Hierarchical Models and Shrinkage Estimation

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Example: Uber Trip Data

- 37,961 Uber trips for 1,909 users
- Goal: Identify users who on average take high value trips
- This seems like an easy exercise right?

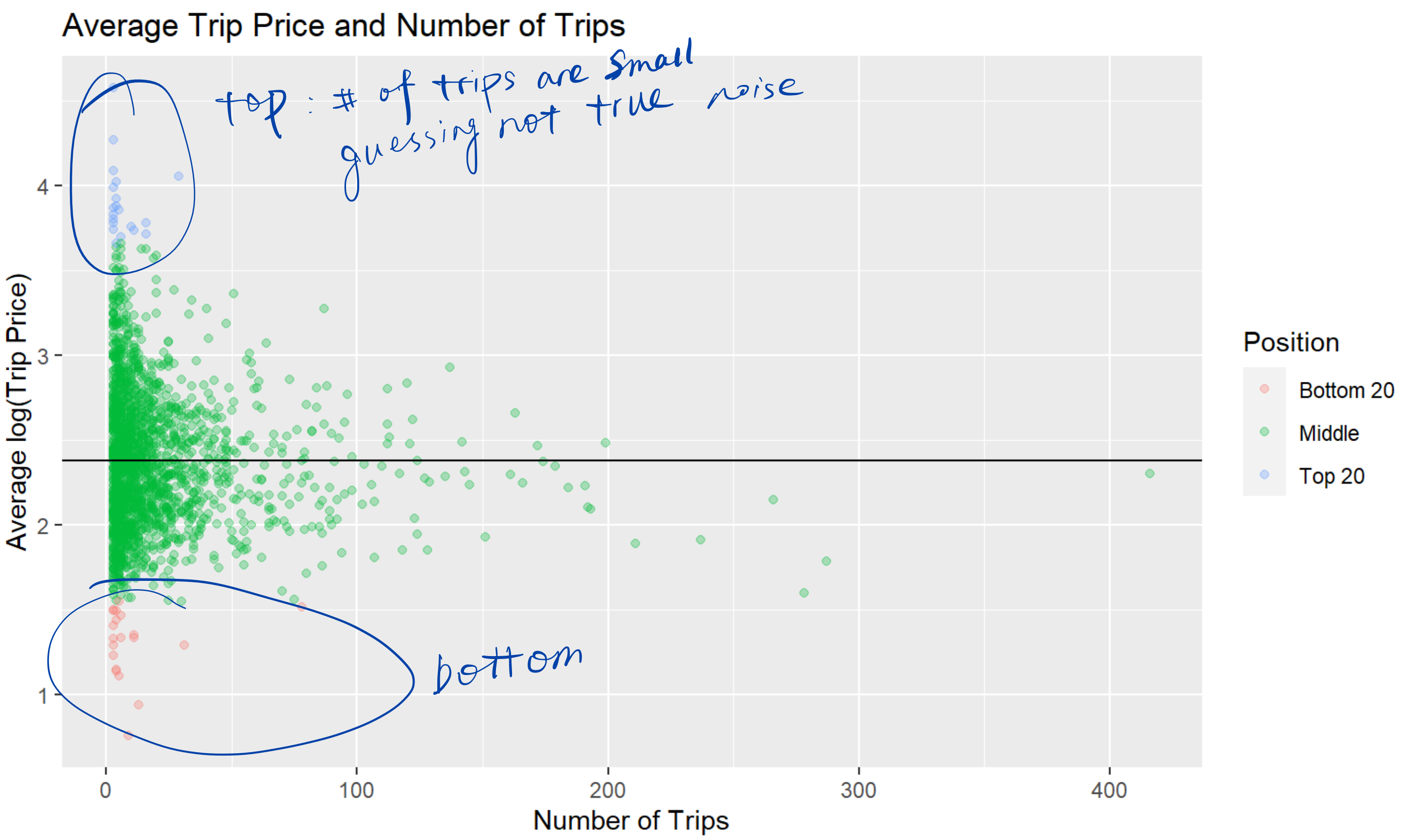
Top 20

```
## # A tibble: 20 x 7
##    userId nTrips avgLogPrice userIdF userIndex empiricalRank Position
##     <dbl>  <int>       <dbl> <fct>       <int>         <dbl> <chr>
##  1  13907      3        4.57 13907        1316             1 Top 20
##  2   9755      3        4.27 9755         1071             2 Top 20
##  3   1849      3        4.09 1849          233             3 Top 20
##  4  13138     29        4.05 13138        1282             4 Top 20
##  5  15994      4        4.02 15994        1414             5 Top 20
##  6   2868      3        3.99 2868          390             6 Top 20
##  7    641      4        3.92 641            48             7 Top 20
##  8   9108      4        3.88 9108         1025             8 Top 20
##  9   3402      3        3.87 3402          465             9 Top 20
## 10  20446      5        3.86 20446        1567            10 Top 20
## 11  15664      3        3.83 15664        1397            11 Top 20
## 12   6885      3        3.80 6885          841            12 Top 20
## 13  18739      3        3.78 18739        1501            13 Top 20
## 14    442     16        3.78 442            36            14 Top 20
## 15  28191     10        3.76 28191        1799            15 Top 20
## 16  15796      3        3.74 15796        1403            16 Top 20
## 17  20822     11        3.74 20822        1579            17 Top 20
## 18   4094     16        3.71 4094          552            18 Top 20
## 19   1837      6        3.70 1837          228            19 Top 20
## 20   9656      4        3.66 9656         1067            20 Top 20
```

Bottom 20

```
## # A tibble: 20 x 7
##    userId nTrips avgLogPrice userIdF userIndex empiricalRank Position 
##     <dbl>  <int>       <dbl> <fct>       <int>         <dbl> <chr>    
##  1  19288      9       0.759 19288        1526          1909 Bottom 20
##  2   5880     13       0.942 5880          749          1908 Bottom 20
##  3  17527      5       1.11  17527        1459          1907 Bottom 20
##  4   1052      4       1.13  1052          147          1906 Bottom 20
##  5   1106      4       1.15  1106          155          1905 Bottom 20
##  6  12035      3       1.23  12035        1217          1904 Bottom 20
##  7   7851     31       1.29  7851          934          1903 Bottom 20
##  8  26257      3       1.29  26257        1747          1902 Bottom 20
##  9    726      3       1.33  726            68          1901 Bottom 20
## 10  16178     11       1.33  16178        1424          1900 Bottom 20
## 11   2495      6       1.33  2495          328          1899 Bottom 20
## 12   6744     11       1.35  6744          823          1898 Bottom 20
## 13  14161      3       1.41  14161        1330          1897 Bottom 20
## 14   5861      4       1.44  5861          747          1896 Bottom 20
## 15  13819      6       1.47  13819        1313          1895 Bottom 20
## 16   6632      3       1.49  6632          814          1894 Bottom 20
## 17  29331      4       1.49  29331        1838          1893 Bottom 20
## 18   7816      3       1.50  7816          929          1892 Bottom 20
## 19    892     78       1.52  892           110          1891 Bottom 20
## 20  22802      5       1.55  22802        1643          1890 Bottom 20
```

Raw Averages



top : # of trips are small
guessing not true noise

bottom

Introduce a Bayesian that will consider the outlier is not useful

General Problem!

- Whenever you want to compare averages of something, this problem will show up:

  Averages based on a small number of observations will always be more noisy than averages over a large number of observations

- This means that the extremes will almost always be made up of the averages based on a small number of observations

- Solution: Users are similar but different from each other. We should use this information to our advantage.

- Idea: We should "shrink" the raw means toward the overall mean in a way so that the noisy averages are shrunk more than the non-noisy averages:

$$\hat{\alpha}_i = F_i(\bar{Y}_i)$$

- Standard approach takes $F_i(\bar{Y}_i) = \bar{Y}_i$ but as we have seen that is a bad idea!

- We need $F_i(\bar{Y}_i) < \bar{Y}_i$ for large noisy averages ($F_i(\bar{Y}_i) > \bar{Y}_i$ for small noisy averages) and $F_i(\bar{Y}_i) \approx \bar{Y}_i$ for non-noisy averages

Hiearchical Model

each user has a parameter, wanna learn it

$$
\begin{aligned}
y_{ij}|\overset{\text{mean}}{\alpha_i}, \sigma &\sim \mathrm{N}(\alpha_i, \sigma^2), \qquad j = 1, \ldots, \overset{\text{\# of trips}}{N_i};\; i = 1, \ldots, N, \\
\alpha_i|\mu, \sigma_\alpha &\sim \mathrm{N}(\mu, \sigma_\alpha^2),
\end{aligned}
$$

plus a prior distribution for $\sigma, \mu, \sigma_\alpha$.

- This is also called a Multilevel Model or a model with partial pooling
- We are primarily interested in the posterior distribution of $\{\alpha_i\}_{i=1}^N$ but also in σ_α.
- What happens when $\sigma_\alpha \to 0$ and $\sigma_\alpha \to \infty$?

Example

- Suppose initially that $\sigma, \mu, \sigma_\alpha$ are known. What is the posterior for α_i?
- We can derive this analytically:

$$p(\alpha_i | Y_i, \mu, \sigma, \sigma_\alpha) = \mathrm{N}(\alpha_i | \mu_{\alpha_i}, \tau_{\alpha_i}^{-1}),$$

where

$$\begin{aligned}
\tau_{\alpha_i} &\equiv \frac{N_i}{\sigma^2} + \frac{1}{\sigma_\alpha^2}, \\
\mu_{\alpha_i} &\equiv \tau_{\alpha_i}^{-1} \left(\frac{N_i}{\sigma^2} \bar{Y}_i + \frac{1}{\sigma_\alpha^2} \mu \right),
\end{aligned}$$

and $\bar{Y}_i \equiv N_i^{-1} \sum_j Y_{ij}$

Special Cases

- We can consider two special cases: $\sigma_\alpha \to 0$ and $\sigma_\alpha \to \infty$
- The posterior mean of α_i in these two special cases is

$$\mathrm{E}[\alpha_i | Y_i, \mu, \sigma_\alpha, \sigma] \to \begin{cases} \bar{Y}_i & \text{for } \sigma_\alpha \to \infty, \\ \mu & \text{for } \sigma_\alpha \to 0. \end{cases}$$

- These cases are also referred to as
  - $\sigma_\alpha \to \infty$ = "No Pooling" (everyone is completely different)
  - $\sigma_\alpha \to 0$ = "Complete Pooling" (everyone is the same)
  - $0 < \sigma_\alpha < \infty$ = "Partial pooling" (everyone is different but similar) ✓ always this case
- Preferred approach: Let the data help in determining size of σ_α!
  - If there is evidence in the data that all users are very similar, then we should learn that σ_α is small
  - If there is evidence in the data that users are very different, then we should learn that σ_α is large

