On BB-tilting-cotilting equivalence of permanently representation-finite algebras

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Abstract

We introduce the notion of BB-tilting-cotilting equivalence and show that it coincides with the classical notion of tilting-cotilting equivalence for the class of permanently representation-finite algebras. We use this result to re-classify piecewise hereditary algebras of Dynkin types in a uniformized way. We show that for gentle one-cycle algebras, being derived equivalent is equivalent to being (BB-)tilting-cotilting equivalent.

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§1. Introduction

One of the significant achievements of the 20th-century algebraic representation theory is the classification of iterated tilted algebras of Dynkin types. According to their derived categories, these algebras are divided into classes indexed by Dynkin diagrams A_n $(n \geqslant 1)$, D_n $(n \geqslant 4)$ and E_n $(n = 6, 7, 8)$. Historically, this classification was accomplished separately for Dynkin types A, D and E respectively in [5], [9] and [22]. Their methods are, however, mutually different. For type A, [5] demonstrates that for gentle tree algebras, tilting processes preserve their local gentleness. For type D, [9] eliminates various non-acceptable subquivers to force the shape of algebras; the proof is quite demanding on preliminaries. Finally, for type E, [22] processes a computer programme, beginning with path algebras and recursively constructing algebras tilted from those already obtained.

It turns out that BB-tilting modules

This idea may have first appeared in [16, Section IV.6], where it is shown that if Δ is a Dynkin quiver, then any iterated tilted algebra A of $\Bbbk\Delta$ is connected to $\Bbbk\Delta$ via a sequence of algebras $A = A_0, A_1, \cdots, A_s = \Bbbk\Delta$ such that each successive algebra A_{i+1} is tilted from A_i by a splitting APR-tilting module. See also [6] for a generalization, where the authors tried to connect a path algebra of Euclidean type to its representation-finite iterated tilted algebra via a sequence of APR-tilting and APR-cotilting modules, not necessarily separating. Our result is, in some sense, both weaker and stronger, for it is formulated by the wider class of BB-(co)tilting modules without caring about splittingness or separatingness, but it is applicable to any algebra such that representation-finiteness is preserved throughout its derived equivalence class; see Section 6 for such an application. Moreover, the proof of our result merely employs elementary knowledge limited to the scope of tilting theory.

Gentle algebras have been fairly well-studied in the past 30 years. Among this class of algebras, the recognition problem of derived equivalence has been completely resolved nowadays; see [11, 3] and literature cited therein. Particular sub-classes like gentle one-cycle algebras and gentle two-cycle algebras are dealt with respectively in [8, 26] and [10, 13]. Worth-mentioning is that, certain "elementary transformations" on the bound quiver that preserve derived equivalence, originally realised in [18] by two-term tilting complexes, are in fact realised by BB-tilting modules. This suggests the possibility that for some gentle algebras, its derived equivalence class could coincide with its tilting-cotilting equivalence class. Indeed, this is true for gentle one-cycle algebras, as we shall prove in Section 6, and non-degenerate gentle two-cycle algebras, as proved in [10, 13]. The general case remains to be investigated.

The paper is organised as follows. After recalling basic knowledge and fixing notations in section 2, we characterise BB-(co)tilting modules and use them to define mutations of algebras in section 3. Then, in section 4, we investigate two partial orders among tilting modules and prove, under some finiteness conditions, the technique of replacing a step of tilting by a sequence of steps of BB-tiltings. We apply our techniques, first in section 4 to reclassify piecewise hereditary algebras of Dynkin types, then in section 5 to gentle one-cycle algebras. It turns out that, in all of our examples, the derived classificaction coincides with the BB-tilting-cotilting classification.

§2. Preliminaries

2.1. Representation theory of finite-dimensional algeras

Let us recall some basic notations. Throughout this subsection, let k be an algebraically closed field.

A **quiver** is a quintuple $Q = (Q_0, Q_1, s, t)$ consisting of a set of **vertices** Q_0, a set of **arrows** Q_1, two mappings $s, t : Q_1 \to Q_0$ that assign to each arrow α its **starting vertex** $s(\alpha)$ and its **terminating vertex** $t(\alpha)$. A **path** of **length** n ($n \geqslant 1$) is a sequence $p = \alpha_1 \cdots \alpha_n$ of arrows such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leqslant i \leqslant n-1$; we define its starting vertex $s(p) = s(\alpha_1)$ and its terminating vertex $t(p) = t(\alpha_n)$. For each $v \in Q_0$ we assign a **trivial path** e_v of length 0, such that $s(e_v) = t(e_v) = v$. The **concatenation** pq of two paths p, q is defined when $t(p) = s(q)$.

The **path algebra** kQ of the quiver Q has as basis the set of all paths in Q, and its multiplication $\cdot$ is induced by concatenation: $p \cdot q = pq$ if $t(p) = s(q)$, and $p \cdot q = 0$ otherwise. In what follows we always assume that kQ is finite-dimensional, which is the case precisely when Q is a **finite quiver**, i.e., Q_0 and Q_1 are both finite sets.

An ideal $I \subseteq kQ$ is called **admissible** if there exists an integer $m \geqslant 2$ such that $(\operatorname{rad} kQ)^m \subseteq I \subseteq (\operatorname{rad} kQ)^2$, where $\operatorname{rad} kQ$ denotes the **radical** of kQ, which equals the ideal generated by all arrows (or length-1 paths) in Q. A **bound quiver algebra** is an algebra of the form kQ/I for some quiver Q and admissible ideal I.

Theorem 2.1. ([24, Corollary 6.10, Theorem 3.7]) *Every finite-dimensional k-algebra A is Morita equivalent to a bound quiver algebra kQ/I, where Q is a finite quiver and I is an admissible ideal of kQ.* □

Let A be a finite-dimensional k-algebra. Algebraic representation theory studies modules categories of such algebras as A. The module category A-mod is a Krull–Schmidt abelian category with enough projectives and injectives, so the central goal is to classify the indecomposable modules of A and the morphisms between them. By Theorem 2.1, to the aim of this project it suffices to consider bound quiver algebras. In particular, for path algebras we have the following fundamental result:

Theorem 2.2. *The representation-finite path algebras are precisely path algebras of Dynkin quivers [24, Section VII.2(a)]. The representation-tame path algebras are precisely path algebras of Euclidean quivers [24, Section VII.2(b)]. The representation-wild path algebras are precisely path algebras of quivers not in the above two classes (called wild quivers).* □

For the description of module categories of path algebras of Dynkin and Euclidean quivers, see [24, Chapter VII] and [25], respectively.

In 1970s Auslander–Reiten theory prevails in algbraic representation theory. It provides algebraic or combinatorial information about indecomposable objects and morphisms in A-mod. Recall that a morphism $f : X \to Y$ is **irreducible** if f is neither a section nor a retraction, but in any factorization $f = gh$, either h is a section or g is a retraction. An exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in A-mod is called an **Auslander–Reiten sequence** or an **almost splitting sequence** if both f and g are irreducible morphisms. In such exact sequences X and Z must be indecomposable, and in fact the whole sequence is determined up to isomorphism by the first term X or the last term Z. We define the **Auslander–Reiten quiver** $\Gamma(A\text{-mod})$ to be the quiver whose vertices correspond to representatives of isomorphism classes of indecomposable A-modules, whose arrows correspond to indecomposable morphisms. On $\Gamma(A\text{-mod})$ we define the **Auslander–Reiten translation** τ, which assigns every non-projective indecomposable module Z to the module X in the Auslander–Reiten sequence $0 \to A \to Y \to Z \to 0$. For details, see [24, Chapter IV].

2.2. Tilting theory

Tilting theory is formulated in 1980s. It is an essential tool in alegbraic representation theory, capable of comparing two module categories via the tilting functors. Let A be a finite-dimensional algebra, and let T be an A-module. Recall that

Definition 2.3. *T is a* **partial tilting module** *if it satisfies:*

(1) $\mathrm{pd}_A T \leqslant 1$;

(2) $\mathrm{Ext}^1_A(T, T) = 0$.

Moreover, T is a **tilting module** *if it also satisfies:*

(3) *There is an exact sequence $0 \to A \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathsf{add}\, T$.*

And dually,

Definition 2.4. *T is a* **partial cotilting module** *if it satisfies:*

(1) $\mathrm{id}_A T \leqslant 1$;

(2) $\mathrm{Ext}^1_A(T, T) = 0$;

Moreover, T is a **cotilting module** *if it also satisfies:*

(3) *There is an exact sequence $0 \to T_1 \to T_0 \to DA \to 0$ with $T_0, T_1 \in \mathsf{add}\, T$.*

In what follows, we will only talk about tilting modules and invite the reader to formulate the dual version for cotilting modules.

Let T be a tilting A-module and $B = \mathrm{End}_A(T)$. The key observation of tilting theory is that the module category of A and B, although non-equivalent in most cases, are related by two tilting functors. Consider two full subcategories $\mathcal{T}(T), \mathcal{F}(T)$ of A-mod and two full subcategories $\mathcal{X}(T), \mathcal{Y}(T)$ of B-mod, where

$$\begin{aligned}
\mathcal{T}(T) &= \{M_A \mid \exists \text{surjection } T^r \to M\} &&= \{M_A \mid \mathrm{Ext}^1_A(T, M) = 0\},\\
\mathcal{F}(T) &= \{M_A \mid \exists \text{injection } M \to (\tau_A T)^r\} &&= \{M_A \mid \mathrm{Hom}_A(T, M) = 0\},\\
\mathcal{X}(T) &= D\{{}_B N \mid \exists \text{injection } N \to (\tau_B T)^r\} &&= \{N_B \mid N \otimes_B T = 0\},\\
\mathcal{Y}(T) &= D\{{}_B N \mid \exists \text{injection } T^r \to N\} &&= \{N_B \mid \mathrm{Tor}^1_A(N, T) = 0\}.
\end{aligned}$$

Lemma 2.5. *If T is a tilting module, then $(\mathcal{T}(T), \mathcal{F}(T))$ is a torsion pair of A-mod and $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion pair of B-mod.* □

A tilting A-module T is said to be **separating** (resp. **splitting**) if we have $\mathcal{T}(T) \cup \mathcal{F}(T) = A$-$\mathsf{mod}$ (resp. $\mathcal{X}(T) \cup \mathcal{Y}(T) = B$-$\mathsf{mod}$).

