the points that are equidistant from G and c form an ellipse e. From a point P outside e the two tangents t1 and t2 to the ellipse are drawn. Let the points where t1 and t2 touch the ellipse be R and S. For how many lattice points P is angle RPS greater than 45 degrees?

247	Squares under a hyperbola	Consider the region constrained by 1 ≤ x and 0 ≤ y ≤ 1/x. Let S1 be the largest square that can fit under the curve. Let S2 be the largest square that fits in the remaining area, and so on. Let the index of Sn be the pair (left, below) indicating the number of squares to the left of Sn and the number of squares below Sn. The diagram shows some such squares labelled by number. S2 has one square to its left and none below, so the index of S2 is (1,0). It can be seen that the index of S32 is (1,1) as is the index of S50. 50 is the largest n for which the index of Sn is (1,1). What is the largest n for which the index of Sn is (3,3)?
248	ers for which Euler’s totient function equ	The first number n for which φ(n)=13! is 6227180929. Find the 150,000th such number.
249	Prime Subset Sums	Let S = {2, 3, 5, ..., 4999} be the set of prime numbers less than 5000. Find the number of subsets of S, the sum of whose elements is a prime number. Enter the rightmost 16 digits as your answer.
250	250250	Find the number of non-empty subsets of {11, 22, 33,..., 250250250250}, the sum of whose elements is divisible by 250. Enter the rightmost 16 digits as your answer.

251	Cardano Triplets	A triplet of positive integers (a,b,c) is called a Cardano Triplet if it satisfies the condition: a+b c √ − − − − − − − √ 3 + a−b c √ − − − − − −

252	Convex Holes	Given a set of points on a plane, we define a convex hole to be a convex polygon having as vertices any of the given points and not containing any of the given points in its interior (in addition to the vertices, other given points may lie on the perimeter of the polygon). As an example, the image below shows a set of twenty points and a few such convex holes. The convex hole shown as a red heptagon has an area equal to 1049694.5 square units, which is the highest possible area for a convex hole on the given set of points. For our example, we used the first 20 points (T2k−1, T2k), for k = 1,2,…,20, produced with the pseudo-random number generator: S0 = 290797 Sn+1 = Sn2 mod 50515093 Tn = (Sn mod 2000) − 1000 i.e. (527, 144), (−488, 732), (−454, −947), … What is the maximum area for a

253	Tidying up	A small child has a "number caterpillar" consisting of forty jigsaw pieces, each with one number on it, which, when connected together in a line, reveal the numbers 1 to 40 in order. Every night, the child's father has to pick up the pieces of the caterpillar that have been scattered across the play room. He picks up the pieces at random and places them in the correct order. As the caterpillar is built up in this way, it forms distinct segments that gradually merge together. The number of segments starts at zero (no pieces placed), generally increases up to about eleven or twelve, then tends to drop again before finishing at a single segment (all pieces placed). For example: Piece Placed Segments So Far 12 1 4 2 29 3 6 4
254	Sums of Digit Factorials	Define f(n) as the sum of the factorials of the digits of n. For example, f(342) = 3! + 4! + 2! = 32. Define sf(n) as the sum of the digits of f(n). So sf(342) = 3 + 2 = 5. Define g(i) to be the smallest positive integer n such that sf(n) = i. Though sf(342) is 5, sf(25) is also 5, and it can be verified that g(5) is 25. Define sg(i) as the sum of the digits of g(i). So sg(5) = 2 + 5 = 7. Further, it can be verified that g(20) is 267 and ∑ sg(i) for 1 ≤ i ≤ 20 is 156. What is ∑ sg(i) for 1 ≤ i ≤ 150?

255	Rounded Square Roots	We define the rounded-square-root of a positive integer n as the square root of n rounded to the nearest integer. The following procedure (essentially Heron's method adapted to integer arithmetic) finds the rounded-square-root of n: Let d be the number of digits of the number n. If d is odd, set $x_0 = 2\times 10^{(d-1)/2}$ x . If d is even, set $x_0 = 7\times 10^{(d-2)/2}$ x .

256	Tatami-Free Rooms	Tatami are rectangular mats, used to completely cover the floor of a room, without overlap. Assuming that the only type of available tatami has dimensions 1×2, there are obviously some limitations for the shape and size of the rooms that can be covered. For this problem, we consider only rectangular rooms with integer dimensions a, b and even size s = a·b. We use the term 'size' to denote the floor surface area of the room, and — without loss of generality — we add the condition a ≤ b. There is one rule to follow when laying out tatami: there must be no points where corners of four different mats meet. For example, consider the two arrangements below for a 4×4 room: The arrangement on the left is acceptable, whereas the one on the right is not: a red "X" in the middle,

257	Angular Bisectors	Given is an integer sided triangle ABC with sides a ≤ b ≤ c. (AB = c, BC = a and AC = b). The angular bisectors of the triangle intersect the sides at points E, F and G (see picture below). The segments EF, EG and FG partition the triangle ABC into four smaller triangles: AEG, BFE, CGF and EFG. It can be proven that for each of these four triangles the ratio area(ABC)/area(subtriangle) is rational. However, there exist triangles for which some or all of these ratios are integral. How many triangles ABC with perimeter≤100,000,000 exist so that the ratio area(ABC)/area(AEG) is integral?
258	A lagged Fibonacci sequence	A sequence is defined as: gk = 1, for 0 ≤ k ≤ 1999 gk = gk-2000 + gk-1999, for k ≥ 2000. Find gk mod 20092010 for k = 1018.

259	Reachable Numbers	A positive integer will be called reachable if it can result from an arithmetic expression obeying the following rules: Uses the digits 1 through 9, in that order and exactly once each. Any successive digits can be concatenated (for example, using the digits 2, 3 and 4 we obtain the number 234). Only the four usual binary arithmetic operations (addition, subtraction, multiplication and division) are allowed. Each operation can be used any number of times, or not at all. Unary minus is not allowed. Any number of (possibly nested) parentheses may be used to define the order of operations. For example, 42 is reachable, since (1/23) * ((4*5)-6) * (78-9) = 42. What is the sum of all positive reachable integers?

260	Stone Game	A game is played with three piles of stones and two players. At her turn, a player removes one or more stones from the piles. However, if she takes stones from more than one pile, she must remove the same number of stones from each of the selected piles. In other words, the player chooses some N>0 and removes: N stones from any single pile; or N stones from each of any two piles (2N total); or N stones from each of the three piles (3N total). The player taking the last stone(s) wins the game. A winning configuration is one where the first player can force a win. For example, (0,0,13), (0,11,11) and (5,5,5) are winning configurations because the first player can immediately remove all stones. A losing configuration is one where the second player can force a win, no

261	Pivotal Square Sums	Let us call a positive integer k a square-pivot, if there is a pair of integers m > 0 and n ≥ k, such that the sum of the (m+1) consecutive squares up to k equals the sum of the m consecutive squares from (n+1) on: (k-m)2 + ... + k2 = (n+1)2 + ... + (n+m)2. Some small square-pivots are 4: 32 + 42 = 52 21: 202 + 212 = 292 24: 212 + 222 + 232 + 242 = 252 + 262 + 272 110: 1082 + 1092 + 1102 = 1332 + 1342 Find the sum of all distinct square-pivots ≤ 1010.

262	Mountain Range	The following equation represents the continuous topography of a mountainous region, giving the elevation h at any point (x,y): A mosquito intends to fly from A(200,200) to B(1400,1400), without leaving the area given by 0 ≤ x, y ≤ 1600. Because of the intervening mountains, it first rises straight up to a point A', having elevation f. Then, while remaining at the same elevation f, it flies around any obstacles until it arrives at a point B' directly above B. First, determine fmin which is the minimum constant elevation allowing such a trip from A to B, while remaining in the specified area. Then, find the length of the shortest path between A' and B', while flying at that constant elevation fmin. Give that length as your answer, rounded to three decimal places. Note: For convenience, the elevation

263	An engineers' dream come true	Consider the number 6. The divisors of 6 are: 1,2,3 and 6. Every number from 1 up to and including 6 can be written as a sum of distinct divisors of 6: 1=1, 2=2, 3=1+2, 4=1+3, 5=2+3, 6=6. A number n is called a practical number if every number from 1 up to and including n can be expressed as a sum of distinct divisors of n. A pair of consecutive prime numbers with a difference of six is called a sexy pair (since "sex" is the Latin word for "six"). The first sexy pair is (23, 29). We may occasionally find a triple-pair, which means three consecutive sexy prime pairs, such that the second member of each pair is the first member of the next pair. We shall call a number n such that : (n-9, n-3), (n-3,n+3), (n+3, n+9) form a triple-pair, and the numbers n-8, n-4, n, n+4 and n+8 are all practical, an engineers’ paradise.

264	Triangle Centres	Consider all the triangles having: All their vertices on lattice points. Circumcentre at the origin O. Orthocentre at the point H(5, 0). There are nine such triangles having a perimeter ≤ 50. Listed and shown in ascending order of their perimeter, they are: A(-4, 3), B(5, 0), C(4, -3) A(4, 3), B(5, 0), C(-4, -3) A(-3, 4), B(5, 0), C(3, -4) A(3, 4), B(5, 0), C(-3, -4) A(0, 5), B(5, 0), C(0, -5) A(1, 8), B(8, -1), C(-4, -7) A(8, 1), B(1, -8), C(-4, 7) A(2, 9), B(9, -2), C(-6, -7) A(9, 2), B(2, -9), C(-6, 7) The sum of their perimeters, rounded to four decimal places, is 291.0089. Find all such triangles with a perimeter ≤ 105.

265	Binary Circles	2N binary digits can be placed in a circle so that all the N-digit clockwise subsequences are distinct. For N=3, two such circular arrangements are possible, ignoring rotations: For the first arrangement, the 3-digit subsequences, in clockwise order, are: 000, 001, 010, 101, 011, 111, 110 and 100. Each circular arrangement can be encoded as a number by concatenating the binary digits starting with the subsequence of all zeros as the most significant bits and proceeding clockwise. The two arrangements for N=3 are thus represented as 23 and 29: 00010111 2 = 23 00011101 2 = 29 Calling S(N) the sum of the unique numeric representations, we can see that S(3) = 23 + 29 = 52. Find S(5).
266	Pseudo Square Root	The divisors of 12 are: 1,2,3,4,6 and 12. The largest divisor of 12 that does not exceed the square root of 12 is 3. We shall call the largest divisor of an integer n that does not exceed the square root of n the pseudo square root (PSR) of n. It can be seen that PSR(3102)=47. Let p be the product of the primes below 190. Find PSR(p) mod 1016.

267	Billionaire	You are given a unique investment opportunity. Starting with £1 of capital, you can choose a fixed proportion, f, of your capital to bet on a fair coin toss repeatedly for 1000 tosses. Your return is double your bet for heads and you lose your bet for tails. For example, if f = 1/4, for the first toss you bet £0.25, and if heads comes up you win £0.5 and so then have £1.5. You then bet £0.375 and if the second toss is tails, you have £1.125. Choosing f to maximize your chances of having at least £1,000,000,000 after 1,000 flips, what is the chance that you become a billionaire? All computations are assumed to be exact (no rounding), but give your answer rounded to 12 digits behind the decimal point in the form 0.abcdefghijkl.
268	ers with at least four distinct prime facto	It can be verified that there are 23 positive integers less than 1000 that are divisible by at least four distinct primes less than 100. Find how many positive integers less than 1016 are divisible by at least four distinct primes less than 100.

269	Polynomials with at least one integer roo	A root or zero of a polynomial P(x) is a solution to the equation P(x) = 0. Define Pn as the polynomial whose coefficients are the digits of n. For example, P5703(x) = 5x3 + 7x2 + 3. We can see that: Pn(0) is the last digit of n, Pn(1) is the sum of the digits of n, Pn(10) is n itself. Define Z(k) as the number of positive integers, n, not exceeding k for which the polynomial Pn has at least one integer root. It can be verified that Z(100 000) is 14696. What is Z(1016)?
270	Cutting Squares	A square piece of paper with integer dimensions N×N is placed with a corner at the origin and two of its sides along the x- and y-axes. Then, we cut it up respecting the following rules: We only make straight cuts between two points lying on different sides of the square, and having integer coordinates. Two cuts cannot cross, but several cuts can meet at the same border point. Proceed until no more legal cuts can be made. Counting any reflections or rotations as distinct, we call C(N) the number of ways to cut an N×N square. For example, C(1) = 2 and C(2) = 30 (shown below). What is C(30) mod 108 ?

271	Modular Cubes, part 1	For a positive number n, define S(n) as the sum of the integers x, for which $1<x<n$ and $x3\equiv 1 \mod n$. When n=91, there are 8 possible values for x, namely : 9, 16, 22, 29, 53, 74, 79, 81. Thus, S(91)=9+16+22+29+53+74+79+81=363. Find S(13082761331670030).
272	Modular Cubes, part 2	For a positive number n, define C(n) as the number of the integers x, for which $1<x<n$ and $x3\equiv 1 \mod n$. When n=91, there are 8 possible values for x, namely : 9, 16, 22, 29, 53, 74, 79, 81. Thus, C(91)=8. Find the sum of the positive numbers $n\le 1011$ for which C(n)=242.
273	Sum of Squares	Consider equations of the form: a2 + b2 = N, $0 \le a \le b$, a, b and N integer. For N=65 there are two solutions: a=1, b=8 and a=4, b=7. We call S(N) the sum of the values of a of all solutions of a2 + b2 = N, $0 \le a \le b$, a, b and N integer. Thus S(65) = 1 + 4 = 5. Find ∑ S(N), for all squarefree N only divisible by primes of the form 4k+1 with 4k+1 < 150.

274	Divisibility Multipliers	For each integer $p > 1$ coprime to 10 there is a positive divisibility multiplier $m < p$ which preserves divisibility by p for the following function on any positive integer, n: f(n) = (all but the last digit of n) + (the last digit of n) * m That is, if m is the divisibility multiplier for p, then f(n) is divisible by p if and only if n is divisible by p. (When n is much larger than p, f(n) will be less than n and repeated application of f provides a multiplicative divisibility test for p.) For example, the divisibility multiplier for 113 is 34. f(76275) = 7627 + 5 * 34 = 7797 : 76275 and 7797 are both divisible by 113 f(12345) = 1234 + 5 * 34 = 1404 : 12345 and 1404 are both not divisible by 113 The sum of the divisibility multipliers for the primes that are coprime to 10 and less than 1000 is 39517. What is

275	Balanced Sculptures	Let us define a balanced sculpture of order n as follows: A polyomino made up of n+1 tiles known as the blocks (n tiles) and the plinth (remaining tile); the plinth has its centre at position (x = 0, y = 0); the blocks have y-coordinates greater than zero (so the plinth is the unique lowest tile); the centre of mass of all the blocks, combined, has x-coordinate equal to zero. When counting the sculptures, any arrangements which are simply reflections about the y-axis, are not counted as distinct. For example, the 18 balanced sculptures of order 6 are shown below; note that each pair of mirror images (about the y-axis) is counted as one sculpture: There are 964 balanced sculptures of order 10 and 360505 of order 15. How many balanced sculptures are there of order 18?
276	Primitive Triangles	Consider the triangles with integer sides a, b and c with a ≤ b ≤ c. An integer sided triangle (a,b,c) is called primitive if gcd(a,b,c)=1. How many primitive integer sided triangles exist with a perimeter not exceeding 10 000 000?

277	A Modified Collatz sequence	A modified Collatz sequence of integers is obtained from a starting value a 1 a in the following way: a n+1 = a n 3 a if a n a is divisible by 3 3 . We shall denote this as a large downward step, "D". a n+1

278	Linear Combinations of Semiprimes	Given the values of integers 1< a 1 < a 2 <⋯< a n 1 , consider the linear combination q 1 a 1 + q 2 a 2 +⋯+ q n a
279	gles with integral sides and an integral a	How many triangles are there with integral sides, at least one integral angle (measured in degrees), and a perimeter that does not exceed 108?

280	Ant and seeds	A laborious ant walks randomly on a 5x5 grid. The walk starts from the central square. At each step, the ant moves to an adjacent square at random, without leaving the grid; thus there are 2, 3 or 4 possible moves at each step depending on the ant's position. At the start of the walk, a seed is placed on each square of the lower row. When the ant isn't carrying a seed and reaches a square of the lower row containing a seed, it will start to carry the seed. The ant will drop the seed on the first empty square of the upper row it eventually reaches. What's the expected number of steps until all seeds have been dropped in the top row? Give your answer rounded to 6 decimal places.
281	Pizza Toppings	You are given a pizza (perfect circle) that has been cut into m·n equal pieces and you want to have exactly one topping on each slice. Let f(m,n) denote the number of ways you can have toppings on the pizza with m different toppings (m ≥ 2), using each topping on exactly n slices (n ≥ 1). Reflections are considered distinct, rotations are not. Thus, for instance, f(2,1) = 1, f(2,2) = f(3,1) = 2 and f(3,2) = 16. f(3,2) is shown below: Find the sum of all f(m,n) such that f(m,n) ≤ 1015.

282	The Ackermann function	For non-negative integers m m , n n , the Ackermann function A(m,n) A is defined as follows: A(m,n)={ n+1 A(m−1,1) A(m−1,A(m,n−1)) if m=0 if m>0 and n=0 if m>0 and n>0 A(m,n)={n+1 if m=0A(m−1,1) if m>0 and n=0A(m−1,A(m,n−1)) if m> For example A(1,0)=2 A , A(2,2)=7 A
283	triangles for which the area/perimeter ra	Consider the triangle with sides 6, 8 and 10. It can be seen that the perimeter and the area are both equal to 24. So the area/perimeter ratio is equal to 1. Consider also the triangle with sides 13, 14 and 15. The perimeter equals 42 while the area is equal to 84. So for this triangle the area/perimeter ratio is equal to 2. Find the sum of the perimeters of all integer sided triangles for which the area/perimeter ratios are equal to positive integers not exceeding 1000.

284	Steady Squares	The 3-digit number 376 in the decimal numbering system is an example of numbers with the special property that its square ends with the same digits: 3762 = 141376. Let's call a number with this property a steady square. Steady squares can also be observed in other numbering systems. In the base 14 numbering system, the 3-digit number c37 is also a steady square: c372 = aa0c37, and the sum of its digits is c+3+7=18 in the same numbering system. The letters a, b, c and d are used for the 10, 11, 12 and 13 digits respectively, in a manner similar to the hexadecimal numbering system. For 1 ≤ n ≤ 9, the sum of the digits of all the n-digit steady squares in the base 14 numbering system is 2d8 (582 decimal). Steady squares with leading 0's are not allowed. Find the sum of the digits of all the n-digit steady squares in the base 14 numbering system for

285	Pythagorean odds	Albert chooses a positive integer k, then two real numbers a, b are randomly chosen in the interval [0,1] with uniform distribution. The square root of the sum (k·a+1)2 + (k·b+1)2 is then computed and rounded to the nearest integer. If the result is equal to k, he scores k points; otherwise he scores nothing. For example, if k = 6, a = 0.2 and b = 0.85, then (k·a+1)2 + (k·b+1)2 = 42.05. The square root of 42.05 is 6.484... and when rounded to the nearest integer, it becomes 6. This is equal to k, so he scores 6 points. It can be shown that if he plays 10 turns with k = 1, k = 2, ..., k = 10, the expected value of his total score, rounded to five decimal places, is 10.20914. If he plays 105 turns with k = 1, k = 2, k = 3, ..., k = 105, what is the expected value of his total score, rounded to five
286	Scoring probabilities	Barbara is a mathematician and a basketball player. She has found that the probability of scoring a point when shooting from a distance x is exactly (1 - x/q), where q is a real constant greater than 50. During each practice run, she takes shots from distances x = 1, x = 2, ..., x = 50 and, according to her records, she has precisely a 2 % chance to score a total of exactly 20 points. Find q and give your answer rounded to 10 decimal places.

287	ree encoding (a simple compression algo	The quadtree encoding allows us to describe a 2N×2N black and white image as a sequence of bits (0 and 1). Those sequences are to be read from left to right like this: the first bit deals with the complete 2N×2N region; "0" denotes a split: the current 2n×2n region is divided into 4 sub-regions of dimension 2n-1×2n-1, the next bits contains the description of the top left, top right, bottom left and bottom right sub-regions - in that order; "10" indicates that the current region contains only black pixels; "11" indicates that the current region contains only white pixels. Consider the following 4×4 image (colored marks denote places where a split can occur): This image can be described by several sequences, for example : "001010101001011111011010101010"
288	An enormous factorial	For any prime p the number N(p,q) is defined by N(p,q) = ∑n=0 to q Tn*pn with Tn generated by the following random number generator: S0 = 290797 Sn+1 = Sn2 mod 50515093 Tn = Sn mod p Let Nfac(p,q) be the factorial of N(p,q). Let NF(p,q) be the number of factors p in Nfac(p,q). You are given that NF(3,10000) mod 320=624955285. Find NF(61,107) mod 6110

289	Eulerian Cycles	Let C(x,y) be a circle passing through the points (x, y), (x, y+1), (x+1, y) and (x+1, y+1). For positive integers m and n, let E(m,n) be a configuration which consists of the m·n circles: { C(x,y): 0 ≤ x < m, 0 ≤ y < n, x and y are integers } An Eulerian cycle on E(m,n) is a closed path that passes through each arc exactly once. Many such paths are possible on E(m,n), but we are only interested in those which are not self-crossing: A non-crossing path just touches itself at lattice points, but it never crosses itself. The image below shows E(3,3) and an example of an Eulerian non-crossing path. Let L(m,n) be the number of Eulerian non-crossing paths on E(m,n). For example, L(1,2) = 2, L(2,2) = 37 and L(3,3) = 104290. Find L(6,10) mod 1010.
290	Digital Signature	How many integers 0 ≤ n < 1018 have the property that the sum of the digits of n equals the sum of digits of 137n?

291	Panaitopol Primes	A prime number p p is called a Panaitopol prime if p= x 4 − y 4 x 3 + y 3 p for some positive integers x x and y y . Find how many Panaitopol primes are less than 5×1015.

292	Pythagorean Polygons	We shall define a pythagorean polygon to be a convex polygon with the following properties: there are at least three vertices, no three vertices are aligned, each vertex has integer coordinates, each edge has integer length. For a given integer n, define P(n) as the number of distinct pythagorean polygons for which the perimeter is ≤ n. Pythagorean polygons should be considered distinct as long as none is a translation of another. You are given that P(4) = 1, P(30) = 3655 and P(60) = 891045. Find P(120).
293	Pseudo-Fortunate Numbers	An even positive integer N will be called admissible, if it is a power of 2 or its distinct prime factors are consecutive primes. The first twelve admissible numbers are 2,4,6,8,12,16,18,24,30,32,36,48. If N is admissible, the smallest integer M > 1 such that N+M is prime, will be called the pseudo-Fortunate number for N. For example, N=630 is admissible since it is even and its distinct prime factors are the consecutive primes 2,3,5 and 7. The next prime number after 631 is 641; hence, the pseudo-Fortunate number for 630 is M=11. It can also be seen that the pseudo-Fortunate number for 16 is 3. Find the sum of all distinct pseudo-Fortunate numbers for admissible numbers N less than 109.

294	Sum of digits - experience #23	For a positive integer k, define d(k) as the sum of the digits of k in its usual decimal representation. Thus d(42) = 4+2 = 6. For a positive integer n, define S(n) as the number of positive integers k < 10n with the following properties : k is divisible by 23 and d(k) = 23. You are given that S(9) = 263626 and S(42) = 6377168878570056. Find S(1112) and give your answer mod 109.
295	Lenticular holes	We call the convex area enclosed by two circles a lenticular hole if: The centres of both circles are on lattice points. The two circles intersect at two distinct lattice points. The interior of the convex area enclosed by both circles does not contain any lattice points. Consider the circles: C0: x2+y2=25 C1: (x+4)2+(y-4)2=1 C2: (x-12)2+(y-4)2=65 The circles C0, C1 and C2 are drawn in the picture below. C0 and C1 form a lenticular hole, as well as C0 and C2. We call an ordered pair of positive real numbers (r1, r2) a lenticular pair if there exist two circles with radii r1 and r2 that form a lenticular hole. We can verify that (1, 5) and (5, √65) are the lenticular pairs of the example above. Let L(N) be the number of distinct lenticular pairs (r1, r2) for which 0 < r1

296	Angular Bisector and Tangent	Given is an integer sided triangle ABC with BC ≤ AC ≤ AB. k is the angular bisector of angle ACB. m is the tangent at C to the circumscribed circle of ABC. n is a line parallel to m through B. The intersection of n and k is called E. How many triangles ABC with a perimeter not exceeding 100 000 exist such that BE has integral length?
297	Zeckendorf Representation	Each new term in the Fibonacci sequence is generated by adding the previous two terms. Starting with 1 and 2, the first 10 terms will be: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89. Every positive integer can be uniquely written as a sum of nonconsecutive terms of the Fibonacci sequence. For example, 100 = 3 + 8 + 89. Such a sum is called the Zeckendorf representation of the number. For any integer n>0, let z(n) be the number of terms in the Zeckendorf representation of n. Thus, z(5) = 1, z(14) = 2, z(100) = 3 etc. Also, for 0<n<106, ∑ z(n) = 7894453. Find ∑ z(n) for 0<n<1017.

298	Selective Amnesia	Larry and Robin play a memory game involving of a sequence of random numbers between 1 and 10, inclusive, that are called out one at a time. Each player can remember up to 5 previous numbers. When the called number is in a player's memory, that player is awarded a point. If it's not, the player adds the called number to his memory, removing another number if his memory is full. Both players start with empty memories. Both players always add new missed numbers to their memory but use a different strategy in deciding which number to remove: Larry's strategy is to remove the number that hasn't been called in the longest time. Robin's strategy is to remove the number that's been in the memory the longest time. Example game: Turn Called number Larry's

299	Three similar triangles	Four points with integer coordinates are selected: A(a, 0), B(b, 0), C(0, c) and D(0, d), with $0 < a < b$ and $0 < c < d$. Point P, also with integer coordinates, is chosen on the line AC so that the three triangles ABP, CDP and BDP are all similar. It is easy to prove that the three triangles can be similar, only if a=c. So, given that a=c, we are looking for triplets (a,b,d) such that at least one point P (with integer coordinates) exists on AC, making the three triangles ABP, CDP and BDP all similar. For example, if (a,b,d)=(2,3,4), it can be easily verified that point P(1,1) satisfies the above condition. Note that the triplets (2,3,4) and (2,4,3) are considered as distinct, although point P(1,1) is common for both. If $b+d < 100$, there are 92 distinct triplets (a,b,d) such that point P exists. If $b+d < 100\ 000$, there are 320471 distinct triplets (a,b,d) such that point P

300	Protein folding	In a very simplified form, we can consider proteins as strings consisting of hydrophobic (H) and polar (P) elements, e.g. HHPPHHHPHHPH. For this problem, the orientation of a protein is important; e.g. HPP is considered distinct from PPH. Thus, there are 2n distinct proteins consisting of n elements. When one encounters these strings in nature, they are always folded in such a way that the number of H-H contact points is as large as possible, since this is energetically advantageous. As a result, the H-elements tend to accumulate in the inner part, with the P-elements on the outside. Natural proteins are folded in three dimensions of course, but we will only consider protein folding in two dimensions. The figure below shows two possible ways that our example protein could be folded (H-H contact points are shown with red dots).