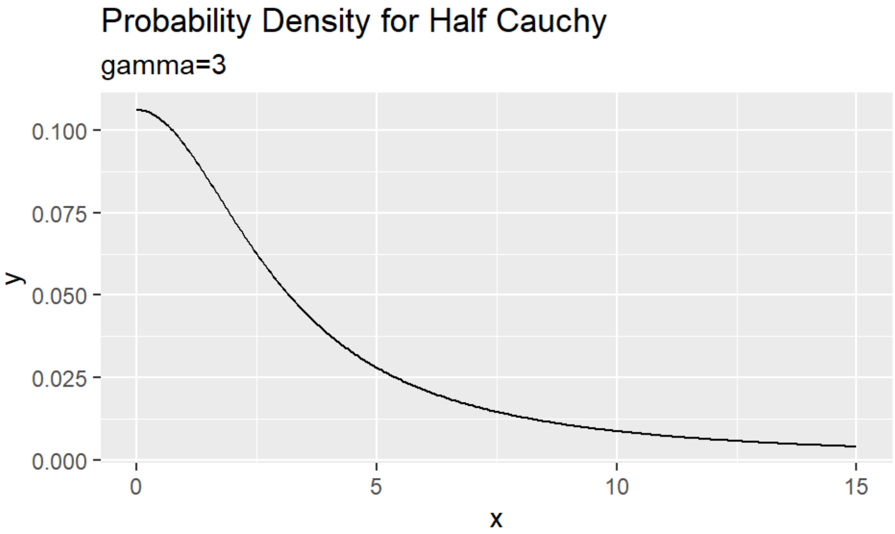
Full Model

不一定是完全对的 model

$$\begin{aligned} y_{ij} | \alpha_i, \sigma &\sim \mathrm{N}(\alpha_i, \sigma^2), \qquad j = 1, \ldots, N_i; i = 1, \ldots, N, \\ \alpha_i | \mu, \sigma_\alpha &\sim \mathrm{N}(\mu, \sigma_\alpha^2), \\ \mu &\sim \mathrm{N}(0, 5^2), \longrightarrow \\ \sigma &\sim \mathrm{Cauchy}_+(0, 3), \\ \sigma_\alpha &\sim \mathrm{Cauchy}_+(0, 3) \end{aligned}$$

Half Cauchy

$$p(x|\gamma) = \frac{1}{\pi\gamma\left(1 + (\frac{x}{\gamma})^2\right)}$$



Finding the Posterior

- This model - while simple - it already too complex to easily solve analytically
- Note that the joint posterior is a distribution of

  mean of α

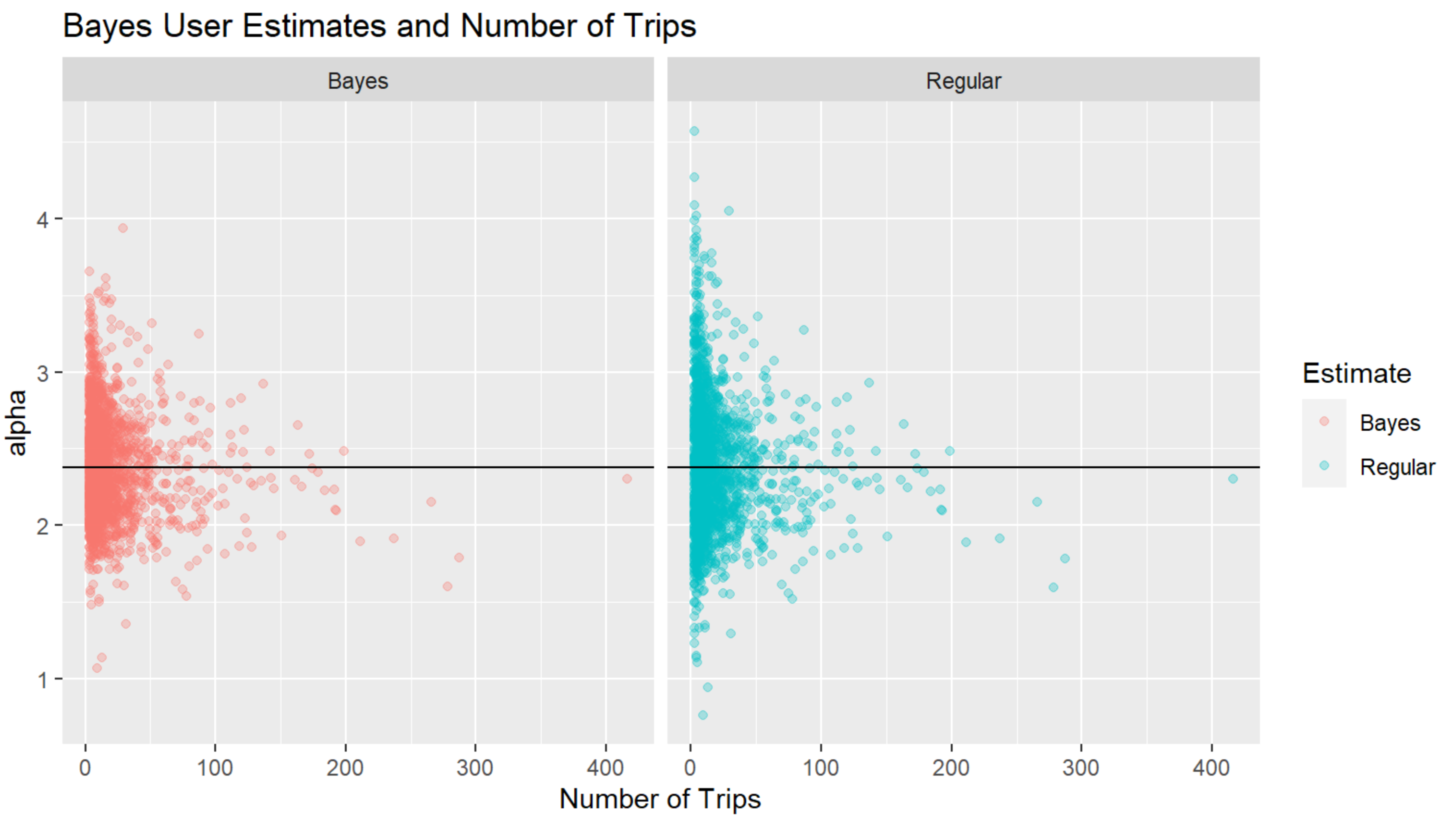
  $$(\alpha_1, \ldots, \alpha_N, \mu, \sigma, \sigma_\alpha),$$ very high dimension

  i.e., $N + 3$ parameters where N is the number of users. Therefore this is a probability distribution in close to 2,000 dimensions!
- We will use a Monte Carlo algorithm to numerically simulate this posterior distribution
- This algorithm will simulate the joint posterior by drawing a sequence of S pseudo-random numbers from the posterior distribution: $\{\tilde{\alpha}_s, \tilde{\mu}_s, \tilde{\sigma}_{\alpha,s}, \tilde{\sigma}_s\}_s^S$
- We will map the model into the probabilistic language `Stan`
- We will not go into the details here - more on that next class!

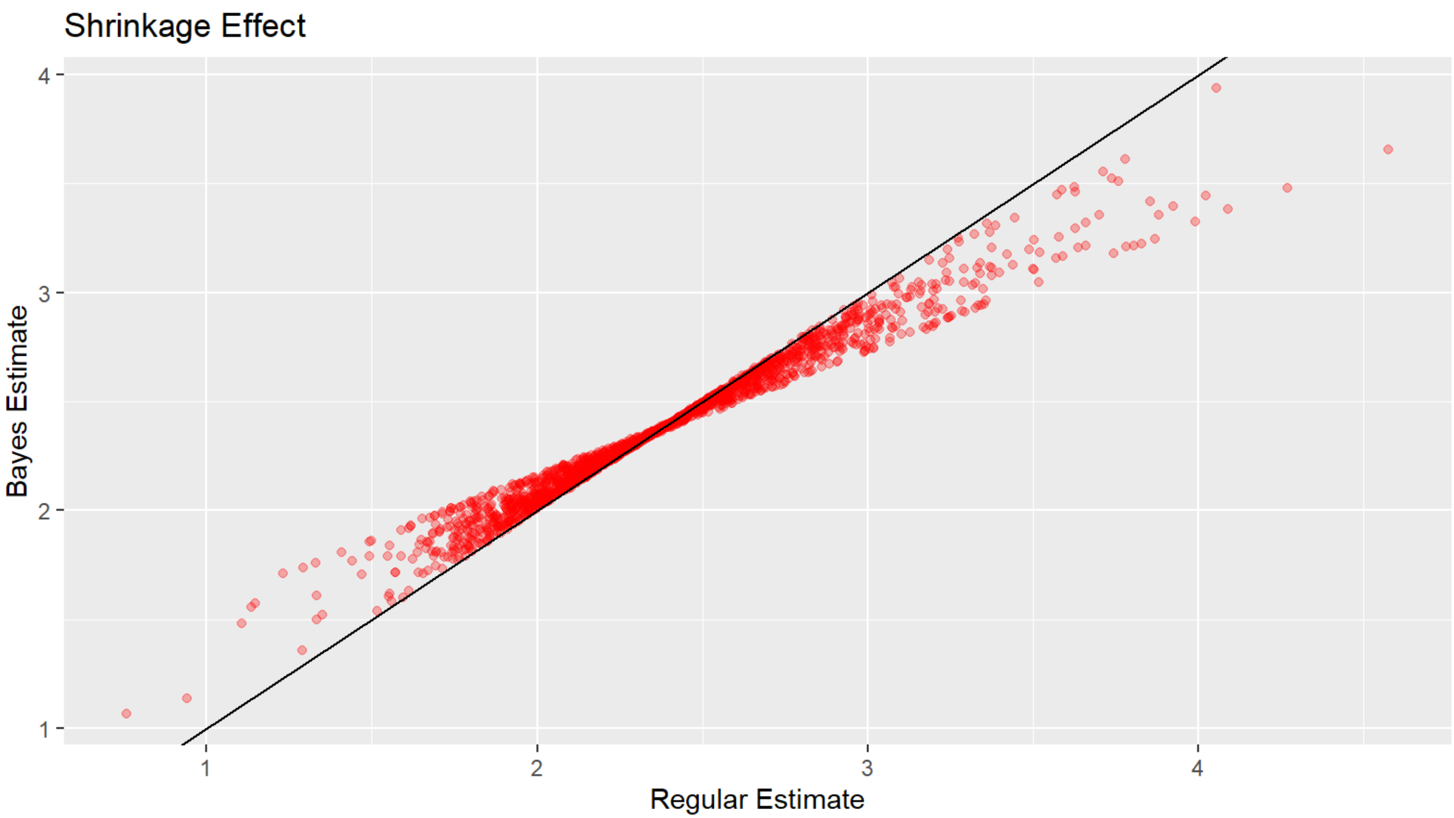
Stan code

```
data {
  int<lower=0> nObs;                        // number of rows in full data
  int<lower=0> nUsers;                      // number of users
  int<lower=1,upper=nUsers> userID[nObs];   // user index for each row
  vector[nObs] y;                           // log amount
}

parameters {
  real<lower=0> sigma;         // sd alpha
  real mu;                     // mean alpha
  vector[nUsers] alpha;        // user effects
  real<lower=0> sigma_y;       // sd data
}

model {
  sigma ~ cauchy(0, 2.5);
  mu ~ normal(0,5);
  alpha ~ normal(mu, sigma);
  sigma_y ~ cauchy(0, 2.5);

  y ~ normal(alpha[userID], sigma_y);
}
```