The tilting theorem is first formed by S. Brenner and M. C. R. Butler, and later reformulated by D. Happel in the following way:

Lemma 2.6. *Let T be a tilting A-module and $B = \mathrm{End}_A(T)$. Then there are two mutually quasi-inverse triangulated equivalences induced by the derived functors*

$$\mathbf{R}\mathrm{Hom}_A(T, -) : \mathbf{D}^b(A) \to \mathbf{D}^b(B),$$
$$T \otimes^{\mathbf{L}}_B - : \mathbf{D}^b(B) \to \mathbf{D}^b(A).$$

We identify A-mod as a full subcategory of $\mathbf{D}^b(A)$ consisting of objects concentrated in degree 0 and similarly for B-mod in $\mathbf{D}^b(B)$. Then

$$\begin{aligned}
\mathcal{T}(T) &= \{M \in A\text{-}\mathsf{mod} \mid \mathbf{R}\mathrm{Hom}_A(T, M) \in B\text{-}\mathsf{mod}[0]\},\\
\mathcal{F}(T) &= \{M \in A\text{-}\mathsf{mod} \mid \mathbf{R}\mathrm{Hom}_A(T, M) \in B\text{-}\mathsf{mod}[1]\},\\
\mathcal{X}(T) &= \{N \in B\text{-}\mathsf{mod} \mid T \otimes^{\mathbf{L}}_B N \in A\text{-}\mathsf{mod}[1]\},\\
\mathcal{Y}(T) &= \{N \in B\text{-}\mathsf{mod} \mid T \otimes^{\mathbf{L}}_B N \in A\text{-}\mathsf{mod}[0]\},
\end{aligned}$$

and $\mathbf{R}\mathrm{Hom}_A(T, -)$ and $T \otimes^{\mathbf{L}}_B -$ restricts to two pairs of equivalences:

$$\mathrm{Hom}_R(T, -) : \mathcal{T}(T) \longleftrightarrow \mathcal{Y}(T) : T \otimes_B -,$$
$$\mathrm{Ext}^1_R(T, -) : \mathcal{F}(T) \longleftrightarrow \mathcal{X}(T) : \mathrm{Tor}^R_1(\mathcal{T}, -).$$

Proof. See [24, Theorem 3.8] and [16, Section III.3]. □

We remark that there is a notion of "generalized tilting modules", which will not be used in our paper. For its definition, see [16, Chapter III].

Another topic in tilting theory is the completion of partial tilting modules. For any module T, denote by $\#T$ the number of pairwise non-isomorphic indecomposable direct summands of T. Denote by $K_0(A)$ the Grothendieck group of A.

Definition 2.7. *A partial tilting module T is called* **almost complete** *if $\#T = \operatorname{rank} K_0(A) - 1$.*

The following lemmata is well-known.

Lemma 2.8. *If T is a partial tilting module, then there exists a module E such that $T \oplus E$ is a tilting module.*

Proof. See [24, Lemma 2.4]. □

Remark. E is called a completion of T. We note that the completion constructed in [loc. cit.] is called the Bongartz completion of T.

Lemma 2.9. *If T is a partial (co)tilting module, then $\#T \leqslant \operatorname{rank} K_0(A)$. Moreover, T is a (co)tilting module if and only if $\#T = \operatorname{rank} K_0(A)$.*

Proof. See [24, Corollary VI.4.4]. □

2.3. Iterated tilted algebras

Let Δ be a finite acyclic quiver and x be a vertex. The **reflection** of Δ at x is defined in two cases: when x is a sink, it is the new quiver $\sigma_x^+(\Delta)$ obtained from Δ by reversing all arrows directing in x; when x is a source, it is the new quiver $\sigma_x^-(\Delta)$ obtained from Δ by reversing all arrows directing out of x. We define an equivalence relation on the class of finite acyclic quivers as follows: $\Delta \sim \Delta'$ if and only if there is a sequence of finite acyclic quivers $\Delta = \Delta_0, \Delta_1, \cdots, \Delta_s = \Delta'$, such that Δ_{i+1} is obtained as a reflection of Δ_i.

Definition 2.10. ([16, IV.1.1, IV.4.1, IV.4.4]) *Let Δ be a finite acyclic quiver and A be a finite-dimensional algebra.*

(1) *A is a* **piecewise hereditary algebra** *of type Δ if A is derived equivalent to $\Bbbk\Delta$.*

(2) *A is* **tiltable** *to $\Bbbk\Delta$ if there exists a sequence of algebras $A = A_0, A_1, \cdots, A_s = \Bbbk\Delta$ and a tilting B_i-module M_i for each i, such that $B_{i+1} \cong \operatorname{End}_{B_i}(M_i)$.*

(3) *A is an* **iterated tilted algebra** *of type Δ if there exists a sequence of algebras $A = A_0, A_1, \cdots, A_s = \Bbbk\Delta$ and a splitting tilting B_i-module M_i for each i, such that $B_{i+1} \cong \operatorname{End}_{B_i}(M_i)$.*

Lemma 2.11. ([16, Corollary IV.5.5]) *The three concepts are equivalent.* □

Historically, the classification of iterated tilted algebras have been accomplished in several cases: [5] for type A, [9] for type D, [22] for type E_6, E_7, E_8 and [8] for type $\tilde{\mathrm{A}}$.

2.4. Gentle algebras

Let A be a finite-dimensional basic algebra.

Definition 2.12. *A is called a* **gentle algebra** *if $A \cong \Bbbk Q/I$, in which:*

(G1) *Each vertex is the source of at most two arrows and the target of at most two arrows.*

(G2) *For each arrow $\alpha : x \to y$, there is at most one arrow β with source y such that $\alpha\beta \in I$ and at most one arrow γ with target x such that $\gamma\alpha \in I$.*

(G3) *For each arrow $\alpha : x \to y$, there is at most one arrow β with source y such that $\alpha\beta \notin I$ and at most one arrow γ with target x such that $\gamma\alpha \notin I$.*

(G4) *The ideal I is generated by paths of length 2.*

A is called a **gentle tree algebra** (*resp.* **gentle n-cycle algebra**) *if the underlying unoriented graph $|Q|$ of the quiver Q is a tree (resp. a graph with precisely n cycles, or equivalently, $\#$edges $-$ $\#$vertices $= n - 1$).*

Gentle algebras first arose in the context of classification of iterated tilted algebras of type $\tilde{\mathrm{A}}$, see [8]. They are a specific subclass of string algebras, whose module categories have been described in [15, Section 3] in terms of string objects and band objects. Their derived categories are also determined in [12, 4, 27] and described via graded arcs and closed curves on the associated dissected surfaces; see [20]. The derived equivalence class of a gentle algebra is completely determined by the homotopy class of the foliation associated to the dissected surface; see [3]. This recovers some of the long-known combinatorical invariants, like the AG-invariant [11].

2.5. The Hasse quivers of a partially ordered set

Let $(P, <)$ be a partially ordered set. For elements $x, y \in P$, we denote by the "interval" notation $[x, y]$ the subset $\{z \in P \mid x \leqslant z \leqslant y\}$.

The Hasse quiver associated to $(P, <)$ is a quiver $\mathcal{H}_{(P,<)}$ whose vertices bijectively correspond to elements of P and for two elements $x, y \in P$, there is an arrow $x \to y$ in $\mathcal{H}_{(P,<)}$ if and only if $x < y$ and there exists no element lying strictly between x and y, i.e., $[x, y] = \{x, y\}$.

We record two elementary facts about Hasse quivers for later usage:

(1) A finite partially ordered set is completely determined by its Hasse quiver.

(2) The Hasse quiver of the "interval" subset $([x, y], <)$ equals the full subquiver of $\mathcal{H}_{(P,<)}$ consisting of vertices in $[x, y]$.

§3. BB-(co)tilting modules and mutations of algebras

In this section, we introduce the notion of left and right mutations of algebras via BB-(co)tilting modules and show some of their properties. We introduce the notion of BB-tilting-cotilting equivalence among finite-dimensional basic algebras.

To begin with, let us recall the definition and some characterizations of BB-(co)tilting modules.

Definition 3.1. *Let A be a finite-dimensional basic algebra.*

(1) *A* **BB-tilting module** *is a tilting module with exactly one non-projective indecomposable direct summand.*

(2) *A* **BB-cotilting module** *is a cotilting module with exactly one non-injective indecomposable direct summand.*

Lemma 3.2. *Let A be a finite-dimensional basic algebra and $A = P \oplus Q$ with P indecomposable. Then the following statements are equivalent:*

(1) *There is a BB-tilting module T with Q as a direct summand;*

(2) *$f : P \to Q'$, the left minimal $(\mathsf{add}\, Q)$-approximation of P, is injective.*

(3) *Q is a faithful A-module;*

(4) *$S^+ := D(A/A(1-e)A)$ is non-injective and $\mathrm{pd}_A(\tau^{-1}S^+) \leqslant 1$, where $D : A\text{-}\mathsf{mod} \to \mathsf{mod}\text{-}A$ is the standard duality functor, e is the primitive idempotent corresponding to P and τ^{-1} is the inverse Auslander–Reiten translation of $A\text{-}\mathsf{mod}$.*

Moreover, if all these conditions hold, then $T \cong Q \oplus \mathrm{Coker}\, f$ and $\mathrm{Coker}\, f \cong \tau^{-1}S^+$.

Proof. (1)⇒(2): P is clearly the Bongartz completion of Q. So if there exists another completion X of Q, by [21, Lemma 2.1] there will be an exact sequence $0 \to P \xrightarrow{f} Q' \xrightarrow{g} X \to 0$, where f is the left minimal $(\mathsf{add}\, Q)$-approximation of P. Clearly f must be an injection.

(2)⇒(3): By assumption, there is an injection $P \to Q^m$ for some integer m. Since $P \oplus Q^m$ contains A as a submodule, it is faithful. Since Q^m contains P as a submodule, Q^m is fathful. Finally, Q is faithful.

(3)⇒(1): By [1, Corollary 2.24], Q admits two different completions. One being P, the other one must be non-projective.