301	Nim	Nim is a game played with heaps of stones, where two players take it in turn to remove any number of stones from any heap until no stones remain. We'll consider the three-heap normal-play version of Nim, which works as follows: - At the start of the game there are three heaps of stones. - On his turn the player removes any positive number of stones from any single heap. - The first player unable to move (because no stones remain) loses. If (n1,n2,n3) indicates a Nim position consisting of heaps of size n1, n2 and n3 then there is a simple function X(n1,n2,n3) — that you may look up or attempt to deduce for yourself — that returns: zero if, with perfect strategy, the player about to move will eventually lose; or non-zero if, with perfect strategy, the player about to move will eventually win.

302	Strong Achilles Numbers	A positive integer n is powerful if p2 is a divisor of n for every prime factor p in n. A positive integer n is a perfect power if n can be expressed as a power of another positive integer. A positive integer n is an Achilles number if n is powerful but not a perfect power. For example, 864 and 1800 are Achilles numbers: 864 = 25·33 and 1800 = 23·32·52. We shall call a positive integer S a Strong Achilles number if both S and φ(S) are Achilles numbers.1 For example, 864 is a Strong Achilles number: φ(864) = 288 = 25·32. However, 1800 isn't a Strong Achilles number because: φ(1800) = 480 = 25·31·51. There are 7 Strong Achilles numbers below 104 and 656 below 108. How many Strong Achilles numbers are there below 1018? 1 φ denotes Euler's totient function.

303	Multiples with small digits	For a positive integer n, define f(n) as the least positive multiple of n that, written in base 10, uses only digits ≤ 2. Thus f(2)=2, f(3)=12, f(7)=21, f(42)=210, f(89)=1121222. Also, ∑ n=1 100 f(n) n =11363107 ∑ . Find ∑ n=1 10000 f(n) n ∑ .
304	Primonacci	For any positive integer n the function next_prime(n) returns the smallest prime p such that p>n. The sequence a(n) is defined by: a(1)=next_prime(1014) and a(n)=next_prime(a(n-1)) for n>1. The fibonacci sequence f(n) is defined by: f(0)=0, f(1)=1 and f(n)=f(n-1)+f(n-2) for n>1. The sequence b(n) is defined as f(a(n)). Find ∑ b(n) for 1≤n≤100 000. Give your answer mod 1234567891011.

305	Reflexive Position	Let's call S the (infinite) string that is made by concatenating the consecutive positive integers (starting from 1) written down in base 10. Thus, S = 1234567891011121314151617181920212223242... It's easy to see that any number will show up an infinite number of times in S. Let's call f(n) the starting position of the nth occurrence of n in S. For example, f(1)=1, f(5)=81, f(12)=271 and f(7780)=111111365. Find ∑ f(3k) for 1≤k≤13.

306	Paper-strip Game	The following game is a classic example of Combinatorial Game Theory: Two players start with a strip of n white squares and they take alternate turns. On each turn, a player picks two contiguous white squares and paints them black. The first player who cannot make a move loses. If n = 1, there are no valid moves, so the first player loses automatically. If n = 2, there is only one valid move, after which the second player loses. If n = 3, there are two valid moves, but both leave a situation where the second player loses. If n = 4, there are three valid moves for the first player; she can win the game by painting the two middle squares. If n = 5, there are four valid moves for the first player (shown below in red); but no matter what she does, the second player (blue) wins. So, for 1 ≤ n ≤ 5, there are 3 values of
307	Chip Defects	k defects are randomly distributed amongst n integrated-circuit chips produced by a factory (any number of defects may be found on a chip and each defect is independent of the other defects). Let p(k,n) represent the probability that there is a chip with at least 3 defects. For instance p(3,7) ≈ 0.0204081633. Find p(20 000, 1 000 000) and give your answer rounded to 10 decimal places in the form 0.abcdefghij

308	\n amazing Prime-generating Automato	A program written in the programming language Fractran consists of a list of fractions. The internal state of the Fractran Virtual Machine is a positive integer, which is initially set to a seed value. Each iteration of a Fractran program multiplies the state integer by the first fraction in the list which will leave it an integer. For example, one of the Fractran programs that John Horton Conway wrote for prime-generation consists of the following 14 fractions: 17 91 , 78 85 , 19 51 , 23 38

309	Integer Ladders	In the classic "Crossing Ladders" problem, we are given the lengths x and y of two ladders resting on the opposite walls of a narrow, level street. We are also given the height h above the street where the two ladders cross and we are asked to find the width of the street (w). Here, we are only concerned with instances where all four variables are positive integers. For example, if x = 70, y = 119 and h = 30, we can calculate that w = 56. In fact, for integer values x, y, h and 0 < x < y < 200, there are only five triplets (x,y,h) producing integer solutions for w: (70, 119, 30), (74, 182, 21), (87, 105, 35), (100, 116, 35) and (119, 175, 40). For integer values x, y, h and 0 < x < y < 1 000 000, how many triplets (x,y,h) produce integer solutions for w?
310	Nim Square	Alice and Bob play the game Nim Square. Nim Square is just like ordinary three-heap normal play Nim, but the players may only remove a square number of stones from a heap. The number of stones in the three heaps is represented by the ordered triple (a,b,c). If 0≤a≤b≤c≤29 then the number of losing positions for the next player is 1160. Find the number of losing positions for the next player if 0≤a≤b≤c≤100 000.

311	Biclinic Integral Quadrilaterals	ABCD is a convex, integer sided quadrilateral with 1 ≤ AB < BC < CD < AD. BD has integer length. O is the midpoint of BD. AO has integer length. We'll call ABCD a biclinic integral quadrilateral if AO = CO ≤ BO = DO. For example, the following quadrilateral is a biclinic integral quadrilateral: AB = 19, BC = 29, CD = 37, AD = 43, BD = 48 and AO = CO = 23. Let B(N) be the number of distinct biclinic integral quadrilaterals ABCD that satisfy AB2+BC2+CD2+AD2 ≤ N. We can verify that B(10 000) = 49 and B(1 000 000) = 38239. Find B(10 000 000 000).
312	Cyclic paths on Sierpiński graphs	- A Sierpiński graph of order-1 (S1) is an equilateral triangle. - Sn+1 is obtained from Sn by positioning three copies of Sn so that every pair of copies has one common corner. Let C(n) be the number of cycles that pass exactly once through all the vertices of Sn. For example, C(3) = 8 because eight such cycles can be drawn on S3, as shown below: It can also be verified that : C(1) = C(2) = 1 C(5) = 71328803586048 C(10 000) mod 108 = 37652224 C(10 000) mod 138 = 617720485 Find C(C(C(10 000))) mod 138.

313	Sliding game	In a sliding game a counter may slide horizontally or vertically into an empty space. The objective of the game is to move the red counter from the top left corner of a grid to the bottom right corner; the space always starts in the bottom right corner. For example, the following sequence of pictures show how the game can be completed in five moves on a 2 by 2 grid. Let S(m,n) represent the minimum number of moves to complete the game on an m by n grid. For example, it can be verified that S(5,4) = 25. There are exactly 5482 grids for which S(m,n) = p2, where p < 100 is prime. How many grids does S(m,n) = p2, where p < 106 is prime?

314	The Mouse on the Moon	The moon has been opened up, and land can be obtained for free, but there is a catch. You have to build a wall around the land that you stake out, and building a wall on the moon is expensive. Every country has been allotted a 500 m by 500 m square area, but they will possess only that area which they wall in. 251001 posts have been placed in a rectangular grid with 1 meter spacing. The wall must be a closed series of straight lines, each line running from post to post. The bigger countries of course have built a 2000 m wall enclosing the entire 250 000 m2 area. The Duchy of Grand Fenwick, has a tighter budget, and has asked you (their Royal Programmer) to compute what shape would get best maximum enclosed-area/wall-length ratio. You have done some preliminary calculations on a sheet of paper. For a 2000 meter wall enclosing the 250 000 m2 area the enclosed-area/wall-length

315	Digital root clocks	Sam and Max are asked to transform two digital clocks into two "digital root" clocks. A digital root clock is a digital clock that calculates digital roots step by step. When a clock is fed a number, it will show it and then it will start the calculation, showing all the intermediate values until it gets to the result. For example, if the clock is fed the number 137, it will show: "137" → "11" → "2" and then it will go black, waiting for the next number. Every digital number consists of some light segments: three horizontal (top, middle, bottom) and four vertical (top-left, top-right, bottom-left, bottom-right). Number "1" is made of vertical top-right and bottom-right, number "4" is made by middle horizontal and vertical top-left, top-right and bottom-right. Number "8" lights them all.

316	Numbers in decimal expansions	Let p = p1 p2 p3 ... be an infinite sequence of random digits, selected from {0,1,2,3,4,5,6,7,8,9} with equal probability. It can be seen that p corresponds to the real number 0.p1 p2 p3 .... It can also be seen that choosing a random real number from the interval [0,1) is equivalent to choosing an infinite sequence of random digits selected from {0,1,2,3,4,5,6,7,8,9} with equal probability. For any positive integer n with d decimal digits, let k be the smallest index such that pk, pk+1, ...pk+d-1 are the decimal digits of n, in the same order. Also, let g(n) be the expected value of k; it can be proven that g(n) is always finite and, interestingly, always an integer number. For example, if n = 535, then for p = 31415926535897...., we get k = 9 for p =
317	Firecracker	A firecracker explodes at a height of 100 m above level ground. It breaks into a large number of very small fragments, which move in every direction; all of them have the same initial velocity of 20 m/s. We assume that the fragments move without air resistance, in a uniform gravitational field with g=9.81 m/s2. Find the volume (in m3) of the region through which the fragments move before reaching the ground. Give your answer rounded to four decimal places.

318	2011 nines	Consider the real number 2 − √ + 3 − √ 2 . When we calculate the even powers of 2 − √ + 3 − √ 2 we get: (2 − √ +

319	Bounded Sequences	Let x1, x2,..., xn be a sequence of length n such that: x1 = 2 for all 1 < i ≤ n : xi-1 < xi for all i and j with 1 ≤ i, j ≤ n : (xi) j < (xj + 1)i There are only five such sequences of length 2, namely: {2,4}, {2,5}, {2,6}, {2,7} and {2,8}. There are 293 such sequences of length 5; three examples are given below: {2,5,11,25,55}, {2,6,14,36,88}, {2,8,22,64,181}. Let t(n) denote the number of such sequences of length n. You are given that t(10) = 86195 and t(20) = 5227991891. Find t(1010) and give your answer modulo 109.
320	Factorials divisible by a huge integer	Let N(i) be the smallest integer n such that n! is divisible by (i!)1234567890 Let S(u)=∑ N(i) for 10 ≤ i ≤ u. S(1000)=614538266565663. Find S(1 000 000) mod 1018.

321	Swapping Counters	A horizontal row comprising of 2n + 1 squares has n red counters placed at one end and n blue counters at the other end, being separated by a single empty square in the centre. For example, when n = 3. A counter can move from one square to the next (slide) or can jump over another counter (hop) as long as the square next to that counter is unoccupied. Let M(n) represent the minimum number of moves/actions to completely reverse the positions of the coloured counters; that is, move all the red counters to the right and all the blue counters to the left. It can be verified M(3) = 15, which also happens to be a triangle number. If we create a sequence based on the values of n for which M(n) is a triangle number then the first five terms would be: 1, 3, 10, 22, and 63, and their sum would be 99.
322	Binomial coefficients divisible by 10	Let T(m, n) be the number of the binomial coefficients iCn that are divisible by 10 for n ≤ i < m(i, m and n are positive integers). You are given that T(109, 107-10) = 989697000. Find T(1018, 1012-10).

323	itwise-OR operations on random intege	Let y0, y1, y2,... be a sequence of random unsigned 32 bit integers (i.e. 0 ≤ yi < 232, every value equally likely). For the sequence xi the following recursion is given: x0 = 0 and xi = xi-1| yi-1, for i > 0. (| is the bitwise-OR operator) It can be seen that eventually there will be an index N such that xi = 232 -1 (a bit-pattern of all ones) for all i ≥ N. Find the expected value of N. Give your answer rounded to 10 digits after the decimal point.
324	Building a tower	Let f(n) represent the number of ways one can fill a 3×3×n tower with blocks of 2×1×1. You're allowed to rotate the blocks in any way you like; however, rotations, reflections etc of the tower itself are counted as distinct. For example (with q = 100000007) : f(2) = 229, f(4) = 117805, f(10) mod q = 96149360, f(103) mod q = 24806056, f(106) mod q = 30808124. Find f(1010000) mod 100000007.

325	Stone Game II	A game is played with two piles of stones and two players. At her turn, a player removes a number of stones from the larger pile. The number of stones she removes must be a positive multiple of the number of stones in the smaller pile. E.g., let the ordered pair(6,14) describe a configuration with 6 stones in the smaller pile and 14 stones in the larger pile, then the first player can remove 6 or 12 stones from the larger pile. The player taking all the stones from a pile wins the game. A winning configuration is one where the first player can force a win. For example, (1,5), (2,6) and (3,12) are winning configurations because the first player can immediately remove all stones in the second pile. A losing configuration is one where the second player can force a win, no matter what the first player does. For example, (2,3) and (3,4) are losing configurations: any legal move leaves a

326	Modulo Summations	Let a_n be a sequence recursively defined by: $a_1=1$, $a_n=\left(\sum_{k=1}^{n-1} k\cdot a_k\right) \bmod n$. So the first 10 elements of a_n are: 1,1,0,3,0,3,5,4,1,9.

327	Rooms of Doom	A series of three rooms are connected to each other by automatic doors. Each door is operated by a security card. Once you enter a room the door automatically closes and that security card cannot be used again. A machine at the start will dispense an unlimited number of cards, but each room (including the starting room) contains scanners and if they detect that you are holding more than three security cards or if they detect an unattended security card on the floor, then all the doors will become permanently locked. However, each room contains a box where you may safely store any number of security cards for use at a later stage. If you simply tried to travel through the rooms one at a time then as you entered room 3 you would have used all three cards and would be trapped in that room forever! However, if you make use of the storage boxes, then escape is possible.

328	Lowest-cost Search	We are trying to find a hidden number selected from the set of integers {1, 2, ..., n} by asking questions. Each number (question) we ask, has a cost equal to the number asked and we get one of three possible answers: "Your guess is lower than the hidden number", or "Yes, that's it!", or "Your guess is higher than the hidden number". Given the value of n, an optimal strategy minimizes the total cost (i.e. the sum of all the questions asked) for the worst possible case. E.g. If n=3, the best we can do is obviously to ask the number "2". The answer will immediately lead us to find the hidden number (at a total cost = 2). If n=8, we might decide to use a "binary search" type of strategy: Our first question would be "4" and if the hidden number is higher than 4 we will need one or two additional questions. Let our second question be "6". If the

329	Prime Frog	Susan has a prime frog. Her frog is jumping around over 500 squares numbered 1 to 500. He can only jump one square to the left or to the right, with equal probability, and he cannot jump outside the range [1;500]. (if it lands at either end, it automatically jumps to the only available square on the next move.) When he is on a square with a prime number on it, he croaks 'P' (PRIME) with probability 2/3 or 'N' (NOT PRIME) with probability 1/3 just before jumping to the next square. When he is on a square with a number on it that is not a prime he croaks 'P' with probability 1/3 or 'N' with probability 2/3 just before jumping to the next square. Given that the frog's starting position is random with the same probability for every square, and given that she listens to his first 15 croaks, what is the probability that she hears the

330	Euler's Number	An infinite sequence of real numbers a(n) is defined for all integers n as follows: a(n)= ⎧ ⎨ ⎩ 1 ∑ i=1 ∞ a(n−i) i! n<0 n≥0 $a(n)=\{1n<0\sum i=1\infty a(n-i)i!n\geq 0$ For example, a(0) = 1 1! + 1 2!

331	Cross flips	N×N disks are placed on a square game board. Each disk has a black side and white side. At each turn, you may choose a disk and flip all the disks in the same row and the same column as this disk: thus 2×N-1 disks are flipped. The game ends when all disks show their white side. The following example shows a game on a 5×5 board. It can be proven that 3 is the minimal number of turns to finish this game. The bottom left disk on the N×N board has coordinates (0,0); the bottom right disk has coordinates (N-1,0) and the top left disk has coordinates (0,N-1). Let CN be the following configuration of a board with N×N disks: A disk at (x,y) satisfying N−1≤ x 2 + y

332	Spherical triangles	A spherical triangle is a figure formed on the surface of a sphere by three great circular arcs intersecting pairwise in three vertices. Let C(r) be the sphere with the centre (0,0,0) and radius r. Let Z(r) be the set of points on the surface of C(r) with integer coordinates. Let T(r) be the set of spherical triangles with vertices in Z(r). Degenerate spherical triangles, formed by three points on the same great arc, are not included in T(r). Let A(r) be the area of the smallest spherical triangle in T(r). For example A(14) is 3.294040 rounded to six decimal places. Find ∑ r=1 50 A(r) ∑ . Give your answer rounded to six

333	Special partitions	All positive integers can be partitioned in such a way that each and every term of the partition can be expressed as 2ix3j, where i,j ≥ 0. Let's consider only those such partitions where none of the terms can divide any of the other terms. For example, the partition of 17 = 2 + 6 + 9 = (21x30 + 21x31 + 20x32) would not be valid since 2 can divide 6. Neither would the partition 17 = 16 + 1 = (24x30 + 20x30) since 1 can divide 16. The only valid partition of 17 would be 8 + 9 = (23x30 + 20x32). Many integers have more than one valid partition, the first being 11 having the following two partitions. 11 = 2 + 9 = (21x30 + 20x32) 11 = 8 + 3 = (23x30 + 20x31) Let's define P(n) as the number of valid partitions of n. For example, P(11) = 2. Let's consider only the prime integers q which would have a single valid partition such as P(17). The sum of the primes q <100 such

334	Spilling the beans	In Plato's heaven, there exist an infinite number of bowls in a straight line. Each bowl either contains some or none of a finite number of beans. A child plays a game, which allows only one kind of move: removing two beans from any bowl, and putting one in each of the two adjacent bowls. The game ends when each bowl contains either one or no beans. For example, consider two adjacent bowls containing 2 and 3 beans respectively, all other bowls being empty. The following eight moves will finish the game: You are given the following sequences: t 0 t i b i =123456, ={ t

335	Gathering the beans	Whenever Peter feels bored, he places some bowls, containing one bean each, in a circle. After this, he takes all the beans out of a certain bowl and drops them one by one in the bowls going clockwise. He repeats this, starting from the bowl he dropped the last bean in, until the initial situation appears again. For example with 5 bowls he acts as follows: So with 5 bowls it takes Peter 15 moves to return to the initial situation. Let M(x) represent the number of moves required to return to the initial situation, starting with x bowls. Thus, M(5) = 15. It can also be verified that M(100) = 10920. Find M(2k+1). Give your answer modulo 79.

336	Maximix Arrangements	A train is used to transport four carriages in the order: ABCD. However, sometimes when the train arrives to collect the carriages they are not in the correct order. To rearrange the carriages they are all shunted on to a large rotating turntable. After the carriages are uncoupled at a specific point the train moves off the turntable pulling the carriages still attached with it. The remaining carriages are rotated 180 degrees. All of the carriages are then rejoined and this process is repeated as often as necessary in order to obtain the least number of uses of the turntable. Some arrangements, such as ADCB, can be solved easily: the carriages are separated between A and D, and after DCB are rotated the correct order has been achieved. However, Simple Simon, the train driver, is not known for his efficiency, so he always solves the problem by
337	Totient Stairstep Sequences	Let {a1, a2,..., an} be an integer sequence of length n such that: a1 = 6 for all 1 ≤ i < n : φ(ai) < φ(ai+1) < ai < ai+11 Let S(N) be the number of such sequences with an ≤ N. For example, S(10) = 4: {6}, {6, 8}, {6, 8, 9} and {6, 10}. We can verify that S(100) = 482073668 and S(10 000) mod 108 = 73808307. Find S(20 000 000) mod 108. 1 φ denotes Euler's totient function.

338	Cutting Rectangular Grid Paper	A rectangular sheet of grid paper with integer dimensions w × h is given. Its grid spacing is 1. When we cut the sheet along the grid lines into two pieces and rearrange those pieces without overlap, we can make new rectangles with different dimensions. For example, from a sheet with dimensions 9 × 4 , we can make rectangles with dimensions 18 × 2, 12 × 3 and 6 × 6 by cutting and rearranging as below: Similarly, from a sheet with dimensions 9 × 8 , we can make rectangles with dimensions 18 × 4 and 12 × 6 . For a pair w and h, let F(w,h) be the number of distinct rectangles that can be made from a sheet with dimensions w × h . For example, F(2,1) = 0, F(2,2) = 1, F(9,4) = 3 and F(9,8) = 2. Note that rectangles congruent to the initial one are not counted in F(w,h).

339	Peredur fab Efrawg	"And he came towards a valley, through which ran a river; and the borders of the valley were wooded, and on each side of the river were level meadows. And on one side of the river he saw a flock of white sheep, and on the other a flock of black sheep. And whenever one of the white sheep bleated, one of the black sheep would cross over and become white; and when one of the black sheep bleated, one of the white sheep would cross over and become black." en.wikisource.org Initially each flock consists of n sheep. Each sheep (regardless of colour) is equally likely to be the next sheep to bleat. After a sheep has bleated and a sheep from the other flock has crossed over, Peredur may remove a number of white sheep in order to maximize the expected final number of black sheep. Let E(n) be the expected final number of black sheep if Peredur uses an optimal strategy.

340	Crazy Function	For fixed integers a, b, c, define the crazy function F(n) as follows: F(n) = n - c for all n > b F(n) = F(a + F(a + F(a + F(a + n)))) for all n ≤ b. Also, define S(a,b,c)= ∑ n=0 b F(n) S . For example, if a = 50, b = 2000 and c = 40, then F(0) = 3240 and F(2000) = 2040. Also, S(50, 2000, 40) = 5204240. Find the last 9 digits of S(217, 721, 127).

341	Golomb's self-describing sequence	The Golomb's self-describing sequence (G(n)) (is the only nondecreasing sequence of natural numbers such that n n appears exactly G(n) G times in the sequence. The values of G(n) G for the first few n n are n G(n) 1 1 2 2 3 2
342	The totient of a square is a cube	Consider the number 50. 502 = 2500 = 22 × 54, so φ(2500) = 2 × 4 × 53 = 8 × 53 = 23 × 53. 1 So 2500 is a square and φ(2500) is a cube. Find the sum of all numbers n, 1 < n < 1010 such that φ(n2) is a cube. 1 φ denotes Euler's totient function.

343	Fractional Sequences	For any positive integer k, a finite sequence ai of fractions xi/yi is defined by: a1 = 1/k and ai = (xi-1+1)/(yi-1-1) reduced to lowest terms for i>1. When ai reaches some integer n, the sequence stops. (That is, when yi=1.) Define f(k) = n. For example, for k = 20: 1/20 → 2/19 → 3/18 = 1/6 → 2/5 → 3/4 → 4/3 → 5/2 → 6/1 = 6 So f(20) = 6. Also f(1) = 1, f(2) = 2, f(3) = 1 and ∑ f(k3) = 118937 for 1 ≤ k ≤ 100. Find ∑ f(k3) for 1 ≤ k ≤ 2×106.

344	Silver dollar game	One variant of N.G. de Bruijn's silver dollar game can be described as follows: On a strip of squares a number of coins are placed, at most one coin per square. Only one coin, called the silver dollar, has any value. Two players take turns making moves. At each turn a player must make either a regular or a special move. A regular move consists of selecting one coin and moving it one or more squares to the left. The coin cannot move out of the strip or jump on or over another coin. Alternatively, the player can choose to make the special move of pocketing the leftmost coin rather than making a regular move. If no regular moves are possible, the player is forced to pocket the leftmost coin. The winner is the player who pockets the silver dollar. A winning configuration is an

345	Matrix Sum	We define the Matrix Sum of a matrix as the maximum sum of matrix elements with each element being the only one in his row and column. For example, the Matrix Sum of the matrix below equals 3315 (= 863 + 383 + 343 + 959 + 767): 7 53 183 439 863 497 383 563 79 973 287 63 343 169 583 627 343 773 959 943 767 473 103 699 303 Find the Matrix Sum of: 7 53 183 439 863 497 383 563 79 973 287 63 343 169 583 627 343 773 959 943 767 473 103 699 303 957 703 583 639 913 447 283 463 29 23 487 463 993 119 883 327 493 423 159 743 217 623 3 399 853 407 103 983 89 463 290 516 212 462 350 960 376 682 962 300 780 486 502 912 800 250 346 172 812 350 870 456 192 162 593 473 915 45 989 873 823 965 425 329 803
346	Strong Repunits	The number 7 is special, because 7 is 111 written in base 2, and 11 written in base 6 (i.e. 710 = 116 = 1112). In other words, 7 is a repunit in at least two bases b > 1. We shall call a positive integer with this property a strong repunit. It can be verified that there are 8 strong repunits below 50: {1,7,13,15,21,31,40,43}. Furthermore, the sum of all strong repunits below 1000 equals 15864. Find the sum of all strong repunits below 1012.

347	Largest integer divisible by two primes	The largest integer ≤ 100 that is only divisible by both the primes 2 and 3 is 96, as 96=32*3=25*3. For two distinct primes p and q let M(p,q,N) be the largest positive integer ≤N only divisible by both p and q and M(p,q,N)=0 if such a positive integer does not exist. E.g. M(2,3,100)=96. M(3,5,100)=75 and not 90 because 90 is divisible by 2 ,3 and 5. Also M(2,73,100)=0 because there does not exist a positive integer ≤ 100 that is divisible by both 2 and 73. Let S(N) be the sum of all distinct M(p,q,N). S(100)=2262. Find S(10 000 000).
348	Sum of a square and a cube	Many numbers can be expressed as the sum of a square and a cube. Some of them in more than one way. Consider the palindromic numbers that can be expressed as the sum of a square and a cube, both greater than 1, in exactly 4 different ways. For example, 5229225 is a palindromic number and it can be expressed in exactly 4 different ways: 22852 + 203 22232 + 663 18102 + 1253 11972 + 1563 Find the sum of the five smallest such palindromic numbers.

349	Langton's ant	An ant moves on a regular grid of squares that are coloured either black or white. The ant is always oriented in one of the cardinal directions (left, right, up or down) and moves from square to adjacent square according to the following rules: - if it is on a black square, it flips the colour of the square to white, rotates 90 degrees counterclockwise and moves forward one square. - if it is on a white square, it flips the colour of the square to black, rotates 90 degrees clockwise and moves forward one square. Starting with a grid that is entirely white, how many squares are black after 1018 moves of the ant?