Result


Bayes User Estimates and Number of Trips
Bayes
Regular
alpha
4
3
2
1
0
100
200
300
400
Number of Trips
Estimate
Bayes
Regular

Result

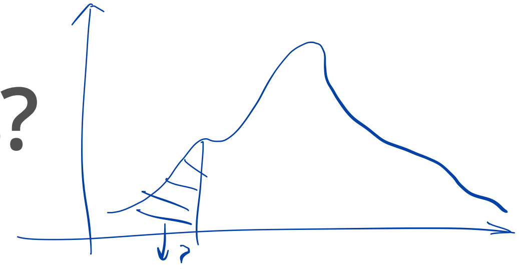


small avg

we pull them up

higher avg, pull them down

How did we make this?



$$E[X] = \int x p(x) = dx \ ?$$

$$\approx \frac{1}{5000} \sum_{i=1}^{5000} \tilde{X}_i$$

$$V[X] = \int (X - E[X])^2 p(x)\, d(x)$$

$$\approx \frac{1}{5000} \sum_{i=1}^{5000} (\bar{X}_i - \bar{x})^2$$

Monte Carlo Estimate

- The algorithm generates a stream of pseudo random numbers from the distribution we are interested in

- Why is this useful? Note that if $\{\tilde{x}_s\}_{s=1}^{S}$ are random draws from a distribution π then

$$\mathrm{E}_\pi[f(x)] \approx \frac{1}{S} \sum_{s=1}^{S} f(\tilde{x}_s),$$

  for a function f, where we can control the approximation error by choosing large enough enough M.

- This is called Monte Carlo simulation

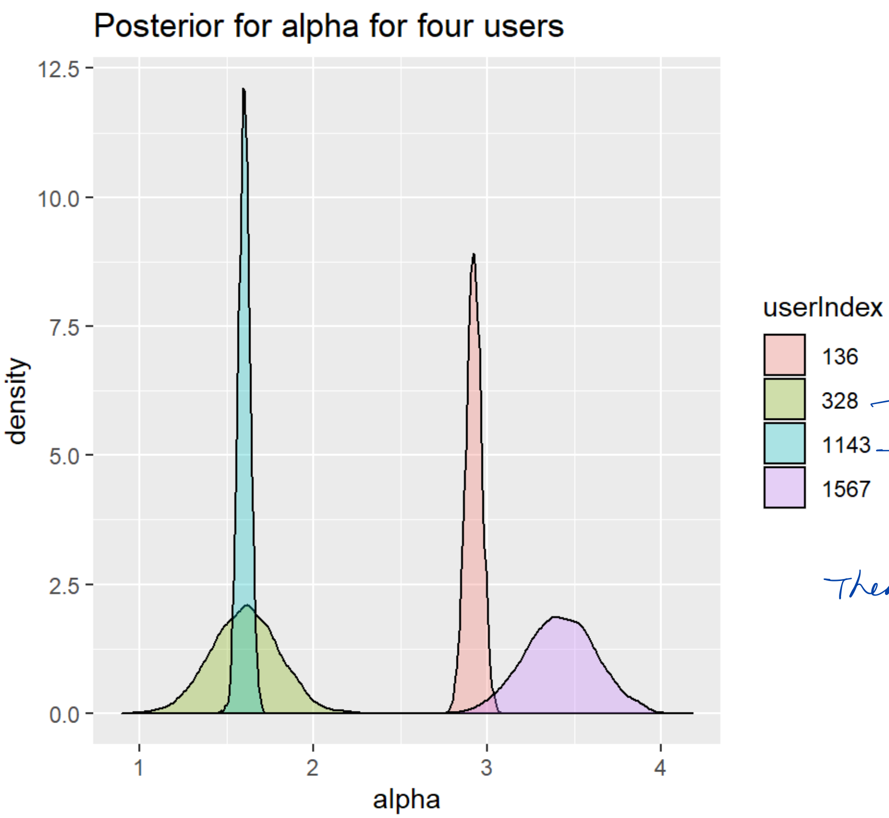
- Almost all Bayesian models are trained this way

- Next week we will look at the details of how these algorithms are designed

Using Draws

- Posterior average: $\bar{\tilde{x}} = \frac{1}{S} \sum_{s=1}^{S} \tilde{x}_s$
- Posterior standard deviation: $\sqrt{\frac{1}{S} \sum_{s=1}^{S} (\tilde{x}_s - \bar{\tilde{x}})^2}$
- Posterior quantiles: Empirical quantiles of $\{\tilde{x}_s\}_{s=1}^{S}$
- Posterior distribution: histogram or density of $\{\tilde{x}_s\}_{s=1}^{S}$

Using Draws



```
## # A tibble: 4 x 6
##   userIndex value .lower .upper avgLogPrice nTrips
##       <int> <dbl>  <dbl>  <dbl>       <dbl>  <int>
## 1       136  2.92   2.84   3.01        2.93    137
## 2       328  1.61   1.23   1.97        1.33      6
## 3      1143  1.60   1.54   1.67        1.60    278
## 4      1567  3.42   3.02   3.82        3.86      5
```

→ posterior, has a valuation 1~2, has uncertainty

→ quite confident

Therefore we can manage risk.

Using Draws: Posterior Ranks

- Suppose want to identify the top 15 users in terms of value (measured as average spend per trip)
- This is a question about rank. For example the top ranked user is

$$user_1(\alpha_1, \ldots, \alpha_N) \equiv \{i : \alpha_i \geq \alpha_j,\ \forall j \in \{1, \ldots, N\}\}$$

- Note that the rank depends on the unknown parameters $\alpha_1, \ldots, \alpha_N$.
- We can use the posterior draws of the α vector to simulate posterior ranks:
  - For each draw $\tilde{\alpha}$ find the rank of each user: $\tilde{r}_1, \ldots, \tilde{r}_N$
  - This creates S draws of each user's ranking
  - Then we can summarize these S draws and find the mean rank, min rank, max rank etc for each user

No.	Samples
1	$\alpha_1^1, \alpha_2^1 \cdots \alpha_{1909}^1$
2	$\alpha_1^2, \alpha_2^2 \cdots - \alpha_{1909}^2$
3	
$\vdots$	
2000	$\alpha_1^{2000}, \alpha_2^{2000}, \cdots -, \alpha_{1909}^{2000}$

Each α will have a rank in this draw.

Calculate the avg rank for 1909 α.

Posterior Ranks

```
## # A tibble: 15 x 7
##    userIndex meanPostRank minPostRank maxPostRank nTrips avgLogPrice
##        <int>        <dbl>       <dbl>       <dbl>  <int>       <dbl>
##  1      1282         1.35           1           7     29        4.05
##  2        36         7.43           1          51     16        3.78
##  3      1316         9.89           1         154      3        4.57
##  4       552        10.1            1          78     16        3.71
##  5      1579        12.6            1          85     11        3.74
##  6      1799        13.8            1         103     10        3.76
##  7       877        14.4            1          79     16        3.63
##  8      1129        14.7            1          66     20        3.59
##  9       879        16.5            2         107     14        3.63
## 10      1743        16.6            1          75     19        3.57
## 11      1071        21.5            1         361      3        4.27
## 12      1414        22.4            1         218      4        4.02
## 13      1567        24.7            1         327      5        3.86
## 14       271        27.1            2         161     20        3.44
## 15      1375        28.2            6          68     51        3.36
## # ... with 1 more variable: empiricalRank <dbl>
```

higher

Multilevel Regression Model

- The basic idea of shrinkage estimation can be applied to any model
- Let’s consider a regression model with a multilevel/hierarchical structure:

$$\begin{aligned}
y_{ij}|\alpha_i, \beta_i, \sigma &\sim \mathrm{N}(\alpha_i + \beta_i x_i, \sigma^2), \qquad j = 1, \ldots, N_i; i = 1, \ldots, N, \\
\alpha_i, \beta_i|\mu, \Sigma &\sim \mathrm{N}(\mu, \Sigma),
\end{aligned}$$

- Note that without the second stage, this would be like training an independent regression model for each i

- The model is closed by specifying a prior for σ, μ, Σ.
- Note that

$$\Sigma \to \begin{cases} 0 & \text{One pooled regression,} \\ \infty & N \text{ independent regressions} \end{cases}$$

Case Study: Multilevel Demand Model

- Weekly sales and prices of Frito Lay Pretzels for 76 Stores
- 3 Years of data
- We want to allow stores to have different baseline sales and different demand price effects

$$\begin{aligned} \log y_{sw} &= \alpha_s + \beta_s \log p_{sw} + \varepsilon_{sw}, \\ \varepsilon_{sw} | \sigma &\sim \mathrm{N}(0, \sigma^2), \\ \alpha_s, \beta_s | \mu, \Sigma &\sim \mathrm{N}(\mu, \Sigma), \end{aligned}$$

where y_{sw} is sales volume for store s in week w, and p_{sw} is the brand price for store s in week w

- Let's try two different priors:
  - A shrinkage prior where we allow the model to learn the degree to which stores are similar
  - An independence prior with zero pooling, i.e., we treat the 76 stores as independent

Priors

- To specify a prior on the covariance matrix Σ, we use the decomposition

$$\Sigma \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix} \times \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \times \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix},$$