(1)–(3)⇒(4): By [21, Lemma 2.1], the BB-tilting module containing Q as a direct summand must be $Q \oplus \mathrm{Coker}\, f$. To deduce that $\mathrm{Coker}\, f \cong \tau^{-1}(S^+)$ we modify the proof of [19, Proposition 7.4].

To the short exact sequence $0 \to P \xrightarrow{f} Q' \to X \to 0$ where $X = \mathrm{Coker}\, f$ we apply the Nakayama functor ν to obtain the exact sequence

$$0 \to Y \to \nu P \xrightarrow{\nu f} \nu Q' \to \nu X \to 0$$

where $Y = \mathrm{Ker}\, \nu f$. Since $\nu : A\text{-}\mathsf{proj} \to A\text{-}\mathsf{inj}$ is an equivalence, νf is the left minimal $(\mathsf{add}\, \nu Q)$-approximation of νP. By chasing a diagram, one can show that Y satisfies $\mathrm{Hom}_A(Y, \nu Q) = 0$, so $Y \subseteq S^+$ since the latter is the maximal submodule of νA satisfying the same property. We show the inverse inclusion $S^+ \subseteq Y$. Since S^+ has a simple socle, there is an injective envelope $i : S^+ \to \nu P$. Since $\mathrm{Hom}_A(S^+, \nu Q) = 0$, i factors through $Y \to \nu P$, making S^+ a submodule of Y. So $Y \cong S^+$ and $\tau X \cong S^+$ by the construction of τ.

One can easily see that $Q\oplus\tau^{-1}S^+$ is a BB-tilting module: S^+ must be non-injective, otherwise $\tau^{-1}S^+=0$ and f would become a ridiculous isomorphism; $\mathrm{pd}_A(\tau^{-1}S^+)\leqslant 1$ is obvious.

(4)⇒(1): It follows from [2, Theorem 2.32]. □

We remark that our definition of BB-(co)tilting modules follows [19, Definition 7.3]. This is a generalization of the classical definition in [14, Theorem IX], which requires that $\mathrm{Ext}^1_A(S,S)=0$ where S is the simple module corresponding to the primitive idempotent e.

There is a dual version of Lemma 3.2 for BB-cotilting modules.

Now we introduce the notion of mutations of algebras via BB-(co)tilting modules.

Definition 3.3. *Let A be a finite-dimensional basic algebra, $A=P\oplus Q$ where P is an indecomposable projective module, e is the primitive idempotent corresponding to P.*

(1) *If the BB-tilting module T containing Q as a direct summand exists, we say that A is* **left mutable** *at e and define the* **left mutation** *of A at e to be $\mu_e^+(A):=\mathrm{End}_A(T)$.*

(2) *If the BB-cotilting module T containing νQ as a direct summand exists, we say that A is* **right mutable** *at e and define the* **right mutation** *of A at e to be $\mu_e^-(A):=\mathrm{End}_A(T)$.*

In practice, when dealing with an algebra A given by a bound quiver, we shall say that A is "left mutable at vertex i" if it is left mutable at the idempotent associated with vertex i, and write $\mu_i^+(A)$ for the resulting algebra under mutation. Other terminologies and notations modify in the same manner.

We shall show some properties of left and right mutations of algebras.

Proposition 3.4. *Let A be a finite-dimensional basic algebra, $A=P\oplus Q$ where P is an indecomposable projective module, e is the primitive idempotent corresponding to P.*

(1) *A is left mutable at e if and only if $\mu_e^+(A)$ is right mutable at e', where e' is the primitive idempotent corresponding to $\mathrm{Hom}_A(T,X)$, T is the BB-tilting module containing Q and X is its unique non-projective summand. Moreover, $\mu_{e'}^-(\mu_e^+(A))\cong A$.*

(2) *A is right mutable at e if and only if $\mu_e^-(A)$ is left mutable at e', where e' is the primitive idempotent corresponding to $\mathrm{Hom}_A(X,T)$, T is the BB-cotilting module containing νQ and X is its unique non-injective summand. Moreover, $\mu_{e'}^-(\mu_e^-(A))\cong A$.*

Proof. Up to duality, we only show (1).

Denote by $T=Q\oplus X$ the BB-tilting module containing X as the unique non-projective summand. Then T is also a left tilting module of $B=\mu_e^+(A)$ [24, Lemma 3.3(b)] and we have an isomorphism of algebras $A\cong\mathrm{End}({}_BT)^{\mathrm{op}}$ [24, Lemma 3.3(c)]. By applying the duality functor D we have $A\cong\mathrm{End}((DT)_B)$ where DT is clearly a BB-cotilting B-module. The idempotent of B corresponding to the unique non-injective summand $\mathrm{Hom}_{A^{\mathrm{op}}}(DT, DX)$ is clearly the idempotent e' corresponding to the projective summand $\mathrm{Hom}_A(T, X)$. So we conclude that $\mu^-_{e'}(\mu^+_e(A)) \cong A$. $\square$

We can thus consider the equivalence relation among finite-dimensional basic algebras defined as follows: $A \sim B$ if and only if there exists a sequence of algebras $A = B_0, B_1, \cdots, B_s = B$, such that $B_{i+1} \cong \mathrm{End}_{B_i}(T_i)$ where T_i is a BB-tilting or BB-cotilting B_i-module. We refer to A and B as being **BB-tilting-cotilting equivalent** or being **mutation equivalent** if one likes.

There is another well-known notion called **tilting-cotilting equivalence**, which is defined as above, but using arbitrary tilting and cotilting modules. In the next section, we will investigate under what conditions these two equivalence relations coincide.

§4. BB-tilting-cotilting equivalence

We shall introduce the class of permanently representation-finite algebras, since for algebras in this class, it will be proved that BB-tilting-cotilting equivalence is equivalent with tilting-cotilting equivalence.

Definition 4.1. *An algebra A is called* **permanently representation-finite** *if any algebra derived equivalent to A is representation-finite.*

Such algebras are not as rare as the reader might suggest. Familiar examples include piecewise hereditary algebras of Dynkin types, gentle one-cycle algebras that fail to satisfy the clock condition (i.e., algebras with discrete derived category [26]), as well as some non-degenerate gentle two-cycle algebras [13]. These examples share the common feature that their derived equivalence classes are known, thus allowing further subdivision into BB-tilting-cotilting equivalence classes. These will be investigated in the following sections.

To show that for permanently representation-finite algebras, BB-tilting-cotilting equivalence coincide with tilting-cotilting equivalence, we need techniques on partial orders among tilting modules. Using these, we can realize a step of (co)tilting as a sequence of left (right) mutations.

4.1. Two partial orders

Let A be a finite-dimensional algebra. On the set $\mathrm{tilt}(A)$ of isomorphism classes of basic tilting A-modules, there exist two partial orders whose definitions we now recall. For details see [21, 17].

The mutation order. If M is an almost complete partial tilting module, there exist at most two non-isomorphic indecomposable partial tilting modules X, Y such that $T_1 := M \oplus X$ and $T_2 := M \oplus Y$ are tilting modules. We let $T_1 > T_2$ if X is the Bongartz completion of M and say that T_2 is a **tilting mutation** of T_1 at X. It's proved in [21, Lemma 2.1] that the mutation relation “$>$” on $\mathrm{tilt}(A)$ cannot form oriented cycles, so it extends to a partial order on $\mathrm{tilt}(A)$, denoted by “$\gg$”. To be precise, two tilting modules T, T' satisfies $T \gg T'$ if and only if there exists a chain of tilting mutations $T = T_0 > T_1 > \cdots > T_s = T'$ in $\mathrm{tilt}(A)$. Three facts are needed later:

Facts. (1) If X is the Bongartz completion of M, then there is an injection $X \to M'$ for some $M' \in \mathsf{add}\, M$.

(2) If Y is not the Bongartz completion of M, then there is a surjection $M' \to Y$ for some $M' \in \mathsf{add}\, M$.

(3) If both completions X and Y exist, then there is a short exact sequence $0 \to X \xrightarrow{f} M' \xrightarrow{g} Y \to 0$, with $M' \in \mathsf{add}\, M$, such that f is a minimal left $(\mathsf{add}\, M)$-approximation and g is a minimal right $(\mathsf{add}\, M)$-approximation.

The torsion class order. This partial order on $\mathrm{tilt}(A)$, denoted by “$\succ$”, is induced by inclusion between torsion classes generated by tilting modules. To be precise, two tilting modules T, T' satisfy $T \succ T'$ if and only if there is an inclusion $\mathrm{Gen}(T) \supseteq \mathrm{Gen}(T')$, where $\mathrm{Gen}(T)$ is the torsion class generated by T.

These two partial orders, although different in general, are connected by the following lemma.

Lemma 4.2. ([17, Theorem 2.1]) *The Hasse quivers of* $\gg$ *and* $\succ$ *coincide.*

Proof. The proof of [17, loc. cit.] for generalized tilting modules works perfectly in our case after some obvious and mild modifications. □

By a **chain of tilting mutations** in $\mathrm{tilt}(A)$, we mean a finite sequence of tilting mutations $T_0 > T_1 > \cdots > T_s$. The following lemma gives conditions for the existence of a chain between two tilting modules.

Lemma 4.3. *Suppose* $T \succ T'$ *in* $\mathrm{tilt}(A)$. *If the interval* $[T', T]$ *is finite, then there exists a chain of tilting mutations* $T = T_0 > T_1 > \cdots > T_s = T'$.

Proof. Either partial order on $[T', T]$ is the restriction of the respective partial order on $\mathrm{tilt}(A)$, so the two associated Hasse quivers on $[T', T]$ must coincide, both being the restriction of two identical Hasse quivers on $\mathrm{tilt}(A)$. Since any finite partially ordered set is uniquely determined by its Hasse quiver, we conclude that “$\gg$” and “$\succ$” coincide on $[T', T]$. We thus have $T \gg T'$ by assumption. Since $\gg$ is extended by tilting mutations, we conclude that there exists a chain of tilting mutations $T = T_0 > T_1 > \cdots > T_s = T'$ as required. □

Corollary 4.4. *If* $\mathrm{tilt}(A)$ *is a finite set, which is the case when* A *is representation-finite, the same conclusion as in Lemma 4.3 holds.* □

4.2. BB-tilting-cotilting equivalence

Corollary 4.4 have the following consequence.