350	aining the least greatest and the greates	A list of size n is a sequence of n natural numbers. Examples are (2,4,6), (2,6,4), (10,6,15,6), and (11). The greatest common divisor, or gcd, of a list is the largest natural number that divides all entries of the list. Examples: gcd(2,6,4) = 2, gcd(10,6,15,6) = 1 and gcd(11) = 11. The least common multiple, or lcm, of a list is the smallest natural number divisible by each entry of the list. Examples: lcm(2,6,4) = 12, lcm(10,6,15,6) = 30 and lcm(11) = 11. Let f(G, L, N) be the number of lists of size N with gcd ≥ G and lcm ≤ L. For example: f(10, 100, 1) = 91. f(10, 100, 2) = 327. f(10, 100, 3) = 1135. f(10, 100, 1000) mod 1014 = 3286053. Find f(106, 1012, 1018) mod 1014.

351	Hexagonal orchards	A hexagonal orchard of order n is a triangular lattice made up of points within a regular hexagon with side n. The following is an example of a hexagonal orchard of order 5: Highlighted in green are the points which are hidden from the center by a point closer to it. It can be seen that for a hexagonal orchard of order 5, 30 points are hidden from the center. Let H(n) be the number of points hidden from the center in a hexagonal orchard of order n. H(5) = 30. H(10) = 138. H(1 000) = 1177848. Find H(100 000 000).

352	Blood tests	Each one of the 25 sheep in a flock must be tested for a rare virus, known to affect 2% of the sheep population. An accurate and extremely sensitive PCR test exists for blood samples, producing a clear positive / negative result, but it is very time-consuming and expensive. Because of the high cost, the vet-in-charge suggests that instead of performing 25 separate tests, the following procedure can be used instead: The sheep are split into 5 groups of 5 sheep in each group. For each group, the 5 samples are mixed together and a single test is performed. Then, If the result is negative, all the sheep in that group are deemed to be virus-free. If the result is positive, 5 additional tests will be performed (a separate test for each animal) to determine the affected individual(s).

353	Risky moon	A moon could be described by the sphere C(r) C with centre (0,0,0) (and radius r r . There are stations on the moon at the points on the surface of C(r) C with integer coordinates. The station at (0,0,r) (is called North Pole station, the station at (0,0,−r) (is called South Pole station. All stations are connected with each other via the shortest road on the

354	Distances in a bee's honeycomb	Consider a honey bee's honeycomb where each cell is a perfect regular hexagon with side length 1 1 . One particular cell is occupied by the queen bee. For a positive real number L L , let B(L) B count the cells with distance L L from the queen bee cell (all distances are measured from centre to centre); you may assume that the honeycomb is large enough to accommodate for any distance we wish to consider. For example, B(3
355	Maximal coprime subset	Define Co(n) to be the maximal possible sum of a set of mutually co-prime elements from {1, 2, ..., n}. For example Co(10) is 30 and hits that maximum on the subset {1, 5, 7, 8, 9}. You are given that Co(30) = 193 and Co(100) = 1356. Find Co(200000).

356	Largest roots of cubic polynomials	Let an be the largest real root of a polynomial g(x) = x3 - 2n·x2 + n. For example, a2 = 3.86619826... Find the last eight digits of ∑ i=1 30 ⌊ a 987654321 i ⌋ ∑ . Note: ⌊a⌋ ⌊ represents the floor function.
357	Prime generating integers	Consider the divisors of 30: 1,2,3,5,6,10,15,30. It can be seen that for every divisor d of 30, d+30/d is prime. Find the sum of all positive integers n not exceeding 100 000 000 such that for every divisor d of n, d+n/d is prime.

358	Cyclic numbers	A cyclic number with n digits has a very interesting property: When it is multiplied by 1, 2, 3, 4, ... n, all the products have exactly the same digits, in the same order, but rotated in a circular fashion! The smallest cyclic number is the 6-digit number 142857 : 142857 × 1 = 142857 142857 × 2 = 285714 142857 × 3 = 428571 142857 × 4 = 571428 142857 × 5 = 714285 142857 × 6 = 857142 The next cyclic number is 0588235294117647 with 16 digits : 0588235294117647 × 1 = 0588235294117647 0588235294117647 × 2 = 1176470588235294 0588235294117647 × 3 = 1764705882352941 ... 0588235294117647 × 16 = 9411764705882352

359	Hilbert's New Hotel	An infinite number of people (numbered 1, 2, 3, etc.) are lined up to get a room at Hilbert's newest infinite hotel. The hotel contains an infinite number of floors (numbered 1, 2, 3, etc.), and each floor contains an infinite number of rooms (numbered 1, 2, 3, etc.). Initially the hotel is empty. Hilbert declares a rule on how the nth person is assigned a room: person n gets the first vacant room in the lowest numbered floor satisfying either of the following: the floor is empty the floor is not empty, and if the latest person taking a room in that floor is person m, then m + n is a perfect square Person 1 gets room 1 in floor 1 since floor 1 is empty. Person 2 does not get room 2 in floor 1 since 1 + 2 = 3 is not a perfect square. Person 2 instead gets room 1 in floor 2
360	Scary Sphere	Given two points (x1,y1,z1) and (x2,y2,z2) in three dimensional space, the Manhattan distance between those points is defined as |x1-x2|+|y1-y2|+|z1-z2|. Let C(r) be a sphere with radius r and center in the origin O(0,0,0). Let I(r) be the set of all points with integer coordinates on the surface of C(r). Let S(r) be the sum of the Manhattan distances of all elements of I(r) to the origin O. E.g. S(45)=34518. Find S(1010).

361	Subsequence of Thue-Morse sequence	The Thue-Morse sequence {Tn} is a binary sequence satisfying: T0 = 0 T2n = Tn T2n+1 = 1 - Tn The first several terms of {Tn} are given as follows: 01101001100101101001011001101001.... We define {An} as the sorted sequence of integers such that the binary expression of each element appears as a subsequence in {Tn}. For example, the decimal number 18 is expressed as 10010 in binary. 10010 appears in {Tn} (T8 to T12), so 18 is an element of {An}. The decimal number 14 is expressed as 1110 in binary. 1110 never appears in {Tn}, so 14 is not an element of {An}. The first several terms of An are given as follows: n 0 1 2 3 4 5 6 7 8 9 10 11 12 ... An 0 1 2 3 4 5 6 9 10 11 12 13 18 ... We can also verify that A100 = 3251

362	Squarefree factors	Consider the number 54. 54 can be factored in 7 distinct ways into one or more factors larger than 1: 54, 2×27, 3×18, 6×9, 3×3×6, 2×3×9 and 2×3×3×3. If we require that the factors are all squarefree only two ways remain: 3×3×6 and 2×3×3×3. Let's call Fsf(n) the number of ways n can be factored into one or more squarefree factors larger than 1, so Fsf(54)=2. Let S(n) be ∑ Fsf(k) for k=2 to n. S(100)=193. Find S(10 000 000 000).

363	Bézier Curves	A cubic Bézier curve is defined by four points: P0, P1, P2 and P3. The curve is constructed as follows: On the segments P0P1, P1P2 and P2P3 the points Q0,Q1 and Q2 are drawn such that P0Q0 / P0P1 = P1Q1 / P1P2 = P2Q2 / P2P3 = t (t in [0,1]). On the segments Q0Q1 and Q1Q2 the points R0 and R1 are drawn such that Q0R0 / Q0Q1 = Q1R1 / Q1Q2 = t for the same value of t. On the segment R0R1 the point B is drawn such that R0B / R0R1 = t for the same value of t. The Bézier curve defined by the points P0, P1, P2, P3 is the locus of B as Q0 takes all possible positions on the segment P0P1. (Please note that for all points the value of t is the same.) At this (external) web address you will find an applet that allows you to drag the points P0, P1, P2 and P3 to see what the Bézier curve (green curve)

364	Comfortable distance	There are N seats in a row. N people come after each other to fill the seats according to the following rules: If there is any seat whose adjacent seat(s) are not occupied take such a seat. If there is no such seat and there is any seat for which only one adjacent seat is occupied take such a seat. Otherwise take one of the remaining available seats. Let T(N) be the number of possibilities that N seats are occupied by N people with the given rules. The following figure shows T(4)=8. We can verify that T(10) = 61632 and T(1 000) mod 100 000 007 = 47255094. Find T(1 000 000) mod 100 000 007.

365	A huge binomial coefficient	The binomial coefficient (10 18 10 9) (is a number with more than 9 billion (9× 10 9 9) digits. Let M(n,k,m) M denote the binomial coefficient (n k) (modulo m

366	Stone Game III	Two players, Anton and Bernhard, are playing the following game. There is one pile of n stones. The first player may remove any positive number of stones, but not the whole pile. Thereafter, each player may remove at most twice the number of stones his opponent took on the previous move. The player who removes the last stone wins. E.g. n=5 If the first player takes anything more than one stone the next player will be able to take all remaining stones. If the first player takes one stone, leaving four, his opponent will take also one stone, leaving three stones. The first player cannot take all three because he may take at most 2x1=2 stones. So let's say he takes also one stone, leaving 2. The second player can take the two remaining stones and wins. So 5 is a losing position for the first

367	Bozo sort	Bozo sort, not to be confused with the slightly less efficient bogo sort, consists out of checking if the input sequence is sorted and if not swapping randomly two elements. This is repeated until eventually the sequence is sorted. If we consider all permutations of the first 4 natural numbers as input the expectation value of the number of swaps, averaged over all 4! input sequences is 24.75. The already sorted sequence takes 0 steps. In this problem we consider the following variant on bozo sort. If the sequence is not in order we pick three elements at random and shuffle these three elements randomly. All 3!=6 permutations of those three elements are equally likely. The already sorted sequence will take 0 steps. If we consider all permutations of the first 4 natural numbers as input the

368	A Kempner-like series	The harmonic series 1+ 1 2 + 1 3 + 1 4 +... 1 is well known to be divergent. If we however omit from this series every term where the denominator has a 9 in it, the series remarkably enough converges to approximately 22.9206766193. This modified harmonic series is called the Kempner series. Let us now consider another modified harmonic series by omitting from the harmonic series every term where the denominator has 3 or more equal consecutive digits. One can verify that
369	Badugi	In a standard 52 card deck of playing cards, a set of 4 cards is a Badugi if it contains 4 cards with no pairs and no two cards of the same suit. Let f(n) be the number of ways to choose n cards with a 4 card subset that is a Badugi. For example, there are 2598960 ways to choose five cards from a standard 52 card deck, of which 514800 contain a 4 card subset that is a Badugi, so f(5) = 514800. Find ∑ f(n) for 4 ≤ n ≤ 13.

370	Geometric triangles	Let us define a geometric triangle as an integer sided triangle with sides a ≤ b ≤ c so that its sides form a geometric progression, i.e. b2 = a · c . An example of such a geometric triangle is the triangle with sides a = 144, b = 156 and c = 169. There are 861805 geometric triangles with perimeter ≤ 106 . How many geometric triangles exist with perimeter ≤ 2.5·1013 ?
371	Licence plates	Oregon licence plates consist of three letters followed by a three digit number (each digit can be from [0..9]). While driving to work Seth plays the following game: Whenever the numbers of two licence plates seen on his trip add to 1000 that's a win. E.g. MIC-012 and HAN-988 is a win and RYU-500 and SET-500 too. (as long as he sees them in the same trip). Find the expected number of plates he needs to see for a win. Give your answer rounded to 8 decimal places behind the decimal point. Note: We assume that each licence plate seen is equally likely to have any three digit number on it.

372	Pencils of rays	Let R(M,N) R be the number of lattice points (x,y) (which satisfy M<x≤N M , M<y≤N M and ⌊ y 2 x 2 ⌋ ⌊ is odd. We can verify that R(0,100)=3019 R and
373	Circumscribed Circles	Every triangle has a circumscribed circle that goes through the three vertices. Consider all integer sided triangles for which the radius of the circumscribed circle is integral as well. Let S(n) be the sum of the radii of the circumscribed circles of all such triangles for which the radius does not exceed n. S(100)=4950 and S(1200)=1653605. Find S(107).

374	Maximum Integer Partition Product	An integer partition of a number n is a way of writing n as a sum of positive integers. Partitions that differ only in the order of their summands are considered the same. A partition of n into distinct parts is a partition of n in which every part occurs at most once. The partitions of 5 into distinct parts are: 5, 4+1 and 3+2. Let f(n) be the maximum product of the parts of any such partition of n into distinct parts and let m(n) be the number of elements of any such partition of n with that product. So f(5)=6 and m(5)=2. For n=10 the partition with the largest product is 10=2+3+5, which gives f(10)=30 and m(10)=3. And their product, f(10)·m(10) = 30·3 = 90 It can be verified that ∑ f(n)·m(n) for 1 ≤ n ≤ 100 = 1683550844462.

375	Minimum of subsequences	Let S n S be an integer sequence produced with the following pseudo-random number generator: S 0 S n+1 =290797 = S 2 n mod50515093 S0=290797Sn+1=Sn2mod50515093 Let A(i,j) A be the minimum of the numbers S i ,

376	Nontransitive sets of dice	Consider the following set of dice with nonstandard pips: Die A: 1 4 4 4 4 4 Die B: 2 2 2 5 5 5 Die C: 3 3 3 3 3 6 A game is played by two players picking a die in turn and rolling it. The player who rolls the highest value wins. If the first player picks die A and the second player picks die B we get P(second player wins) = 7/12 > 1/2 If the first player picks die B and the second player picks die C we get P(second player wins) = 7/12 > 1/2 If the first player picks die C and the second player picks die A we get P(second player wins) = 25/36 > 1/2 So whatever die the first player picks, the second player can pick another die and have a larger than 50% chance of winning. A set of dice having this property is called a nontransitive set of dice. We wish to investigate how many sets of nontransitive dice exist. We will

377	Sum of digits, experience 13	There are 16 positive integers that do not have a zero in their digits and that have a digital sum equal to 5, namely: 5, 14, 23, 32, 41, 113, 122, 131, 212, 221, 311, 1112, 1121, 1211, 2111 and 11111. Their sum is 17891. Let f(n) be the sum of all positive integers that do not have a zero in their digits and have a digital sum equal to n. Find ∑ i=1 17 f(13 i) ∑ . Give the last 9 digits as your answer.
378	Triangle Triples	Let T(n) be the nth triangle number, so T(n) = n (n+1) 2 . Let dT(n) be the number of divisors of T(n). E.g.: T(7) = 28 and dT(7) = 6. Let Tr(n) be the number of triples (i, j, k) such that 1 ≤ i < j < k ≤ n and dT(i) > dT(j) > dT(k). Tr(20) = 14, Tr(100) = 5772 and Tr(1000) = 11174776. Find Tr(60 000 000). Give the last 18 digits of your answer.

379	Least common multiple count	Let f(n) be the number of couples (x,y) with x and y positive integers, $x \le y$ and the least common multiple of x and y equal to n. Let g be the summatory function of f, i.e.: $g(n) = \sum f(i)$ for $1 \le i \le n$. You are given that g(106) = 37429395. Find g(1012).
380	Amazing Mazes!	An m×n maze is an m×n rectangular grid with walls placed between grid cells such that there is exactly one path from the top-left square to any other square. The following are examples of a 9×12 maze and a 15×20 maze: Let C(m,n) be the number of distinct m×n mazes. Mazes which can be formed by rotation and reflection from another maze are considered distinct. It can be verified that C(1,1) = 1, C(2,2) = 4, C(3,4) = 2415, and C(9,12) = 2.5720e46 (in scientific notation rounded to 5 significant digits). Find C(100,500) and write your answer in scientific notation rounded to 5 significant digits. When giving your answer, use a lowercase e to separate mantissa and exponent. E.g. if the answer is 1234567891011 then the answer format would be 1.2346e12.

381	(prime-k) factorial	For a prime p let S(p) = (∑ (p-k)!) mod(p) for 1 ≤ k ≤ 5. For example, if p=7, (7-1)! + (7-2)! + (7-3)! + (7-4)! + (7-5)! = 6! + 5! + 4! + 3! + 2! = 720+120+24+6+2 = 872. As 872 mod(7) = 4, S(7) = 4. It can be verified that ∑ S(p) = 480 for 5 ≤ p < 100. Find ∑ S(p) for 5 ≤ p < 108.
382	Generating polygons	A polygon is a flat shape consisting of straight line segments that are joined to form a closed chain or circuit. A polygon consists of at least three sides and does not self-intersect. A set S of positive numbers is said to generate a polygon P if: no two sides of P are the same length, the length of every side of P is in S, and S contains no other value. For example: The set {3, 4, 5} generates a polygon with sides 3, 4, and 5 (a triangle). The set {6, 9, 11, 24} generates a polygon with sides 6, 9, 11, and 24 (a quadrilateral). The sets {1, 2, 3} and {2, 3, 4, 9} do not generate any polygon at all. Consider the sequence s, defined as follows: s1 = 1, s2 = 2, s3 = 3 sn = sn-1 + sn-3 for n > 3. Let Un be the set {s1, s2, ..., sn}. For example, U10 = {1, 2, 3, 4, 6, 9, 13, 19,

383	Divisibility comparison between factorial	Let f5(n) be the largest integer x for which 5x divides n. For example, f5(625000) = 7. Let T5(n) be the number of integers i which satisfy f5((2·i-1)!) < 2·f5(i!) and 1 ≤ i ≤ n. It can be verified that T5(103) = 68 and T5(109) = 2408210. Find T5(1018).
384	Rudin-Shapiro sequence	Define the sequence a(n) as the number of adjacent pairs of ones in the binary expansion of n (possibly overlapping). E.g.: a(5) = a(1012) = 0, a(6) = a(1102) = 1, a(7) = a(1112) = 2 Define the sequence b(n) = (-1)a(n). This sequence is called the Rudin-Shapiro sequence. Also consider the summatory sequence of b(n): s(n)= ∑ i=0 n b(i) s . The first couple of values of these sequences are: n 0 1 2 3 4 5 6 7 a(n) 0 0 0 1 0 0 1 2 b(n) 1 1 1 -1 1 1 -1 1 s(n) 1 2 3 2 3 4 3 4 The sequence s(n) has the remarkable

385	Ellipses inside triangles	For any triangle T in the plane, it can be shown that there is a unique ellipse with largest area that is completely inside T. For a given n, consider triangles T such that: - the vertices of T have integer coordinates with absolute value ≤ n, and - the foci1 of the largest-area ellipse inside T are (√13,0) and (-√13,0). Let A(n) be the sum of the areas of all such triangles. For example, if n = 8, there are two such triangles. Their vertices are (-4,-3),(-4,3),(8,0) and (4,3),(4,-3),(-8,0), and the area of each triangle is 36. Thus A(8) = 36 + 36 = 72. It can be verified that A(10) = 252, A(100) = 34632 and A(1000) = 3529008. Find A(1 000 000 000). 1The foci (plural of focus) of an ellipse are two points A and B such that for every point P on the boundary of the
386	Maximum length of an antichain	Let n be an integer and S(n) be the set of factors of n. A subset A of S(n) is called an antichain of S(n) if A contains only one element or if none of the elements of A divides any of the other elements of A. For example: S(30) = {1, 2, 3, 5, 6, 10, 15, 30} {2, 5, 6} is not an antichain of S(30). {2, 3, 5} is an antichain of S(30). Let N(n) be the maximum length of an antichain of S(n). Find ∑ N(n) for 1 ≤ n ≤ 108

387	Harshad Numbers	A Harshad or Niven number is a number that is divisible by the sum of its digits. 201 is a Harshad number because it is divisible by 3 (the sum of its digits.) When we truncate the last digit from 201, we get 20, which is a Harshad number. When we truncate the last digit from 20, we get 2, which is also a Harshad number. Let's call a Harshad number that, while recursively truncating the last digit, always results in a Harshad number a right truncatable Harshad number. Also: 201/3=67 which is prime. Let's call a Harshad number that, when divided by the sum of its digits, results in a prime a strong Harshad number. Now take the number 2011 which is prime. When we truncate the last digit from it we get 201, a strong Harshad number that is also right truncatable.
388	Distinct Lines	Consider all lattice points (a,b,c) with 0 ≤ a,b,c ≤ N. From the origin O(0,0,0) all lines are drawn to the other lattice points. Let D(N) be the number of distinct such lines. You are given that D(1 000 000) = 831909254469114121. Find D(1010). Give as your answer the first nine digits followed by the last nine digits.

389	Platonic Dice	An unbiased single 4-sided die is thrown and its value, T, is noted. T unbiased 6-sided dice are thrown and their scores are added together. The sum, C, is noted. C unbiased 8-sided dice are thrown and their scores are added together. The sum, O, is noted. O unbiased 12-sided dice are thrown and their scores are added together. The sum, D, is noted. D unbiased 20-sided dice are thrown and their scores are added together. The sum, I, is noted. Find the variance of I, and give your answer rounded to 4 decimal places.

390	gles with non rational sides and integral	Consider the triangle with sides 5 – √ 5 , 65 – – √ 65 and 68 – – √ 68 . It can be shown that this triangle has area 9 9 . S(n) S is the sum of the areas of all triangles

391	Hopping Game	Let sk be the number of 1's when writing the numbers from 0 to k in binary. For example, writing 0 to 5 in binary, we have 0, 1, 10, 11, 100, 101. There are seven 1's, so s5 = 7. The sequence S = {sk : k ≥ 0} starts {0, 1, 2, 4, 5, 7, 9, 12, ...}. A game is played by two players. Before the game starts, a number n is chosen. A counter c starts at 0. At each turn, the player chooses a number from 1 to n (inclusive) and increases c by that number. The resulting value of c must be a member of S. If there are no more valid moves, the player loses. For example: Let n = 5. c starts at 0. Player 1 chooses 4, so c becomes 0 + 4 = 4. Player 2 chooses 5, so c becomes 4 + 5 = 9. Player 1 chooses 3, so c becomes 9 + 3 = 12. etc.

392	Enmeshed unit circle	A rectilinear grid is an orthogonal grid where the spacing between the gridlines does not have to be equidistant. An example of such grid is logarithmic graph paper. Consider rectilinear grids in the Cartesian coordinate system with the following properties: The gridlines are parallel to the axes of the Cartesian coordinate system. There are N+2 vertical and N+2 horizontal gridlines. Hence there are (N+1) x (N+1) rectangular cells. The equations of the two outer vertical gridlines are x = -1 and x = 1. The equations of the two outer horizontal gridlines are y = -1 and y = 1. The grid cells are colored red if they overlap with the unit circle, black otherwise. For this problem we would like you to find the positions of the remaining N inner horizontal and N inner vertical
393	Migrating ants	An n×n grid of squares contains n2 ants, one ant per square. All ants decide to move simultaneously to an adjacent square (usually 4 possibilities, except for ants on the edge of the grid or at the corners). We define f(n) to be the number of ways this can happen without any ants ending on the same square and without any two ants crossing the same edge between two squares. You are given that f(4) = 88. Find f(10).

394	Eating pie	Jeff eats a pie in an unusual way. The pie is circular. He starts with slicing an initial cut in the pie along a radius. While there is at least a given fraction F of pie left, he performs the following procedure: - He makes two slices from the pie centre to any point of what is remaining of the pie border, any point on the remaining pie border equally likely. This will divide the remaining pie into three pieces. - Going counterclockwise from the initial cut, he takes the first two pie pieces and eats them. When less than a fraction F of pie remains, he does not repeat this procedure. Instead, he eats all of the remaining pie. For $x \geq 1$, let $E(x)$ be the expected number of times Jeff repeats the procedure above with $F = 1/x$. It can be verified that $E(1) = 1$, $E(2) \approx 1.2676536759$, and $E(7.5) \approx 2.1215732071$.

395	Pythagorean tree	The Pythagorean tree is a fractal generated by the following procedure: Start with a unit square. Then, calling one of the sides its base (in the animation, the bottom side is the base): Attach a right triangle to the side opposite the base, with the hypotenuse coinciding with that side and with the sides in a 3-4-5 ratio. Note that the smaller side of the triangle must be on the 'right' side with respect to the base (see animation). Attach a square to each leg of the right triangle, with one of its sides coinciding with that leg. Repeat this procedure for both squares, considering as their bases the sides touching the triangle. The resulting figure, after an infinite number of iterations, is the Pythagorean tree. It can be shown that there exists at least one rectangle, whose sides are

396	Weak Goodstein sequence	For any positive integer n, the nth weak Goodstein sequence {g1, g2, g3, ...} is defined as: g1 = n for k > 1, gk is obtained by writing gk-1 in base k, interpreting it as a base k + 1 number, and subtracting 1. The sequence terminates when gk becomes 0. For example, the 6th weak Goodstein sequence is {6, 11, 17, 25, ...}: g1 = 6. g2 = 11 since 6 = 1102, 1103 = 12, and 12 - 1 = 11. g3 = 17 since 11 = 1023, 1024 = 18, and 18 - 1 = 17. g4 = 25 since 17 = 1014, 1015 = 26, and 26 - 1 = 25. and so on. It can be shown that every weak Goodstein sequence terminates. Let G(n) be the number of nonzero elements in the nth weak Goodstein sequence. It can be verified that G(2) = 3, G(4) =
397	Triangle on parabola	On the parabola y = x2/k, three points A(a, a2/k), B(b, b2/k) and C(c, c2/k) are chosen. Let F(K, X) be the number of the integer quadruplets (k, a, b, c) such that at least one angle of the triangle ABC is 45-degree, with 1 ≤ k ≤ K and -X ≤ a < b < c ≤ X. For example, F(1, 10) = 41 and F(10, 100) = 12492. Find F(106, 109).

398	Cutting rope	Inside a rope of length n, n-1 points are placed with distance 1 from each other and from the endpoints. Among these points, we choose m-1 points at random and cut the rope at these points to create m segments. Let E(n, m) be the expected length of the second-shortest segment. For example, E(3, 2) = 2 and E(8, 3) = 16/7. Note that if multiple segments have the same shortest length the length of the second-shortest segment is defined as the same as the shortest length. Find E(107, 100). Give your answer rounded to 5 decimal places behind the decimal point.

399	Squarefree Fibonacci Numbers	The first 15 fibonacci numbers are: 1,1,2,3,5,8,13,21,34,55,89,144,233,377,610. It can be seen that 8 and 144 are not squarefree: 8 is divisible by 4 and 144 is divisible by 4 and by 9. So the first 13 squarefree fibonacci numbers are: 1,1,2,3,5,13,21,34,55,89,233,377 and 610. The 200th squarefree fibonacci number is: 971183874599339129547649988289594072811608739584170445. The last sixteen digits of this number are: 1608739584170445 and in scientific notation this number can be written as 9.7e53. Find the 100 000 000th squarefree fibonacci number. Give as your answer its last sixteen digits followed by a comma followed by the number in scientific notation (rounded to one digit after the decimal point).