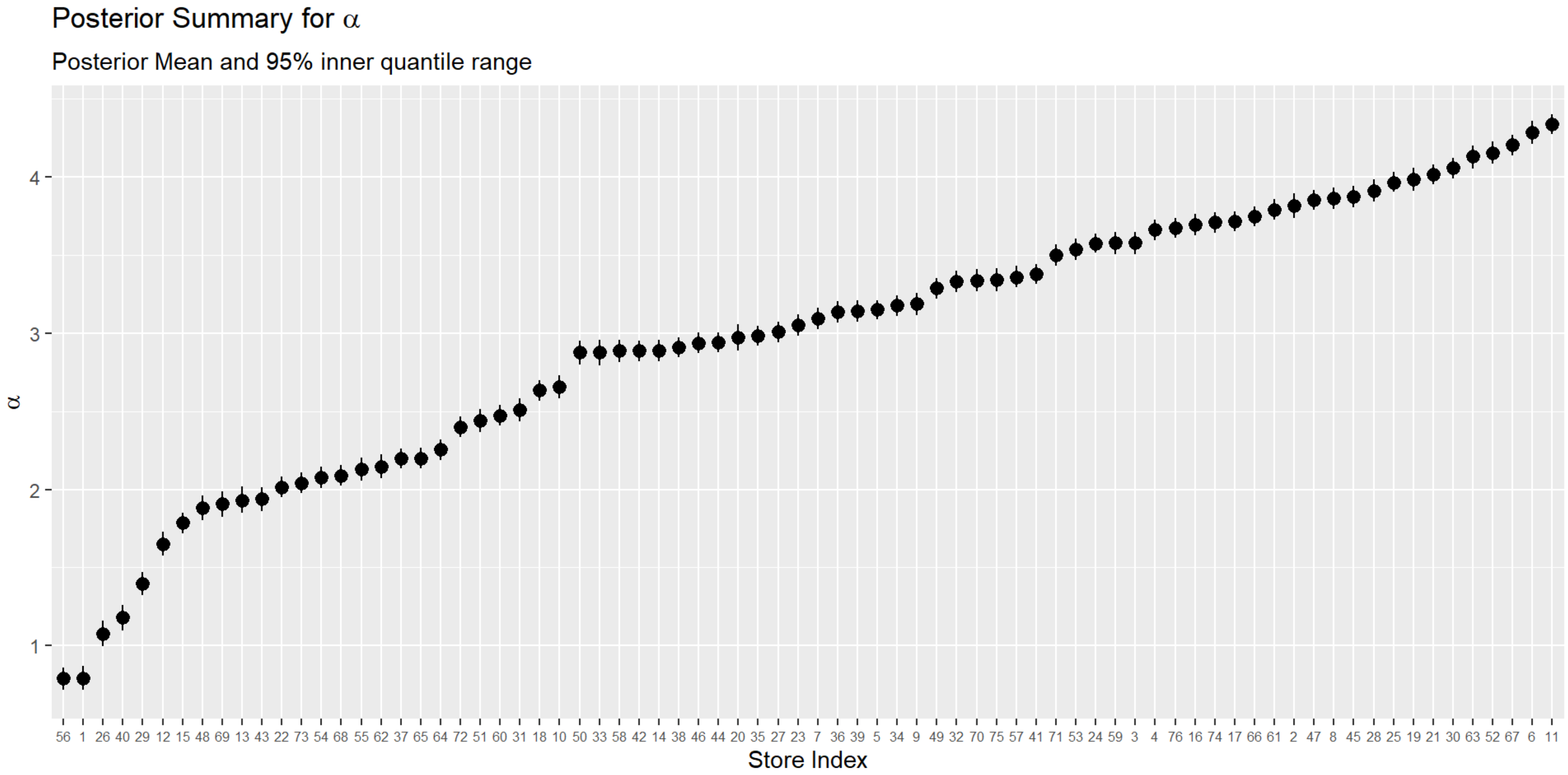
  where ρ is the correlation coefficient between α_i and β_i.

- Priors are then assigned to τ_1, τ_2 and ρ:

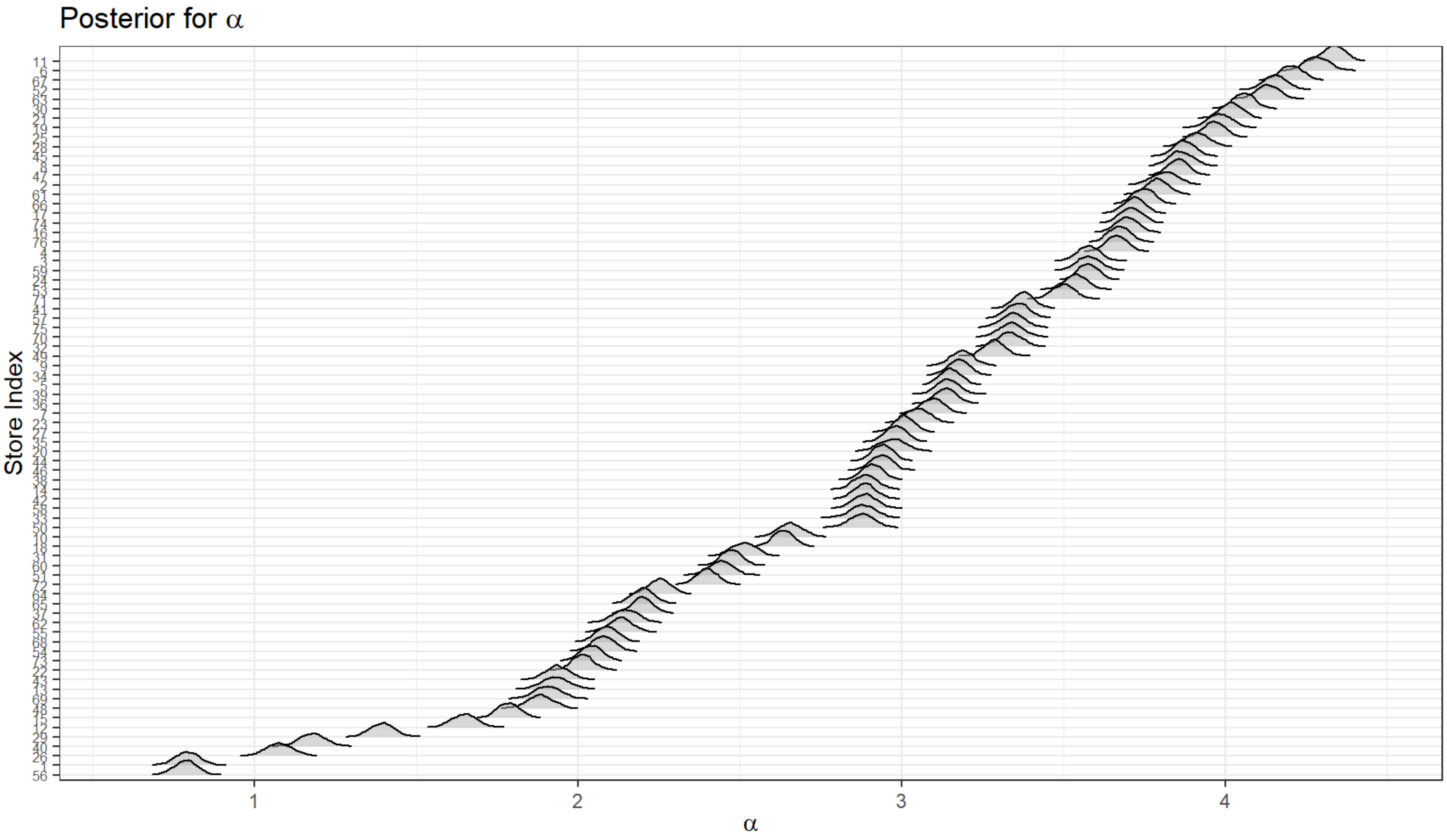
$$\begin{aligned} &\tau_1 \sim \mathrm{Cauchy}_+(0, 2.5), \ \ \tau_2 \sim \mathrm{Cauchy}_+(0, 2.5) \\ &\rho \sim \mathrm{U}(-1, 1). \end{aligned}$$

- For the independence prior we just assign independent diffuse (meaning large variance) normal distributions for α_s and β_s, e.g., $\alpha_s \sim \mathrm{N}(0, 10), \ \beta_s \sim \mathrm{N}(0, 10)$

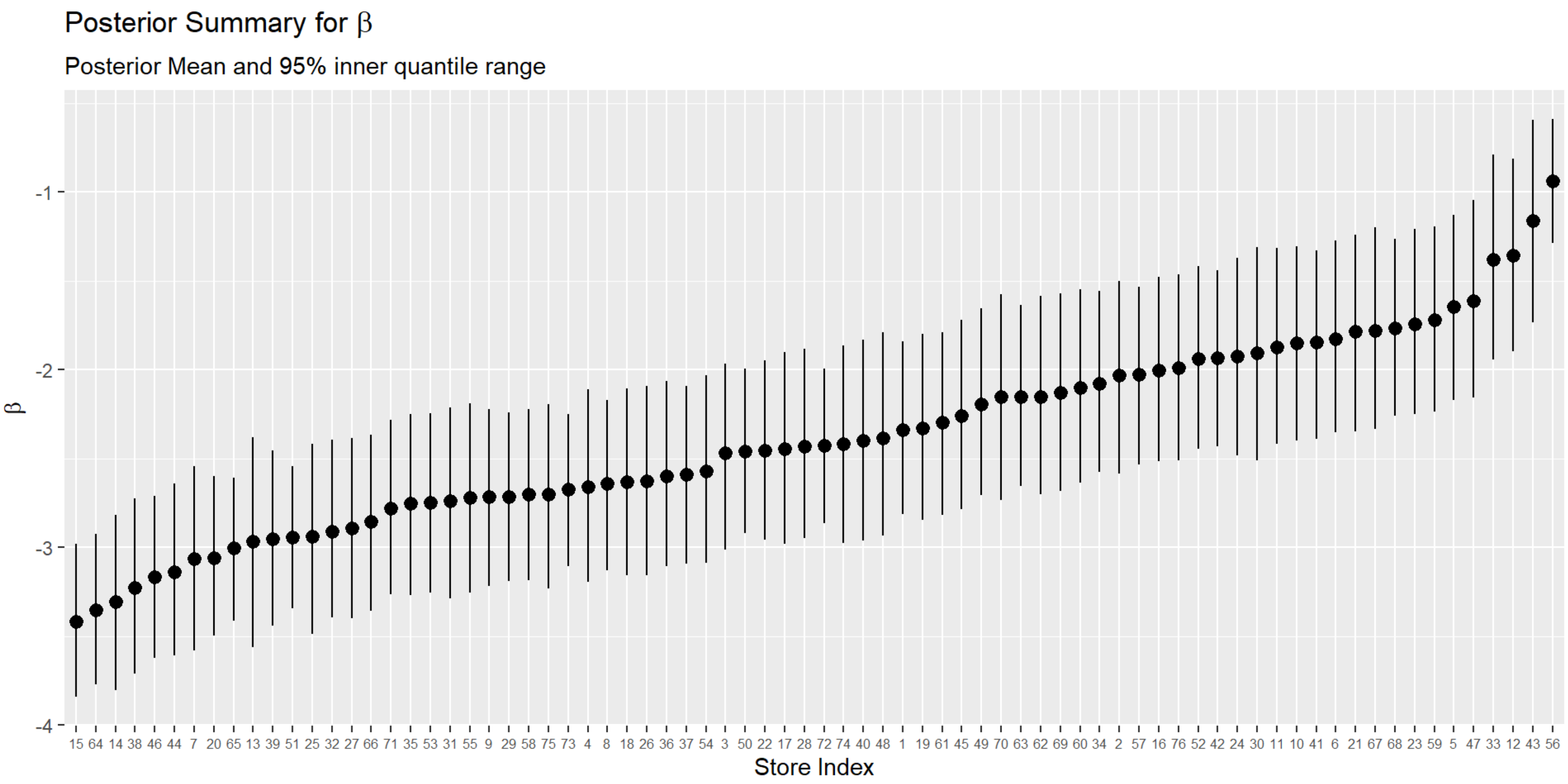
Results (With Shrinkage)



