January 2021

WORKING PAPER SERIES

2021-EQM-01

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Generalised Commensurability Properties of Efficiency Measures: Implications for Productivity Indicators

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December 19, 2020

Abstract

We analyse the role of new weak and strong commensurability conditions on efficiency measures and especially on productivity measurement. If strong commensurability fails, then a productivity index (indicator) may exhibit a homogeneity bias yielding inconsistent and contradictory results. In particular, we show that the Luenberger productivity indicator (LPI) is sensitive to proportional changes in the input-output quantities, while the Malmquist productivity index is not affected by such changes. This is due to the homogeneity degree of the directional distance function under constant returns to scale. In particular, the directional distance function only satisfies the weak commensurability axiom in general. However, if the directional distance function is a diagonally homogeneous function of the technology, then the directional distance function satisfies strong commensurability. This explains why the direction of an arithmetic mean of the observed data works well. Numerical examples and an empirical illustration are proposed. Under a translation homothetic technology, the LPI is not affected by any additive directional transformation of the observations.

Keywords: Malmquist and Luenberger productivity, Directional and proportional distance function, Weak and strong commensurability.

JEL: C43, C67, D24

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1 Introduction

The purpose of this contribution is to point out some particular properties of a recent generalization of Shephard (1970) distance function, known as the directional distance function. Distance functions are employed in consumption and production theory. Luenberger (1992 a,b) introduces the benefit function as a directional representation of preferences, which generalizes Shephard's (1970) input distance function defined in terms of the utility function. Luenberger (1995) introduces the shortage function as a transposition of the benefit function in a production context. Chambers, Chung and Färe (1996) relabel this same function as a directional distance function and since then it is commonly known by this name. The directional distance function generalizes existing distance functions by accounting for both input reductions and output expansions and it is dual to the profit function (see Chambers, Chung and Färe (1998) for details). Furthermore, the directional distance function offers flexibility due to the variety of direction vectors it allows for (see, e.g., Chambers, Färe and Grosskopf (1996)). Chambers, Chung and Färe (1996) analyze the benefit function as well as the directional distance function in detail and extend the composition rules of McFadden (1978) to these new concepts.

These distance functions have been extensively used in the economic literature to measure productivity. Based upon Shephardian distance functions as general representations of technology, discrete-time Malmquist input- and output-oriented productivity indexes - introduced by Caves, Christensen and Diewert (1982)- have been made empirically tractable by Färe et al. (1995). Meanwhile, more general primal productivity indicators have been proposed. Chambers and Pope (1996) define a Luenberger productivity indicator in terms of differences between directional distance functions (see also Chambers (2002)). Note that traditional "indexes" denote productivity measures based on ratios, while "indicators" use differences (see Diewert (2005) for a detailed discussion).

Russell (1988) introduces an important property that any technical efficiency measure should satisfy: the commensurability condition. This means that an efficiency measure should be invariant with respect to any change in the units of measurement. This condition is very natural and most of the existing technical efficiency measures (or distance functions) satisfy it. This is the case for all the Shephardian measures as well as the Färe and Lovell (1978) measure, perhaps the first non-radial measure in the literature.

Many of the new efficiency measures proposed in the literature involve some parameters in their definitions. This is the case of the measures proposed by Chambers, Chung and Färe (1996), Chavas and Cox (1999), Mehdiloozad, Sahoo and Roshdi (2014), and Briec (1999), among others. Therefore, the notion of commensurability proposed by Russell (1988) must be modified to take into account these generalized structures. A first purpose of this contribution is then to generalise the commensurability notion to account for efficiency measures involving some parameters.

The second purpose of this contribution is to show that that there may exist some problems for measuring productivity with a measure that fails to satisfy the commensurability property independently of the parameters it is depending on. For instance, the Luenberger productivity indicator that is related to the axiomatic properties of the directional distance function may yield some irrelevant and contradictory results depending on the direction that is chosen. Briec, Dervaux and Leleu (2003) show that the directional distance function satisfies a special version of the commensurability condition when the direction g is "pre-assigned". Hence, the Russell (1988) commensurability condition cannot be applied to the directional distance function. To overcome this problem, we introduce a slight modification of the commensurability condition and we distinguish between two notions called weak and strong commensurability, respectively. Strong commensurability extends the original Russell (1988) commensurability notion to the case where distance functions involve specific parameters. It is shown that the directional distance function satisfies the former but fails to satisfy the latter. However, most of the existing efficiency measures do satisfy the strong version of the commensurability condition.

