Recursive Feasibility of Stochastic Model Predictive Control with Mission-Wide Probabilistic Constraints

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Overview

Background

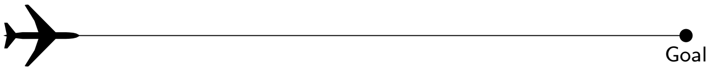
- Stochastic Optimal Control
- Mission-Wide Chance-Constrained Optimal Control
- Chance Constraints: mission-wide & stage-wise

Contributions

- Remaining MWPSs & Initially Prescribed MWPS
  (MWPS: Mission-Wide Probability of Safety)
- Recursive Feasibility of MWPS Constraints
- Stochastic Linear MPC with MWPS guarantee

Stochastic Optimal Control

Mission: plan an optimal trajectory for the drone



State transition: $\underbrace{\mathbf{s}, \mathbf{u}}_{\text{current state\&input}} \xrightarrow{\text{s.t. disturbance } \mathbf{w}} \underbrace{\mathbf{s}_+}_{\text{successor state}}$

- $\mathbf{s}, \mathbf{u}$ and $\mathbf{w}$ are all continuous
- we will describe this transition via conditional distribution:

$$\rho\left[\mathbf{s}_+ \,|\, \mathbf{s}, \mathbf{u}\right] \qquad \text{(depending on the distribution of } \mathbf{w}\text{)}$$

- In classic control, it is modeled as $\mathbf{s}_+ = \mathbf{f}(\mathbf{s}, \mathbf{u}, \mathbf{w})$ for some function $\mathbf{f}$

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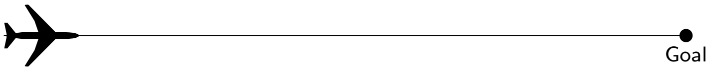

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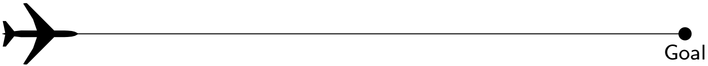

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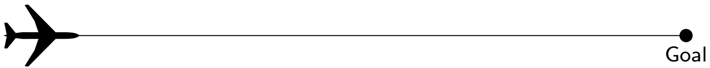

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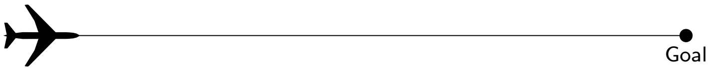

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Stochastic Optimal Control

Mission: plan an optimal trajectory for the drone within N stages

A simplified diagram above: the drone may reach the goal point before stage N.

Performance assessment:

- N: mission horizon or number of times control is applied (typically very large)
- stage cost: $L(\mathbf{s}, \mathbf{u}) \in \mathbb{R}$, terminal cost: $M(\mathbf{s}_N) \in \mathbb{R}$
- policy sequence is $\boldsymbol{\pi} := \{\boldsymbol{\pi}_0, \ldots, \boldsymbol{\pi}_{N-1}\}$ such that $\mathbf{u}_k = \boldsymbol{\pi}_k(\mathbf{s}_k)$
- total cost:

$$\mathbb{E}\left[M(\mathbf{s}_N) + \sum_{k=0}^{N-1} L(\mathbf{s}_k, \boldsymbol{\pi}_k(\mathbf{s}_k))\right]$$

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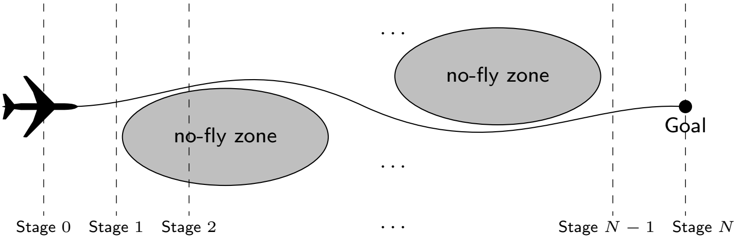
Mission: plan an optimal trajectory for the drone within N stages and subject to safety constraints



Safety considerations:

- A mission to be safe if: $\mathbf{s}_{1,\dots,N} \in \mathbb{S}$.(may be impossible!!!)
- Mission-Wide Probability of Safety (MWPS):

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- MWPS constraint, i.e., mission-wide chance constraint:

$$\mathbb{P}[\mathbf{s}_{1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_0, \boldsymbol{\pi}] \geq S, \quad \text{with } S \in [0,1]$$