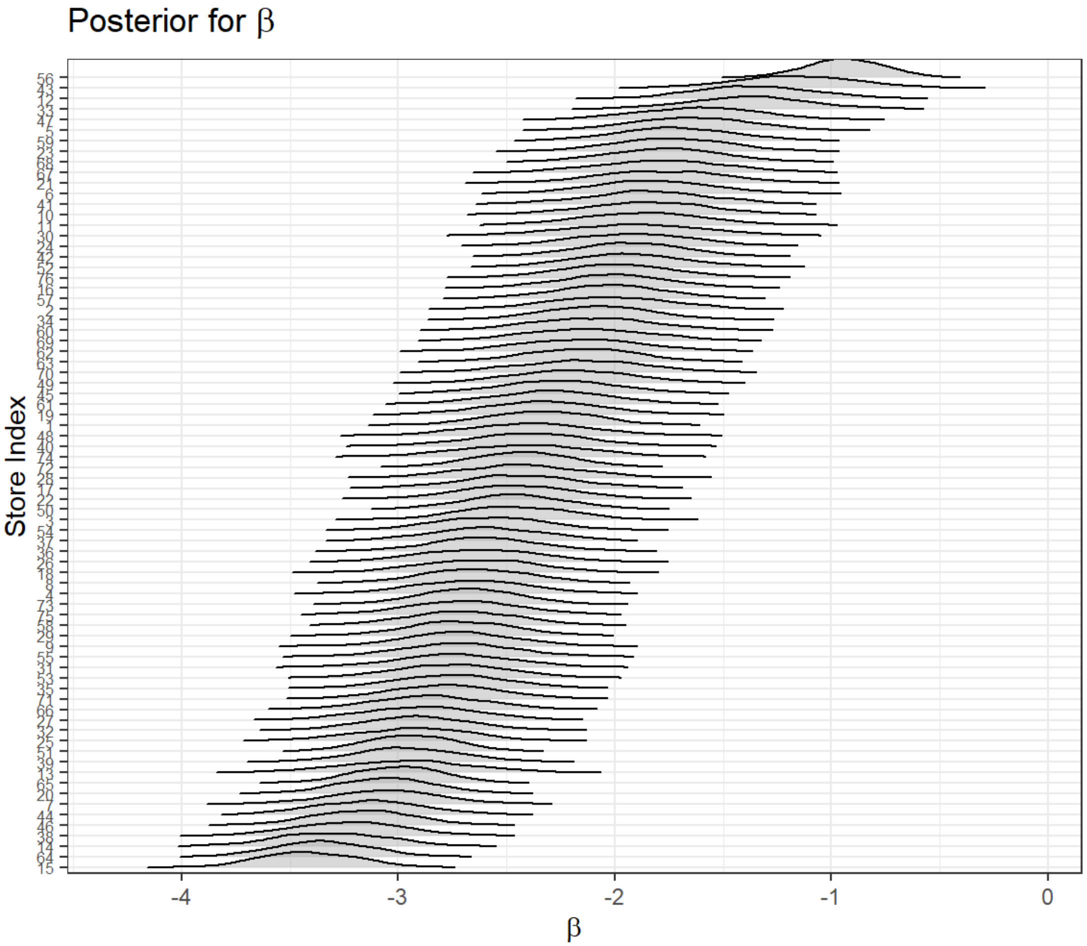
Results (With Shrinkage)



Results (With Shrinkage)



Results (With Shrinkage)



Covariation

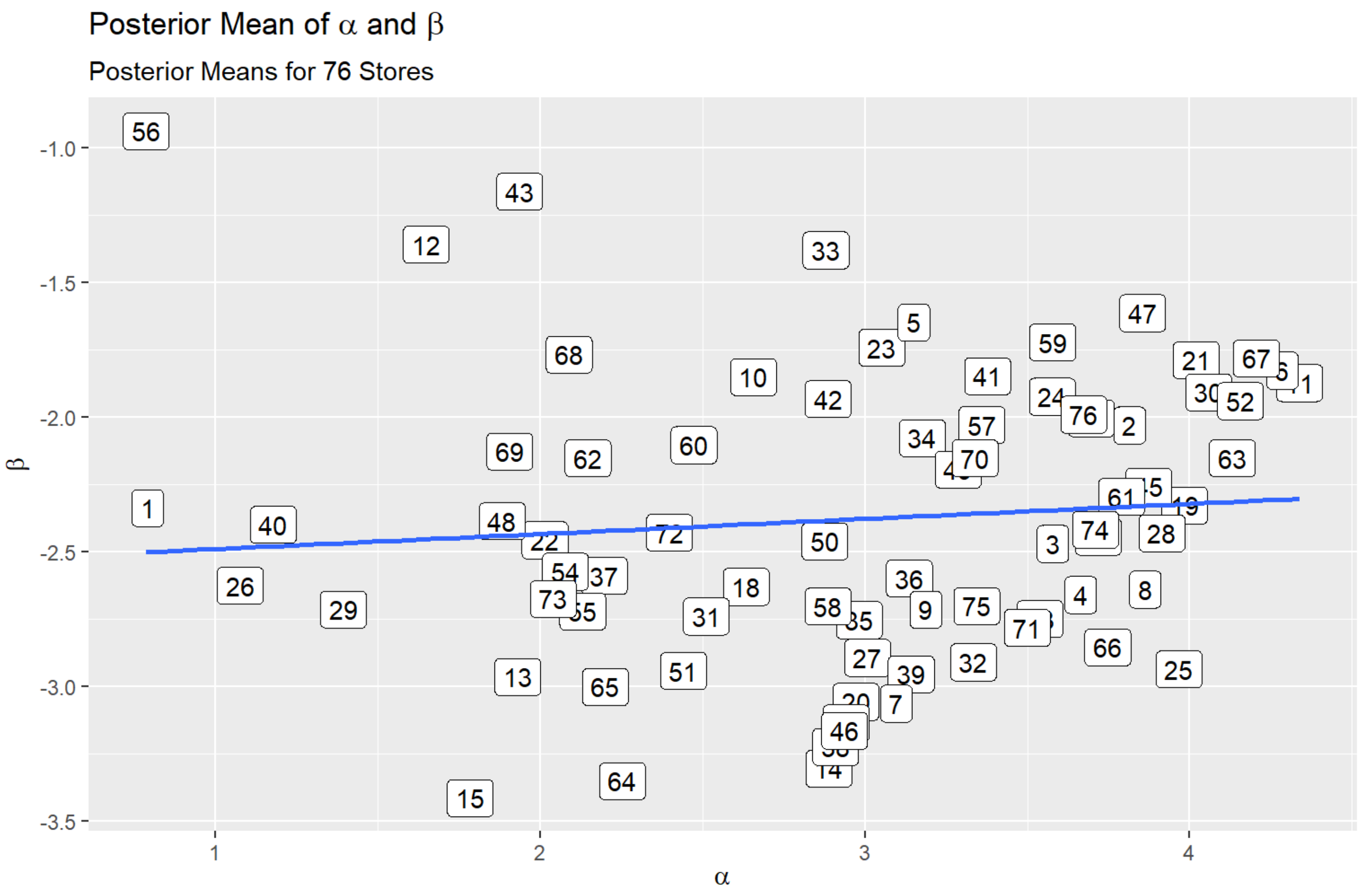
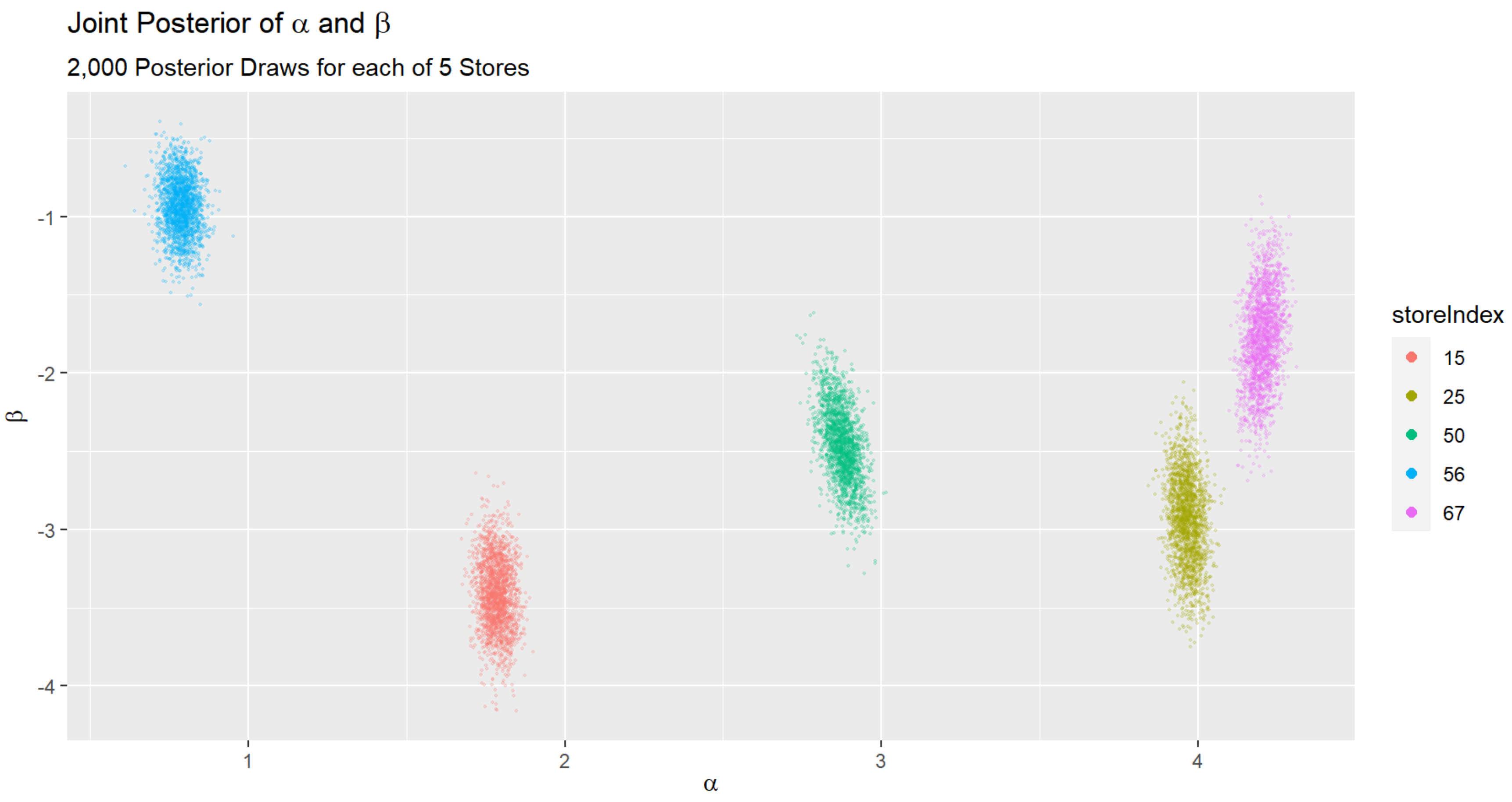
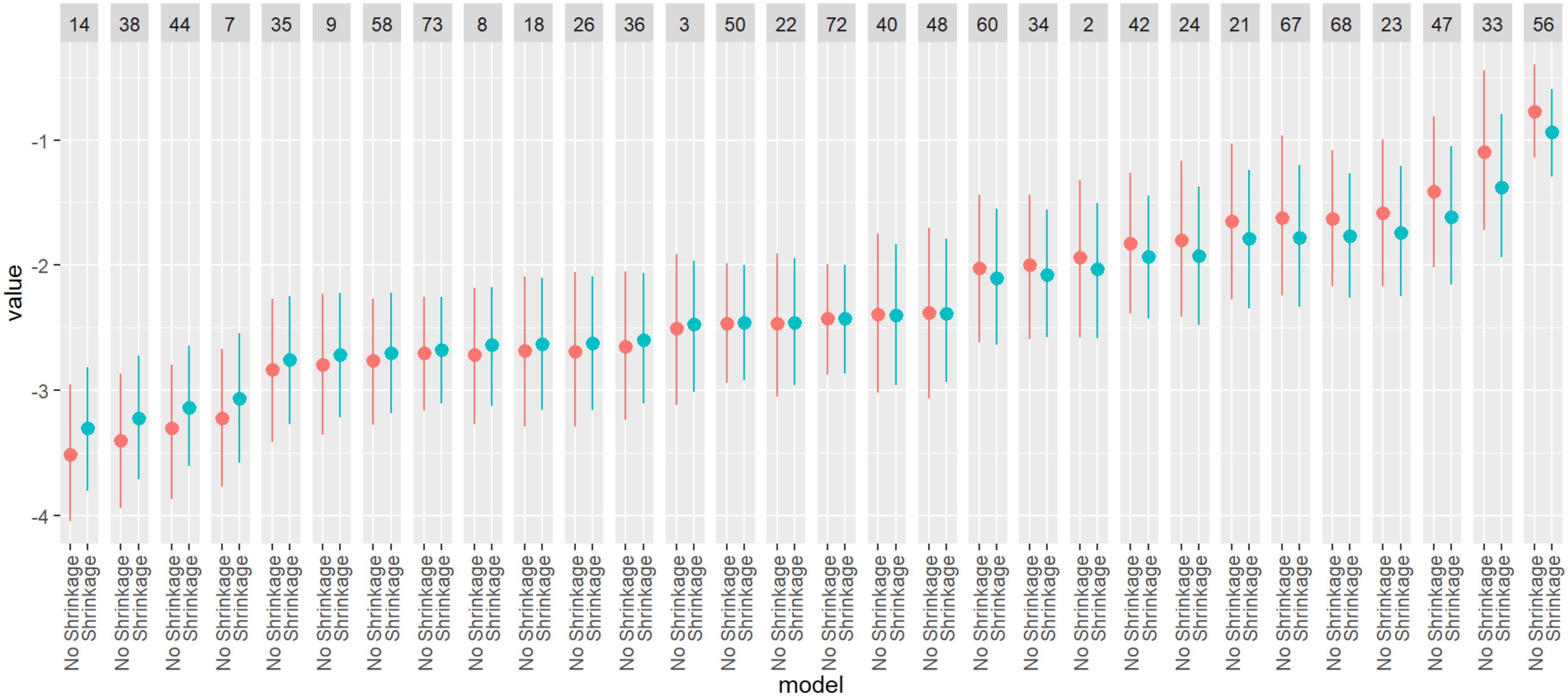