400	Fibonacci tree game	A Fibonacci tree is a binary tree recursively defined as: T(0) is the empty tree. T(1) is the binary tree with only one node. T(k) consists of a root node that has T(k-1) and T(k-2) as children. On such a tree two players play a take-away game. On each turn a player selects a node and removes that node along with the subtree rooted at that node. The player who is forced to take the root node of the entire tree loses. Here are the winning moves of the first player on the first turn for T(k) from k=1 to k=6. Let f(k) be the number of winning moves of the first player (i.e. the moves for which the second player has no winning strategy) on the first turn of the game when this game is played on T(k). For example, f(5) = 1 and f(10) = 17. Find f(10000). Give the last 18 digits of
401	Sum of squares of divisors	The divisors of 6 are 1,2,3 and 6. The sum of the squares of these numbers is 1+4+9+36=50. Let sigma2(n) represent the sum of the squares of the divisors of n. Thus sigma2(6)=50. Let SIGMA2 represent the summatory function of sigma2, that is SIGMA2(n)=∑ sigma2(i) for i=1 to n. The first 6 values of SIGMA2 are: 1,6,16,37,63 and 113. Find SIGMA2(1015) modulo 109.

402	Integer-valued polynomials	It can be shown that the polynomial n4 + 4n3 + 2n2 + 5n is a multiple of 6 for every integer n. It can also be shown that 6 is the largest integer satisfying this property. Define M(a, b, c) as the maximum m such that n4 + an3 + bn2 + cn is a multiple of m for all integers n. For example, M(4, 2, 5) = 6. Also, define S(N) as the sum of M(a, b, c) for all 0 < a, b, c ≤ N. We can verify that S(10) = 1972 and S(10000) = 2024258331114. Let Fk be the Fibonacci sequence: F0 = 0, F1 = 1 and Fk = Fk-1 + Fk-2 for k ≥ 2. Find the last 9 digits of ∑ S(Fk) for 2 ≤ k ≤ 1234567890123.
403	ttice points enclosed by parabola and li	For integers a and b, we define D(a, b) as the domain enclosed by the parabola y = x2 and the line y = a·x + b: D(a, b) = { (x, y) | x2 ≤ y ≤ a·x + b }. L(a, b) is defined as the number of lattice points contained in D(a, b). For example, L(1, 2) = 8 and L(2, -1) = 1. We also define S(N) as the sum of L(a, b) for all the pairs (a, b) such that the area of D(a, b) is a rational number and |a|,|b| ≤ N. We can verify that S(5) = 344 and S(100) = 26709528. Find S(1012). Give your answer mod 108.

404	Crisscross Ellipses	Ea is an ellipse with an equation of the form x2 + 4y2 = 4a2. Ea' is the rotated image of Ea by θ degrees counterclockwise around the origin O(0, 0) for 0° < θ < 90°. b is the distance to the origin of the two intersection points closest to the origin and c is the distance of the two other intersection points. We call an ordered triplet (a, b, c) a canonical ellipsoidal triplet if a, b and c are positive integers. For example, (209, 247, 286) is a canonical ellipsoidal triplet. Let C(N) be the number of distinct canonical ellipsoidal triplets (a, b, c) for a ≤ N. It can be verified that C(103) = 7, C(104) = 106 and C(106) = 11845. Find C(1017).
405	A rectangular tiling	We wish to tile a rectangle whose length is twice its width. Let T(0) be the tiling consisting of a single rectangle. For n > 0, let T(n) be obtained from T(n-1) by replacing all tiles in the following manner: The following animation demonstrates the tilings T(n) for n from 0 to 5: Let f(n) be the number of points where four tiles meet in T(n). For example, f(1) = 0, f(4) = 82 and f(109) mod 177 = 126897180. Find f(10k) for k = 1018, give your answer modulo 177.

406	Guessing Game	We are trying to find a hidden number selected from the set of integers {1, 2, ..., n} by asking questions. Each number (question) we ask, we get one of three possible answers: "Your guess is lower than the hidden number" (and you incur a cost of a), or "Your guess is higher than the hidden number" (and you incur a cost of b), or "Yes, that's it!" (and the game ends). Given the value of n, a, and b, an optimal strategy minimizes the total cost for the worst possible case. For example, if n = 5, a = 2, and b = 3, then we may begin by asking "2" as our first question. If we are told that 2 is higher than the hidden number (for a cost of b=3), then we are sure that "1" is the hidden number (for a total cost of 3). If we are told that 2 is lower than the hidden number (for a cost of a=2), then our next question will be "4". If we are told that 4 is higher than the hidden number (for a cost of b=3),
407	Idempotents	If we calculate a2 mod 6 for 0 ≤ a ≤ 5 we get: 0,1,4,3,4,1. The largest value of a such that a2 ≡ a mod 6 is 4. Let's call M(n) the largest value of a < n such that a2 ≡ a (mod n). So M(6) = 4. Find ∑ M(n) for 1 ≤ n ≤ 107.

408	Admissible paths through a grid	Let's call a lattice point (x, y) inadmissible if x, y and x + y are all positive perfect squares. For example, (9, 16) is inadmissible, while (0, 4), (3, 1) and (9, 4) are not. Consider a path from point (x1, y1) to point (x2, y2) using only unit steps north or east. Let's call such a path admissible if none of its intermediate points are inadmissible. Let P(n) be the number of admissible paths from (0, 0) to (n, n). It can be verified that P(5) = 252, P(16) = 596994440 and P(1000) mod 1 000 000 007 = 341920854. Find P(10 000 000) mod 1 000 000 007.
409	Nim Extreme	Let n be a positive integer. Consider nim positions where: There are n non-empty piles. Each pile has size less than 2n. No two piles have the same size. Let W(n) be the number of winning nim positions satisfying the above conditions (a position is winning if the first player has a winning strategy). For example, W(1) = 1, W(2) = 6, W(3) = 168, W(5) = 19764360 and W(100) mod 1 000 000 007 = 384777056. Find W(10 000 000) mod 1 000 000 007.

410	Circle and tangent line	Let C be the circle with radius r, x2 + y2 = r2. We choose two points P(a, b) and Q(-a, c) so that the line passing through P and Q is tangent to C. For example, the quadruplet (r, a, b, c) = (2, 6, 2, -7) satisfies this property. Let F(R, X) be the number of the integer quadruplets (r, a, b, c) with this property, and with 0 < r ≤ R and 0 < a ≤ X. We can verify that F(1, 5) = 10, F(2, 10) = 52 and F(10, 100) = 3384. Find F(108, 109) + F(109, 108).
411	Uphill paths	Let n be a positive integer. Suppose there are stations at the coordinates (x, y) = (2i mod n, 3i mod n) for 0 ≤ i ≤ 2n. We will consider stations with the same coordinates as the same station. We wish to form a path from (0, 0) to (n, n) such that the x and y coordinates never decrease. Let S(n) be the maximum number of stations such a path can pass through. For example, if n = 22, there are 11 distinct stations, and a valid path can pass through at most 5 stations. Therefore, S(22) = 5. The case is illustrated below, with an example of an optimal path: It can also be verified that S(123) = 14 and S(10000) = 48. Find ∑ S(k5) for 1 ≤ k ≤ 30.

412	Gnomon numbering	For integers m, n ($0 \le n < m$), let L(m, n) be an $m \times m$ grid with the top-right $n \times n$ grid removed. For example, L(5, 3) looks like this: We want to number each cell of L(m, n) with consecutive integers 1, 2, 3, ... such that the number in every cell is smaller than the number below it and to the left of it. For example, here are two valid numberings of L(5, 3): Let LC(m, n) be the number of valid numberings of L(m, n). It can be verified that LC(3, 0) = 42, LC(5, 3) = 250250, LC(6, 3) = 406029023400 and LC(10, 5) mod 76543217 = 61251715. Find LC(10000, 5000) mod 76543217.
413	One-child Numbers	We say that a d-digit positive number (no leading zeros) is a one-child number if exactly one of its sub-strings is divisible by d. For example, 5671 is a 4-digit one-child number. Among all its sub-strings 5, 6, 7, 1, 56, 67, 71, 567, 671 and 5671, only 56 is divisible by 4. Similarly, 104 is a 3-digit one-child number because only 0 is divisible by 3. 1132451 is a 7-digit one-child number because only 245 is divisible by 7. Let F(N) be the number of the one-child numbers less than N. We can verify that F(10) = 9, F(103) = 389 and F(107) = 277674. Find F(1019).

414	Kaprekar constant	6174 is a remarkable number; if we sort its digits in increasing order and subtract that number from the number you get when you sort the digits in decreasing order, we get 7641-1467=6174. Even more remarkable is that if we start from any 4 digit number and repeat this process of sorting and subtracting, we'll eventually end up with 6174 or immediately with 0 if all digits are equal. This also works with numbers that have less than 4 digits if we pad the number with leading zeroes until we have 4 digits. E.g. let's start with the number 0837: 8730-0378=8352 8532-2358=6174 6174 is called the Kaprekar constant. The process of sorting and subtracting and repeating this until either 0 or the Kaprekar constant is reached is called the Kaprekar routine. We can consider the Kaprekar routine

415	Titanic sets	A set of lattice points S is called a titanic set if there exists a line passing through exactly two points in S. An example of a titanic set is S = {(0, 0), (0, 1), (0, 2), (1, 1), (2, 0), (1, 0)}, where the line passing through (0, 1) and (2, 0) does not pass through any other point in S. On the other hand, the set {(0, 0), (1, 1), (2, 2), (4, 4)} is not a titanic set since the line passing through any two points in the set also passes through the other two. For any positive integer N, let T(N) be the number of titanic sets S whose every point (x, y) satisfies 0 ≤ x, y ≤ N. It can be verified that T(1) = 11, T(2) = 494, T(4) = 33554178, T(111) mod 108 = 13500401 and T(105) mod 108 = 63259062. Find T(1011) mod 108.
416	A frog's trip	A row of n squares contains a frog in the leftmost square. By successive jumps the frog goes to the rightmost square and then back to the leftmost square. On the outward trip he jumps one, two or three squares to the right, and on the homeward trip he jumps to the left in a similar manner. He cannot jump outside the squares. He repeats the round-trip travel m times. Let F(m, n) be the number of the ways the frog can travel so that at most one square remains unvisited. For example, F(1, 3) = 4, F(1, 4) = 15, F(1, 5) = 46, F(2, 3) = 16 and F(2, 100) mod 109 = 429619151. Find the last 9 digits of F(10, 1012).

417	Reciprocal cycles II	A unit fraction contains 1 in the numerator. The decimal representation of the unit fractions with denominators 2 to 10 are given: 1/2 = 0.5 1/3 = 0.(3) 1/4 = 0.25 1/5 = 0.2 1/6 = 0.1(6) 1/7 = 0.(142857) 1/8 = 0.125 1/9 = 0.(1) 1/10 = 0.1 Where 0.1(6) means 0.166666..., and has a 1-digit recurring cycle. It can be seen that 1/7 has a 6-digit recurring cycle. Unit fractions whose denominator has no other prime factors than 2 and/or 5 are not considered to have a recurring cycle. We define the length of the recurring cycle of those unit fractions as 0. Let L(n) denote the length of the recurring cycle of 1/n. You are given
418	Factorisation triples	Let n be a positive integer. An integer triple (a, b, c) is called a factorisation triple of n if: 1 ≤ a ≤ b ≤ c a·b·c = n. Define f(n) to be a + b + c for the factorisation triple (a, b, c) of n which minimises c / a. One can show that this triple is unique. For example, f(165) = 19, f(100100) = 142 and f(20!) = 4034872. Find f(43!).

419	Look and say sequence	The look and say sequence goes 1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, ... The sequence starts with 1 and all other members are obtained by describing the previous member in terms of consecutive digits. It helps to do this out loud: 1 is 'one one' → 11 11 is 'two ones' → 21 21 is 'one two and one one' → 1211 1211 is 'one one, one two and two ones' → 111221 111221 is 'three ones, two twos and one one' → 312211 ... Define A(n), B(n) and C(n) as the number of ones, twos and threes in the n'th element of the sequence respectively. One can verify that A(40) = 31254, B(40) = 20259 and C(40) = 11625. Find A(n), B(n) and C(n) for n = 1012. Give your answer modulo 230 and separate your values for A, B and C by

420	2x2 positive integer matrix	A positive integer matrix is a matrix whose elements are all positive integers. Some positive integer matrices can be expressed as a square of a positive integer matrix in two different ways. Here is an example: (40 48 12 40)= (2 12 3 2) 2 = (6 4 1
421	Prime factors of n15+1	Numbers of the form n15+1 are composite for every integer n > 1. For positive integers n and m let s(n,m) be defined as the sum of the distinct prime factors of n15+1 not exceeding m. E.g. 215+1 = 3×3×11×331. So s(2,10) = 3 and s(2,1000) = 3+11+331 = 345. Also 1015+1 = 7×11×13×211×241×2161×9091. So s(10,100) = 31 and s(10,1000) = 483. Find ∑ s(n,108) for 1 ≤ n ≤ 1011.

422	Sequence of points on a hyperbola	Let H be the hyperbola defined by the equation 12x2 + 7xy - 12y2 = 625. Next, define X as the point (7, 1). It can be seen that X is in H. Now we define a sequence of points in H, {Pi : i ≥ 1}, as: P1 = (13, 61/4). P2 = (-43/6, -4). For i > 2, Pi is the unique point in H that is different from Pi-1 and such that line PiPi-1 is parallel to line Pi-2X. It can be shown that Pi is well-defined, and that its coordinates are always rational. You are given that P3 = (-19/2, -229/24), P4 = (1267/144, -37/12) and P7 = (17194218091/143327232, 274748766781/1719926784). Find Pn for n = 1114 in the following format: If Pn = (a/b, c/d) where the fractions are in lowest terms and the denominators are positive, then the answer is (a + b + c + d) mod 1 000 000 007.

423	Consecutive die throws	Let n be a positive integer. A 6-sided die is thrown n times. Let c be the number of pairs of consecutive throws that give the same value. For example, if n = 7 and the values of the die throws are (1,1,5,6,6,6,3), then the following pairs of consecutive throws give the same value: (1,1,5,6,6,6,3) (1,1,5,6,6,6,3) (1,1,5,6,6,6,3) Therefore, c = 3 for (1,1,5,6,6,6,3). Define C(n) as the number of outcomes of throwing a 6-sided die n times such that c does not exceed π(n).1 For example, C(3) = 216, C(4) = 1290, C(11) = 361912500 and C(24) = 4727547363281250000. Define S(L) as ∑ C(n) for 1 ≤ n ≤ L. For example, S(50) mod 1 000 000 007 = 832833871. Find S(50 000 000) mod 1 000 000 007. 1 π denotes the prime-counting function, i.e. π(n) is the number of

424	Kakuro	The above is an example of a cryptic kakuro (also known as cross sums, or even sums cross) puzzle, with its final solution on the right. (The common rules of kakuro puzzles can be found easily on numerous internet sites. Other related information can also be currently found at krazydad.com whose author has provided the puzzle data for this challenge.) The downloadable text file (kakuro200.txt) contains the description of 200 such puzzles, a mix of 5x5 and 6x6 types. The first puzzle in the file is the above example which is coded as follows: 6,X,X,(vCC),(vI),X,X,X,(hH),B,O,(vCA),(vJE),X,(hFE,vD),O,O,O,O,(hA),O,I,(hJC,vB),O,O,(hJC),H,O,O,O,X,X,X,(hJE),O,O,X The first character is a numerical digit indicating the size of the information grid. It would be either a 6 (for a 5x5 kakuro puzzle) or a 7 (for a 6x6 puzzle) followed by a comma (,). The extra top line and left column are needed to

425	Prime connection	Two positive numbers A and B are said to be connected (denoted by "A ↔ B") if one of these conditions holds: (1) A and B have the same length and differ in exactly one digit; for example, 123 ↔ 173. (2) Adding one digit to the left of A (or B) makes B (or A); for example, 23 ↔ 223 and 123 ↔ 23. We call a prime P a 2's relative if there exists a chain of connected primes between 2 and P and no prime in the chain exceeds P. For example, 127 is a 2's relative. One of the possible chains is shown below: 2 ↔ 3 ↔ 13 ↔ 113 ↔ 103 ↔ 107 ↔ 127 However, 11 and 103 are not 2's relatives. Let F(N) be the sum of the primes ≤ N which are not 2's relatives. We can verify that $F(10^3) = 431$ and $F(10^4) = 78728$. Find $F(10^7)$.

426	Box-ball system	Consider an infinite row of boxes. Some of the boxes contain a ball. For example, an initial configuration of 2 consecutive occupied boxes followed by 2 empty boxes, 2 occupied boxes, 1 empty box, and 2 occupied boxes can be denoted by the sequence (2, 2, 2, 1, 2), in which the number of consecutive occupied and empty boxes appear alternately. A turn consists of moving each ball exactly once according to the following rule: Transfer the leftmost ball which has not been moved to the nearest empty box to its right. After one turn the sequence (2, 2, 2, 1, 2) becomes (2, 2, 1, 2, 3) as can be seen below; note that we begin the new sequence starting at the first occupied box. A system like this is called a Box-Ball System or BBS for short. It can be shown that after a sufficient number of turns, the system evolves to a state where the consecutive numbers

427	n-sequences	A sequence of integers S = {si} is called an n-sequence if it has n elements and each element si satisfies 1 ≤ si ≤ n. Thus there are nn distinct n-sequences in total. For example, the sequence S = {1, 5, 5, 10, 7, 7, 7, 2, 3, 7} is a 10-sequence. For any sequence S, let L(S) be the length of the longest contiguous subsequence of S with the same value. For example, for the given sequence S above, L(S) = 3, because of the three consecutive 7's. Let f(n) = ∑ L(S) for all n-sequences S. For example, f(3) = 45, f(7) = 1403689 and f(11) = 481496895121. Find f(7 500 000) mod 1 000 000 009.

428	Necklace of circles	Let a, b and c be positive numbers. Let W, X, Y, Z be four collinear points where |WX| = a, |XY| = b, |YZ| = c and |WZ| = a + b + c. Let Cin be the circle having the diameter XY. Let Cout be the circle having the diameter WZ. The triplet (a, b, c) is called a necklace triplet if you can place k ≥ 3 distinct circles C1, C2, ..., Ck such that: Ci has no common interior points with any Cj for 1 ≤ i, j ≤ k and i ≠ j, Ci is tangent to both Cin and Cout for 1 ≤ i ≤ k, Ci is tangent to Ci+1 for 1 ≤ i < k, and Ck is tangent to C1. For example, (5, 5, 5) and (4, 3, 21) are necklace triplets, while it can be shown that (2, 2, 5) is not. Let T(n) be the number of necklace triplets (a, b, c) such that a, b and c are positive integers, and b ≤ n. For example, T(1) = 9, T(20) = 732 and T(3000) = 438106.
429	Sum of squares of unitary divisors	A unitary divisor d of a number n is a divisor of n that has the property gcd(d, n/d) = 1. The unitary divisors of 4! = 24 are 1, 3, 8 and 24. The sum of their squares is 12 + 32 + 82 + 242 = 650. Let S(n) represent the sum of the squares of the unitary divisors of n. Thus S(4!)=650. Find S(100 000 000!) modulo 1 000 000 009.

430	Range flips	N disks are placed in a row, indexed 1 to N from left to right. Each disk has a black side and white side. Initially all disks show their white side. At each turn, two, not necessarily distinct, integers A and B between 1 and N (inclusive) are chosen uniformly at random. All disks with an index from A to B (inclusive) are flipped. The following example shows the case N = 8. At the first turn A = 5 and B = 2, and at the second turn A = 4 and B = 6. Let E(N, M) be the expected number of disks that show their white side after M turns. We can verify that E(3, 1) = 10/9, E(3, 2) = 5/3, E(10, 4) ≈ 5.157 and E(100, 10) ≈ 51.893. Find E(1010, 4000). Give your answer rounded to 2 decimal places behind the decimal point.

431	Square Space Silo	Fred the farmer arranges to have a new storage silo installed on his farm and having an obsession for all things square he is absolutely devastated when he discovers that it is circular. Quentin, the representative from the company that installed the silo, explains that they only manufacture cylindrical silos, but he points out that it is resting on a square base. Fred is not amused and insists that it is removed from his property. Quick thinking Quentin explains that when granular materials are delivered from above a conical slope is formed and the natural angle made with the horizontal is called the angle of repose. For example if the angle of repose, α=30 α degrees, and grain is delivered at the centre of the silo then a perfect cone will form towards the top of the cylinder. In the case of this silo, which has a diameter of 6m, the amount of
432	Totient sum	Let S(n,m) = ∑φ(n × i) for 1 ≤ i ≤ m. (φ is Euler's totient function) You are given that S(510510,106)= 45480596821125120. Find S(510510,1011). Give the last 9 digits of your answer.

433	Steps in Euclid's algorithm	Let E(x0, y0) be the number of steps it takes to determine the greatest common divisor of x0 and y0 with Euclid's algorithm. More formally: x1 = y0, y1 = x0 mod y0 xn = yn-1, yn = xn-1 mod yn-1 E(x0, y0) is the smallest n such that yn = 0. We have E(1,1) = 1, E(10,6) = 3 and E(6,10) = 4. Define S(N) as the sum of E(x,y) for 1 ≤ x,y ≤ N. We have S(1) = 1, S(10) = 221 and S(100) = 39826. Find S(5·106).
434	Rigid graphs	Recall that a graph is a collection of vertices and edges connecting the vertices, and that two vertices connected by an edge are called adjacent. Graphs can be embedded in Euclidean space by associating each vertex with a point in the Euclidean space. A flexible graph is an embedding of a graph where it is possible to move one or more vertices continuously so that the distance between at least two nonadjacent vertices is altered while the distances between each pair of adjacent vertices is kept constant. A rigid graph is an embedding of a graph which is not flexible. Informally, a graph is rigid if by replacing the vertices with fully rotating hinges and the edges with rods that are unbending and inelastic, no parts of the graph can be moved independently from the rest of the graph. The grid graphs embedded in the

435	Polynomials of Fibonacci numbers	The Fibonacci numbers { f n ,n≥0} { are defined recursively as f n = f n−1 + f n−2 f with base cases f 0 =0 f and f 1 =1

436	Unfair wager	Julie proposes the following wager to her sister Louise. She suggests they play a game of chance to determine who will wash the dishes. For this game, they shall use a generator of independent random numbers uniformly distributed between 0 and 1. The game starts with S = 0. The first player, Louise, adds to S different random numbers from the generator until S > 1 and records her last random number 'x'. The second player, Julie, continues adding to S different random numbers from the generator until S > 2 and records her last random number 'y'. The player with the highest number wins and the loser washes the dishes, i.e. if y > x the second player wins. For example, if the first player draws 0.62 and 0.44, the first player turn ends since 0.62+0.44 > 1 and x = 0.44. If the second players draws 0.1, 0.27

437	Fibonacci primitive roots	When we calculate 8n modulo 11 for n=0 to 9 we get: 1, 8, 9, 6, 4, 10, 3, 2, 5, 7. As we see all possible values from 1 to 10 occur. So 8 is a primitive root of 11. But there is more: If we take a closer look we see: 1+8=9 8+9=17≡6 mod 11 9+6=15≡4 mod 11 6+4=10 4+10=14≡3 mod 11 10+3=13≡2 mod 11 3+2=5 2+5=7 5+7=12≡1 mod 11. So the powers of 8 mod 11 are cyclic with period 10, and 8n + 8n+1 ≡ 8n+2 (mod 11). 8 is called a Fibonacci primitive root of 11. Not every prime has a Fibonacci primitive root. There are 323 primes less than 10000 with one or more Fibonacci primitive

438	;ger part of polynomial equation's soluti	For an n-tuple of integers t = (a1, ..., an), let (x1, ..., xn) be the solutions of the polynomial equation xn + a1xn-1 + a2xn-2 + ... + an-1x + an = 0. Consider the following two conditions: x1, ..., xn are all real. If x1, ..., xn are sorted, ⌊xi⌋ = i for 1 ≤ i ≤ n. (⌊·⌋: floor function.) In the case of n = 4, there are 12 n-tuples of integers which satisfy both conditions. We define S(t) as the sum of the absolute values of the integers in t. For n = 4 we can verify that ∑ S(t) = 2087 for all n-tuples t which satisfy both conditions. Find ∑ S(t) for n = 7.
439	Sum of sum of divisors	Let d(k) be the sum of all divisors of k. We define the function S(N) = ∑ N i=1 ∑ N j=1 d(i·j) ∑ . For example, S(3) = d(1) + d(2) + d(3) + d(2) + d(4) + d(6) + d(3) + d(6) + d(9) = 59. You are given that S(103) = 563576517282 and S(105) mod 109 = 215766508. Find S(1011) mod 109.

440	GCD and Tiling	We want to tile a board of length n and height 1 completely, with either 1 × 2 blocks or 1 × 1 blocks with a single decimal digit on top: For example, here are some of the ways to tile a board of length n = 8: Let T(n) be the number of ways to tile a board of length n as described above. For example, T(1) = 10 and T(2) = 101. Let S(L) be the triple sum ∑a,b,c gcd(T(ca), T(cb)) for 1 ≤ a, b, c ≤ L. For example: S(2) = 10444 S(3) = 1292115238446807016106539989 S(4) mod 987 898 789 = 670616280. Find S(2000) mod 987 898 789.
441	he inverse summation of coprime couple	For an integer M, we define R(M) as the sum of 1/(p·q) for all the integer pairs p and q which satisfy all of these conditions: 1 ≤ p < q ≤ M p + q ≥ M p and q are coprime. We also define S(N) as the sum of R(i) for 2 ≤ i ≤ N. We can verify that S(2) = R(2) = 1/2, S(10) ≈ 6.9147 and S(100) ≈ 58.2962. Find S(107). Give your answer rounded to four decimal places.

442	Eleven-free integers	An integer is called eleven-free if its decimal expansion does not contain any substring representing a power of 11 except 1. For example, 2404 and 13431 are eleven-free, while 911 and 4121331 are not. Let E(n) be the nth positive eleven-free integer. For example, E(3) = 3, E(200) = 213 and E(500 000) = 531563. Find E(1018).
443	GCD sequence	Let g(n) be a sequence defined as follows: g(4) = 13, g(n) = g(n-1) + gcd(n, g(n-1)) for n > 4. The first few values are: n 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 ... g(n) 13 14 16 17 18 27 28 29 30 31 32 33 34 51 54 55 60 ... You are given that g(1 000) = 2524 and g(1 000 000) = 2624152. Find g(1015).

444	The Roundtable Lottery	A group of p people decide to sit down at a round table and play a lottery-ticket trading game. Each person starts off with a randomly-assigned, unscratched lottery ticket. Each ticket, when scratched, reveals a whole-pound prize ranging anywhere from £1 to £p, with no two tickets alike. The goal of the game is for each person to maximize his ticket winnings upon leaving the game. An arbitrary person is chosen to be the first player. Going around the table, each player has only one of two options: 1. The player can scratch his ticket and reveal its worth to everyone at the table. 2. The player can trade his unscratched ticket for a previous player's scratched ticket, and then leave the game with that ticket. The previous player then scratches his newly-acquired ticket and reveals its worth to everyone at the table.