Lemma 4.5. *Suppose that* $\mathrm{tilt}(A)$ *is finite. For any tilting module* T, *fix a chain of tilting mutations* $A = T_0 > T_1 > \cdots > T_s = T$ *from* A *to* T *and let*

$B_i = \mathrm{End}_A(T_i)$. *Then B_{i+1} is tilted from B_i, using $M_i := \mathrm{Hom}_A(T_i, T_{i+1})$ as a tilting B_i-module. Moreover, M_i is a BB-tilting module.*

Proof. By Corollary 4.4, the two partial orders on $\mathrm{tilt}(A)$ coincide. Hence, from $T_i > T_{i+1}$ we get $T_{i+1} \in \mathrm{Gen}(T_i)$. Using [24, Lemma 3.2] we obtain

$$\mathrm{Hom}_{B_i}(M_i, M_i) \cong \mathrm{Hom}_A(T_{i+1}, T_{i+1}) = B_{i+1},$$
$$\mathrm{Ext}^1_{B_i}(M_i, M_i) \cong \mathrm{Ext}^1_A(T_{i+1}, T_{i+1}) = 0.$$

Since there is exactly one indecomposable summand of T_{i+1} which is not one of T_i, we see that M_i has exactly one non-projective direct summand. To conclude that M_i is a BB-tilting B_i-module, we must show that $\mathrm{pd}_{B_i} M_i \leqslant 1$.

Since T_{i+1} is a tilting mutation of T_i, there exists an almost complete partial tilting module T' and two indecomposable partial tilting modules U_1, U_2 such that $T_i = T' \oplus U_1$ and $T_{i+1} = T' \oplus U_2$, and there is a short exact sequence $0 \to U_1 \to T'' \to U_2 \to 0$ with $T'' \in \mathsf{add}\, T'$. Applying $\mathrm{Hom}_A(T_i, -)$ we get an exact sequence of B_i-modules

$$0 \to \mathrm{Hom}_A(T_i, U_1) \to \mathrm{Hom}_A(T_i, T'') \to \mathrm{Hom}_A(T_i, U_2) \to \mathrm{Ext}^1_A(T_i, U_1) = 0.$$

Clearly the third term is the required summand X in M, and the first term is the required indecomposable projective B_i-module P_1. Since P_1 is clearly the Bongartz completion of $B_i \setminus P_1$ (the complement of P_1), we conclude that X is obtained by mutating B at P_1. The above exact sequence also shows that $\mathrm{pd}_{B_i} M_i \leqslant 1$. □

In short, we have shown that if A is representation-finite, a step of ordinary tilting from A to B can be realized as a sequence of left mutations $A \rightsquigarrow B_1 \rightsquigarrow \cdots \rightsquigarrow B_s = B$. There is also a dual result on replacing a step of cotilting from A to B by a sequence of right mutations, under the assumption that the set of cotilting modules $\mathrm{cotilt}(A)$ is finite, or even that A is representation-finite.

Now we can state and prove the asserted result.

Proposition 4.6. *Let A be a permanently representation-finite algebra. Then any algebra tilting-cotilting equivalent to A is also BB-tilting-cotilting equivalent to A.*

Proof. The condition guarantees that if B is connected to A by a sequence of tiltings and cotiltings as $A = B_0 \rightsquigarrow B_1 \rightsquigarrow \cdots \rightsquigarrow B_s = B$, then every B_i is representation-finite and each step of (co)tilting $B_i \rightsquigarrow B_{i+1}$ can be replaced by a sequence of BB-(co)tiltings by (the dual of) Lemma 4.5. □

We remark that, for general algebras, derived equivalence is coarser than tilting-cotilting equivalence, which is also coarser than BB-tilting-cotilting equivalence. We also remark that our result holds for "permanently tilting-cotilting-finite" algebras, meaning that any algebra derived equivalent to it has a finite number of tilting modules and cotilting modules. However, such a condition is not very handy and we do not delve into it in depth.

§5. Classification of piecewise hereditary algebras of Dynkin types

The "BB-tilting-cotilting" toolkit established in the previous section can be applied to determine the class of piecewise hereditary algebras of Dynkin types. In this section, based on general knowledge of piecewise hereditary algebras in [16], we build a method applicable to each Dynkin type.

Let Δ be a Dynkin quiver.

Lemma 5.1. *Suppose $\mathcal{A}_\Delta$ is a set of pairwise non-isomorphic algebras. Then $\mathcal{A}_\Delta$ forms a set of representatives of piecewise hereditary algebras of type Δ up to isomorphism if and only if the following conditions hold:*

(1) Each $A \in \mathcal{A}_\Delta$ admits a sequence of left or right mutations to $\Bbbk\Delta$;

(2) $\mathcal{A}_\Delta$ is closed under left mutations and taking opposite algebras.

Condition (2) can be replaced by

(2′) $\mathcal{A}_\Delta$ is closed under left or right mutations.

Proof. Necessity follows from general facts that piecewise hereditary algebras are precisely iterated tilted algebras (thus tiltable to path algebras) [16, Corollary IV.5.5], are closed under taking opposites [16, Corollary IV.5.6] and are permanently representation-finite when Δ is Dynkin since we know by direct calculation that $\Bbbk\Delta$ is derived-finite, i.e., $\mathbf{D}^b(\Bbbk\Delta)$ admits finitely many nonisomorphic indecomposable objects up to isomorphism.

Sufficiency: Condition (1) shows that each element of $\mathcal{A}_\Delta$ is derived equivalent to $\Bbbk\Delta$, hence piecewise hereditary of type Δ. In particular, they are permanently representation-finite. It is well-known that B is tilted from A if and only if A^{op} is tilted from B^{op}. Hence, to show that an piecewise hereditary algebra A of type Δ belongs to $\mathcal{A}_\Delta$, it suffices to show that A^{op} can be iterated tilted from $\Bbbk\Delta^{\mathrm{op}}$. But this follows immediately from (2) that $\mathcal{A}_\Delta$ is closed under left mutations. Similarly, using that B is tilted from A if and only if A is cotilted from B, to show that a piecewise hereditary algebra A of type Δ belongs to $\mathcal{A}_\Delta$, it suffices to show that A can be iterated cotilted from $\Bbbk\Delta$. But this follows immediately from (2′) that $\mathcal{A}_\Delta$ is closed under right mutations. □

In practice, Condition (2) is handier than Condition (2′) since forming opposite algebras takes almost no efforts. Also, right mutations in (1) could be omitted without harm, but nevertheless we allow right mutations in order to add flexibility in our proofs.

Note that our proof does not work for non-Dynkin quivers since it relies on Lemma 4.5, which needs the representation-finite assumption.

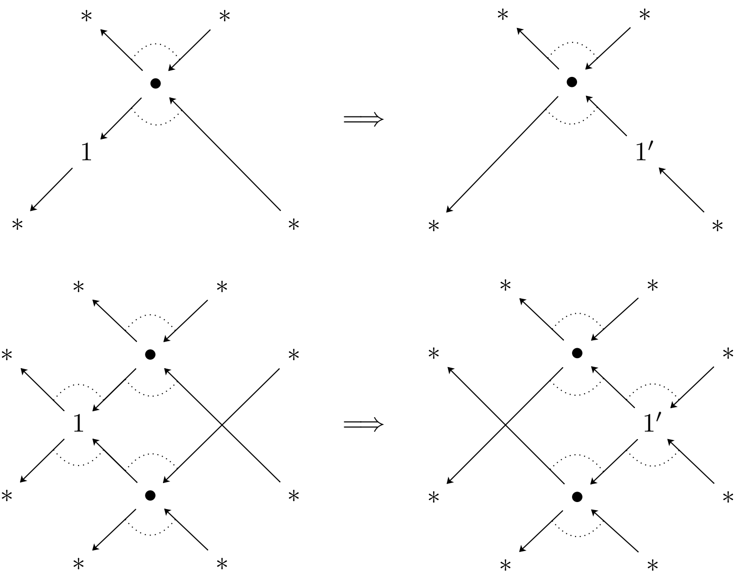
5.1. Type A

Now we deal with the case of Dynkin type A.

Proposition 5.2. *The set $\mathcal{A}_{\mathrm{A}_n}$ of representatives of isomorphism classes of gentle tree algebras with n vertices forms a set of representatives of isomorphism classes of piecewise hereditary algebras of type A_n.*

Proof. We verify the two conditions in Lemma 5.1.

(2) The class of gentle tree algebras is clearly closed under taking opposites. For the other propose, we explicitly show how a gentle tree algebra $A = \Bbbk Q/I$ changes under left mutations. Suppose that 1 is a mutable vertex of A. The only possibilities of left mutations at vertex 1 are illustrated below.



In the above figure, each "$*$" represents a branch attached to a given vertex and may be empty, in which case all arrows and relations containing this "$*$" should be erased. Note that if vertex 1 has only one immediate predecessor, it cannot be located in the middle of a relation, since otherwise $P(1)$ would be non-left mutable; whereas if 1 has two immediate predecessors, it is always left mutable. The effect of a left mutation can thus be understood as transforming vertex 1 into vertex $1'$ and modifying arrows and relations as shown above.

Since the proof of both cases are rather easy, we omit them and note that for arbitrary gentle algebras a formula for left mutations is proved in Lemma 6.3.