Illustration of Posterior Uncertainty



Posterior Summaries with Shrinkage and No Shrinkage

30 random stores



Expected Demand

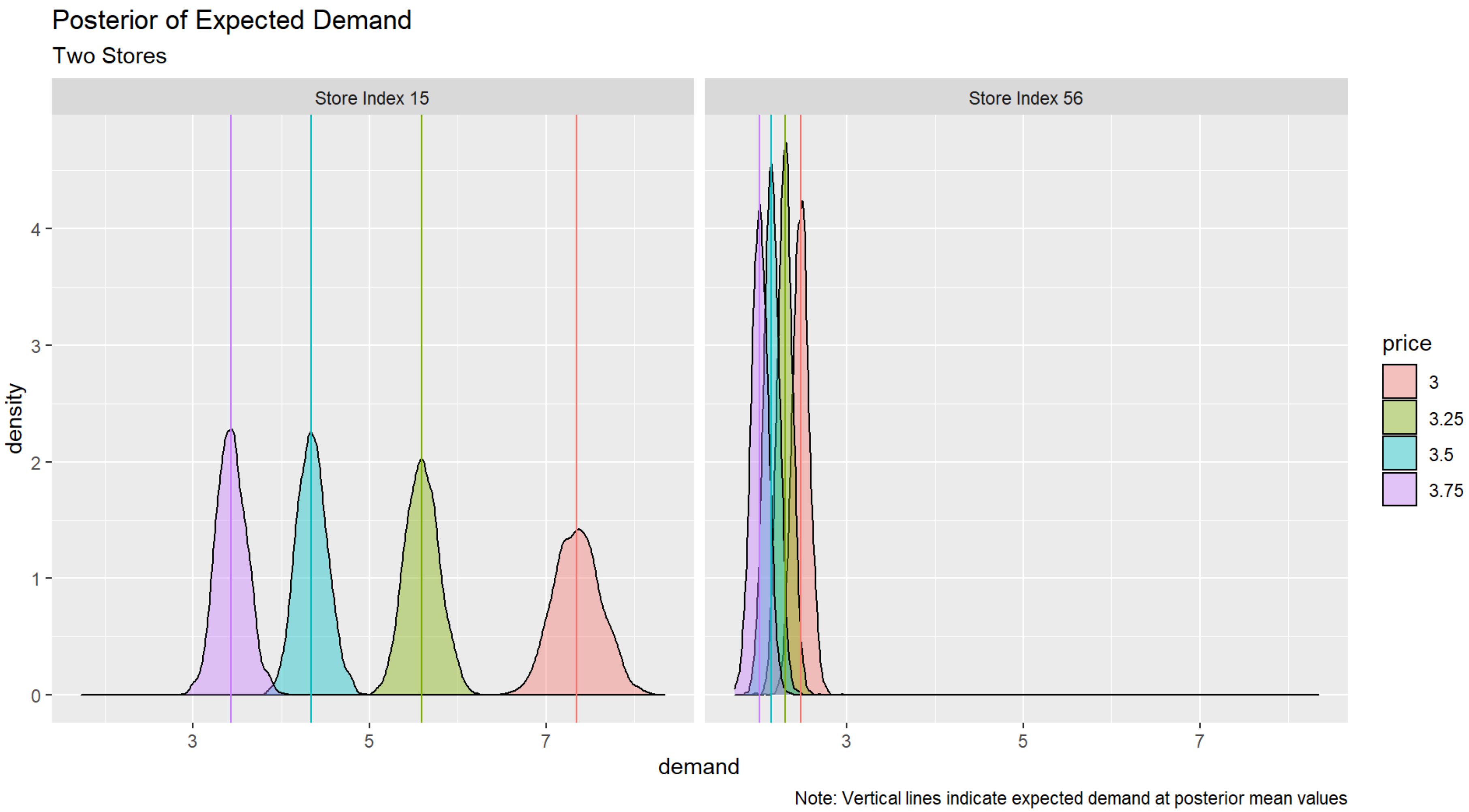
- Let’s try to use the model to predict expected future demand $Y_s{}^*$ for a store s
- Note that the generative model of sales is a log-normal distribution. Therefore,

$$\mathrm{E}[Y_s{}^* | \theta_s, \sigma] = \exp\left\{ \alpha_s + \beta_s \log p + \frac{\sigma^2}{2} \right\} \equiv g(\theta_s, \sigma; p)$$

- Notice that this is a simple function of $\theta_s \equiv (\alpha_s, \beta_s)$ and p.
- Since our algorithm already provides us draws from the posterior for θ_s, σ, we can easily generated draws of g simply as

$$\left\{ g(\tilde{\theta}_{sd}, \tilde{\sigma}_d; p) \right\}_{d=1}^{D}$$

- By varying p we can then easily trace out the effect of price changes on expected demand



Posterior of Expected Demand

Two Stores

Note: Vertical lines indicate expected demand at posterior mean values

Price Setting?