445	Retractions A	For every integer $n>1$ n , the family of functions f n,a,b f is defined by f n,a,b $(x) \equiv ax+b \bmod n$ f for a,b,x a integer and $0<a<n, 0\le b<n, 0\le x<n$ 0 . We will call f n,a,b f a retraction if f

446	Retractions B	For every integer n>1 n , the family of functions f n,a,b f is defined by f n,a,b (x)≡ax+bmodn f for a,b,x a integer and 0<a<n,0≤b<n,0≤x<n 0 . We will call f n,a,b f a retraction if f

447	Retractions C	For every integer n>1 n , the family of functions f n,a,b f is defined by f n,a,b (x)≡ax+bmodn f for a,b,x a integer and 0<a<n,0≤b<n,0≤x<n 0 . We will call f n,a,b f a retraction if f
448	Average least common multiple	The function lcm(a,b) denotes the least common multiple of a and b. Let A(n) be the average of the values of lcm(n,i) for 1≤i≤n. E.g: A(2)=(2+2)/2=2 and A(10)=(10+10+30+20+10+30+70+40+90+10)/10=32. Let S(n)=∑ A(k) for 1≤k≤n. S(100)=122726. Find S(99999999019) mod 999999017.

449	Chocolate covered candy	Phil the confectioner is making a new batch of chocolate covered candy. Each candy centre is shaped like an ellipsoid of revolution defined by the equation: b2x2 + b2y2 + a2z2 = a2b2. Phil wants to know how much chocolate is needed to cover one candy centre with a uniform coat of chocolate one millimeter thick. If a=1 mm and b=1 mm, the amount of chocolate required is 28 3 π mm3 If a=2 mm and b=1 mm, the amount of chocolate required is approximately 60.35475635 mm3. Find the amount of chocolate in mm3 required if a=3 mm and b=1 mm. Give your answer as the number rounded to 8 decimal places behind the decimal point.

450	Hypocycloid and Lattice points	A hypocycloid is the curve drawn by a point on a small circle rolling inside a larger circle. The parametric equations of a hypocycloid centered at the origin, and starting at the right most point is given by: x(t)=(R−r)cos(t)+rcos(R−r r t) x y(t)=(R−r)sin(t)−rsin(R−r r t) y Where R is the radius of the large circle and r the radius of the small circle. Let C(R,r) C be the set of distinct points with integer coordinates on the hypocycloid with radius R and r and for which there

451	Modular inverses	Consider the number 15. There are eight positive numbers less than 15 which are coprime to 15: 1, 2, 4, 7, 8, 11, 13, 14. The modular inverses of these numbers modulo 15 are: 1, 8, 4, 13, 2, 11, 7, 14 because $1 \cdot 1$ mod 15=1 $2 \cdot 8=16$ mod 15=1 $4 \cdot 4=16$ mod 15=1 $7 \cdot 13=91$ mod 15=1 $11 \cdot 11=121$ mod 15=1 $14 \cdot 14=196$ mod 15=1 Let I(n) be the largest positive number m smaller than n-1 such that the modular inverse of m modulo n equals m itself. So I(15)=11. Also I(100)=51 and I(7)=1. Find $\sum$ I(n) for $3 \leq n \leq 2 \times 107$
452	Long Products	Define F(m,n) as the number of n-tuples of positive integers for which the product of the elements doesn't exceed m. F(10, 10) = 571. F(106, 106) mod 1 234 567 891 = 252903833. Find F(109, 109) mod 1 234 567 891.

453	Lattice Quadrilaterals	A simple quadrilateral is a polygon that has four distinct vertices, has no straight angles and does not self-intersect. Let Q(m, n) be the number of simple quadrilaterals whose vertices are lattice points with coordinates (x,y) satisfying $0 \le x \le m$ and $0 \le y \le n$. For example, Q(2, 2) = 94 as can be seen below: It can also be verified that Q(3, 7) = 39590, Q(12, 3) = 309000 and Q(123, 45) = 70542215894646. Find Q(12345, 6789) mod 135707531.
454	Diophantine reciprocals III	In the following equation x, y, and n are positive integers. 1 x + 1 y = 1 n 1x+1y=1n For a limit L we define F(L) as the number of solutions which satisfy $x < y \le L$. We can verify that F(15) = 4 and F(1000) = 1069. Find F(1012).

455	Powers With Trailing Digits	Let f(n) be the largest positive integer x less than 109 such that the last 9 digits of nx form the number x (including leading zeros), or zero if no such integer exists. For example: f(4) = 411728896 (4411728896 = ...490411728896) f(10) = 0 f(157) = 743757 (157743757 = ...567000743757) ∑ f(n), 2 ≤ n ≤ 103 = 442530011399 Find ∑ f(n), 2 ≤ n ≤ 106.
456	Triangles containing the origin II	Define: xn = (1248n mod 32323) - 16161 yn = (8421n mod 30103) - 15051 Pn = {(x1, y1), (x2, y2), ..., (xn, yn)} For example, P8 = {(-14913, -6630), (-10161, 5625), (5226, 11896), (8340, -10778), (15852, -5203), (-15165, 11295), (-1427, -14495), (12407, 1060)}. Let C(n) be the number of triangles whose vertices are in Pn which contain the origin in the interior. Examples: C(8) = 20 C(600) = 8950634 C(40 000) = 2666610948988 Find C(2 000 000).
457	polynomial modulo the square of a prin	Let f(n) = n2 - 3n - 1. Let p be a prime. Let R(p) be the smallest positive integer n such that f(n) mod p2 = 0 if such an integer n exists, otherwise R(p) = 0. Let SR(L) be ∑ R(p) for all primes not exceeding L. Find SR(107).

458	Permutations of Project	Consider the alphabet A made out of the letters of the word "project": A={c,e,j,o,p,r,t}. Let T(n) be the number of strings of length n consisting of letters from A that do not have a substring that is one of the 5040 permutations of "project". T(7)=77-7!=818503. Find T(1012). Give the last 9 digits of your answer.
459	Flipping game	The flipping game is a two player game played on a N by N square board. Each square contains a disk with one side white and one side black. The game starts with all disks showing their white side. A turn consists of flipping all disks in a rectangle with the following properties: the upper right corner of the rectangle contains a white disk the rectangle width is a perfect square (1, 4, 9, 16, ...) the rectangle height is a triangular number (1, 3, 6, 10, ...) Players alternate turns. A player wins by turning the grid all black. Let W(N) be the number of winning moves for the first player on a N by N board with all disks white, assuming perfect play. W(1) = 1, W(2) = 0, W(5) = 8 and W(102) = 31395. For N=5, the first player's eight winning first moves are:

460	An ant on the move	On the Euclidean plane, an ant travels from point A(0, 1) to point B(d, 1) for an integer d. In each step, the ant at point (x0, y0) chooses one of the lattice points (x1, y1) which satisfy x1 ≥ 0 and y1 ≥ 1 and goes straight to (x1, y1) at a constant velocity v. The value of v depends on y0 and y1 as follows: If y0 = y1, the value of v equals y0. If y0 ≠ y1, the value of v equals (y1 - y0) / (ln(y1) - ln(y0)). The left image is one of the possible paths for d = 4. First the ant goes from A(0, 1) to P1(1, 3) at velocity (3 - 1) / (ln(3) - ln(1)) ≈ 1.8205. Then the required time is sqrt(5) / 1.8205 ≈ 1.2283. From P1(1, 3) to P2(3, 3) the ant travels at velocity 3 so the required time is 2 / 3 ≈ 0.6667. From P2(3, 3) to B(4, 1) the ant travels at velocity (1 - 3) / (ln(1) - ln(3)) ≈ 1.8205 so the required time is sqrt(5) / 1.8205 ≈ 1.2283. Thus the total required time is 1.2283

461	Almost Pi	Let fn(k) = ek/n - 1, for all non-negative integers k. Remarkably, f200(6) + f200(75) + f200(89) + f200(226) = 3.141592644529… ≈ π. In fact, it is the best approximation of π of the form fn(a) + fn(b) + fn(c) + fn(d) for n = 200. Let g(n) = a2 + b2 + c2 + d 2 for a, b, c, d that minimize the error: | fn(a) + fn(b) + fn(c) + fn(d) - π| (where |x| denotes the absolute value of x). You are given g(200) = 62 + 752 + 892 + 2262 = 64658. Find g(10000).

462	Permutation of 3-smooth numbers	A 3-smooth number is an integer which has no prime factor larger than 3. For an integer N, we define S(N) as the set of 3-smooth numbers less than or equal to N . For example, S(20) = { 1, 2, 3, 4, 6, 8, 9, 12, 16, 18 }. We define F(N) as the number of permutations of S(N) in which each element comes after all of its proper divisors. This is one of the possible permutations for N = 20. - 1, 2, 4, 3, 9, 8, 16, 6, 18, 12. This is not a valid permutation because 12 comes before its divisor 6. - 1, 2, 4, 3, 9, 8, 12, 16, 6, 18. We can verify that F(6) = 5, F(8) = 9, F(20) = 450 and F(1000) ≈ 8.8521816557e21. Find F(1018). Give as your answer its scientific notation rounded to ten digits after the decimal point. When giving your answer, use a lowercase e to separate mantissa and exponent. E.g. if the answer is

463	A weird recurrence relation	The function f f is defined for all positive integers as follows: f(1)=1 f f(3)=3 f f(2n)=f(n) f f(4n+1)=2f(2n+1)−f(n) f f(4n+3)=3f(2n+1)−2f(n) f The function S(n) S is defined as ∑ n i=1 f(i) ∑ .

464	Möbius function and intervals	The Möbius function, denoted μ(n), is defined as: μ(n) = (-1)ω(n) if n is squarefree (where ω(n) is the number of distinct prime factors of n) μ(n) = 0 if n is not squarefree. Let P(a,b) be the number of integers n in the interval [a,b] such that μ(n) = 1. Let N(a,b) be the number of integers n in the interval [a,b] such that μ(n) = -1. For example, P(2,10) = 2 and N(2,10) = 4. Let C(n) be the number of integer pairs (a,b) such that: 1 ≤ a ≤ b ≤ n, 99·N(a,b) ≤ 100·P(a,b), and 99·P(a,b) ≤ 100·N(a,b). For example, C(10) = 13, C(500) = 16676 and C(10 000) = 20155319. Find C(20 000 000).

465	Polar polygons	The kernel of a polygon is defined by the set of points from which the entire polygon's boundary is visible. We define a polar polygon as a polygon for which the origin is strictly contained inside its kernel. For this problem, a polygon can have collinear consecutive vertices. However, a polygon still cannot have self-intersection and cannot have zero area. For example, only the first of the following is a polar polygon (the kernels of the second, third, and fourth do not strictly contain the origin, and the fifth does not have a kernel at all): Notice that the first polygon has three consecutive collinear vertices. Let P(n) be the number of polar polygons such that the vertices (x, y) have integer coordinates whose absolute values are not greater than n. Note that polygons should be counted as different if they have different set of edges, even if they enclose the same
466	Distinct terms in a multiplication table	Let P(m,n) be the number of distinct terms in an m×n multiplication table. For example, a 3×4 multiplication table looks like this: × 1 2 3 4 1 1 2 3 4 2 2 4 6 8 3 3 6 9 12 There are 8 distinct terms {1,2,3,4,6,8,9,12}, therefore P(3,4) = 8. You are given that: P(64,64) = 1263, P(12,345) = 1998, and P(32,1015) = 13826382602124302. Find P(64,1016).

467	Superinteger	An integer s is called a superinteger of another integer n if the digits of n form a subsequence of the digits of s. For example, 2718281828 is a superinteger of 18828, while 314159 is not a superinteger of 151. Let p(n) be the nth prime number, and let c(n) be the nth composite number. For example, p(1) = 2, p(10) = 29, c(1) = 4 and c(10) = 18. {p(i) : i ≥ 1} = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, ...} {c(i) : i ≥ 1} = {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, ...} Let PD the sequence of the digital roots of {p(i)} (CD is defined similarly for {c(i)}): PD = {2, 3, 5, 7, 2, 4, 8, 1, 5, 2, ...} CD = {4, 6, 8, 9, 1, 3, 5, 6, 7, 9, ...} Let Pn be the integer formed by concatenating the first n elements of PD (Cn is defined similarly for CD). P10 = 2357248152 C10 = 4689135679 Let f(n) be the smallest positive integer

468	Smooth divisors of binomial coefficients	An integer is called B-smooth if none of its prime factors is greater than B B . Let S B (n) S be the largest B B -smooth divisor of n n . Examples: S 1 (10)=1 S S 4

469	Empty chairs	In a room N chairs are placed around a round table. Knights enter the room one by one and choose at random an available empty chair. To have enough elbow room the knights always leave at least one empty chair between each other. When there aren't any suitable chairs left, the fraction C of empty chairs is determined. We also define E(N) as the expected value of C. We can verify that E(4) = 1/2 and E(6) = 5/9. Find E(1018). Give your answer rounded to fourteen decimal places in the form 0.abcdefghijklmn.

470	Super Ramvok	Consider a single game of Ramvok: Let t represent the maximum number of turns the game lasts. If t = 0, then the game ends immediately. Otherwise, on each turn i, the player rolls a die. After rolling, if i < t the player can either stop the game and receive a prize equal to the value of the current roll, or discard the roll and try again next turn. If i = t, then the roll cannot be discarded and the prize must be accepted. Before the game begins, t is chosen by the player, who must then pay an up-front cost ct for some constant c. For c = 0, t can be chosen to be infinite (with an up-front cost of 0). Let R(d, c) be the expected profit (i.e. net gain) that the player receives from a single game of optimally-played Ramvok, given a fair d-sided die and cost constant c. For example, R(4, 0.2) = 2.65. Assume that the player has sufficient funds for paying any/all up-front costs. Now consider a game of Super

471	Triangle inscribed in ellipse	The triangle ΔABC is inscribed in an ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ x , $0 < 2b < a$, a and b integers. Let r(a,b) be the radius of the incircle of ΔABC when the incircle has center (2b, 0) and A has coordinates $(\frac{a}{2}, \frac{\sqrt{3}}{2}b)$

472	Comfortable Distance II	There are N seats in a row. N people come one after another to fill the seats according to the following rules: No person sits beside another. The first person chooses any seat. Each subsequent person chooses the seat furthest from anyone else already seated, as long as it does not violate rule 1. If there is more than one choice satisfying this condition, then the person chooses the leftmost choice. Note that due to rule 1, some seats will surely be left unoccupied, and the maximum number of people that can be seated is less than N (for N > 1). Here are the possible seating arrangements for N = 15: We see that if the first person chooses correctly, the 15 seats can seat up to 7 people. We can also see that the first person has 9 choices to maximize the number of people that may be seated. Let f(N) be the number of choices the first person has to maximize the

473	Phigital number base	Let φ φ be the golden ratio: φ= 1+ 5 √ 2 . φ Remarkably it is possible to write every positive integer as a sum of powers of φ φ even if we require that every power of φ φ is used at most once in this sum. Even then this representation is not unique. We can make it unique by requiring that no powers with consecutive exponents are used and that the
474	Last digits of divisors	For a positive integer n and digits d, we define F(n, d) as the number of the divisors of n whose last digits equal d. For example, F(84, 4) = 3. Among the divisors of 84 (1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84), three of them (4, 14, 84) have the last digit 4. We can also verify that F(12!, 12) = 11 and F(50!, 123) = 17888. Find F(106!, 65432) modulo (1016 + 61).

475	Music festival	12n musicians participate at a music festival. On the first day, they form 3n quartets and practice all day. It is a disaster. At the end of the day, all musicians decide they will never again agree to play with any member of their quartet. On the second day, they form 4n trios, each musician avoiding his previous quartet partners. Let f(12n) be the number of ways to organize the trios amongst the 12n musicians. You are given f(12) = 576 and f(24) mod 1 000 000 007 = 509089824. Find f(600) mod 1 000 000 007.
476	Circle Packing II	Let R(a, b, c) be the maximum area covered by three non-overlapping circles inside a triangle with edge lengths a, b and c. Let S(n) be the average value of R(a, b, c) over all integer triplets (a, b, c) such that $1 \le a \le b \le c < a + b \le n$ You are given S(2) = R(1, 1, 1) ≈ 0.31998, S(5) ≈ 1.25899. Find S(1803) rounded to 5 decimal places behind the decimal point.

477	Number Sequence Game	The number sequence game starts with a sequence S of N numbers written on a line. Two players alternate turns. At his turn, a player must select and remove either the first or the last number remaining in the sequence. The player score is the sum of all the numbers he has taken. Each player attempts to maximize his own sum. If N = 4 and S = {1, 2, 10, 3}, then each player maximizes his score as follows: Player 1: removes the first number (1) Player 2: removes the last number from the remaining sequence (3) Player 1: removes the last number from the remaining sequence (10) Player 2: removes the remaining number (2) Player 1 score is 1 + 10 = 11. Let F(N) be the score of player 1 if both players follow the optimal strategy for the sequence S = {s1, s2, ..., sN} defined as: s1 = 0

478	Mixtures	Let us consider mixtures of three substances: A, B and C. A mixture can be described by a ratio of the amounts of A, B, and C in it, i.e., (a : b : c). For example, a mixture described by the ratio (2 : 3 : 5) contains 20% A, 30% B and 50% C. For the purposes of this problem, we cannot separate the individual components from a mixture. However, we can combine different amounts of different mixtures to form mixtures with new ratios. For example, say we have three mixtures with ratios (3 : 0 : 2), (3 : 6 : 11) and (3 : 3 : 4). By mixing 10 units of the first, 20 units of the second and 30 units of the third, we get a new mixture with ratio (6 : 5 : 9), since: $(10\cdot3/5 + 20\cdot3/20 + 30\cdot3/10 : 10\cdot0/5 + 20\cdot6/20 + 30\cdot3/10 : 10\cdot2/5 + 20\cdot11/20 + 30\cdot4/10) = (18 : 15 : 27) = (6 : 5 : 9)$ However, with the same three mixtures, it is impossible to form the ratio (3 : 2 : 1), since the amount of B is

479	Roots on the Rise	Let a k a , b k b , and c k c represent the three solutions (real or complex numbers) to the equation 1 x =(k x) 2 (k+ x 2)−kx

480	The Last Question	Consider all the words which can be formed by selecting letters, in any order, from the phrase: thereisasyetinsufficientdataforameaningfulanswer Suppose those with 15 letters or less are listed in alphabetical order and numbered sequentially starting at 1. The list would include: 1 : a 2 : aa 3 : aaa 4 : aaaa 5 : aaaaa 6 : aaaaaa 7 : aaaaaac 8 : aaaaaacd 9 : aaaaaacde 10 : aaaaaacdee 11 : aaaaaacdeee 12 : aaaaaacdeeee 13 : aaaaaacdeeeee 14 : aaaaaacdeeeeee 15 : aaaaaacdeeeeeef 16 : aaaaaacdeeeeeeg

481	Chef Showdown	A group of chefs (numbered #1, #2, etc) participate in a turn-based strategic cooking competition. On each chef's turn, he/she cooks up a dish to the best of his/her ability and gives it to a separate panel of judges for taste-testing. Let S(k) represent chef #k's skill level (which is publicly known). More specifically, S(k) is the probability that chef #k's dish will be assessed favorably by the judges (on any/all turns). If the dish receives a favorable rating, then the chef must choose one other chef to be eliminated from the competition. The last chef remaining in the competition is the winner. The game always begins with chef #1, with the turn order iterating sequentially over the rest of the chefs still in play. Then the cycle repeats from the lowest-numbered chef. All chefs aim to optimize their chances of winning within the rules as stated, assuming that the other chefs behave
482	The incenter of a triangle	ABC is an integer sided triangle with incenter I and perimeter p. The segments IA, IB and IC have integral length as well. Let L = p + |IA| + |IB| + |IC|. Let S(P) = ∑ L for all such triangles where p ≤ P. For example, S(103) = 3619. Find S(107).

483	Repeated permutation	We define a permutation as an operation that rearranges the order of the elements {1, 2, 3, ..., n}. There are n! such permutations, one of which leaves the elements in their initial order. For n = 3 we have 3! = 6 permutations: - P1 = keep the initial order - P2 = exchange the 1st and 2nd elements - P3 = exchange the 1st and 3rd elements - P4 = exchange the 2nd and 3rd elements - P5 = rotate the elements to the right - P6 = rotate the elements to the left If we select one of these permutations, and we re-apply the same permutation repeatedly, we eventually restore the initial order. For a permutation Pi, let f(Pi) be the number of steps required to restore the initial order by applying the permutation Pi repeatedly. For n = 3, we obtain:
484	Arithmetic Derivative	The arithmetic derivative is defined by p' = 1 for any prime p (ab)' = a'b + ab' for all integers a, b (Leibniz rule) For example, 20' = 24. Find ∑ gcd(k,k') for 1 < k ≤ 5×1015. Note: gcd(x,y) denotes the greatest common divisor of x and y.

485	Maximum number of divisors	Let d(n) be the number of divisors of n. Let M(n,k) be the maximum value of d(j) for n ≤ j ≤ n+k-1. Let S(u,k) be the sum of M(n,k) for 1 ≤ n ≤ u-k+1. You are given that S(1000,10)=17176. Find S(100 000 000,100 000).
486	Palindrome-containing strings	Let F5(n) be the number of strings s such that: s consists only of '0's and '1's, s has length at most n, and s contains a palindromic substring of length at least 5. For example, F5(4) = 0, F5(5) = 8, F5(6) = 42 and F5(11) = 3844. Let D(L) be the number of integers n such that 5 ≤ n ≤ L and F5(n) is divisible by 87654321. For example, D(107) = 0 and D(5·109) = 51. Find D(1018).
487	Sums of power sums	Let fk(n) be the sum of the kth powers of the first n positive integers. For example, f2(10) = 12 + 22 + 32 + 42 + 52 + 62 + 72 + 82 + 92 + 102 = 385. Let Sk(n) be the sum of fk(i) for 1 ≤ i ≤ n. For example, S4(100) = 35375333830. What is ∑ (S10000(1012) mod p) over all primes p between 2 · 109 and 2 · 109 + 2000?

488	Unbalanced Nim	Alice and Bob have enjoyed playing Nim every day. However, they finally got bored of playing ordinary three-heap Nim. So, they added an extra rule: - Must not make two heaps of the same size. The triple (a,b,c) indicates the size of three heaps. Under this extra rule, (2,4,5) is one of the losing positions for the next player. To illustrate: - Alice moves to (2,4,3) - Bob moves to (0,4,3) - Alice moves to (0,2,3) - Bob moves to (0,2,1) Unlike ordinary three-heap Nim, (0,1,2) and its permutations are the end states of this game. For an integer N, we define F(N) as the sum of a+b+c for all the losing positions for the next player, with 0 < a < b < c < N. For example, F(8) = 42, because there are 4 losing positions for the next
489	Common factors between two sequence	Let G(a, b) be the smallest non-negative integer n for which gcd(n3 + b, (n + a)3 + b) is maximized. For example, G(1, 1) = 5 because gcd(n3 + 1, (n + 1)3 + 1) reaches its maximum value of 7 for n = 5, and is smaller for 0 ≤ n < 5. Let H(m, n) = ∑ G(a, b) for 1 ≤ a ≤ m, 1 ≤ b ≤ n. You are given H(5, 5) = 128878 and H(10, 10) = 32936544. Find H(18, 1900).

490	Jumping frog	There are n stones in a pond, numbered 1 to n. Consecutive stones are spaced one unit apart. A frog sits on stone 1. He wishes to visit each stone exactly once, stopping on stone n. However, he can only jump from one stone to another if they are at most 3 units apart. In other words, from stone i, he can reach a stone j if 1 ≤ j ≤ n and j is in the set {i-3, i-2, i-1, i+1, i+2, i+3}. Let f(n) be the number of ways he can do this. For example, f(6) = 14, as shown below: 1 → 2 → 3 → 4 → 5 → 6 1 → 2 → 3 → 5 → 4 → 6 1 → 2 → 4 → 3 → 5 → 6 1 → 2 → 4 → 5 → 3 → 6 1 → 2 → 5 → 3 → 4 → 6 1 → 2 → 5 → 4 → 3 → 6 1 → 3 → 2 → 4 → 5 → 6 1 → 3 → 2 → 5 → 4 → 6 1 → 3 → 4 → 2 → 5 → 6 1 → 3 → 5 → 2 → 4 → 6 1 → 4 → 2 → 3 → 5 → 6
491	Double pandigital number divisible by 11	We call a positive integer double pandigital if it uses all the digits 0 to 9 exactly twice (with no leading zero). For example, 40561817703823564929 is one such number. How many double pandigital numbers are divisible by 11?

492	Exploding sequence	Define the sequence a1, a2, a3, ... as: a1 = 1 an+1 = 6an2 + 10an + 3 for n ≥ 1. Examples: a3 = 2359 a6 = 269221280981320216750489044576319 a6 mod 1 000 000 007 = 203064689 a100 mod 1 000 000 007 = 456482974 Define B(x,y,n) as ∑ (an mod p) for every prime p such that x ≤ p ≤ x+y. Examples: B(109, 103, 103) = 23674718882 B(109, 103, 1015) = 20731563854 Find B(109, 107, 1015).
493	Under The Rainbow	70 coloured balls are placed in an urn, 10 for each of the seven rainbow colours. What is the expected number of distinct colours in 20 randomly picked balls? Give your answer with nine digits after the decimal point (a.bcdefghij).

494	Collatz prefix families	The Collatz sequence is defined as: $a_{i+1}=\begin{cases} \frac{a_i}{2} & \text{if } a_i \text{ is even} \\ 3a_i+1 & \text{if } a_i \text{ is odd} \end{cases}$ a . The Collatz conjecture states that starting from any positive integer, the sequence eventually reaches the cycle 1,4,2,1....

495	g n as the product of k distinct positive ir	Let W(n,k) be the number of ways in which n can be written as the product of k distinct positive integers. For example, W(144,4) = 7. There are 7 ways in which 144 can be written as a product of 4 distinct positive integers: 144 = 1×2×4×18 144 = 1×2×8×9 144 = 1×2×3×24 144 = 1×2×6×12 144 = 1×3×4×12 144 = 1×3×6×8 144 = 2×3×4×6 Note that permutations of the integers themselves are not considered distinct. Furthermore, W(100!,10) modulo 1 000 000 007 = 287549200. Find W(10000!,30) modulo 1 000 000 007.
496	Incenter and circumcenter of triangle	Given an integer sided triangle ABC: Let I be the incenter of ABC. Let D be the intersection between the line AI and the circumcircle of ABC (A ≠ D). We define F(L) as the sum of BC for the triangles ABC that satisfy AC = DI and BC ≤ L. For example, F(15) = 45 because the triangles ABC with (BC,AC,AB) = (6,4,5), (12,8,10), (12,9,7), (15,9,16) satisfy the conditions. Find F(109).