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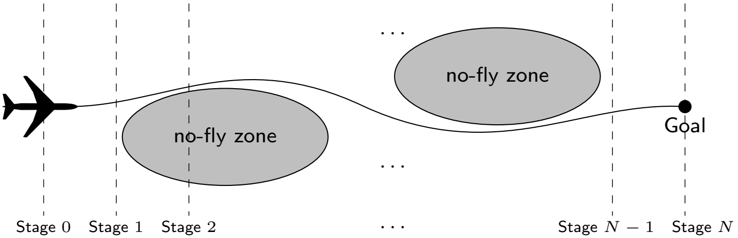

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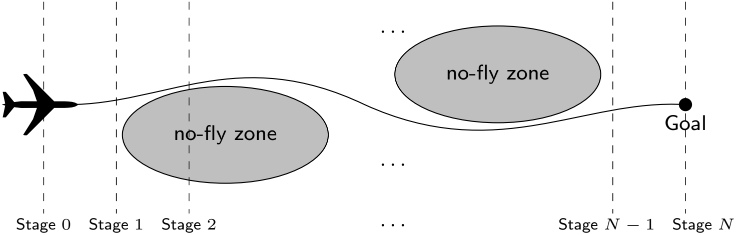

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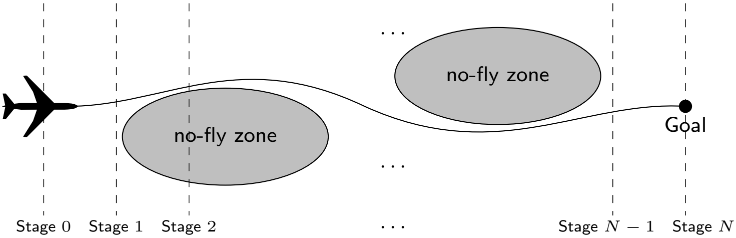

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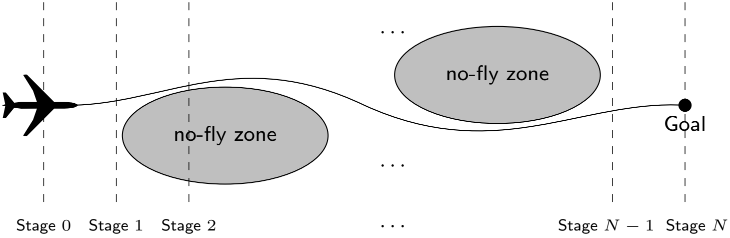

Find a policy sequence solution of:

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\begin{aligned}
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Notice: minimization over an infinite dimensional function space. Cumbersome!!!

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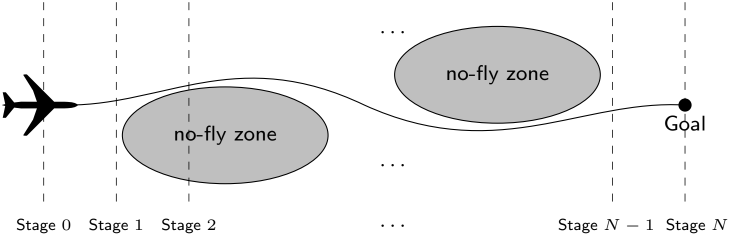

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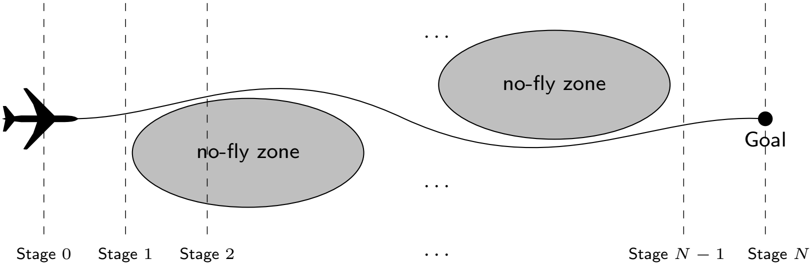