Full Uncertainty

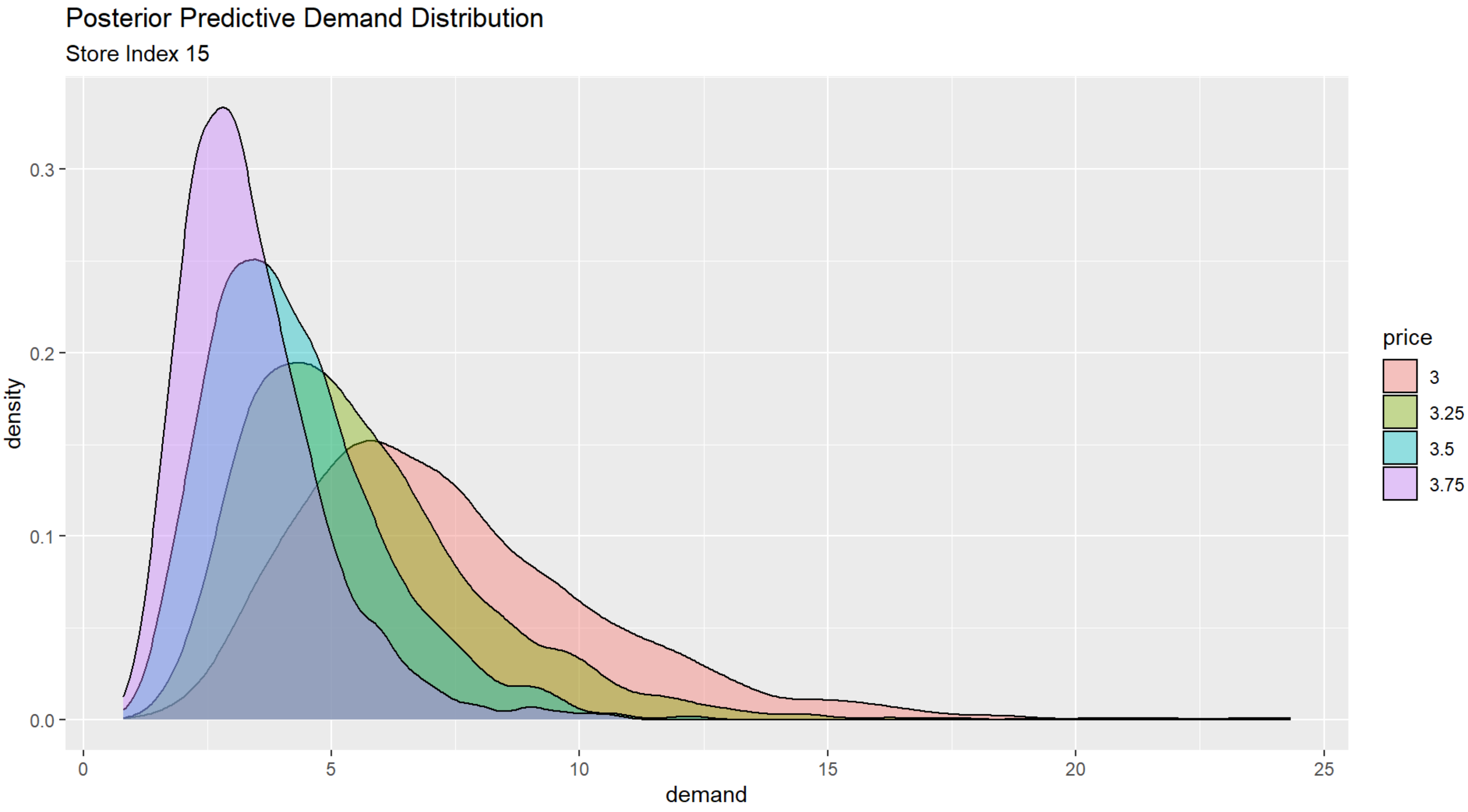
- What is the full uncertainty facing the store about next week's demand?
- This involves two sources: model uncertainty and the specific draw of demand that will materialize conditional on a specific model
- The answer is the posterior predictive distribution:

$$p(Y_s{}^*|p, \mathrm{data}) = \int p(Y_s{}^*|p, \theta_s, \sigma) p(\theta_s, \sigma|\mathrm{data}) d\theta_s d\sigma$$

- We can simulate this quite easily:

1. For each simulated draw $\tilde{\theta}_s, \tilde{\sigma}$,
2. Sample $\tilde{Y}_s^*$ as

$$\tilde{Y}^* \sim \mathrm{LogNormal}(\tilde{\alpha}_s + \tilde{\beta}_s \log p, \tilde{\sigma})$$


Posterior Predictive Demand Distribution
Store Index 15
density
0.0
0.1
0.2
0.3
demand
0
5
10
15
20
25
price
3
3.25
3.5
3.75

Model 2: Explain variation

- Our model above naturally incorporates variation in parameters across stores
- Can we explain this variation? Why are some stores price sensitive and others not? Why do some stores have low baseline sales?
- We could do some simple correlations/regressions of parameter estimates on store characteristics….BUT…a much better approach is to incorporate store characteristics explicitly in the model and then see if we can learn any dependencies
- All we have to do is modify the prior distribution of $\theta_s = (\alpha_s, \beta_s)$

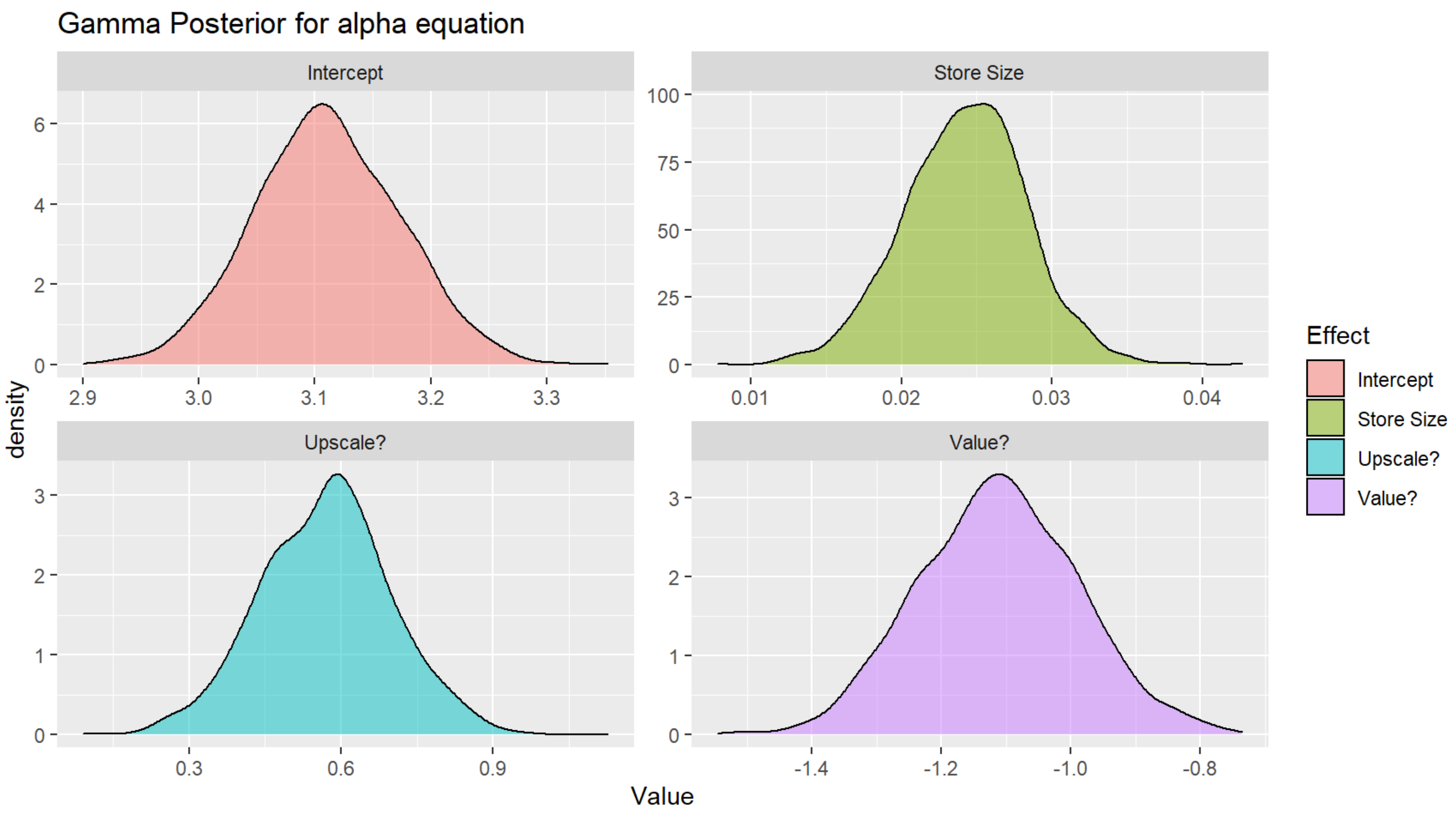
Model 2

$$\begin{aligned}
\log y_{sw} &= \alpha_s + \beta_s \log p_{sw} + \varepsilon_{sw}, \\
\varepsilon_{sw}|\sigma &\sim \mathrm{N}(0, \sigma^2), \\
\alpha_s &= \gamma'_\alpha Z_s + \psi_{\alpha,s}, \\
\beta_s &= \gamma'_\beta Z_s + \psi_{\beta,s}, \\
\psi_s \equiv (\psi_{\alpha,s}, \psi_{s,\beta})|\Sigma &\sim \mathrm{N}(0, \Sigma),
\end{aligned}$$

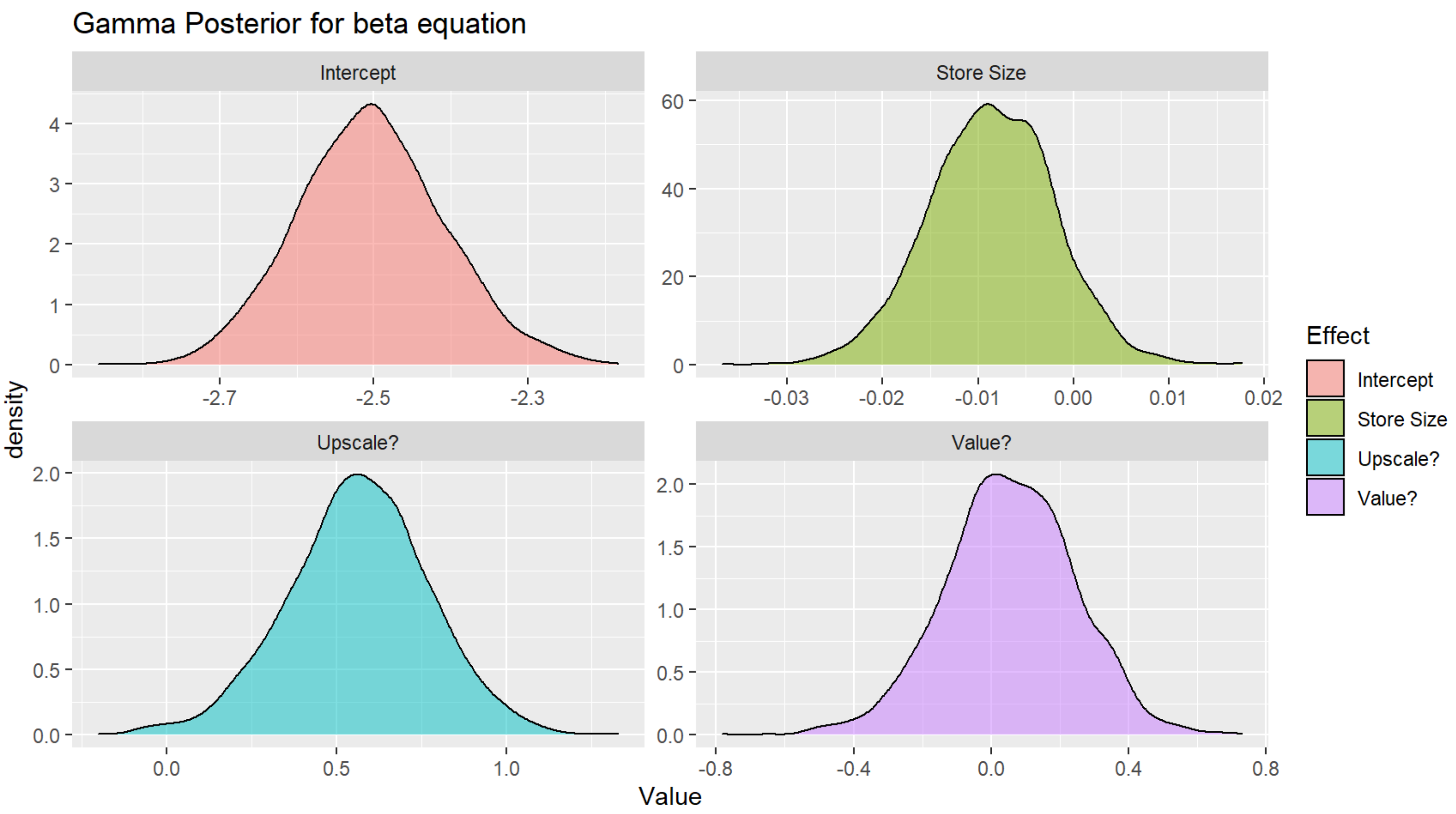
- Here Z_s is a vector store characteristics for store s
- The previous model is a special case of this with $Z_s = 1$
- We can use the same priors as the previous mmodel plus a prior on the γ parameters, e.g.,