497	Drunken Tower of Hanoi	Bob is very familiar with the famous mathematical puzzle/game, "Tower of Hanoi," which consists of three upright rods and disks of different sizes that can slide onto any of the rods. The game begins with a stack of n disks placed on the leftmost rod in descending order by size. The objective of the game is to move all of the disks from the leftmost rod to the rightmost rod, given the following restrictions: Only one disk can be moved at a time. A valid move consists of taking the top disk from one stack and placing it onto another stack (or an empty rod). No disk can be placed on top of a smaller disk. Moving on to a variant of this game, consider a long room k units (square tiles) wide, labeled from 1 to k in ascending order. Three rods are placed at squares a, b, and c, and a stack of n disks is placed on the rod at square a. Bob begins the game standing at
498	Remainder of polynomial division	For positive integers n and m, we define two polynomials Fn(x) = xn and Gm(x) = (x-1)m. We also define a polynomial Rn,m(x) as the remainder of the division of Fn(x) by Gm(x). For example, R6,3(x) = 15x2 - 24x + 10. Let C(n, m, d) be the absolute value of the coefficient of the d-th degree term of Rn,m(x). We can verify that C(6, 3, 1) = 24 and C(100, 10, 4) = 227197811615775. Find C(1013, 1012, 104) mod 999999937.

499	St. Petersburg Lottery	A gambler decides to participate in a special lottery. In this lottery the gambler plays a series of one or more games. Each game costs m pounds to play and starts with an initial pot of 1 pound. The gambler flips an unbiased coin. Every time a head appears, the pot is doubled and the gambler continues. When a tail appears, the game ends and the gambler collects the current value of the pot. The gambler is certain to win at least 1 pound, the starting value of the pot, at the cost of m pounds, the initial fee. The gambler cannot continue to play if his fortune falls below m pounds. Let pm(s) denote the probability that the gambler will never run out of money in this lottery given his initial fortune s and the cost per game m. For example p2(2) ≈ 0.2522, p2(5) ≈ 0.6873 and p6(10 000) ≈ 0.9952 (note: pm(s) = 0 for s < m). Find p15(109) and give your answer
500	Problem 500!!!	The number of divisors of 120 is 16. In fact 120 is the smallest number having 16 divisors. Find the smallest number with 2500500 divisors. Give your answer modulo 500500507.

501	Eight Divisors	The eight divisors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24. The ten numbers not exceeding 100 having exactly eight divisors are 24, 30, 40, 42, 54, 56, 66, 70, 78 and 88. Let f(n) be the count of numbers not exceeding n with exactly eight divisors. You are given f(100) = 10, f(1000) = 180 and f(106) = 224427. Find f(1012).
502	Counting Castles	We define a block to be a rectangle with a height of 1 and an integer-valued length. Let a castle be a configuration of stacked blocks. Given a game grid that is w units wide and h units tall, a castle is generated according to the following rules: Blocks can be placed on top of other blocks as long as nothing sticks out past the edges or hangs out over open space. All blocks are aligned/snapped to the grid. Any two neighboring blocks on the same row have at least one unit of space between them. The bottom row is occupied by a block of length w. The maximum achieved height of the entire castle is exactly h. The castle is made from an even number of blocks. The following is a sample castle for w=8 and h=5: Let F(w,h) represent the number of

503	Compromise or persist	Alice is playing a game with n cards numbered 1 to n. A game consists of iterations of the following steps. (1) Alice picks one of the cards at random. (2) Alice cannot see the number on it. Instead, Bob, one of her friends, sees the number and tells Alice how many previously-seen numbers are bigger than the number which he is seeing. (3) Alice can end or continue the game. If she decides to end, the number becomes her score. If she decides to continue, the card is removed from the game and she returns to (1). If there is no card left, she is forced to end the game. Let F(n) be the Alice's expected score if she takes the optimized strategy to minimize her score. For example, F(3) = 5/3. At the first iteration, she should continue the game. At the second iteration, she should end the game if Bob says that
504	Square on the Inside	Let ABCD be a quadrilateral whose vertices are lattice points lying on the coordinate axes as follows: A(a, 0), B(0, b), C(−c, 0), D(0, −d), where 1 ≤ a, b, c, d ≤ m and a, b, c, d, m are integers. It can be shown that for m = 4 there are exactly 256 valid ways to construct ABCD. Of these 256 quadrilaterals, 42 of them strictly contain a square number of lattice points. How many quadrilaterals ABCD strictly contain a square number of lattice points for m = 100?

505	Bidirectional Recurrence	Let: x(0) x(1) x(2k) x(2k+1) y n (k) A(n) =0 =1 =(3x(k)+2x(⌊ k 2 ⌋)) mod 2 60 for k≥1, where ⌊ ⌋ is the floor function =(2x(k)+3x(⌊ k 2 ⌋)) mod 2 60 for k≥1

506	Clock sequence	Consider the infinite repeating sequence of digits: 1234321234321234321... Amazingly, you can break this sequence of digits into a sequence of integers such that the sum of the digits in the n'th value is n. The sequence goes as follows: 1, 2, 3, 4, 32, 123, 43, 2123, 432, 1234, 32123, ... Let vn be the n'th value in this sequence. For example, v2 = 2, v5 = 32 and v11 = 32123. Let S(n) be v1 + v2 + ... + vn. For example, S(11) = 36120, and S(1000) mod 123454321 = 18232686. Find S(1014) mod 123454321.

507	Shortest Lattice Vector	Let t n t be the tribonacci numbers defined as: t 0 = t 1 =0 t ; t 2 =1 t ; t n = t n−1 + t

508	Integers in base i-1	Consider the Gaussian integer i-1. A base i-1 representation of a Gaussian integer a+bi is a finite sequence of digits dn-1dn-2...d1d0 such that: a+bi = dn-1(i-1)n-1 + dn-2(i-1)n-2 + ... + d1(i-1) + d0 Each dk is in {0,1} There are no leading zeroes, i.e. dn-1 ≠ 0, unless a+bi is itself 0 Here are base i-1 representations of a few Gaussian integers: 11+24i → 111010110001101 24-11i → 110010110011 8+0i → 111000000 -5+0i → 11001101 0+0i → 0 Remarkably, every Gaussian integer has a unique base i-1 representation! Define f(a+bi) as the number of 1s in the unique base i-1 representation of a+bi. For example, f(11+24i) = 9 and f(24-11i) = 7.

509	Divisor Nim	Anton and Bertrand love to play three pile Nim. However, after a lot of games of Nim they got bored and changed the rules somewhat. They may only take a number of stones from a pile that is a proper divisor of the number of stones present in the pile. E.g. if a pile at a certain moment contains 24 stones they may take only 1,2,3,4,6,8 or 12 stones from that pile. So if a pile contains one stone they can't take the last stone from it as 1 isn't a proper divisor of 1. The first player that can't make a valid move loses the game. Of course both Anton and Bertrand play optimally. The triple (a,b,c) indicates the number of stones in the three piles. Let S(n) be the number of winning positions for the next player for $1 \le a, b, c \le n$. S(10) = 692 and S(100) = 735494.
510	Tangent Circles	Circles A and B are tangent to each other and to line L at three distinct points. Circle C is inside the space between A, B and L, and tangent to all three. Let rA, rB and rC be the radii of A, B and C respectively. Let $S(n) = \sum rA + rB + rC$, for $0 < rA \le rB \le n$ where rA, rB and rC are integers. The only solution for $0 < rA \le rB \le 5$ is rA = 4, rB = 4 and rC = 1, so S(5) = 4 + 4 + 1 = 9. You are also given S(100) = 3072. Find S(109).

511	Sequences with nice divisibility propertie	Let Seq(n,k) be the number of positive-integer sequences {ai}1≤i≤n of length n such that: n is divisible by ai for 1 ≤ i ≤ n, and n + a1 + a2 + ... + an is divisible by k. Examples: Seq(3,4) = 4, and the 4 sequences are: {1, 1, 3} {1, 3, 1} {3, 1, 1} {3, 3, 3} Seq(4,11) = 8, and the 8 sequences are: {1, 1, 1, 4} {1, 1, 4, 1} {1, 4, 1, 1} {4, 1, 1, 1} {2, 2, 2, 1} {2, 2, 1, 2} {2, 1, 2, 2} {1, 2, 2, 2} The last nine digits of Seq(1111,24) are 840643584. Find the last nine digits of Seq(1234567898765,4321).

512	Sums of totients of powers	Let φ(n) φ be Euler's totient function. Let f(n)=(∑ n i=1 φ(n i)) mod (n+1) f . Let g(n)= ∑ n i=1 f(i) g . g(100)=2007 g
513	Integral median	ABC is an integral sided triangle with sides a≤b≤c. mc is the median connecting C and the midpoint of AB. F(n) is the number of such triangles with c≤n for which mc has integral length as well. F(10)=3 and F(50)=165. Find F(100000).

514	Geoboard Shapes	A geoboard (of order N) is a square board with equally-spaced pins protruding from the surface, representing an integer point lattice for coordinates 0 ≤ x,y ≤ N. John begins with a pinless geoboard. Each position on the board is a hole that can be filled with a pin. John decides to generate a random integer between 1 and N+1 (inclusive) for each hole in the geoboard. If the random integer is equal to 1 for a given hole, then a pin is placed in that hole. After John is finished generating numbers for all (N+1)2 holes and placing any/all corresponding pins, he wraps a tight rubberband around the entire group of pins protruding from the board. Let S represent the shape that is formed. S can also be defined as the smallest convex shape that contains all the pins. The above image depicts a sample layout for N = 4. The green markers indicate positions where pins have

515	Dissonant Numbers	Let d(p,n,0) be the multiplicative inverse of n modulo prime p, defined as n × d(p,n,0) = 1 mod p. Let d(p,n,k) = ∑ n i=1 ∑ d(p,i,k−1) for k ≥ 1. Let D(a,b,k) = ∑ ∑ (d(p,p-1,k) mod p) for all primes a ≤ p < a + b. You are given: D(101,1,10) = 45 D(103,102,102) = 8334 D(106,103,103) = 38162302 Find D(109,105,105).
516	5-smooth totients	5-smooth numbers are numbers whose largest prime factor doesn't exceed 5. 5-smooth numbers are also called Hamming numbers. Let S(L) be the sum of the numbers n not exceeding L such that Euler's totient function φ(n) is a Hamming number. S(100)=3728. Find S(1012). Give your answer modulo 232.

517	A real recursion	For every real number a>1 a is given the sequence g a g by: g a (x)=1 g for x<a x g a (x)= g a (x−1)+ g a (x−a)
518	Prime triples and geometric sequences	Let S(n) = Σ a+b+c over all triples (a,b,c) such that: a, b, and c are prime numbers. a < b < c < n. a+1, b+1, and c+1 form a geometric sequence. For example, S(100) = 1035 with the following triples: (2, 5, 11), (2, 11, 47), (5, 11, 23), (5, 17, 53), (7, 11, 17), (7, 23, 71), (11, 23, 47), (17, 23, 31), (17, 41, 97), (31, 47, 71), (71, 83, 97) Find S(108).

519	Tricoloured Coin Fountains	An arrangement of coins in one or more rows with the bottom row being a block without gaps and every coin in a higher row touching exactly two coins in the row below is called a fountain of coins. Let f(n) be the number of possible fountains with n coins. For 4 coins there are three possible arrangements: Therefore f(4) = 3 while f(10) = 78. Let T(n) be the number of all possible colourings with three colours for all f(n) different fountains with n coins, given the condition that no two touching coins have the same colour. Below you see the possible colourings for one of the three valid fountains for 4 coins: You are given that T(4) = 48 and T(10) = 17760. Find the last 9 digits of T(20000).
520	Simbers	We define a simber to be a positive integer in which any odd digit, if present, occurs an odd number of times, and any even digit, if present, occurs an even number of times. For example, 141221242 is a 9-digit simber because it has three 1's, four 2's and two 4's. Let Q(n) be the count of all simbers with at most n digits. You are given Q(7) = 287975 and Q(100) mod 1 000 000 123 = 123864868. Find (∑1≤u≤39 Q(2u)) mod 1 000 000 123.

521	Smallest prime factor	Let smpf(n) be the smallest prime factor of n. smpf(91)=7 because 91=7×13 and smpf(45)=3 because 45=3×3×5. Let S(n) be the sum of smpf(i) for 2 ≤ i ≤ n. E.g. S(100)=1257. Find S(1012) mod 109.
522	Hilbert's Blackout	Despite the popularity of Hilbert's infinite hotel, Hilbert decided to try managing extremely large finite hotels, instead. To cut costs, Hilbert wished to power the new hotel with his own special generator. Each floor would send power to the floor above it, with the top floor sending power back down to the bottom floor. That way, Hilbert could have the generator placed on any given floor (as he likes having the option) and have electricity flow freely throughout the entire hotel. Unfortunately, the contractors misinterpreted the schematics when they built the hotel. They informed Hilbert that each floor sends power to another floor at random, instead. This may compromise Hilbert's freedom to have the generator placed anywhere, since blackouts could occur on certain floors. For example, consider a sample flow diagram for a three-story hotel:

523	First Sort I	Consider the following algorithm for sorting a list: 1. Starting from the beginning of the list, check each pair of adjacent elements in turn. 2. If the elements are out of order: a. Move the smallest element of the pair at the beginning of the list. b. Restart the process from step 1. 3. If all pairs are in order, stop. For example, the list { 4 1 3 2 } is sorted as follows: 4 1 3 2 (4 and 1 are out of order so move 1 to the front of the list) 1 4 3 2 (4 and 3 are out of order so move 3 to the front of the list) 3 1 4 2 (3 and 1 are out of order so move 1 to the front of the list) 1 3 4 2 (4 and 2 are out of order so move 2 to the front of the list) 2 1 3 4 (2 and 1 are out of order so move 1 to the front of the list) 1 2 3 4 (The list is now sorted) Let F(L) be the number of times step 2a is executed to sort list L. For example,

524	First Sort II	Consider the following algorithm for sorting a list: 1. Starting from the beginning of the list, check each pair of adjacent elements in turn. 2. If the elements are out of order: a. Move the smallest element of the pair at the beginning of the list. b. Restart the process from step 1. 3. If all pairs are in order, stop. For example, the list { 4 1 3 2 } is sorted as follows: 4 1 3 2 (4 and 1 are out of order so move 1 to the front of the list) 1 4 3 2 (4 and 3 are out of order so move 3 to the front of the list) 3 1 4 2 (3 and 1 are out of order so move 1 to the front of the list) 1 3 4 2 (4 and 2 are out of order so move 2 to the front of the list) 2 1 3 4 (2 and 1 are out of order so move 1 to the front of the list) 1 2 3 4 (The list is now sorted) Let F(L) be the number of times step 2a is executed to sort list L. For example,

525	Rolling Ellipse	An ellipse E(a, b) is given at its initial position by equation: x 2 a 2 + (y−b) 2 b 2 =1 x The ellipse rolls without slipping along the x axis for one complete turn. Interestingly, the length of the curve generated by a focus is independent from the size of the minor axis: F(a,b)=2π max(a,b) F This is not true for the curve generated by the ellipse center. Let C(a,b) be the length of the curve generated by the center of the ellipse as it rolls without

526	rgest prime factors of consecutive numb	Let f(n) be the largest prime factor of n. Let g(n) = f(n) + f(n+1) + f(n+2) + f(n+3) + f(n+4) + f(n+5) + f(n+6) + f(n+7) + f(n+8), the sum of the largest prime factor of each of nine consecutive numbers starting with n. Let h(n) be the maximum value of g(k) for 2 ≤ k ≤ n. You are given: f(100) = 5 f(101) = 101 g(100) = 409 h(100) = 417 h(109) = 4896292593 Find h(1016).

527	Randomized Binary Search	A secret integer t is selected at random within the range $1 \le t \le n$. The goal is to guess the value of t by making repeated guesses, via integer g. After a guess is made, there are three possible outcomes, in which it will be revealed that either $g < t$, $g = t$, or $g > t$. Then the process can repeat as necessary. Normally, the number of guesses required on average can be minimized with a binary search: Given a lower bound L and upper bound H (initialized to L = 1 and H = n), let $g = \lfloor (L+H)/2 \rfloor$ where $\lfloor \cdot \rfloor$ is the integer floor function. If g = t, the process ends. Otherwise, if $g < t$, set $L = g+1$, but if $g > t$ instead, set $H = g-1$. After setting the new bounds, the search process repeats, and ultimately ends once t is found. Even if t can be deduced without searching, assume that a search will be required anyway to confirm the value. Your friend Bob believes that the
528	Constrained Sums	Let S(n,k,b) represent the number of valid solutions to $x1 + x2 + ... + xk \le n$, where $0 \le xm \le bm$ for all $1 \le m \le k$. For example, S(14,3,2) = 135, S(200,5,3) = 12949440, and S(1000,10,5) mod 1 000 000 007 = 624839075. Find ($\sum 10 \le k \le 15$ S(10k,k,k)) mod 1 000 000 007.

529	10-substrings	A 10-substring of a number is a substring of its digits that sum to 10. For example, the 10-substrings of the number 3523014 are: 3523014 3523014 3523014 3523014 A number is called 10-substring-friendly if every one of its digits belongs to a 10-substring. For example, 3523014 is 10-substring-friendly, but 28546 is not. Let T(n) be the number of 10-substring-friendly numbers from 1 to 10n (inclusive). For example T(2) = 9 and T(5) = 3492. Find T(1018) mod 1 000 000 007.

530	GCD of Divisors	Every divisor d of a number n has a complementary divisor n/d. Let f(n) be the sum of the greatest common divisor of d and n/d over all positive divisors d of n, that is f(n)= ∑ d|n gcd(d, n d) f . Let F be the summatory function of f, that is F(k)= ∑ n=1 k f(n) F . You are given that F(10)=32 and F(1000)=12776.
531	Chinese leftovers	Let g(a,n,b,m) be the smallest non-negative solution x to the system: x = a mod n x = b mod m if such a solution exists, otherwise 0. E.g. g(2,4,4,6)=10, but g(3,4,4,6)=0. Let φ(n) be Euler's totient function. Let f(n,m)=g(φ(n),n,φ(m),m) Find ∑ f(n,m) for 1000000 ≤ n < m < 1005000

532	Nanobots on Geodesics	Bob is a manufacturer of nanobots and wants to impress his customers by giving them a ball coloured by his new nanobots as a present. His nanobots can be programmed to select and locate exactly one other bot precisely and, after activation, move towards this bot along the shortest possible path and draw a coloured line onto the surface while moving. Placed on a plane, the bots will start to move towards their selected bots in a straight line. In contrast, being placed on a ball, they will start to move along a geodesic as the shortest possible path. However, in both cases, whenever their target moves they will adjust their direction instantaneously to the new shortest possible path. All bots will move at the same speed after their simultaneous activation until each bot reaches its goal. Now Bob places n bots on the ball (with radius 1) equidistantly on a small circle with radius 0.999 and programs
533	inimum values of the Carmichael functio	The Carmichael function λ(n) is defined as the smallest positive integer m such that am = 1 modulo n for all integers a coprime with n. For example λ(8) = 2 and λ(240) = 4. Define L(n) as the smallest positive integer m such that λ(k) ≥ n for all k ≥ m. For example, L(6) = 241 and L(100) = 20 174 525 281. Find L(20 000 000). Give the last 9 digits of your answer.

534	Weak Queens	The classical eight queens puzzle is the well known problem of placing eight chess queens on a 8×8 chessboard so that no two queens threaten each other. Allowing configurations to reappear in rotated or mirrored form, a total of 92 distinct configurations can be found for eight queens. The general case asks for the number of distinct ways of placing n queens on a n×n board, e.g. you can find 2 distinct configurations for n=4. Let's define a weak queen on a n×n board to be a piece which can move any number of squares if moved horizontally, but a maximum of $n-1-w$ squares if moved vertically or diagonally, $0 \le w < n$ being the "weakness factor". For example, a weak queen on a n×n board with a weakness factor of w=1 located in the bottom row will not be able to threaten any square in the top row as the weak queen would need to move $n-1$ squares vertically or diagonally to

535	Fractal Sequence	Consider the infinite integer sequence S starting with: S = 1, 1, 2, 1, 3, 2, 4, 1, 5, 3, 6, 2, 7, 8, 4, 9, 1, 10, 11, 5, ... Circle the first occurrence of each integer. S = ①, 1, ②, 1, ③, 2, ④, 1, ⑤, 3, ⑥, 2, ⑦, ⑧, 4, ⑨, 1, ⑩, ⑪, 5, ... The sequence is characterized by the following properties: The circled numbers are consecutive integers starting with 1. Immediately preceding each non-circled numbers ai, there are exactly ⌊√ai⌋ adjacent circled numbers, where ⌊⌋ is the floor function. If we remove all circled numbers, the remaining numbers form a sequence identical to S, so S is a fractal sequence. Let T(n) be the sum of the first n elements of the sequence. You are given T(1) = 1, T(20) = 86, T(103) = 364089 and T(109) = 498676527978348241.
536	Modulo power identity	Let S(n) be the sum of all positive integers m not exceeding n having the following property: a m+4 ≡ a (mod m) for all integers a. The values of m ≤ 100 that satisfy this property are 1, 2, 3, 5 and 21, thus S(100) = 1+2+3+5+21 = 32. You are given S(106) = 22868117. Find S(1012).

537	Counting tuples	Let π(x) be the prime counting function, i.e. the number of prime numbers less than or equal to x. For example, π(1)=0, π(2)=1, π(100)=25. Let T(n,k) be the number of k-tuples (x1,...,xk) which satisfy: 1. every xi is a positive integer; 2. ∑ i=1 k π(x i)=n ∑ For example T(3,3)=19. The 19 tuples are (1,1,5), (1,5,1), (5,1,1), (1,1,6), (1,6,1), (6,1,1), (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (1,2,4), (1,4,2), (2,1,4), (2,4,1), (4,1,2), (4,2,1), (2,2,2). You are given T(10,10) = 869 985 and T(103,103) ≡ 578 270 566 (mod 1 004

538	Maximum quadrilaterals	Consider a positive integer sequence S = (s1, s2, ..., sn). Let f(S) be the perimeter of the maximum-area quadrilateral whose side lengths are 4 elements (si, sj, sk, sl) of S (all i, j, k, l distinct). If there are many quadrilaterals with the same maximum area, then choose the one with the largest perimeter. For example, if S = (8, 9, 14, 9, 27), then we can take the elements (9, 14, 9, 27) and form an isosceles trapezium with parallel side lengths 14 and 27 and both leg lengths 9. The area of this quadrilateral is 127.611470879... It can be shown that this is the largest area for any quadrilateral that can be formed using side lengths from S. Therefore, f(S) = 9 + 14 + 9 + 27 = 59. Let un = 2B(3n) + 3B(2n) + B(n+1), where B(k) is the number of 1 bits of k in base 2. For example, B(6) = 2, B(10) = 2 and B(15) = 4, and u5 = 24 + 32 + 2 = 27. Also, let Un be the sequence (u1, u2, ...,

539	Odd elimination	Start from an ordered list of all integers from 1 to n. Going from left to right, remove the first number and every other number afterward until the end of the list. Repeat the procedure from right to left, removing the right most number and every other number from the numbers left. Continue removing every other numbers, alternating left to right and right to left, until a single number remains. Starting with n = 9, we have: 1 2 3 4 5 6 7 8 9 2 4 6 8 2 6 6 Let P(n) be the last number left starting with a list of length n. Let S(n)= ∑ k=1 n P(k) S

540	Counting primitive Pythagorean triples	A Pythagorean triple consists of three positive integers a,b a and c c satisfying a 2 + b 2 = c 2 a . The triple is called primitive if a,b a and c c are relatively prime.

541	isibility of Harmonic Number Denomina	The nth harmonic number Hn is defined as the sum of the multiplicative inverses of the first n positive integers, and can be written as a reduced fraction an/bn. H n = ∑ k=1 n 1 k = a n b n H , with gcd(a n , b

542	eometric Progression with Maximum Su	Let S(k) be the sum of three or more distinct positive integers having the following properties: No value exceeds k. The values form a geometric progression. The sum is maximal. S(4) = 4 + 2 + 1 = 7 S(10) = 9 + 6 + 4 = 19 S(12) = 12 + 6 + 3 = 21 S(1000) = 1000 + 900 + 810 + 729 = 3439 Let T(n)= ∑ n k=4 (−1) k S(k) T . T(1000) = 2268 Find T(1017).

543	Prime-Sum Numbers	Define function P(n,k) = 1 if n can be written as the sum of k prime numbers (with repetitions allowed), and P(n,k) = 0 otherwise. For example, P(10,2) = 1 because 10 can be written as either 3 + 7 or 5 + 5, but P(11,2) = 0 because no two primes can sum to 11. Let S(n) be the sum of all P(i,k) over 1 ≤ i,k ≤ n. For example, S(10) = 20, S(100) = 2402, and S(1000) = 248838. Let F(k) be the kth Fibonacci number (with F(0) = 0 and F(1) = 1). Find the sum of all S(F(k)) over 3 ≤ k ≤ 44
544	Chromatic Conundrum	Let F(r,c,n) be the number of ways to colour a rectangular grid with r rows and c columns using at most n colours such that no two adjacent cells share the same colour. Cells that are diagonal to each other are not considered adjacent. For example, F(2,2,3) = 18, F(2,2,20) = 130340, and F(3,4,6) = 102923670. Let S(r,c,n) = ∑ n k=1 ∑ F(r,c,k). For example, S(4,4,15) mod 109+7 = 325951319. Find S(9,10,1112131415) mod 109+7.

545	Faulhaber's Formulas	The sum of the kth powers of the first n positive integers can be expressed as a polynomial of degree k+1 with rational coefficients, the Faulhaber's Formulas: 1 k + 2 k +...+ n k = ∑ n i=1 i k = ∑ k+1 i=1 a i

546	The Floor's Revenge	Define fk(n) = ∑ n i=0 ∑ fk(⌊ i k ⌋ ⌊) where fk(0) = 1 and ⌊x⌋ ⌊ denotes the floor function. For example, f5(10) = 18, f7(100) = 1003, and f2(103) = 264830889564. Find (∑ 10 k=2 (fk(1014))

547	e of random points within hollow square	Assuming that two points are chosen randomly (with uniform distribution) within a rectangle, it is possible to determine the expected value of the distance between these two points. For example, the expected distance between two random points in a unit square is about 0.521405, while the expected distance between two random points in a rectangle with side lengths 2 and 3 is about 1.317067. Now we define a hollow square lamina of size n to be an integer sized square with side length n ≥ 3 consisting of n2 unit squares from which a rectangle consisting of x × y unit squares (1 ≤ x,y ≤ n - 2) within the original square has been removed. For n = 3 there exists only one hollow square lamina: For n = 4 you can find 9 distinct hollow square laminae, allowing shapes to reappear in rotated or mirrored form: Let S(n) be the sum of the expected distance between two points chosen
548	Gozinta Chains	A gozinta chain for n is a sequence {1,a,b,...,n} where each element properly divides the next. There are eight gozinta chains for 12: {1,12} ,{1,2,12}, {1,2,4,12}, {1,2,6,12}, {1,3,12}, {1,3,6,12}, {1,4,12} and {1,6,12}. Let g(n) be the number of gozinta chains for n, so g(12)=8. g(48)=48 and g(120)=132. Find the sum of the numbers n not exceeding 1016 for which g(n)=n.