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- Mission-Wide Probability of Safety (MWPS): w.r.t. the whole mission.
- Stage-Wise Probability of Safety (SWPS): w.r.t. a single time stage.
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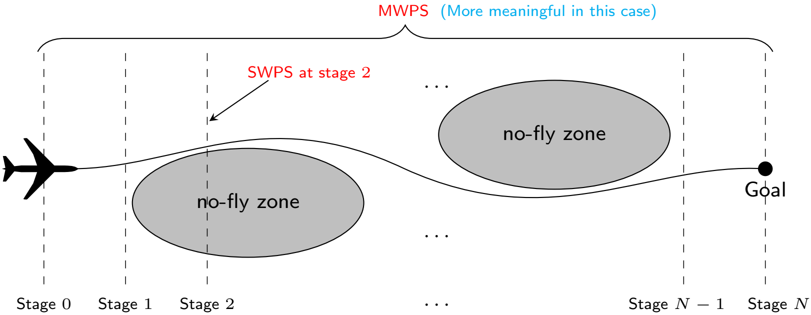

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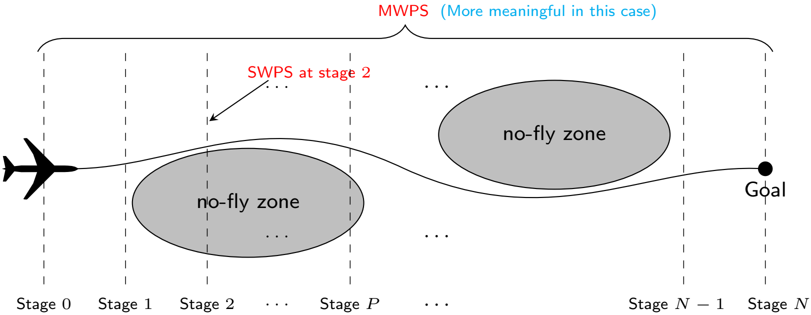

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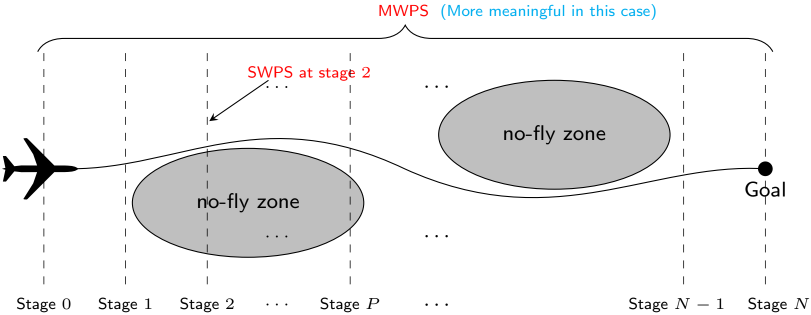

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It is hard to non-conservatively enforce MWPS via (multiple) SWPS because of the correlation between successive states

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$$\mathbb{P}_{0,N} := \mathbb{P}\left[\mathbf{s}_{1,\ldots,N} \in \mathbb{S} \,\middle|\, \mathbf{s}_0, \boldsymbol{\pi}^k\right]$$

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$\mathbb{P}_{0,N}$ is constant, while $\mathbb{P}_{k,N}$ is random, depending on the specific realization of $\mathbf{s}_{1,\ldots,k}$

Any relations between $\mathbb{P}_{0,N}$ and $\mathbb{P}_{k,N}$?

Lemma 1

Given a policy sequence $\boldsymbol{\pi}$, then we observe that

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for all $k = 1, \ldots, N-1$, i.e. the remaining MWPSs are equal to the initially prescribed MWPS in the expected value sense.

This Lemma forms a basis for constructing a recursive feasible SMPC controller

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Recursive Feasibility of MWPS Constraints

SMPC with Shrinking-horizon policy (solving N optimal control problems in an "on-line" fashion:)

- Find policy

$$\tilde{\boldsymbol{\pi}}^{k} = \left\{ \tilde{\boldsymbol{\pi}}^{k}_{k}, \ldots, \tilde{\boldsymbol{\pi}}^{k}_{N-1} \right\}$$

  that solves the problem

$$\min_{\boldsymbol{\pi}^{k}} \quad \mathbb{E}\left[M(\mathbf{s}_N) + \sum_{l=k}^{N-1} L(\mathbf{s}_l, \boldsymbol{\pi}^{k}_{l}(\mathbf{s}_l)) \right]$$

$$\text{s.t.} \quad \mathbb{P}[\mathbf{s}_{k+1,\ldots,N} \in \mathbb{S} \,|\, \mathbf{s}_k, \boldsymbol{\pi}^{k}] \geq S$$

  (π^k are parameterized in practice such that it lies in the subspace of its actual admissible set)

- The control inputs applied to the closed-loop system are given by

$$\mathbf{u}_k = \tilde{\boldsymbol{\pi}}^{k}_{k}(\mathbf{s}_k), \quad k = 0, \ldots, N-1$$

How to ensure recursive feasibility?