$$\gamma_\alpha \sim \mathrm{N}(0, 5^2 I_K), \quad \gamma_\beta \sim \mathrm{N}(0, 5^2 I_K)$$

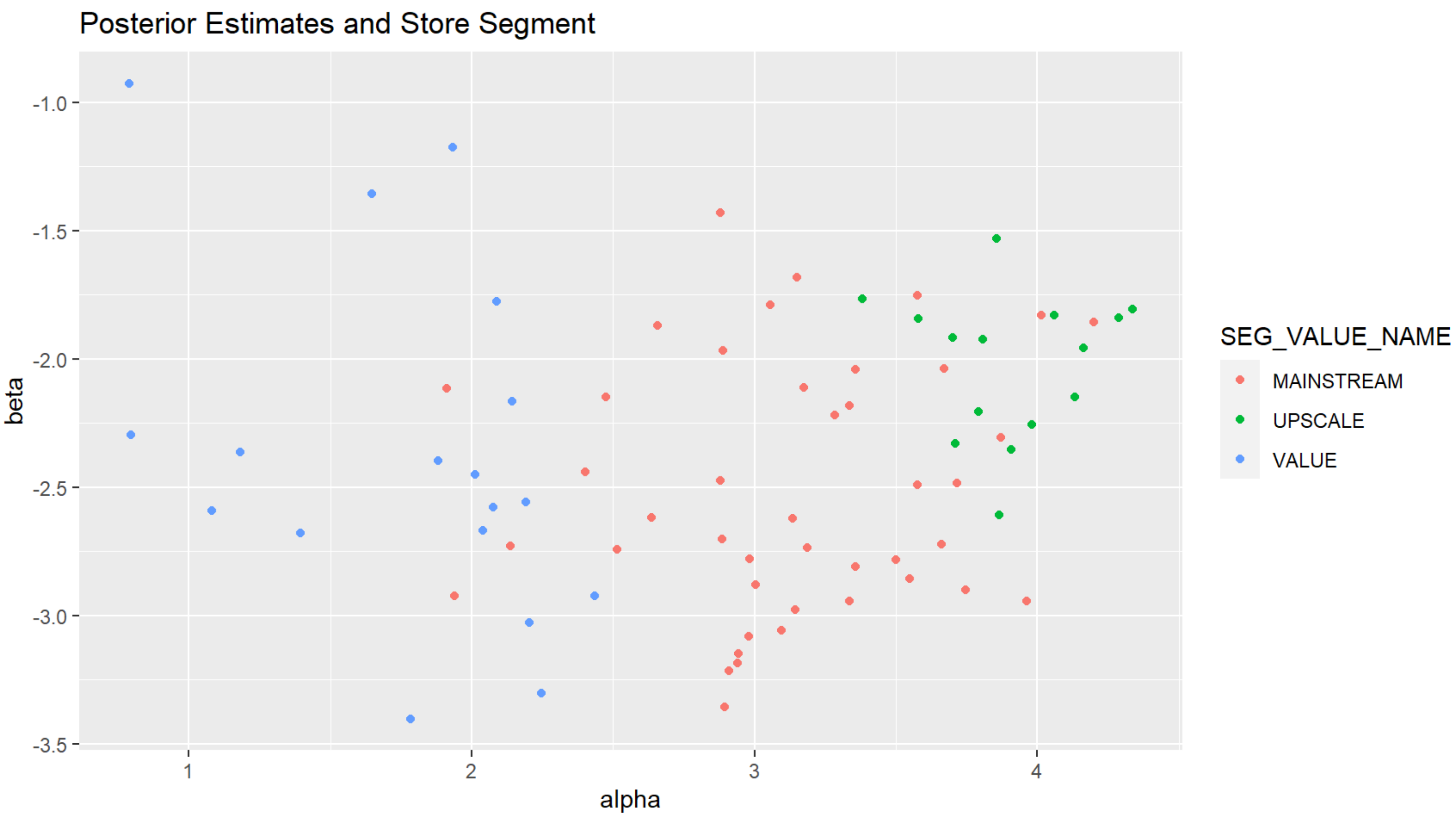
Results


Gamma Posterior for alpha equation
Intercept
Store Size
Upscale?
Value?
density
Value
6
4
2
0
2.9
3.0
3.1
3.2
3.3
100
75
50
25
0
0.01
0.02
0.03
0.04
3
2
1
0
0.3
0.6
0.9
3
2
1
0
-1.4
-1.2
-1.0
-0.8
Effect
Intercept
Store Size
Upscale?
Value?

Results


Gamma Posterior for beta equation
Intercept
Store Size
Upscale?
Value?
density
Value
4
3
2
1
0
-2.7
-2.5
-2.3
60
40
20
0
-0.03
-0.02
-0.01
0.00
0.01
0.02
2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
-0.8
-0.4
0.0
0.4
0.8
Effect
Intercept
Store Size
Upscale?
Value?

Results


Posterior Estimates and Store Segment
beta
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
1
2
3
4
alpha
SEG_VALUE_NAME
MAINSTREAM
UPSCALE
VALUE

Appendix

Deriving Posterior for Normal Model

The model is

$$\begin{aligned} y_{ij}|\alpha_i,\sigma &\sim \mathrm{N}(\alpha_i,\sigma^2), \qquad j=1,\ldots,N_i; i=1,\ldots,N, \\ \alpha_i|\mu,\sigma_\alpha &\sim \mathrm{N}(\mu,\sigma_\alpha^2), \end{aligned}$$

- To get the posterior for α_i conditional on the remaining parameters, we need to calculate

$$p(\alpha_i|y_i) = \frac{p(y_i|\alpha_i)p(\alpha_i)}{\int p(y_i|\alpha_i)p(\alpha_i)d\alpha_i},$$

where the likelihood function is

$$\begin{aligned} p(y_i|\alpha_i) &= \prod_{j=1}^{N_i} \frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{1}{2\sigma^2}(y_{ij}-\alpha_i)^2\Big) \\ &= \Big(\frac{1}{\sqrt{2\pi}\sigma}\Big)^{N_i}\exp\Big(-\frac{1}{2\sigma^2}\sum_{j=1}^{N_i}(y_{ij}-\alpha_i)^2\Big) \end{aligned}$$

- Note that we can rewrite the sum as

$$\begin{aligned}\sum_{j=1}^{N_i}(y_{ij}-\alpha_i)^2 &= \sum_j (Y_{ij}^2+\alpha_i^2-2Y_{ij}\alpha_i)\\ &= N_i(\alpha_i^2-2\alpha_i\bar{y}_i)+\sum_j y_{ij}^2\\ &= N_i(\alpha_i-\bar{y}_i)^2+\sum_j y_{ij}^2-N_i\bar{y}_i^2\end{aligned}$$

- The second term doesn't depend on α_i and will cancel out in the fraction definining the posterior. Therefore, we can write the numerator as

$$\exp\Big(-\frac{N_i}{2\sigma^2}(\alpha_i-\bar{y}_i)^2\Big)\times\exp\Big(-\frac{1}{2\sigma_\alpha^2}(\alpha_i-\mu)^2\Big)=$$
$$\exp\Big(-\frac{1}{2}\Big[\frac{N_i}{\sigma^2}(\alpha_i-\bar{y}_i)^2+\frac{1}{\sigma_\alpha^2}(\alpha_i-\mu)^2\Big]\Big)$$

- Using the “completing the square” result from week 1, slide 30, we can write the term in square brackets as

$$\frac{N_i}{\sigma^2}(\alpha_i - \bar{y}_i)^2 + \frac{1}{\sigma_\alpha^2}(\alpha_i - \mu)^2 =$$

$$\left(\frac{N_i}{\sigma^2} + \frac{1}{\sigma_\alpha^2}\right)\left[\alpha_i - \mu_{\alpha_i}\right]^2 + C,$$

  where

$$\mu_{\alpha_i} \equiv \frac{\frac{N_i}{\sigma^2}\bar{y}_i + \frac{1}{\sigma_\alpha^2}\mu}{\frac{N_i}{\sigma^2} + \frac{1}{\sigma_\alpha^2}},$$

and C is a constant that doesn’t depend on α_i.

- Collecting terms we then have the posterior for α_i:

  $$p(\alpha_i|y_i) = \frac{\exp\big(-\frac{\tau_{\alpha_i}}{2}(\alpha-\mu_{\alpha_i})^2\big)}{\int\exp\big(-\frac{\tau_{\alpha_i}}{2}(\alpha-\mu_{\alpha_i})^2\big)d\alpha_i},$$

  where $\tau_{\alpha_i} = \frac{N_i}{\sigma^2} + \frac{1}{\sigma_\alpha^2}$. We can either solve the integral in the denominator or simply realize that the numerator is proportional to the density for normal distribution. Either way we have

  $$p(\alpha_i|y_i) = \mathrm{N}(\mu_{\alpha_i}, \tau_{\alpha_i}^{-1})$$