(1) This is classical; see, e.g., [24, Proposition IX.6.1] or [5, Theorem 2.3]. Nonetheless, we re-prove it using our mutation toolkit. Let A be a gentle tree

algebra with at least one relation. If A has the following shape:

$$\begin{tikzcd}
 & & & & * & \\
n & \cdots \arrow[l] & 2 \arrow[l] & 1 \arrow[l] & 0 \arrow[l, "\beta"'] \arrow[u] & * \arrow[l] \\
 & & & & * \arrow[u, "\alpha"'] &
\end{tikzcd}$$

where the "$*$" on arrow α is nonempty, we perform left mutations on vertices $1, 2, \cdots, n$ in order to stuff these vertices into arrow α and thus eliminate a relation $\alpha\beta = 0$. Oppositely, if A has the following shape:

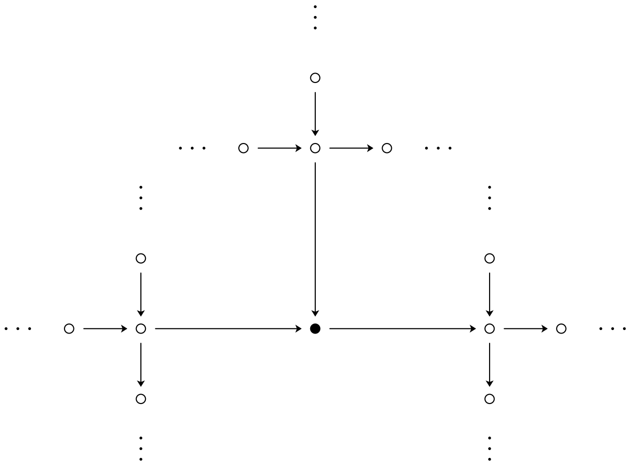
$$\begin{tikzcd}
 & & & & * \arrow[d] & \\
n \arrow[r] & \cdots \arrow[r] & 2 \arrow[r] & 1 \arrow[r, "\beta"] & 0 \arrow[r] \arrow[d, "\alpha"] & * \\
 & & & & * &
\end{tikzcd}$$

where the "$*$" on arrow α is nonempty, we perform right mutations on vertices $1, 2, \cdots, n$ in order to stuff these vertices into arrow α and thus eliminate a relation $\beta\alpha = 0$. Repeating these processes, we end up with a gentle tree algebra with no relations, i.e., a path algebra. $\square$

5.2. Type D

We apply Lemma 5.1 to the classification for Dynkin type D. To describe the set $\mathcal{A}_{\mathrm{D}_n}$ we need to introduce the concepts of branch extensions and branch co-extensions. For details the reader is referred to [7, §XV.1-§XV.3].

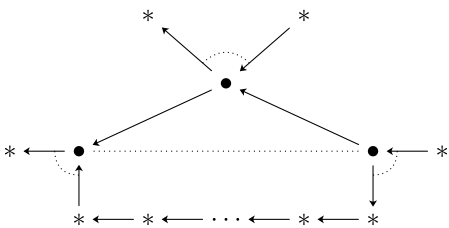
Let $A = \Bbbk Q/I$ be a bound quiver algebra, $A' = \Bbbk Q'/I'$ be a full bound subquiver algebra of A and i be a vertex of Q with at most 2 neighbours in Q'. Denote by Q_i the full subquiver of A consisting of i and its neighbours in Q'. We say that A is obtained from A' by a branch extension at i if the full bound subquiver algebra formed by $(Q \setminus Q') \cup Q_i$ is a **branch**, i.e., a full bound subquiver algebra of the **full branch** shown as below:



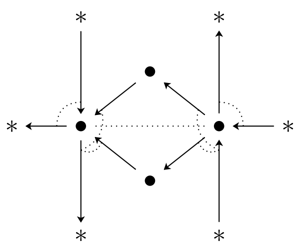
in which every composition of a horizontal arrow and a vertical arrow is zero, vice versa. The bullet vertex "•" is called the **germ** of the branch.

In what follows, we use an asterisk "$*$" to indicate a branch extension.

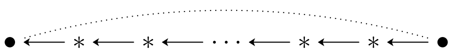
Proposition 5.3. ([9, Théorème 7.2]) *The set $\mathcal{A}_{\mathrm{D}_n}$ of the algebras shown below forms a set of representatives of isomorphism classes of iterated tilted algebras of type* D_n.



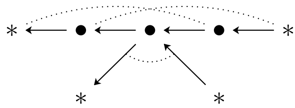
(D1)



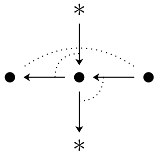
(D2)



(D3)



(D4)



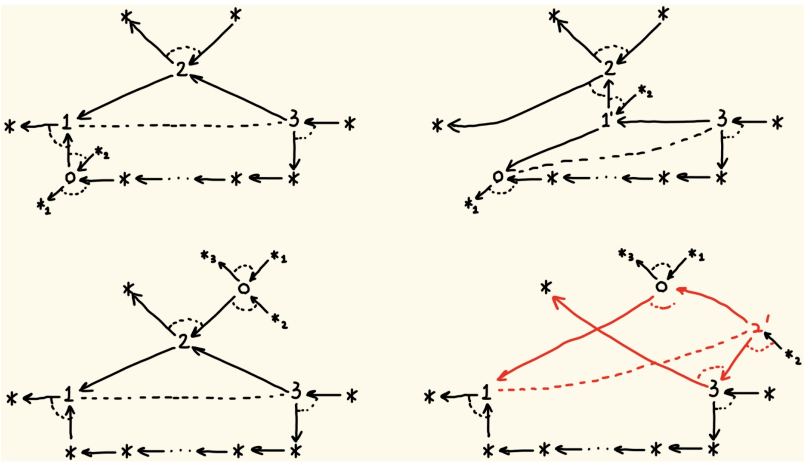
(D5)

Remark. Compare type (D4) with the one shown in [9, Théorème 7.2].

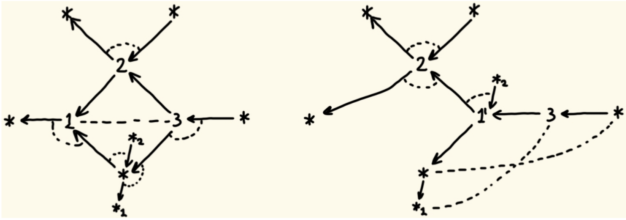
Proof. We verify the two conditions in Lemma 5.1. It can be directly seen from the figure that each of the five classes from (D1) to (D5) is closed under taking opposites, and so is $\mathcal{A}_{\mathrm{D}_n}$. This observation will simplify our proof.

(2) It suffices to work out all left mutations in each case and verify that the newly obtained algebra is still in $\mathcal{A}_{\mathrm{D}_n}$. For conciseness, all proofs are omitted and only non-obvious left mutations shall be exhibited. Since in this part we only do left mutations, to simplify the writing we shall omit all the word "left" in this part.

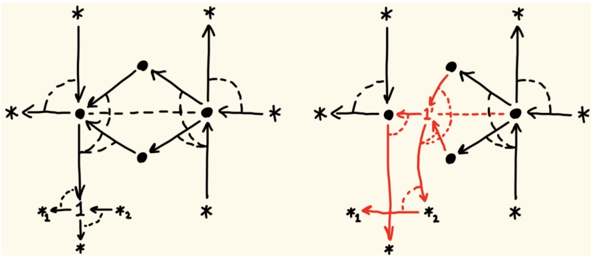
(D1) The only non-obvious mutations are those at three bullet vertices. Observe that the rightmost bullet is never mutable. So there are two cases:



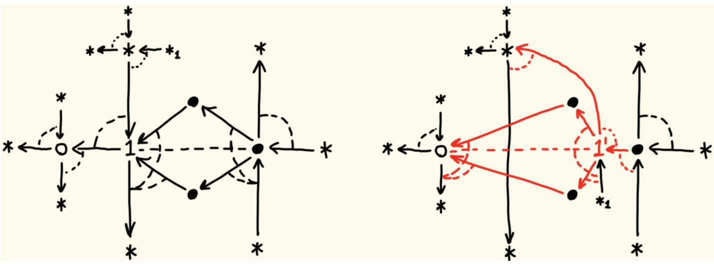
In the boundary case, where both paths from vertex 3 to vertex 1 are of length 2, the mutation at vertex 1 sends the algebra to (D4), as shown below:



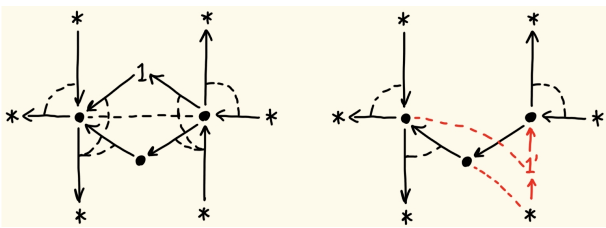
(D2) Four cases deserve consideration. First, the mutation at the lower left "$*$", which we now denote by "1", is shown as follows. Notice that if "$*_1$" is nonempty, so should be "$*_2$", or else vertex 1 would be non-mutable.



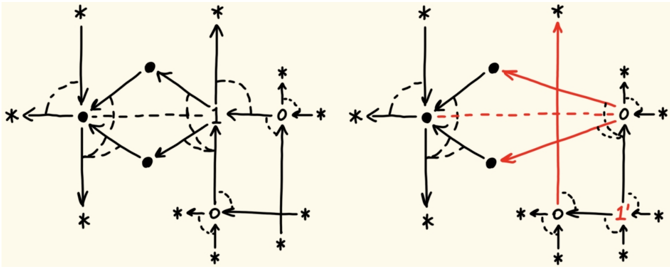
For the leftmost bullet vertex, the mutation is as follows:



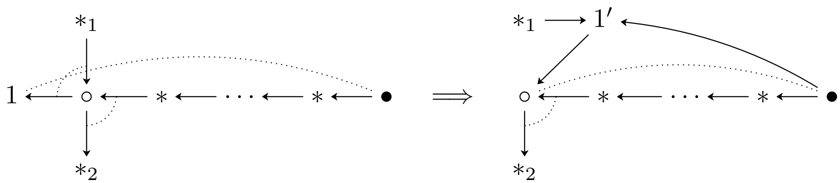
For the two bullet vertices on the middle vertical line, the mutation gives rise to an algebra in (D4) as follows (by symmetry, we only mutate the upper one):



Finally, for the rightmost bullet vertex, the mutation is as follows:

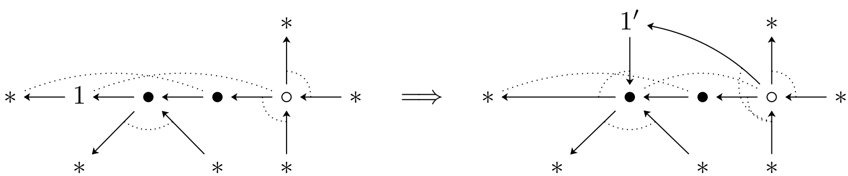


(D3) Every vertex in the branch represented by a "$*$", if mutable, mutates as in gentle tree algebras. Moreover, by 5.2, such a mutation preserves family (D3). We remark that the germ vertex of the rightmost "$*$" is non-mutable. As to the bullet vertices, the right one, being a source, is also non-mutable. Therefore, we only exhibit the mutation at vertex 1:

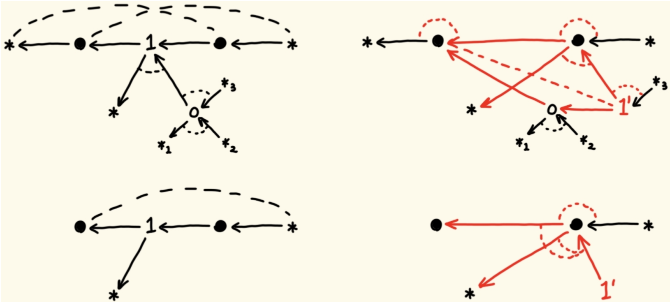


We obtain now an algebra in (D1).