549	Divisibility of factorials	The smallest number m such that 10 divides m! is m=5. The smallest number m such that 25 divides m! is m=10. Let s(n) be the smallest number m such that n divides m!. So s(10)=5 and s(25)=10. Let S(n) be ∑s(i) for 2 ≤ i ≤ n. S(100)=2012. Find S(108).
550	Divisor game	Two players are playing a game. There are k piles of stones. When it is his turn a player has to choose a pile and replace it by two piles of stones under the following two conditions: Both new piles must have a number of stones more than one and less than the number of stones of the original pile. The number of stones of each of the new piles must be a divisor of the number of stones of the original pile. The first player unable to make a valid move loses. Let f(n,k) be the number of winning positions for the first player, assuming perfect play, when the game is played with k piles each having between 2 and n stones (inclusively). f(10,5)=40085. Find f(107,1012). Give your answer modulo 987654321.

551	Sum of digits sequence	Let a0, a1, a2, ... be an integer sequence defined by: a0 = 1; for n ≥ 1, an is the sum of the digits of all preceding terms. The sequence starts with 1, 1, 2, 4, 8, 16, 23, 28, 38, 49, ... You are given a106 = 31054319. Find a1015.
552	Chinese leftovers II	Let An be the smallest positive integer satisfying An mod pi = i for all 1 ≤ i ≤ n, where pi is the i-th prime. For example A2 = 5, since this is the smallest positive solution of the system of equations A2 mod 2 = 1 A2 mod 3 = 2 The system of equations for A3 adds another constraint. That is, A3 is the smallest positive solution of A3 mod 2 = 1 A3 mod 3 = 2 A3 mod 5 = 3 and hence A3 = 23. Similarly, one gets A4 = 53 and A5 = 1523. Let S(n) be the sum of all primes up to n that divide at least one element in the sequence A. For example, S(50) = 69 = 5 + 23 + 41, since 5 divides A2, 23 divides A3 and 41 divides A10 = 5765999453. No other prime number up to 50 divides an element in A. Find S(300000).

553	Power sets of power sets	Let P(n) be the set of the first n positive integers {1, 2, ..., n}. Let Q(n) be the set of all the non-empty subsets of P(n). Let R(n) be the set of all the non-empty subsets of Q(n). An element X ∈ R(n) is a non-empty subset of Q(n), so it is itself a set. From X we can construct a graph as follows: Each element Y ∈ X corresponds to a vertex and labeled with Y; Two vertices Y1 and Y2 are connected if Y1 ∩ Y2 ≠ ∅. For example, X = {{1}, {1,2,3}, {3}, {5,6}, {6,7}} results in the following graph: This graph has two connected components. Let C(n,k) be the number of elements of R(n) that have exactly k connected components in their graph. You are given C(2,1) = 6, C(3,1) = 111, C(4,2) = 486, C(100,10) mod 1 000 000 007 = 728209718. Find C(104,10) mod 1 000 000 007.

554	Centaurs on a chess board	On a chess board, a centaur moves like a king or a knight. The diagram below shows the valid moves of a centaur (represented by an inverted king) on an 8x8 board. It can be shown that at most n2 non-attacking centaurs can be placed on a board of size 2n×2n. Let C(n) be the number of ways to place n2 centaurs on a 2n×2n board so that no centaur attacks another directly. For example C(1) = 4, C(2) = 25, C(10) = 1477721. Let Fi be the ith Fibonacci number defined as F1 = F2 = 1 and Fi = Fi-1 + Fi-2 for i > 2. Find (∑ i=2 90 C(F i

555	McCarthy 91 function	The McCarthy 91 function is defined as follows: M 91 (n)={ n−10 M 91 (M 91 (n+11)) if n>100 if 0≤n≤100 M91(n)={n−10if n>100M91(M91(n+11))if 0≤n≤100 We can generalize this definition by abstracting away the constants into new variables: M m,k,s (n)={ n−s M m,k,s

556	Squarefree Gaussian Integers	A Gaussian integer is a number z = a + bi where a, b are integers and i2 = -1. Gaussian integers are a subset of the complex numbers, and the integers are the subset of Gaussian integers for which b = 0. A Gaussian integer unit is one for which a2 + b2 = 1, i.e. one of 1, i, -1, -i. Let's define a proper Gaussian integer as one for which a > 0 and b ≥ 0. A Gaussian integer z1 = a1 + b1i is said to be divisible by z2 = a2 + b2i if z3 = a3 + b3i = z1/z2 is a Gaussian integer. z 1 z 2 = a 1 + b 1 i

557	Cutting triangles	A triangle is cut into four pieces by two straight lines, each starting at one vertex and ending on the opposite edge. This results in forming three smaller triangular pieces, and one quadrilateral. If the original triangle has an integral area, it is often possible to choose cuts such that all of the four pieces also have integral area. For example, the diagram below shows a triangle of area 55 that has been cut in this way. Representing the areas as a, b, c and d, in the example above, the individual areas are a = 22, b = 8, c = 11 and d = 14. It is also possible to cut a triangle of area 55 such that a = 20, b = 2, c = 24, d = 9. Define a triangle cutting quadruple (a, b, c, d) as a valid integral division of a triangle, where a is the area of the triangle between the two cut vertices, d is the area of the quadrilateral and b and c are the areas of the two other triangles, with the restriction that b ≤

558	Irrational base	Let r be the real root of the equation x3 = x2 + 1. Every positive integer can be written as the sum of distinct increasing powers of r. If we require the number of terms to be finite and the difference between any two exponents to be three or more, then the representation is unique. For example, 3 = r -10 + r -5 + r -1 + r 2 and 10 = r -10 + r -7 + r 6. Interestingly, the relation holds for the complex roots of the equation. Let w(n) be the number of terms in this unique representation of n. Thus w(3) = 4 and w(10) = 3. More formally, for all positive integers n, we have: n = ∑ k=−∞ ∞ ∑ bk rk

559	Permuted Matrices	An ascent of a column j in a matrix occurs if the value of column j is smaller than the value of column j+1 in all rows. Let P(k, r, n) be the number of r x n matrices with the following properties: The rows are permutations of {1, 2, 3, ... , n}. Numbering the first column as 1, a column ascent occurs at column j<n if and only if j is not a multiple of k. For example, P(1, 2, 3) = 19, P(2, 4, 6) = 65508751 and P(7, 5, 30) mod 1000000123 = 161858102. Let Q(n) = ∑ k=1 n ∑ P(k, n, n). For example, Q(5) = 21879393751 and Q(50) mod 1000000123 = 819573537. Find Q(50000) mod 1000000123.

560	Coprime Nim	Coprime Nim is just like ordinary normal play Nim, but the players may only remove a number of stones from a pile that is coprime with the current size of the pile. Two players remove stones in turn. The player who removes the last stone wins. Let L(n, k) be the number of losing starting positions for the first player, assuming perfect play, when the game is played with k piles, each having between 1 and n - 1 stones inclusively. For example, L(5, 2) = 6 since the losing initial positions are (1, 1), (2, 2), (2, 4), (3, 3), (4, 2) and (4, 4). You are also given L(10, 5) = 9964, L(10, 10) = 472400303, L(103, 103) mod 1 000 000 007 = 954021836. Find L(107, 107) mod 1 000 000 007

561	Divisor Pairs	Let S(n) S be the number of pairs (a,b) (of distinct divisors of n n such that a a divides b b . For n=6 n we get the following pairs: (1,2),(1,3),(1,6),(2,6) (and (3,6) (

562	Maximal perimeter	Construct triangle ABC such that: Vertices A, B and C are lattice points inside or on the circle of radius r centered at the origin; the triangle contains no other lattice point inside or on its edges; the perimeter is maximum. Let R be the circumradius of triangle ABC and T(r) = R/r. For r = 5, one possible triangle has vertices (-4,-3), (4,2) and (1,0) with perimeter 13 − − √ + 34 − − √ + 89 − −

563	Robot Welders	A company specialises in producing large rectangular metal sheets, starting from unit square metal plates. The welding is performed by a range of robots of increasing size. Unfortunately, the programming options of these robots are rather limited. Each one can only process up to 25 identical rectangles of metal, which they can weld along either edge to produce a larger rectangle. The only programmable variables are the number of rectangles to be processed (up to and including 25), and whether to weld the long or short edge. For example, the first robot could be programmed to weld together 11 raw unit square plates to make a 11×1 strip. The next could take 10 of these 11×1 strips, and weld them either to make a longer 110×1 strip, or a 11×10 rectangle. Many, but not all, possible dimensions of metal sheets can be constructed in this way. One regular customer has a

564	Maximal polygons	A line segment of length 2n−3 2 is randomly split into n n segments of integer length (n≥3 n). In the sequence given by this split, the segments are then used as consecutive sides of a convex n n -polygon, formed in such a way that its area is maximal. All of the (2n−4 n−1) (possibilities for splitting up the initial line segment occur with the same probability. Let

565	Divisibility of sum of divisors	Let σ(n) σ be the sum of the divisors of n n . E.g. the divisors of 4 are 1, 2 and 4, so σ(4)=7 σ . The numbers n n not exceeding 20 such that 7 divides σ(n) σ are: 4,12,13 and 20, the sum of these numbers being 49. Let S(n,d) S be the sum of the numbers i i

566	Cake Icing Puzzle	Adam plays the following game with his birthday cake. He cuts a piece forming a circular sector of 60 degrees and flips the piece upside down, with the icing on the bottom. He then rotates the cake by 60 degrees counterclockwise, cuts an adjacent 60 degree piece and flips it upside down. He keeps repeating this, until after a total of twelve steps, all the icing is back on top. Amazingly, this works for any piece size, even if the cutting angle is an irrational number: all the icing will be back on top after a finite number of steps. Now, Adam tries something different: he alternates cutting pieces of size x= 360 9 x degrees,

567	Reciprocal games I	Tom has built a random generator that is connected to a row of n n light bulbs. Whenever the random generator is activated each of the n n lights is turned on with the probability of 1 2 1 , independently of its former state or the state of the other light bulbs. While discussing with his friend Jerry how to use his generator, they invent two different games, they call the reciprocal games: Both games consist of n n turns. Each turn is started by choosing a number k

568	Reciprocal games II	Tom has built a random generator that is connected to a row of n n light bulbs. Whenever the random generator is activated each of the n n lights is turned on with the probability of 1 2 1 , independently of its former state or the state of the other light bulbs. While discussing with his friend Jerry how to use his generator, they invent two different games, they call the reciprocal games: Both games consist of n n turns. Each turn is started by choosing a number k

569	Prime Mountain Range	A mountain range consists of a line of mountains with slopes of exactly 45°, and heights governed by the prime numbers, pn. The up-slope of the kth mountain is of height p2k−1, and the downslope is p2k. The first few foot-hills of this range are illustrated below. Tenzing sets out to climb each one in turn, starting from the lowest. At the top of each peak, he looks back and counts how many of the previous peaks he can see. In the example above, the eye-line from the third mountain is drawn in red, showing that he can only see the peak of the second mountain from this viewpoint. Similarly, from the 9th mountain, he can see three peaks, those of the 5th, 7th and 8th mountain. Let P(k) be the number of peaks that are visible looking back from the kth mountain. Hence P(3)=1 and P(9)=3. Also ∑ k=1

570	Snowflakes	A snowflake of order n is formed by overlaying an equilateral triangle (rotated by 180 degrees) onto each equilateral triangle of the same size in a snowflake of order n-1. A snowflake of order 1 is a single equilateral triangle. Some areas of the snowflake are overlaid repeatedly. In the above picture, blue represents the areas that are one layer thick, red two layers thick, yellow three layers thick, and so on. For an order n snowflake, let A(n) be the number of triangles that are one layer thick, and let B(n) be the number of triangles that are three layers thick. Define G(n) = gcd(A(n), B(n)). E.g. A(3) = 30, B(3) = 6, G(3)=6 A(11) = 3027630, B(11) = 19862070, G(11) = 30 Further, G(500) = 186 and ∑ 500 n=3

571	Super Pandigital Numbers	A positive number is pandigital in base b if it contains all digits from 0 to b - 1 at least once when written in base b. A n-super-pandigital number is a number that is simultaneously pandigital in all bases from 2 to n inclusively. For example 978 = 11110100102 = 11000203 = 331024 = 124035 is the smallest 5-super-pandigital number. Similarly, 1093265784 is the smallest 10-super-pandigital number. The sum of the 10 smallest 10-super-pandigital numbers is 20319792309. What is the sum of the 10 smallest 12-super-pandigital numbers?

572	Idempotent matrices	A matrix M M is called idempotent if M 2 =M M . Let M M be a three by three matrix : M= ⎛ ⎜ a d g b e h c f

573	Unfair race	n n runners in very different training states want to compete in a race. Each one of them is given a different starting number k k (1≤k≤n) (according to his (constant) individual racing speed being v k = k n v . In order to give the slower runners a chance to win the race, n n different starting positions are chosen randomly (with uniform distribution)

574	Verifying Primes	Let q q be a prime and $A \geq B > 0$ A be two integers with the following properties: A A and B B have no prime factor in common, that is $gcd(A,B)=1$ gcd . The product AB A is divisible by every prime less than q. It can be shown that, given these conditions, any sum $A+B<$

575	Wandering Robots	It was quite an ordinary day when a mysterious alien vessel appeared as if from nowhere. After waiting several hours and receiving no response it is decided to send a team to investigate, of which you are included. Upon entering the vessel you are met by a friendly holographic figure, Katharina, who explains the purpose of the vessel, Eulertopia. She claims that Eulertopia is almost older than time itself. Its mission was to take advantage of a combination of incredible computational power and vast periods of time to discover the answer to life, the universe, and everything. Hence the resident cleaning robot, Leonhard, along with his housekeeping responsibilities, was built with a powerful computational matrix to ponder the meaning of life as he wanders through a massive 1000 by 1000 square grid of rooms. She goes on to explain that the rooms are numbered sequentially from left to

576	Irrational jumps	A bouncing point moves counterclockwise along a circle with circumference 1 1 with jumps of constant length l<1 l , until it hits a gap of length g<1 g , that is placed in a distance d d counterclockwise from the starting point. The gap does not include the starting point, that is g+d<1 g . Let S(l,g,d) S be the sum of the length of all jumps, until the point falls into the gap. It can

577	Counting hexagons	An equilateral triangle with integer side length n≥3 n is divided into n 2 n equilateral triangles with side length 1 as shown in the diagram below. The vertices of these triangles constitute a triangular lattice with (n+1)(n+2) 2 (lattice points. Let H(n) H be the number of all regular hexagons that can be found by connecting 6 of these points. For example, H(3)=1 H

578	Integers with decreasing prime powers	Any positive integer can be written as a product of prime powers: p1a1 × p2a2 × ... × pkak, where pi are distinct prime integers, ai > 0 and pi < pj if i < j. A decreasing prime power positive integer is one for which ai ≥ aj if i < j. For example, 1, 2, 15=3×5, 360=23×32×5 and 1000=23×53 are decreasing prime power integers. Let C(n) be the count of decreasing prime power positive integers not exceeding n. C(100) = 94 since all positive integers not exceeding 100 have decreasing prime powers except 18, 50, 54, 75, 90 and 98. You are given C(106) = 922052. Find C(1013).

579	Lattice points in lattice cubes	A lattice cube is a cube in which all vertices have integer coordinates. Let C(n) be the number of different lattice cubes in which the coordinates of all vertices range between (and including) 0 and n. Two cubes are hereby considered different if any of their vertices have different coordinates. For example, C(1)=1, C(2)=9, C(4)=100, C(5)=229, C(10)=4469 and C(50)=8154671. Different cubes may contain different numbers of lattice points. For example, the cube with the vertices (0, 0, 0), (3, 0, 0), (0, 3, 0), (0, 0, 3), (0, 3, 3), (3, 0, 3), (3, 3, 0), (3, 3, 3) contains 64 lattice points (56 lattice points on the surface including the 8 vertices and 8 points within the cube). In contrast, the cube with the vertices (0, 2, 2), (1, 4, 4), (2, 0, 3), (2, 3, 0), (3, 2, 5), (3, 5, 2), (4, 1, 1), (5, 3, 3) contains only 40 lattice points (20 points on the surface and 20 points within the cube), although both cubes have the same

580	Squarefree Hilbert numbers	A Hilbert number is any positive integer of the form 4k+1 4 for integer k≥0 k . We shall define a squarefree Hilbert number as a Hilbert number which is not divisible by the square of any Hilbert number other than one. For example, 117 117 is a squarefree Hilbert number, equaling 9×13 9 . However 6237 6237 is a Hilbert number that is not squarefree in this sense, as it is divisible by 9
581	47-smooth triangular numbers	A number is p-smooth if it has no prime factors larger than p. Let T be the sequence of triangular numbers, ie T(n)=n(n+1)/2. Find the sum of all indices n such that T(n) is 47-smooth.
582	Nearly isosceles 120 degree triangles	Let a, b and c be the sides of an integer sided triangle with one angle of 120 degrees, a≤b≤c and b-a≤100. Let T(n) be the number of such triangles with c≤n. T(1000)=235 and T(108)=1245. Find T(10100).

583	Heron Envelopes	A standard envelope shape is a convex figure consisting of an isosceles triangle (the flap) placed on top of a rectangle. An example of an envelope with integral sides is shown below. Note that to form a sensible envelope, the perpendicular height of the flap (BCD) must be smaller than the height of the rectangle (ABDE). In the envelope illustrated, not only are all the sides integral, but also all the diagonals (AC, AD, BD, BE and CE) are integral too. Let us call an envelope with these properties a Heron envelope. Let S(p) be the sum of the perimeters of all the Heron envelopes with a perimeter less than or equal to p. You are given that S(104) = 884680. Find S(107).

584	Birthday Problem Revisited	A long long time ago in a galaxy far far away, the Wimwians, inhabitants of planet WimWi, discovered an unmanned drone that had landed on their planet. On examining the drone, they uncovered a device that sought the answer for the so called "Birthday Problem". The description of the problem was as follows: If people on your planet were to enter a very large room one by one, what will be the expected number of people in the room when you first find 3 people with Birthdays within 1 day from each other. The description further instructed them to enter the answer into the device and send the drone into space again. Startled by this turn of events, the Wimwians consulted their best mathematicians. Each year on Wimwi has 10 days and the mathematicians assumed equally likely birthdays and ignored leap years (leap years in Wimwi have 11 days), and found

585	Nested square roots	Consider the term x+ y √ + z √ − − − − − − − − − − − √ x that is representing a nested square root. x x ,

586	Binary Quadratic Form	The number 209 can be expressed as a 2 +3ab+ b 2 a in two distinct ways: 209= 8 2 +3·8·5+ 5 2 209 209= 13 2 +3·13·1+ 1 2 209 Let f(n,r)

587	Concave triangle	A square is drawn around a circle as shown in the diagram below on the left. We shall call the blue shaded region the L-section. A line is drawn from the bottom left of the square to the top right as shown in the diagram on the right. We shall call the orange shaded region a concave triangle. It should be clear that the concave triangle occupies exactly half of the L-section. Two circles are placed next to each other horizontally, a rectangle is drawn around both circles, and a line is drawn from the bottom left to the top right as shown in the diagram below. This time the concave triangle occupies approximately 36.46% of the L-section. If n circles are placed next to each other horizontally, a rectangle is drawn around the n circles, and a line is drawn from the bottom left to the top right, then it can be shown that the

588	Quintinomial coefficients	The coefficients in the expansion of (x+1) k (are called binomial coefficients. Analoguously the coefficients in the expansion of (x 4 + x 3 + x 2 +x+1) k (are called quintinomial coefficients. (quintus= Latin for fifth). Consider the expansion of (

589	Poohsticks Marathon	Christopher Robin and Pooh Bear love the game of Poohsticks so much that they invented a new version which allows them to play for longer before one of them wins and they have to go home for tea. The game starts as normal with both dropping a stick simultaneously on the upstream side of a bridge. But rather than the game ending when one of the sticks emerges on the downstream side, instead they fish their sticks out of the water, and drop them back in again on the upstream side. The game only ends when one of the sticks emerges from under the bridge ahead of the other one having also 'lapped' the other stick - that is, having made one additional journey under the bridge compared to the other stick. On a particular day when playing this game, the time taken for a stick to travel under the bridge varies between a minimum of 30 seconds, and a maximum of 60 seconds. The time

590	Sets with a given Least Common Multiple	Let H(n) denote the number of sets of positive integers such that the least common multiple of the integers in the set equals n. E.g.: The integers in the following ten sets all have a least common multiple of 6: {2,3}, {1,2,3}, {6}, {1,6}, {2,6} ,{1,2,6}, {3,6}, {1,3,6}, {2,3,6} and {1,2,3,6}. Thus H(6)=10. Let L(n) denote the least common multiple of the numbers 1 through n. E.g. L(6) is the least common multiple of the numbers 1,2,3,4,5,6 and L(6) equals 60. Let HL(n) denote H(L(n)). You are given HL(4)=H(12)=44. Find HL(50000). Give your answer modulo 109.

591	est Approximations by Quadratic Intege	Given a non-square integer d d , any real x x can be approximated arbitrarily close by quadratic integers a+b d − − √ a , where a,b a are integers. For example, the following inequalities approximate π π with precision 10 −13 10
592	Factorial trailing digits 2	For any N, let f(N) be the last twelve hexadecimal digits before the trailing zeroes in N!. For example, the hexadecimal representation of 20! is 21C3677C82B40000, so f(20) is the digit sequence 21C3677C82B4. Find f(20!). Give your answer as twelve hexadecimal digits, using uppercase for the digits A to F.

593	Fleeting Medians	We define two sequences S={S(1),S(2),...,S(n)} S and S 2 ={ S 2 (1), S 2 (2),..., S 2 (n)} S : S(k)=(p k) k S mod

594	Rhombus Tilings	For a polygon P P , let t(P) t be the number of ways in which P P can be tiled using rhombi and squares with edge length 1. Distinct rotations and reflections are counted as separate tilings. For example, if O O is a regular octagon with edge length 1, then t(O)=8 t . As it happens, all these 8 tilings are rotations of one another: Let O a,b

595	Incremental Random Sort	A deck of cards numbered from 1 to n is shuffled randomly such that each permutation is equally likely. The cards are to be sorted into ascending order using the following technique: Look at the initial sequence of cards. If it is already sorted, then there is no need for further action. Otherwise, if any subsequences of cards happen to be in the correct place relative to one another (ascending with no gaps), then those subsequences are fixed by attaching the cards together. For example, with 7 cards initially in the order 4123756, the cards labelled 1, 2 and 3 would be attached together, as would 5 and 6. The cards are 'shuffled' by being thrown into the air, but note that any correctly sequenced cards remain attached, so their orders are maintained. The cards (or bundles of attached cards) are then picked up randomly. You should assume that this
596	Number of lattice points in a hyperball	Let T(r) be the number of integer quadruplets x, y, z, t such that x2 + y2 + z2 + t2 ≤ r2. In other words, T(r) is the number of lattice points in the four-dimensional hyperball of radius r. You are given that T(2) = 89, T(5) = 3121, T(100) = 493490641 and T(104) = 49348022079085897. Find T(108) mod 1000000007.

597	Torpids	The Torpids are rowing races held annually in Oxford, following some curious rules: A division consists of n n boats (typically 13), placed in order based on past performance. All boats within a division start at 40 metre intervals along the river, in order with the highest-placed boat starting furthest upstream. The boats all start rowing simultaneously, upstream, trying to catch the boat in front while avoiding being caught by boats behind. Each boat continues rowing until either it reaches the finish line or it catches up with ("bumps") a boat in front. The finish line is a distance L L metres (the course length, in reality about 1800 metres) upstream from the starting position of the lowest-placed

598	Split Divisibilities	Consider the number 48. There are five pairs of integers a a and b b (a≤b a) such that a×b=48 a : (1,48), (2,24), (3,16), (4,12) and (6,8). It can be seen that both 6 and 8 have 4 divisors. So of those five pairs one consists of two integers with the same number of divisors. In general: Let C(n) C be the number of pairs of positive integers

599	Distinct Colourings of a Rubik's Cube	The well-known Rubik's Cube puzzle has many fascinating mathematical properties. The 2×2×2 variant has 8 cubelets with a total of 24 visible faces, each with a coloured sticker. Successively turning faces will rearrange the cubelets, although not all arrangements of cubelets are reachable without dismantling the puzzle. Suppose that we wish to apply new stickers to a 2×2×2 Rubik's cube in a non-standard colouring. Specifically, we have n n different colours available (with an unlimited supply of stickers of each colour), and we place one sticker on each of the 24 faces in any arrangement that we please. We are not required to use all the colours, and if desired the same colour may appear in more than one face of a single cubelet.
600	Integer sided equiangular hexagons	Let H(n) be the number of distinct integer sided equiangular convex hexagons with perimeter not exceeding n. Hexagons are distinct if and only if they are not congruent. You are given H(6) = 1, H(12) = 10, H(100) = 31248. Find H(55106). Equiangular hexagons with perimeter not exceeding 12

601	Divisibility streaks	For every positive number n n we define the function streak(n)=k s as the smallest positive integer k k such that n+k n is not divisible by k+1 k . E.g: 13 is divisible by 1 14 is divisible by 2 15 is divisible by 3 16 is divisible by 4 17 is NOT divisible by 5 So streak(13)=4 s

602	Product of Head Counts	Alice enlists the help of some friends to generate a random number, using a single unfair coin. She and her friends sit around a table and, starting with Alice, they take it in turns to toss the coin. Everyone keeps a count of how many heads they obtain individually. The process ends as soon as Alice obtains a Head. At this point, Alice multiplies all her friends' Head counts together to obtain her random number. As an illustration, suppose Alice is assisted by Bob, Charlie, and Dawn, who are seated round the table in that order, and that they obtain the sequence of Head/Tail outcomes THHH—TTTT—THHT—H beginning and ending with Alice. Then Bob and Charlie each obtain 2 heads, and Dawn obtains 1 head. Alice's random number is therefore $2\times2\times1=4$ 2 .

603	Substring sums of prime concatenations	Let S(n) S be the sum of all contiguous integer-substrings that can be formed from the integer n n . The substrings need not be distinct. For example, S(2024)=2+0+2+4+20+02+24+202+024+2024=2304 S . Let P(n) P be the integer formed by concatenating the first n n primes together. For example, P(7)=2357111317 P .