How to fulfill the MWPS constraint of the closed-loop system?

Recursive Feasibility of MWPS Constraints

Proposition 1

Assume that there exist feasible $\boldsymbol{\pi}^0$ satisfies the MWPS cosntraint:

$$\mathbb{P}[\mathbf{s}_{1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_0, \boldsymbol{\pi}^0] \geq S_0 \geq S$$

and that each policy sequence $\boldsymbol{\pi}^k$ is built under the constraint:

$$\mathbb{P}[\mathbf{s}_{k+1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_k, \boldsymbol{\pi}^k] \geq S_k \gamma_k \mathbb{P}[\mathbf{s}_{k+1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_k, \boldsymbol{\pi}^{k-1}] \tag{1}$$

with $S_k = \gamma_k \mathbb{P}[\mathbf{s}_{k+1,\dots,N} \in \mathbb{S} \,|\, \mathbf{s}_k, \boldsymbol{\pi}^{k-1}]$, where $\gamma_k \in (0, 1]$, for all k. Then the close-loop MWPS constraints under $\{\tilde{\boldsymbol{\pi}}_0^0, \dots, \tilde{\boldsymbol{\pi}}_{N-1}^{N-1}\}$ reads as:

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Corollary.

- together with the choice of $\prod_{k=1}^{N-1} \gamma_k S_0 = S$ with make the closed-loop MWPS constraint be fulfilled.
- constraint (1) is feasible for $\boldsymbol{\pi}^k = \boldsymbol{\pi}^{k-1}$, i.e., the SMPC controller is recursive feasible.

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Stochastic Linear MPC with MWPS guarantee

The linear system: $\mathbf{s}_{k+1} = A\mathbf{s}_k + B\mathbf{u}_k + \mathbf{w}_k$

The safe set: $\mathbb{S}$ is polytopic, i.e., $\mathbb{S} = \{\, \mathbf{s} \,|\, C\mathbf{s} + \mathbf{c} \leq 0\}$

At current state $\mathbf{s}_k$, for all $t = k, \ldots, N-1$, solve the problem

$$
\begin{aligned}
\min_{\bar{\mathbf{u}}_{k,\ldots,N-1}} \quad & \mathbb{E}\left[\mathbf{s}_N^\top Q_N \mathbf{s}_N + \sum_{t=k}^{N-1}\left(\mathbf{s}_t^\top Q \mathbf{s}_t + \mathbf{u}_t^\top R \mathbf{u}_t\right)\right] \\
\text{s.t.} \quad & \bar{\mathbf{s}}_k = \mathbf{s}_k \\
& \bar{\mathbf{s}}_{t+1} = A\bar{\mathbf{s}}_t + B\bar{\mathbf{u}}_t, \qquad \text{(nominal dynamic model)} \\
& \mathbf{e}_{t+1} = (A + BK)\,\mathbf{e}_t + \mathbf{w}_t, \qquad \text{(error dynamic model)} \\
& \mathbf{s}_{t+1} = \bar{\mathbf{s}}_{t+1} + \mathbf{e}_{t+1}, \\
& \mathbb{P}[C\mathbf{s}_{t+1} + \mathbf{c} \leq 0,\ \forall t] \geq S_k\,,
\end{aligned}
$$

where Q, Q_N are semi-positive definite, R is positive definite.

the policy $\boldsymbol{\pi}^k$ is parameterized as $\boldsymbol{\pi}_t^k(\mathbf{s}_t) := \bar{\mathbf{u}}_t + K\mathbf{e}_t$

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Cost function. assume zero-mean of $\mathbf{w}_k$, cost function reduces to

$$\bar{\mathbf{s}}_N^\top Q_N \bar{\mathbf{s}}_N + \sum_{t=k}^{N-1} \left(\bar{\mathbf{s}}_t^\top Q \bar{\mathbf{s}}_t + \bar{\mathbf{u}}_t^\top R \bar{\mathbf{u}}_t \right) + \sigma,$$

where σ is a constant term.

Chance constraint. reformulated as

$$\mathbb{P}[H(\mathbf{w}_{k,\dots,N-1}) + \mathcal{C}\bar{\mathbf{s}}_{k+1,\dots,N}^\top \leq \mathbf{0}] \geq S_k$$

with appropriate matrices H and $\mathcal{C}$.