(D4) Mutating at the leftmost bullet vertex yields an algebra in (D2):



The middle bullet vertex is mutable if and only if the leftmost "$*$" is empty, or else the lower right "$*$" is nonempty. Mutation in each case is shown below, yielding an algebra in class (D1) or (D5), respectively:



Finally, The rightmost bullet vertex is always non-mutable. The calculation of mutations at other vertices are relatively straightforward.

(D5) This case is analogous to (D3). For an algebra in (D5), a mutation at a "$*$" leaves it in (D5) or leads it to (D3), while a mutation at the leftmost bullet vertex leads it to (D4):

$$\begin{tikzcd}
 & * \arrow[d] & & & * \arrow[r] & 1' \arrow[dl] \\
1 & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] & \Longrightarrow & \bullet \arrow[d] & \bullet \arrow[u] \arrow[dl, dotted, no head, bend right] \\
 & * & & & * &
\end{tikzcd}$$

(1) Suppose A is an algebra in $\mathcal{A}_{\mathrm{D}_n}$. By results of (2), we can reduce the question as follows:

If $A \in$ (D2), by mutating a bullet vertex on the middle vertex line we may transform it into type (D4).

If $A \in$ (D4), by mutating the middle bullet vertex we may transform it into type (D1) or (D5).

If $A \in$ (D5), by tilting and cotilting mutations at vertices in "$*$" we may transform it into type (D3).

If $A \in$ (D3), by mutating the leftmost vertex, it falls into class (D1).

Finally, for $A \in$ (D1), we explicitly construct a sequence of tilting and cotilting mutations to turn it into a path algebra. By left and right mutations, we may stuff all branches into the main path, such that A becomes

$$\begin{tikzcd}
\circ & \cdots \arrow[l] & \circ \arrow[l] & & \\
 & & & \bullet \arrow[ul] \arrow[dll] & \\
 & \bullet \arrow[rrr, dotted, no head] & & & \bullet \arrow[ul] \arrow[d] \\
 & \circ \arrow[u] & \circ \arrow[l] & \cdots \arrow[l] \quad \circ \arrow[l] & \circ \arrow[l]
\end{tikzcd}$$

Repeatedly mutating the leftmost bullet vertex (see case (D1)), we end up with the path algebra of the following quiver:

$$\begin{tikzcd}
\circ & \cdots \arrow[l] & \circ \arrow[l] & & \\
& & & \bullet \arrow[ul] \arrow[dl] & \bullet \arrow[l] \\
& & \bullet & &
\end{tikzcd}$$

The proof is now complete. □

5.3. Remark on type E

We shall not exhibit the the classification of piecewise hereditary algebras of types $\mathrm{E}_6, \mathrm{E}_7$ and E_8 since it should better be done using a computer, the result comprising dozens of pages of long tables of frames. The algorithm is as follows: begin with the path algebra $\Bbbk\Delta$, form all of its left mutations and their opposites, and keep doing the same thing for newly obtained algebras until no new algebra emerges. Our algorithm calculates only BB-tilting modules, while a similar algorithm in [22] calculates almost all tilting modules over algebras already constructed. Although there is a further reduction in workload in [22] by dismissing those tilting modules without projective summand, the task is still tedious. By comparison, our method is more convenient.

§6. BB-tilting-cotilting-equivalence classification of gentle one-cycle algebras

Since gentle algebras are known to be closed under derived equivalence [23], a natural problem is their classification under tilting-cotilting equivalence. In this section we settle the problem for gentle one-cycle algebras.

Let us begin with a review of the derived classification of gentle one-cycle algebras, c.f., [8, 11].

Lemma 6.1. *Any gentle one-cycle algebra A is derived equivalent to precisely one of the following algebras:*

(1) *The path algebra of the Euclidean quiver of type $\tilde{\mathrm{A}}_{m,n}$.*

(2) *The bound quiver algebra* $\Bbbk Q(r,m,n)/I(r,m,n)$ *shown as below:*

$$\begin{tikzcd}
& & & & 1 \arrow[r, "\alpha_1"] & \cdots \arrow[r, "\alpha_{n-r-2}"] & n-r-1 \arrow[dr, "\alpha_{n-r-1}"] & \\
(-m) \arrow[r, "\alpha_{-m}"] & \cdots \arrow[r] & (-1) \arrow[r, "\alpha_{-1}"] & 0 \arrow[ur, "\alpha_0"] & & & & n-r \arrow[dl, "\alpha_{n-r}"] \\
& & & & n-1 \arrow[ul, "\alpha_{n-1}"] & \cdots \arrow[l, "\alpha_{n-2}"] & n-r+1 \arrow[l, "\alpha_{n-r+1}"] &
\end{tikzcd}$$

where I *is generated by* r *paths* $\alpha_{n-r}\alpha_{n-r+1}, \alpha_{n-r+1}\alpha_{n-r+2}, \cdots, \alpha_{n-1}\alpha_0$.

Moreover, case (1) happens if and only if A satisfies the **clock condition**, *i.e., in the unique cycle of A, the number of clockwise relations equals that of counterclockwise relations.* □

In what follows, we shall call the above algebras "standard" ones.

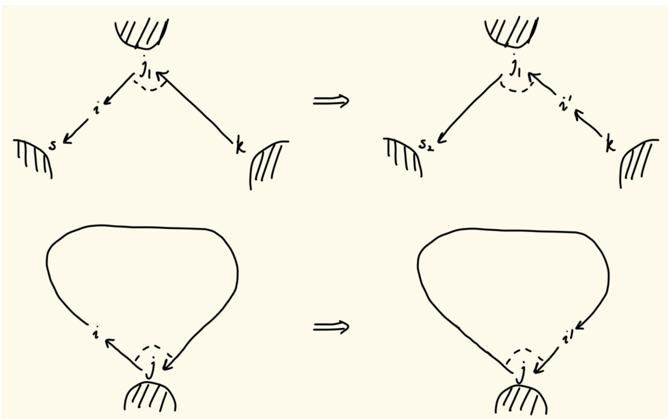
Lemma 6.2. *Gentle one-cycle algebras that do not satisfy the clock condition are permanently representation-finite.*

Proof. A gentle algebra is representation-finite if and only if there exists a band on its bound quiver. As to gentle one-cycle algebras, this condition is further equivalent to that there is at least one relation on the unique cycle of its bound quiver. In view of the above classification, our result follows. □

Thus, by Proposition 4.6, to classify derived-discrete gentle one-cycle algebras under tilting-cotilting equivalence it suffices to classify them under BB-tilting-cotilting equivalence. It is therefore natural to investigate how a gentle algebra modifies under left mutations.

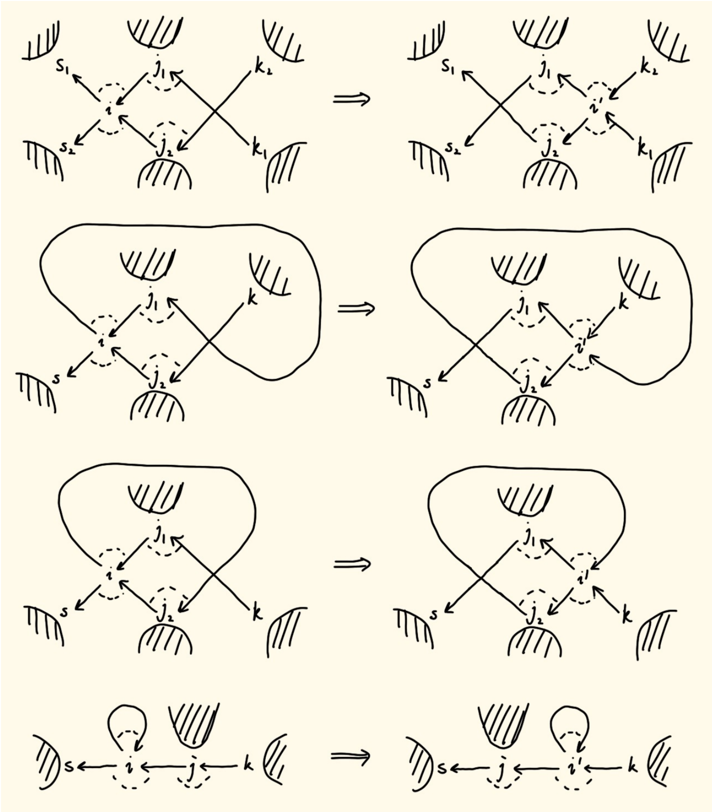
Lemma 6.3. *Consider a gentle algebra* $A = \Bbbk Q/I$ *and fix a vertex* i.

(1) *If there is only one arrow ending with* i, *then all possible left mutations at* i *are*



where in the first case, we allow $s = k$.

(2) *If there are two arrows ending with i, then all possible left mutations at i are*



where in the first case, we allow any of the equalities $s_1 = s_2, j_1 = j_2, k_1 = k_2$ to be true; in the other three cases, we allow $s = k$.