604	Convex path in square	Let F(N) F be the maximum number of lattice points in an axis-aligned N×N N square that the graph of a single strictly convex increasing function can pass through. You are given that F(1)=2 F , F(3)=3 F , F(9)=6 F , F(11)=7 F , F(100)=30 F

605	Pairwise Coin-Tossing Game	Consider an n n -player game played in consecutive pairs: Round 1 1 takes place between players 1 1 and 2 2 , round 2 2 takes place between players 2 2 and 3 3 , and so on and so forth, all the way up to round n
606	Gozinta Chains II	A gozinta chain for n is a sequence {1,a,b,...,n} where each element properly divides the next. For example, there are eight distinct gozinta chains for 12: {1,12}, {1,2,12}, {1,2,4,12}, {1,2,6,12}, {1,3,12}, {1,3,6,12}, {1,4,12} and {1,6,12}. Let S(n) be the sum of all numbers, k, not exceeding n, which have 252 distinct gozinta chains. You are given S(106)=8462952 and S(1012)=623291998881978. Find S(1036), giving the last nine digits of your answer.

607	Marsh Crossing	Frodo and Sam need to travel 100 leagues due East from point A to point B. On normal terrain, they can cover 10 leagues per day, and so the journey would take 10 days. However, their path is crossed by a long marsh which runs exactly South-West to North-East, and walking through the marsh will slow them down. The marsh is 50 leagues wide at all points, and the mid-point of AB is located in the middle of the marsh. A map of the region is shown in the diagram below: The marsh consists of 5 distinct regions, each 10 leagues across, as shown by the shading in the map. The strip closest to point A is relatively light marsh, and can be crossed at a speed of 9 leagues per day. However, each strip becomes progressively harder to navigate, the speeds going down to 8, 7, 6 and finally 5 leagues per day for the final region of marsh, before it ends and the terrain becomes easier again, with the speed going back to 10

608	Divisor Sums	Let D(m,n)= ∑ d|m ∑ k=1 n σ 0 (kd) D where d d runs through all divisors of m m and σ 0 (n) σ is the number of divisors of n n

609	π sequences	For every n≥1 n the prime-counting function π(n) π is equal to the number of primes not exceeding n n . E.g. π(6)=3 π and π(100)=25 π . We say that a sequence of integers u=(u 0 ,…, u m

610	Roman Numerals II	A random generator produces a sequence of symbols drawn from the set {I, V, X, L, C, D, M, #}. Each item in the sequence is determined by selecting one of these symbols at random, independently of the other items in the sequence. At each step, the seven letters are equally likely to be selected, with probability 14% each, but the # symbol only has a 2% chance of selection. We write down the sequence of letters from left to right as they are generated, and we stop at the first occurrence of the # symbol (without writing it). However, we stipulate that what we have written down must always (when non-empty) be a valid Roman numeral representation in minimal form. If appending the next letter would contravene this then we simply skip it and try again with the next symbol generated. Please take careful note of About... Roman Numerals for the definitive

611	Hallway of square steps	Peter moves in a hallway with N+1 doors consecutively numbered from 0 through N. All doors are initially closed. Peter starts in front of door 0, and repeatedly performs the following steps: First, he walks a positive square number of doors away from his position. Then he walks another, larger square number of doors away from his new position. He toggles the door he faces (opens it if closed, closes it if open). And finally returns to door 0. We call an action any sequence of those steps. Peter never performs the exact same action twice, and makes sure to perform all possible actions that don't bring him past the last door. Let F(N) be the number of doors that are open after Peter has performed all possible actions. You are given that F(5) = 1, F(100) = 27, F(1000) = 233 and F(106) = 112168.

612	Friend numbers	Let's call two numbers friend numbers if their representation in base 10 has at least one common digit. E.g. 1123 and 3981 are friend numbers. Let f(n) f be the number of pairs (p,q) (with 1≤p<q<n 1 such that p p and q q are friend numbers. f(100)=1539 f . Find f(

613	Pythagorean Ant	Dave is doing his homework on the balcony and, preparing a presentation about Pythagorean triangles, has just cut out a triangle with side lengths 30cm, 40cm and 50cm from some cardboard, when a gust of wind blows the triangle down into the garden. Another gust blows a small ant straight onto this triangle. The poor ant is completely disoriented and starts to crawl straight ahead in random direction in order to get back into the grass. Assuming that all possible positions of the ant within the triangle and all possible directions of moving on are equiprobable, what is the probability that the ant leaves the triangle along its longest side? Give your answer rounded to 10 digits after the decimal point.

614	Special partitions 2	An integer partition of a number n n is a way of writing n n as a sum of positive integers. Partitions that differ only by the order of their summands are considered the same. We call an integer partition special if 1) all its summands are distinct, and 2) all its even summands are also divisible by 4. For example, the special partitions of 10 10 are: 10=1+4+5=3+7=1+9 10=1+4+5=3+7=1+9 The number 10 10 admits many more integer partitions (a total of 42), but only those three are special.

615	nth number with at least one million pri	Consider the natural numbers having at least 5 prime factors, which don't have to be distinct. Sorting these numbers by size gives a list which starts with: 32=2·2·2·2·2 48=2·2·2·2·3 64=2·2·2·2·2·2 72=2·2·2·3·3 80=2·2·2·2·5 96=2·2·2·2·2·3 ... So, for example, the fifth number with at least 5 prime factors is 80. Find the millionth number with at least one million prime factors. Give your answer modulo 123454321.

616	Creative numbers	Alice plays the following game, she starts with a list of integers L L and on each step she can either: remove two elements a a and b b from L L and add a b a to L L or conversely remove an element c c from

617	Mirror Power Sequence	For two integers n,e>1 n , we define a (n,e) (-MPS (Mirror Power Sequence) to be an infinite sequence of integers (a i) i≥0 (such that for all i≥0 i , a i+1 =min(a e i ,n−

618	Numbers with a given prime factor sum	Consider the numbers 15, 16 and 18: 15=3×5 15 and 3+5=8 3 . 16=2×2×2×2 16 and 2+2+2+2=8 2 . 18=2×3×3 18 and 2+3+3=8 2 . 15, 16 and 18 are the only numbers that have 8 as sum of the prime factors (counted with multiplicity). We define S(k) S

619	Square subsets	For a set of positive integers {a,a+1,a+2,...,b} { , let C(a,b) C be the number of non-empty subsets in which the product of all elements is a perfect square. For example C(5,10)=3 C , since the products of all elements of {5,8,10} { , {5,8,9,10} { and {9} { are perfect squares, and no other subsets of {5,6,7,8,9,10} {

620	Planetary Gears	A circle C C of circumference c c centimetres has a smaller circle S S of circumference s s centimetres lying off-centre within it. Four other distinct circles, which we call "planets", with circumferences p p , p p , q q , q

621	ing an integer as the sum of triangular n	Gauss famously proved that every positive integer can be expressed as the sum of three triangular numbers (including 0 as the lowest triangular number). In fact most numbers can be expressed as a sum of three triangular numbers in several ways. Let G(n) G be the number of ways of expressing n n as the sum of three triangular numbers, regarding different arrangements of the terms of the sum as distinct. For example, G(9)=7 G , as 9 can be expressed as: 3+3+3, 0+3+6, 0+6+3, 3+0+6, 3+6+0, 6+0+3, 6+3+0. You are given G(1000)=78

622	Riffle Shuffles	A riffle shuffle is executed as follows: a deck of cards is split into two equal halves, with the top half taken in the left hand and the bottom half taken in the right hand. Next, the cards are interleaved exactly, with the top card in the right half inserted just after the top card in the left half, the 2nd card in the right half just after the 2nd card in the left half, etc. (Note that this process preserves the location of the top and bottom card of the deck) Let s(n) s be the minimum number of consecutive riffle shuffles needed to restore a deck of size n n to its original configuration, where n n is a positive even number. Amazingly, a standard deck of

623	Lambda Count	The lambda-calculus is a universal model of computation at the core of functional programming languages. It is based on lambda-terms, a minimal programming language featuring only function definitions, function calls and variables. Lambda-terms are built according to the following rules: Any variable x x (single letter, from some infinite alphabet) is a lambda-term. If M M and N N are lambda-terms, then (MN) (is a lambda-term, called the application of M

624	Two heads are better than one	An unbiased coin is tossed repeatedly until two consecutive heads are obtained. Suppose these occur on the (M−1) (th and M M th toss. Let P(n) P be the probability that M M is divisible by n n . For example, the outcomes HH, HTHH, and THTTHH all count towards P(2) P , but THH and HTTHH do not. You are given that P(2)=

625	Gcd sum	G(N)= ∑ N j=1 ∑ j i=1 gcd(i,j) G . You are given: G(10)=122 G . Find G(10 11) G . Give your answer modulo 998244353

626	Counting Binary Matrices	A binary matrix is a matrix consisting entirely of 0s and 1s. Consider the following transformations that can be performed on a binary matrix: Swap any two rows Swap any two columns Flip all elements in a single row (1s become 0s, 0s become 1s) Flip all elements in a single column Two binary matrices A A and B B will be considered equivalent if there is a sequence of such transformations that when applied to A A yields B B . For example, the following two matrices are equivalent:

627	Counting products	Consider the set S S of all possible products of n n positive integers not exceeding m m , that is S={ x 1 x 2 ... x n |1≤ x 1 , x 2 ,...,

628	Open chess positions	A position in chess is an (orientated) arrangement of chess pieces placed on a chessboard of given size. In the following, we consider all positions in which n n pawns are placed on a n×n n board in such a way, that there is a single pawn in every row and every column. We call such a position an open position, if a rook, starting at the (empty) lower left corner and using only moves towards the right or upwards, can reach the upper right corner without moving onto any field occupied by a pawn. Let f(n) f be the number of open positions for a n×n

629	Scatterstone Nim	Alice and Bob are playing a modified game of Nim called Scatterstone Nim, with Alice going first, alternating turns with Bob. The game begins with an arbitrary set of stone piles with a total number of stones equal to n n . During a player's turn, he/she must pick a pile having at least 2 2 stones and perform a split operation, dividing the pile into an arbitrary set of p p non-empty, arbitrarily-sized piles where 2≤p≤k 2 for some fixed constant k k . For example, a pile of size

630	Crossed lines	Given a set, L L , of unique lines, let M(L) M be the number of lines in the set and let S(L) S be the sum over every line of the number of times that line is crossed by another line in the set. For example, two sets of three lines are shown below: In both cases M(L) is 3 and S(L) is 6: each of the three lines is crossed by two other lines. Note that even if the lines cross at a single point, all of the separate crossings of lines are counted. Consider points (T 2k−1 T

631	Constrained Permutations	Let (p 1 p 2 … p k) (denote the permutation of the set 1,…,k 1 that maps p i ↦i p . Define the length of the permutation to be k k ; note that the empty permutation ()

632	Square prime factors	For an integer n n , we define the square prime factors of n n to be the primes whose square divides n n . For example, the square prime factors of 1500= 2 2 ×3× 5 3 1500 are 2 2 and 5 5 .

633	Square prime factors II	For an integer n n , we define the square prime factors of n n to be the primes whose square divides n n . For example, the square prime factors of 1500= 2 2 ×3× 5 3 1500 are 2 2 and 5 5 .

634	Numbers of the forma2b3a	Define F(n) F to be the number of integers x≤n x that can be written in the form x= a 2 b 3 x , where a a and b b are integers not necessarily different and both greater than 1. For example, 32= 2 2

635	Subset sums	Let A q (n) A be the number of subsets, B B , of the set {1,2,...,q·n} { that satisfy two conditions: 1) B B has exactly n n elements; 2) the sum of the elements of B B is divisible by n n

636	Restricted Factorisations	Consider writing a natural number as product of powers of natural numbers with given exponents, additionally requiring different base numbers for each power. For example, 256 256 can be written as a product of a square and a fourth power in three ways such that the base numbers are different. That is, 256= 1 2 × 4 4 = 4 2 × 2 4 =

637	Flexible digit sum	Given any positive integer n n , we can construct a new integer by inserting plus signs between some of the digits of the base B B representation of n n , and then carrying out the additions. For example, from n= 123 10 n (n n in base 10) we can construct the four base 10 integers 123 10 123

638	Weighted lattice paths	Let P a,b P denote a path in a a×b a lattice grid with following properties: The path begins at (0,0) (and ends at (a,b) (. The path consists only of unit moves upwards or to the right; that is the coordinates are increasing with every move. Denote A(P a,b) A

639	Summing a multiplicative function	A multiplicative function f(x) f is a function over positive integers satisfying f(1)=1 f and f(ab)=f(a)f(b) f for any two coprime positive integers a a and b b . For integer k k let f k (n) f

640	Shut the Box	Bob plays a single-player game of chance using two standard 6-sided dice and twelve cards numbered 1 to 12. When the game starts, all cards are placed face up on a table. Each turn, Bob rolls both dice, getting numbers x x and y y respectively, each in the range 1,...,6. He must choose amongst three options: turn over card x x , card y y , or card x+y x . (If the chosen card is already face down, it is turned to face up, and vice

641	A Long Row of Dice	Consider a row of n n dice all showing 1. First turn every second die, (2,4,6,…) (, so that the number showing is increased by 1. Then turn every third die. The sixth die will now show a 3. Then turn every fourth die and so on until every n n th die (only the last die) is turned. If the die to be turned is showing a 6 then it is changed to show a 1. Let f(n) f be the number of dice that are showing a 1 when the process finishes. You are given f(100)=2 f

642	Sum of largest prime factors	Let f(n) f be the largest prime factor of n n and F(n)= ∑ i=2 n f(i) F . For example F(10)=32 F , F(100)=1915 F and F(10000)=10118280 F . Find

643	2-Friendly	Two positive integers a a and b b are 2-friendly when gcd(a,b)= 2 t ,t>0 gcd . For example, 24 and 40 are 2-friendly because gcd(24,40)=8= 2 3 gcd while 24 and 36 are not because gcd(24,36)=12= 2 2 ·3 gcd not a power of 2.

644	Squares on the line	Sam and Tom are trying a game of (partially) covering a given line segment of length L L by taking turns in placing unit squares onto the line segment. As illustrated below, the squares may be positioned in two different ways, either "straight" by placing the midpoints of two opposite sides on the line segment, or "diagonal" by placing two opposite corners on the line segment. Newly placed squares may touch other squares, but are not allowed to overlap any other square laid down before. The player who is able to place the last unit square onto the line segment wins. With Sam starting each game by placing the first square, they quickly realise that Sam can easily win every time by placing the first square in the middle of the line segment, making

645	Every Day is a Holiday	On planet J, a year lasts for D D days. Holidays are defined by the two following rules. At the beginning of the reign of the current Emperor, his birthday is declared a holiday from that year onwards. If both the day before and after a day d d are holidays, then d d also becomes a holiday. Initially there are no holidays. Let E(D) E be the expected number of Emperors to reign before all the days of the year are holidays, assuming that their birthdays are independent and uniformly distributed throughout the D

646	Bounded Divisors	Let n n be a natural number and p α 1 1 ⋅ p α 2 2 … p α k k p its prime factorisation. Define the Liouville function λ(n) λ as λ(n)=(−1

647	ear Transformations of Polygonal Numb	It is possible to find positive integers A A and B B such that given any triangular number, T n T , then A T n +B A is always a triangular number. We define F 3 (N) F to be the sum of (A+B) (

648	Skipping Squares	For some fixed ρ∈[0,1] ρ , we begin a sum s s at 0 0 and repeatedly apply a process: With probability ρ ρ , we add 1 1 to s s , otherwise we add 2 2 to s s

649	Low-Prime Chessboard Nim	Alice and Bob are taking turns playing a game consisting of c c different coins on a chessboard of size n n by n n . The game may start with any arrangement of c c coins in squares on the board. It is possible at any time for more than one coin to occupy the same square on the board at the same time. The coins are distinguishable, so swapping two coins gives a different arrangement if (and only if) they are on different squares. On a given turn, the player must choose a coin and move it either left or up

650	Divisors of Binomial Product	Let B(n)= ∏ k=0 n (n k) B , a product of binomial coefficients. For example, B(5)=(5 0)×(5 1)×(5 2)×(5 3)×(

651	Patterned Cylinders	An infinitely long cylinder has its curved surface fully covered with different coloured but otherwise identical rectangular stickers, without overlapping. The stickers are aligned with the cylinder, so two of their edges are parallel with the cylinder's axis, with four stickers meeting at each corner. Let a>0 a and suppose that the colouring is periodic along the cylinder, with the pattern repeating every a a stickers. (The period is allowed to be any divisor of a a .) Let b b be the number of stickers that fit

652	tinct values of a proto-logarithmic funct	Consider the values of log 2 (8) log , log 4 (64) log and log 3 (27) log . All three are equal to 3 3 . Generally, the function f(m,n)= log m (n) f

653	Frictionless Tube	Consider a horizontal frictionless tube with length L L millimetres, and a diameter of 20 millimetres. The east end of the tube is open, while the west end is sealed. The tube contains N N marbles of diameter 20 millimetres at designated starting locations, each one initially moving either westward or eastward with common speed v v . Since there are marbles moving in opposite directions, there are bound to be some collisions. We assume that the collisions are perfectly elastic, so both marbles involved instantly change direction and continue with speed v v

654	Neighbourly Constraints	Let T(n,m) T be the number of m m -tuples of positive integers such that the sum of any two neighbouring elements of the tuple is ≤n ≤ . For example, T(3,4)=8 T , via the following eight 4 4 -tuples: (1,1,1,1) ((1,1,1,2) (

655	Divisible Palindromes	The numbers 545 545 , 5995 5 and 15151 15 are the three smallest palindromes divisible by 109 109 . There are nine palindromes less than 100000 100 which are divisible by 109 109 . How many palindromes less than 10 32 10 are divisible by

656	Palindromic sequences	Given an irrational number α α , let S α (n) S be the sequence S α (n)=⌊α·n⌋−⌊α·(n−1)⌋ S for n≥1 n . (⌊...⌋ ⌊ is the floor-function.) It can be proven that for any irrational α α there exist infinitely many values of

657	Incomplete words	In the context of formal languages, any finite sequence of letters of a given alphabet Σ Σ is called a word over Σ Σ . We call a word incomplete if it does not contain every letter of Σ Σ . For example, using the alphabet Σ={a,b,c} Σ , ' ab a ', ' abab a ' and ' ' (the empty word) are incomplete words over

658	Incomplete words II	In the context of formal languages, any finite sequence of letters of a given alphabet Σ Σ is called a word over Σ Σ . We call a word incomplete if it does not contain every letter of Σ Σ . For example, using the alphabet Σ={a,b,c} Σ , ' ab a ', ' abab a ' and ' ' (the empty word) are incomplete words over

659	Largest prime	Consider the sequence n 2 +3 n with n≥1 n . If we write down the first terms of this sequence we get: 4,7,12,19,28,39,52,67,84,103,124,147,172,199,228,259,292,327,364, 4 ... . We see that the terms for n=6 n and n=7 n (39 39 and

660	Pandigital Triangles	We call an integer sided triangle n n -pandigital if it contains one angle of 120 degrees and, when the sides of the triangle are written in base n n , together they use all n n digits of that base exactly once. For example, the triangle (217, 248, 403) is 9-pandigital because it contains one angle of 120 degrees and the sides written in base 9 are 261 9 , 305 9 , 487 9 261

661	A Long Chess Match	Two friends A A and B B are great fans of Chess. They both enjoy playing the game, but after each game the player who lost the game would like to continue (to get back at his opponent) and the player who won would prefer to stop (to finish on a high). So they come up with a plan. After every game, they would toss a (biased) coin with probability p p of Heads (and hence probability 1−p 1 of Tails). If they get Tails, they will continue with the next game. Otherwise they end the match. Also, after every game the players make a

662	Fibonacci paths	Alice walks on a lattice grid. She can step from one lattice point A(a,b) A to another B(a+x,b+y) B providing distance AB= x 2 + y 2 − − − − − − √ A is a Fibonacci number {1,2,3,5,8,13,...} {

663	Sums of subarrays	Let t k t be the tribonacci numbers defined as: t 0 = t 1 =0 t ; t 2 =1 t ; t k = t k−1 + t

664	An infinite game	Peter is playing a solitaire game on an infinite checkerboard, each square of which can hold an unlimited number of tokens. Each move of the game consists of the following steps: Choose one token T T to move. This may be any token on the board, as long as not all of its four adjacent squares are empty. Select and discard one token D D from a square adjacent to that of T T . Move T T to any one of its four adjacent squares (even if that square is already occupied).

665	Proportionate Nim	Two players play a game with two piles of stones. On his or her turn, a player chooses a positive integer n n and does one of the following: removes n n stones from one pile; removes n n stones from both piles; or removes n n stones from one pile and 2n 2 stones from the other pile. The player who removes the last stone wins. We denote by

666	Polymorphic Bacteria	Members of a species of bacteria occur in two different types: α α and β β . Individual bacteria are capable of multiplying and mutating between the types according to the following rules: Every minute, each individual will simultaneously undergo some kind of transformation. Each individual A A of type α α will, independently, do one of the following (at random with equal probability): clone itself, resulting in a new bacterium of type α

667	Moving Pentagon	After buying a Gerver Sofa from the Moving Sofa Company, Jack wants to buy a matching cocktail table from the same company. Most important for him is that the table can be pushed through his L-shaped corridor into the living room without having to be lifted from its table legs. Unfortunately, the simple square model offered to him is too small for him, so he asks for a bigger model. He is offered the new pentagonal model illustrated below: Note, while the shape and size can be ordered individually, due to the production process, all edges of the pentagonal table have to have the same length. Given optimal form and size, what is the biggest pentagonal cocktail table (in terms of area) that Jack can buy that still fits through his unit wide L-shaped corridor? Give your answer rounded to 10 digits after the decimal point (if Jack had

668	Square root smooth Numbers	A positive integer is called square root smooth if all of its prime factors are strictly less than its square root. Including the number 1 1 , there are 29 29 square root smooth numbers not exceeding 100 100 . How many square root smooth numbers are there not exceeding 10000000000 10 ?

669	The King's Banquet	The Knights of the Order of Fibonacci are preparing a grand feast for their king. There are n n knights, and each knight is assigned a distinct number from 1 to n n . When the knights sit down at the roundtable for their feast, they follow a peculiar seating rule: two knights can only sit next to each other if their respective numbers sum to a Fibonacci number. When the n n knights all try to sit down around a circular table with n n chairs, they are unable to find a suitable seating arrangement for any

670	Colouring a Strip	A certain type of tile comes in three different sizes - 1×1, 1×2, and 1×3 - and in four different colours: blue, green, red and yellow. There is an unlimited number of tiles available in each combination of size and colour. These are used to tile a 2×n 2 rectangle, where n n is a positive integer, subject to the following conditions: The rectangle must be fully covered by non-overlapping tiles. It is not permitted for four tiles to have their corners meeting at a single point. Adjacent tiles must be of different colours. For example, the following is an acceptable tiling of a 2×12 2 rectangle:

671	Colouring a Loop	A certain type of flexible tile comes in three different sizes - 1×1, 1×2, and 1×3 - and in k k different colours. There is an unlimited number of tiles available in each combination of size and colour. These are used to tile a closed loop of width 2 2 and length (circumference) n n , where n n is a positive integer, subject to the following conditions: The loop must be fully covered by non-overlapping tiles. It is not permitted for four tiles to have their corners meeting at a single point. Adjacent tiles must be of different

672	One more one	Consider the following process that can be applied recursively to any positive integer n n : if n=1 n do nothing and the process stops, if n n is divisible by 7 divide it by 7, otherwise add 1. Define g(n) g to be the number of 1's that must be added before the process ends. For example: 125 − → +1

673	Beds and Desks	At Euler University, each of the n n students (numbered from 1 to n n) occupies a bed in the dormitory and uses a desk in the classroom. Some of the beds are in private rooms which a student occupies alone, while the others are in double rooms occupied by two students as roommates. Similarly, each desk is either a single desk for the sole use of one student, or a twin desk at which two students sit together as desk partners. We represent the bed and desk sharing arrangements each by a list of pairs of student numbers. For example, with n=4 n , if (2,3)

674	SolvingII-equations	We define the I I operator as the function I(x,y)=(1+x+y) 2 +y−x I(x,y)=(1+x+y)2+y−x and I I -expressions as arithmetic expressions built only from variables names and applications of I I . A variable name may consist of one or more letters. For example, the three expressions x x , I(x,y) I

675	2ω(n)2	Let ω(n) ω denote the number of distinct prime divisors of a positive integer n n . So ω(1)=0 ω and ω(360)=ω(2 3 × 3 2 ×5)=3 ω . Let S(n) S be

676	Matching Digit Sums	Let d(i,b) d be the digit sum of the number i i in base b b . For example d(9,2)=2 d , since 9= 1001 2 9 . When using different bases, the respective digit sums most of the time deviate from each other, for example d(9,4)=3≠d(9,2) d . However, for some numbers i

677	Coloured Graphs	Let g(n) g be the number of undirected graphs with n n nodes satisfying the following properties: The graph is connected and has no cycles or multiple edges. Each node is either red, blue, or yellow. A red node may have no more than 4 edges connected to it. A blue or yellow node may have no more than 3 edges connected to it. An edge may not directly connect a yellow node to a yellow node. For example, g(2)=5 g , g(3)=15 g , and

678	Fermat-like Equations	If a triple of positive integers (a,b,c) (satisfies a 2 + b 2 = c 2 a , it is called a Pythagorean triple. No triple (a,b,c) (satisfies a e + b e = c

679	Freefarea	Let S S be the set consisting of the four letters {\`A',\`E',\`F',\`R'} { . For n≥0 n , let S ∗ (n) S denote the set of words of length n n consisting of letters belonging to S S . We designate the words FREE,FARE,AREA,REEF FREE

680	Yarra Gnisrever	Let N N and K K be two positive integers. F n F is the n n -th Fibonacci number: F 1 = F 2 =1 F , F n =

681	Maximal Area	Given positive integers a≤b≤c≤d a , it may be possible to form quadrilaterals with edge lengths a,b,c,d a (in any order). When this is the case, let M(a,b,c,d) M denote the maximal area of such a quadrilateral. For example, M(2,2,3,3)=6 M , attained e.g. by a 2×3 2 rectangle. Let SP(n) S be the sum of a+b+c+d a

682	5-Smooth Pairs	5-smooth numbers are numbers whose largest prime factor doesn't exceed 5. 5-smooth numbers are also called Hamming numbers. Let Ω(a) Ω be the count of prime factors of a a (counted with multiplicity). Let s(a) s be the sum of the prime factors of a a (with multiplicity). For example, Ω(300)=5 Ω and s(300)=2+2+3+5+5=17 s

683	The Chase II	Consider the following variant of "The Chase" game. This game is played between n n players sitting around a circular table using two dice. It consists of n−1 n rounds, and at the end of each round one player is eliminated and has to pay a certain amount of money into a pot. The last player remaining is the winner and receives the entire contents of the pot. At the beginning of a round, each die is given to a randomly selected player. A round then consists of a number of turns. During each turn, each of the two players with a die rolls it. If a player rolls a 1 or a 2, she passes the die to her neighbour on the left; if she rolls a 5 or a 6, she passes the die to her neighbour on the right; otherwise, she