Scenario Approximation. further reformulated via samples

$$H^{(i)}(\mathbf{w}_{k,\dots,N-1}^{(i)}) + \mathcal{C}\bar{\mathbf{s}}_{k+1,\dots,N}^\top \leq \mathbf{0}$$

(for all $i = 1, \dots, N_k$, and N_k is the number of samples [cf. Calafiore 2009])

Pick up a "representative" sample. label:

$$\mathcal{I}_j = \max_{i \in \mathbb{I}_{[1,N_k]}} [H^{(i)}]_j, \quad \forall j \in \mathbb{I}_{[1,n_c]},$$

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Stochastic Linear MPC with MWPS guarantee

Eventually, we derive following QP

$$\min_{\bar{\mathbf{u}}} \bar{\mathbf{s}}_N^{\mathrm{T}} Q_N \bar{\mathbf{s}}_N + \sum_{t=k}^{N-1} \left(\bar{\mathbf{s}}_t^{\mathrm{T}} Q \bar{\mathbf{s}}_t + \bar{\mathbf{u}}_t^{\mathrm{T}} R \bar{\mathbf{u}}_t \right)$$

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A regular QP of the
same complexity as
a normal linear MPC

Algorithm: linear SMPC with MWPS constraint

Initialization: S_0, $\gamma_{1,\ldots,N-1}$, initial state $\mathbf{s}_0$;
while $k = 0 : N-1$ **do**
  Evaluate S_k through Monte Carlo simulation;
  Generate N_k samples;
  Get the solution $\bar{\mathbf{u}}^*_{k,\ldots,N-1}$ by solving the QP above;
  Send $\bar{\mathbf{u}}^*_k$ to the actual system and update state: $\mathbf{s}_{k+1} = A\mathbf{s}_k + B\bar{\mathbf{u}}^*_k + \mathbf{w}_k$;
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Stochastic Linear MPC with MWPS guarantee: a case study

- System matrices:
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$
- Disturbance: $\mathbf{w}_k \sim \mathcal{N}(0, 0.04 \cdot I)$
- Safe set matrices:
$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} -2 \\ -2 \\ -10 \\ -2 \end{bmatrix}$
- $Q = I,\ R = 0.1,$
$K = [-0.6167, -1.2703]$
$Q_N = \begin{bmatrix} 2.0599 & 0.5916 \\ 0.5916 & 1.4228 \end{bmatrix}$

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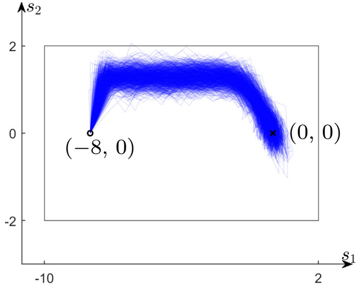

Figure: State trajectories via running Monte Carlo simulations

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- Set $N = 11$, $S_0 = 0.98$ and $\gamma_{1,\ldots,10} = 0.99$, resulting in $S = \prod_{k=1}^{10} \gamma_k S_0 = 0.8863$
- A Monte Carlo simulation that simulates 10^5 missions shows that the resulting ratio of mission success is 99.88%

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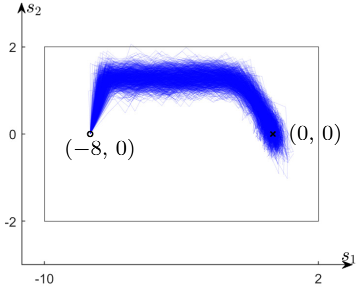

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this discrepancy is due to that the scenario-based method adopted is conservative.

Summary

Conclusions

- Showed that the remaining MWPSs remains constant in the expected value sense
- Proposed a recursively feasible control scheme while ensuring MWPS constraint
- Deployed the idea in the linear case via an efficient scenario-based approach

Future work:

- More advanced methods in place of the classic Monte-Carlo sampling
- Receding-horizon policy in place of the shrinking-horizon policy

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Thank you! Questions?

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- Showed that the remaining MWPSs remains constant in the expected value sense
- Proposed a recursively feasible control scheme while ensuring MWPS constraint
- Deployed the idea in the linear case via an efficient scenario-based approach

Future work:

- More advanced methods in place of the classic Monte-Carlo sampling
- Receding-horizon policy in place of the shrinking-horizon policy

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Thank you! Questions?