Remark. This result is really a generalization of the one appeared in the proof of Proposition 5.2. It is said to have first appeared in an unpublished manuscript by T. Holm, J. Schröer and A. Zimmermann, which is fortunately included in [10, §7.1-§7.3]. It is proved there that these "elementary transformations" are given by tilting complexes and thus preserve derived equivalences. Our result shows that two of them are even realized by left mutations. But in general, the one given in [10, §7.2] is not realized by a left mutation.

Proof. Before beginning, we make some general remarks on how we prove these formulae. Let $A = \Bbbk Q/I$. For a BB-tilting module $T = P' \oplus X$ where P' is the complement of the projective module P corresponding to vertex i, denote $B = \mathrm{End}_A(T)$ and suppose $B = \Bbbk Q'/I'$. Then Q' is the opposite of the quiver of irreducible morphisms of the category $\mathsf{add}\, T$ and I' is given by all vanishing composites of these irreducible morphisms. The calculation of Q' is divided into three steps:

1. Delete from Q the vertex i and all adjacent arrows, and add a new arrow $j \to k$ whenever there exists an arrow $\alpha : j \to i$, an arrow $\beta : i \to k$ such that $\alpha\beta \neq 0$. Here j, k can be equal. This produces $Q \setminus \{i\}$, the opposite of the quiver of irreducible morphisms of $\mathsf{add}\, P'$.

2. Add to $Q \setminus \{i\}$ a new vertex i' and whenever there is an arrow $\alpha : j \to i$ in the original quiver Q, add a new arrow $\alpha' : i' \to j$. Such arrows as α' represents components of g in the almost-split exact sequence $0 \to P \to P'' \xrightarrow{g} X \to 0$ which must be irreducible since g is right minimal almost-split.

3. Analyse which paths of $Q \setminus \{i\}$ factor through one of the paths created in step 2. Usually this requires specific analysis for each case.

We show, as examples, the first case of (1) and the second case of (2).

For the first case of (1), denote by α, β, γ the arrows $i \to s$, $j \to i$, $k \to j$. The left minimal $\mathsf{add}(A \setminus P(i))$-approximation of $P(i)$ produces a short exact sequence $0 \to P(i) \xrightarrow{P(\beta)} P(j) \xrightarrow{h} X \to 0$. The relation $\gamma\beta = 0$ gives a vanishing of composition $P(\beta)P(\gamma) = 0$, so $P(\gamma)$ factors through h as $P(\gamma) = fh$, where $f : X \to P(k)$. Thus, the resulting mutated quiver is shown as stated, where the three arrows $j \to s$, $i' \to j$, $k \to i'$ are, respectively, induced by $\beta\gamma, h, f$.

For the second case of (2), denote $i \xrightarrow{\alpha} s$, $i \xrightarrow{\beta} j_1$, $j_1 \xrightarrow{\gamma} i$, $j_2 \xrightarrow{\delta} i$, $k \xrightarrow{\epsilon} j_2$. The left minimal $\mathsf{add}(A \setminus P(i))$-approximation of $P(i)$ produces a short exact sequence $0 \to P(i) \xrightarrow{\begin{bmatrix} P(\gamma) \\ P(\delta) \end{bmatrix}} P(j_1) \oplus P(j_2) \xrightarrow{[g\ h]} X \to 0$. Since in the following solid diagram

$$\begin{tikzcd}
0 \arrow[r] & P(i) \arrow[r, "\begin{bmatrix} P(\gamma) \\ P(\delta) \end{bmatrix}"] & P(j_1) \oplus P(j_2) \arrow[r, "{[g\ h]}"] \arrow[dr, "{[0\ P(\epsilon)]}"'] & X \arrow[r] \arrow[d, dashed, "f_1"] & 0 \\
 & & & P(k) &
\end{tikzcd}$$

the composition from $P(i)$ to $P(k)$ vanishes, $P(\epsilon)$ factors through h as $P(\epsilon) = f_1 h$. Since in the following solid diagram

$$\begin{tikzcd}
0 \arrow[r] & P(i) \arrow[r, "\begin{bmatrix} P(\gamma) \\ P(\delta) \end{bmatrix}"] & P(j_1) \oplus P(j_2) \arrow[r, "{[g\ h]}"] \arrow[dr, "{[0\ P(\delta\beta)]}"'] & X \arrow[r] \arrow[d, dashed, "f_1"] & 0 \\
 & & & P(j_2) &
\end{tikzcd}$$

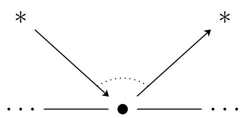
the composition from $P(i)$ to $P(j_2)$ vanishes, $P(\delta\beta)$ factors through g as $P(\delta\beta) = f_2 g$. Thus, the resulting mutated quiver is shown as stated, where the new arrows $j_1 \to s, j_2 \to i, i' \to j_1, i' \to j_2, k \to i'$ are induced by $P(\gamma\alpha), f_2, g, h, f_1$, respectively. $\square$

Now comes to our main result.

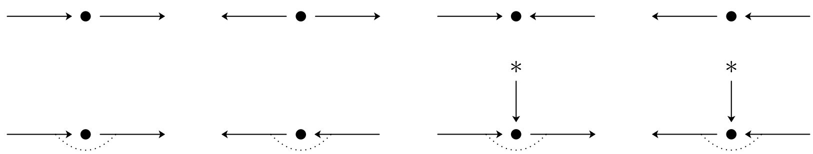
Proposition 6.4. *Any gentle one-cycle algebra admits a sequence of left mutations and right mutations to a derived representative of gentle one-cycle algebras.*

Proof. Let $A = \Bbbk Q/I$ be a gentle one-cycle algebra. Denote by C the unique cycle of Q and fix an orientation of the underlying graph of C such that the number of clockwise relations is not less than the number of counterclockwise relations.

Step 1. To a branch "$*$" attached to C at vertex "$\bullet$":

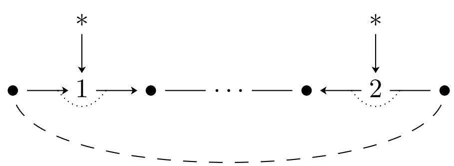


we apply left and right mutations as for gentle tree algebras to turn it into a quiver of type A, equi-oriented as the arrow connecting "$*$" and "$\bullet$"; see part (1) the proof of Proposition 5.2 for details. If two such branches are attached to "$\bullet$", we apply right mutations to the vertices of the directing-out branch in order to stuff them into the directing-in branch, using the same trick as in Proposition 5.2. As a result, we have reduced $A = \Bbbk Q/I$ to a form, in which the local shape around each vertex "$\bullet$" on C is one of the figure below:



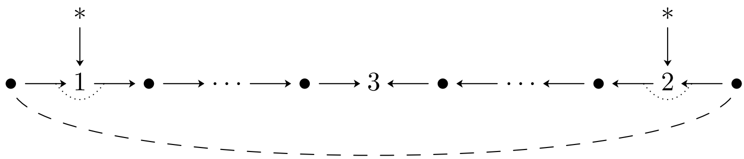
Here "$*$" indicates a branch of equi-oriented type A; the horizontal arrows are on C, drawn obeying the chosen orientation of C. Notice that all relations of A lie on C now.

Step 2. In this step, we devise a procedure to eliminate two relations on C if one is clockwise, the other is counterclockwise and the latter is the relation following immediately after the former on C in the clockwise orientation. Graphically, we are in the following situation:



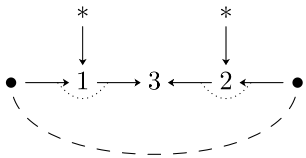
where the relation at 2 is thought of following the one at 1 in the clockwise orientation of C, and there is no other relation on C within. Notice that at "•"'s lying between 1 and 2 there are neither relations nor branches.

To eliminate them, first apply APR-(co)tiltings to sink vertices between 1 and 2 to turn the quiver into the following shape:

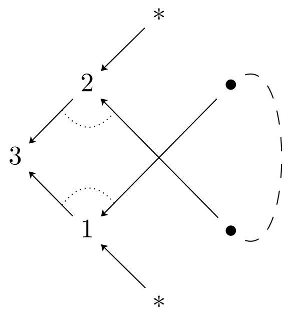


i.e., there is a vertex 3 between 1 and 2 such that all arrows between 1 and 3 direct clockwise and all arrows between 3 and 2 direct counterclockwise. We remark that the number of (counter)clockwise arrows between 1 and 2 does not change during each step of APR-(co)tilting.

Then we apply left mutations to the vertex i immediately succeeding 1 on C until $i = 3$ and apply left mutations to the vertex j immediately succeeding 2 on C until $j = 3$. This turns the quiver into the following shape:

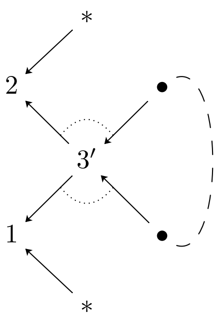


or better

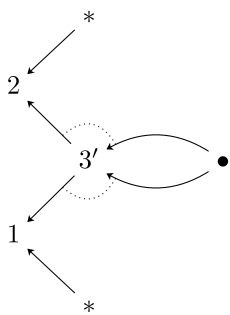


(The two "•"'s shown above can be the same vertex.)

Now we apply a left mutation at vertex 3. Regarding whether or not the two "•"'s shown above are the same vertex, we have two results:



or



Finally, we redo Step 1 to stuff all branches at vertex $3'$ into the cycle C. In the resulting algebra, a pair of relations has been thus eliminated.

Repeating the whole procedure as above, we can reduce A to the form where there are only clockwise relations on the cycle C and A has the local forms as required in the end of Step 1.

Step 3-1. If there is no relation on the cycle C, then A is a path algebra. Applying APR-tiltings to its sink vertices, we can transform it into the path algebra of type $\tilde{\mathrm{A}}_{m,n}$ for some integers m, n.

