GROUP 08: IMAGE RESTORATION BY MATRIX COMPLETION USING SCHATTEN CAPPED P-NORM

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1. Introduction

Matrix completion problems arise in a variety of settings such as recommendation systems, computer vision and signal processing. In this project we focus on low rank matrix completion applied to the case of image restoration. Loss of image information due to corruption, loss due to transmission and noise in the data decreases the quality of the image. Image restoration tries to restore the quality of the image by estimating the lost data.

2. Linear Algebra concepts in the project

In the setting of matrix completion, the data is represented in the form a matrix $M \in \mathbb{R}^{m \times n}$, in which only some of the entries are observed. In case of images, the matrix values will be the pixel intensities. Since we know that most information of a matrix are contributed by the dominant singular values, the underlying latent matrix capturing the true information can be approximated to have a low rank structure. Mathematically, the matrix completion problem is formulated as:

$$\min_X rank(X)$$

$$\text{s.t. } \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(M)$$

where M is the observed incomplete matrix and X is the restored matrix of same dimensions as M and $\Omega \subset \{1, 2, \cdots, m\} \times \{1, 2, \cdots, n\}$ is the observed index set of M. $\mathcal{P}_\Omega$ is the observation operator defined as:

$$[\mathcal{P}_\Omega(M)]_{ij} = \begin{cases} M_{ij}, & (i,j) \in \Omega \\ 0, & otherwise \end{cases}$$

However, the rank minimisation formulation for matrix completion has been proven to be NP-hard [1] due to the non-convexity and discontinuity of the rank function. In order to give a convex relaxation as a surrogate, nuclear norm has been widely seen in literature which is defined as below:

$$\|X\|_* = \sum_{i=1}^{min(m,n)} \sigma_i$$

where σ_i is the i^{th} largest singular value of X.Thus the nuclear norm minimisation is formulated as below:

$$\min_X \|X\|_*$$

$$\text{s.t. } \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(M)$$

It is understood that the latent matrix X is usually approximately low-rank in real world examples. Thus a solution which is exactly of least rank is not considered optimal. Here, the nuclear norm minimisation is seen as a sub-optimal solution since it causes over-shrinkage of the rank. So it is essential that a surrogate function balances between the rank and the nuclear norm of the underlying matrix.

There are several non-convex surrogate functions that have been introduced to approximate the rank function more accurately. One such commonly seen method in this setting is the truncated nuclear norm regularisation (TNNR) which is formulated as below:

$$\|X\|_r = \sum_{i=r+1}^{min(m,n)} \sigma_i$$

$$\min_X \|X\|_r$$

$$\text{s.t. } \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(M)$$

where $\|X\|_r$ is the truncated nuclear norm with r being a parameter. It has been designed so as to penalise just the smallest $min(m,n) - r$ singular values. Another well studied non-convex surrogate is the Schatten p norm. It has been proven that it performs better than the nuclear norm when $p \in (0,1)$[1]. It is defined as below:

$$\|X\|_{S_p} = (\sum_{i=1}^{min(m,n)} \sigma_i^p)^{\frac{1}{p}}$$

The authors of this paper[1] have designed a new non-convex surrogate that acts like a combination of the truncated nuclear norm and the Schatten p norm called the ***Schatten capped*** $\mathbf{p(SC_p)}$**norm** with $p \in (0,1)$ which generalises several existing matrix norms. The same constrained optimisation as before is enforced. The SC_p norm is defined as:

$$\|X\|_{SC_p,\tau} = \left(\sum_{i=1}^{min(m,n)} min(\sigma_i, \tau)^p \right)^{\frac{1}{p}}$$

where $\tau \geq 0$ is a selected parameter which acts as a threshold to filter the singular values. More details on the optimisation framework and results are explained in the video.

References

[1] Guorui Li *et al.*, *Matrix Completion via Schatten Capped p Norm*, IEEE Transactions, Vol.34, No. 1, January 2022