Step 3-2. If there are some relations on C, we will further transform A such that all the arrows on C direct clockwise and are consecutive on C. Label the relations on C by integers $1, \cdots, r$ in clockwise order. Assume temporarily that $r \geqslant 2$. We want to make all the arrows between 1 and 2 orient clockwise. Graphically, we are in the following situation

$$\begin{tikzcd}
 & * \arrow[d] & & & & * \arrow[d] & \\
\bullet \arrow[r] \arrow[rrrrrr, dashed, bend right, no head] & 1 \arrow[r] & \bullet \arrow[r, no head] & \cdots \arrow[r, no head] & \bullet \arrow[r] & 2 \arrow[r] & \bullet
\end{tikzcd}$$

where the segments between 1 and 2 indicate arrows with arbitrary fixed orientations. Applying APR-(co)tiltings to vertices between 1 and 2, we can transform it into

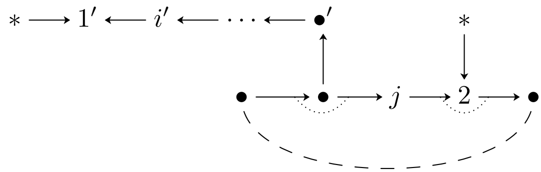
$$\begin{tikzcd}
 & * \arrow[d] & & & & & & & & & * \arrow[d] & \\
\bullet \arrow[r] \arrow[rrrrrrrrrrr, dashed, bend right, no head] & 1 \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & i & \bullet \arrow[l] & \cdots \arrow[l] & \bullet \arrow[l] & j \arrow[l] \arrow[r] & 2 \arrow[r] & \bullet
\end{tikzcd}$$

where arrows between $1, i$ direct clockwise, arrows between i, j direct counterclockwise; here i, j are just symbols, not referring to any actual number. Applying left mutations to vertices immediately succeeding 1 for several times, we may further transform A into

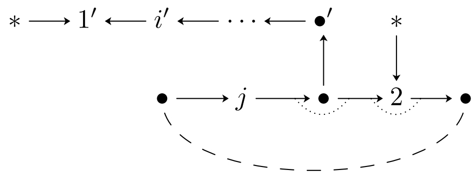
$$\begin{tikzcd}
 & * \arrow[d] & & & & & & * \arrow[d] & \\
\bullet \arrow[r] \arrow[rrrrrrrr, dashed, bend right, no head] & 1 \arrow[r] & i & \bullet \arrow[l] & \cdots \arrow[l] & \bullet \arrow[l] & j \arrow[l] \arrow[r] & 2 \arrow[r] & \bullet
\end{tikzcd}$$

Apply left mutations to vertices from i to the vertex immediately preceding

j, and the result is



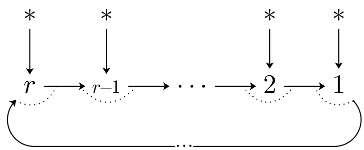
By one more left mutation at j, A is reduced to



Repeat this procedure to the part of C between 2 and 3, $\cdots$, r and 1, and the goal is accomplished.

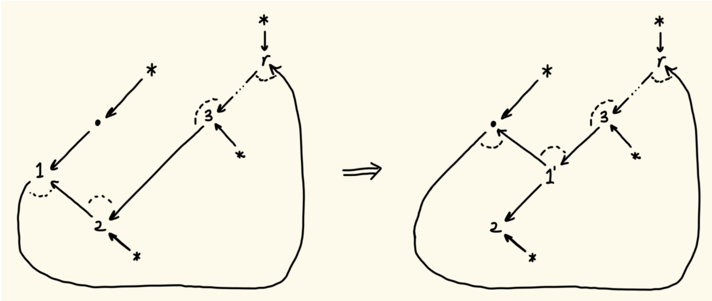
The case $r = 1$ is treated similarly by thinking the two vertices labelled by 1 and 2 in the above figures as the same.

Now the algebra A has the following form:



Before step 4, one should, as usual, sort the branches "$*$" as in Step 1, such that each full subquiver consisting of a branch "$*$" and the $\bullet$ to which it is attached is a path algebra of equi-oriented type A.

Step 4. The algebra A will be transformed into a standard one in 6.1(2). Notice that A is already standard if there is only one relation on its cycle C. So we avoid triviality by assuming there are $r \geqslant 2$ relations on C, labelled $1, 2, \cdots, r$ consecutively in counterclockwise order, as in the above figure. Let $l = (l_1, \cdots, l_r)$ where $l_i =$ the length of the branch "$*$" attached to vertex i. If $l_1 > 0$, we arrange the quiver in the lower left form:



After mutating at vertex 1 and relabelling, l becomes $(l_1-1, l_2+1, l_3, \cdots, l_r)$. Repeating this procedure, we can turn l into $(0, \cdots, 0, \sum_{i=1}^{r} l_i)$. Now there is only one branch on the cycle C, attached to vertex r, and the algebra A is of the standard form. The proof is thus complete. □

Corollary 6.5. *For two gentle one-cycle algebras, being derived equivalent is equivalent to being tilting-cotilting equivalent.*

Proof. Sufficiency is trivial since tilting and cotilting preserve derived categories. Necessity is implied by Proposition 6.4. □

Remark. Even for gentle two-cycle algebras, similar partial results hold. A gentle two-cycle algebra is called **non-degenerate** if $\#\phi_A = 3$ where ϕ is its AG-invariant. In [10, 13] it is shown that for two non-degenerate gentle two-cycle algebras, being derived equivalent is the same as being tilting-cotilting equivalent.

§7. Permanently representation-finite gentle algebras

Tilting-cotilting equivalence classification of gentle algebras has not been accomplished yet. Since our method applies only to the class of permanently representation-finite algebras, it is meaningful to first determine such algebras among gentle algebras.

Corollary 7.1. *Among the gentle algebras satisfying conditions in Theorem ??, being tilting-cotilting equivalent is equivalent to being BB-tilting-cotilting equivalent.*

Proof. This follows directly from Proposition 4.6. □

Acknowledgement

References

[1] Takahide Adachi, Osamu Iyama, and Idun Reiten. τ-tilting theory. *Compositio Mathematica*, 150(3):415–452, 2014.

[2] Takuma Aihara and Osamu Iyama. Silting mutation in triangulated categories. *Journal of the London Mathematical Society*, 85(3):633–668, 2012.

[3] Claire Amiot, Pierre-Guy Plamondon, and Sibylle Schroll. A complete derived invariant for gentle algebras via winding numbers and arf invariants. *Selecta Mathematica*, 29(2):30, 2023.

[4] Kristin Krogh Arnesen, Rosanna Laking, and David Pauksztello. Morphisms between indecomposable complexes in the bounded derived category of a gentle algebra. *Journal of Algebra*, 467:1–46, 2016.

[5] Ibrahim Assem and Dieter Happel. Generalized tilted algebras of type A_n. *Communications in Algebra*, 9(20):2101–2125, 1981.

[6] Ibrahim Assem, Dieter Happel, and Sonia Trepode. Tilting up iterated tilted algebras. *Proceedings of the American Mathematical Society*, 128(8):2223–2232, 2000.

[7] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras: Techniques of representation theory*. Cambridge University Press, 2006.

[8] Ibrahim Assem and Andrzej Skowroński. Iterated tilted algebras of type $\tilde{\mathrm{A}}_n$. *Mathematische Zeitschrift*, 195(2):269–290, 1987.

[9] Ibrahim Assem and Andrzej Skowroński. Algèbres pré-inclinées et catégories dérivées. In *Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin*, pages 1–34, Berlin, Heidelberg, 1989. Springer Berlin Heidelberg.

[10] Diana Avella-Alaminos. Derived classification of gentle algebras with two cycles. *arXiv preprint arXiv:0708.3839*, 2007.

[11] Diana Avella-Alaminos and Christof Geiss. Combinatorial derived invariants for gentle algebras. *Journal of Pure and Applied Algebra*, 212(1):228–243, 2008.

[12] Viktor Bekkert and Héctor A. Merklen. Indecomposables in derived categories of gentle algebras. *Algebras and Representation Theory*, 6(3):285–302, 2003.

[13] Grzegorz Bobinski and Piotr Malicki. On derived equivalence classification of gentle two-cycle algebras. *arXiv preprint arXiv:0708.3755*, 2007.

[14] Sheila Brenner and Michael C. R. Butler. Generalizations of the Bernšteĭn–Gelfand–Ponomarev reflection functors. In *Representation Theory II: Proceedings of the Second International Conference on Representations of Algebras Ottawa, Carleton University, August 13–25, 1979*, pages 103–169. Springer, 1980.

[15] M.C.R. Butler and Claus Michael Ringel. Auslander–Reiten sequences with few middle terms and applications to string algebrass. *Communications in Algebra*, 15(1-2):145–179, 1987.

[16] Dieter Happel. *Triangulated categories in the representation theory of finite dimensional algebras*, volume 119. Cambridge University Press, 1988.

[17] Dieter Happel and Luise Unger. On a partial order of tilting modules: Dedicated to the memory of sheila brenner. *Algebras and Representation Theory*, 8(2):147–156, 2005.

[18] T. Holm, J. Schröer, and A. Zimmermann. Combinatorial derived equivalences between gentle algebras. 2001.

[19] Steffen Koenig and Dong Yang. Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras. *Documenta Mathematica*, 19:403–438, 2014.

[20] Sebastian Opper, Pierre-Guy Plamondon, and Sibylle Schroll. A geometric model for the derived category of gentle algebras. *arXiv preprint arXiv:1801.09659*, 2018.

[21] Christine Riedtmann and Aidan Schofield. On a simplicial complex associated with tilting modules. *Comment. Math. Helv*, 66(1):70–78, 1991.

[22] Bernd Roggon. Selfinjective and iterated tilted algebras of type E_6, E_7, E_8. 1999.

[23] Jan Schröer and Alexander Zimmermann. Stable endomorphism algebras of modules over special biserial algebras. *Mathematische Zeitschrift*, 244(3):515–530, 2003.

[24] Daniel Simson and Andrzej Skowroński. *Elements of the representation theory of associative algebras: Representation-Infinite Tilted Algebras.* Cambridge University Press, 2007.

[25] Daniel Simson and Andrzej Skowronski. Elements of the representation theory of associative algebras: Tubes and concealed algebras of euclidean type. *American history*, 1861(1900), 2007.

[26] Dieter Vossieck. The algebras with discrete derived category. *Journal of Algebra*, 243(1):168–176, 2001.

[27] İlke Çanakçi, David Pauksztello, and Sibylle Schroll. Mapping cones in the bounded derived category of a gentle algebra. *Journal of Algebra*, 530:163–194, 2019.