We apply the formalism suggested by Russell (1988) that associates an efficiency score to any pair of production vector and production technology. In general, a distance function (efficiency measure) is defined given a production technology. If the direction is a diagonally homogeneous function depending on the technology, then a slightly modified formulation of the directional distance function satisfies the strong commensurability condition. This explains why it is useful to consider the direction of an arithmetic mean of the observed data in empirical studies, as already suggested in Chambers, Färe and Grosskopf (1996: p. 185 and 190).

More importantly, under a constant returns to scale assumption, an efficiency measure that does not satisfy the strong commensurability axiom cannot be homogenous of degree 0. In such a case, one can show the existence of a productivity bias when a firm is proportionally re-scaled. In particular, the directional distance function is homogenous of degree 1. This property has some important implications concerning the Luenberger productivity indicator when the direction g is pre-assigned. In such a case, the Luenberger productivity indicator may yield some contradictory results, while the Malmquist productivity index provides very intuitive results in any case. However, one should stress the fact that these properties are intimately related to the returns to scale structure of the production technology. If the production set satisfies a graph translation homotheticity property, then the Luenberger productivity indicator does not exhibit any bias when a firm is translated.

Our empirical study shows that when the direction is proportional under a constant returns to scale assumption, then the results are consistent with those obtained in the Malmquist productivity index case. Some irrelevant and contradictory results appear when the direction is fixed independently of the technology. Interestingly, when the direction is fixed as the arithmetic mean of all the observed data, then the results are comparable to those obtained in the proportional case, with some minor differences. This confirms the interest of the latter specification as already proposed by Chambers, Färe and Grosskopf (1996).

To develop these arguments, this contribution is structured as follows. Section 2 develops the basic definitions of the technology and the various distance functions and efficiency measures. It provides two definitions of the commensurability property refining the axiom proposed by Russell (1988). Section 3 analyzes the implication of the commensurability condition on the consistency of productivity measurement. This we do by introducing a suitable notion of homogeneity bias. Section 3 provides a numerical example reporting some contradiction and irrelevant results. A final section proposes an empirical application comparing the result in the proportional and directional cases. We end with a concluding section 5.

2 Technology and Efficiency Measures: Definitions

2.1 Technology: Definition and Assumptions

A production technology describes how inputs $x = (x_1, ..., x_m) \in \mathbb{R}_+^m$ are transformed into outputs $y = (y_1, ..., y_n) \in \mathbb{R}_+^n$. The production possibility set T is the set of all feasible inputs and outputs vectors and it is defined as follows:

$$T = \left\{(x, y) \in \mathbb{R}_+^{m+n} : \quad x \text{ can produce } y\right\}. \tag{2.1}$$

We suppose that the technology satisfies a series of usual assumptions or axioms:

(A.1) $(0, 0) \in T,\ (0, y) \in T \Rightarrow\ y = 0$ (i.e., no free lunch);

(A.2) For all $x \in \mathbb{R}_+^m$ the subset $A(x) = \{(u, y) \in T : u \leq x\}$ of dominating observations is bounded (i.e., infinite outputs cannot be obtained from a finite input vector);

(A.3) T is closed (i.e., closedness); and

(A.4) $\forall(x, y) \in T,\ (u, v) \in \mathbb{R}_+^{m+n}$ and $(x, -y) \leq (u, -v) \Rightarrow (u, v) \in T$ (i.e., strong input and output disposability).

(A.5) $\forall(x, y) \in T$, and all $\lambda > 0$ $(\lambda x, \lambda y) \in T$ (i.e., constant returns to scale assumption).

The reader can consult Färe, Grosskopf and Lovell (1994) for further comments on these axioms.

2.2 Radial and Directional Efficiency Measures

Distance functions fully characterise technology and for these reason have become standard tools for estimating efficiency and productivity relative to production frontiers. Let $\mathcal{T}$ be the class of all the production technologies satisfying the axioms $(A.1) - (A.4)$.

The Farrell (1957) radial input efficiency measure E_i is the inverse of the Shephard input distance function. It is the map $E_{\text{in}} : \mathbb{R}_+^{m+n} \times \mathcal{T} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ defined as

$$E_{\text{in}}(x, y, T) = \inf_{\lambda}\{\lambda > 0 : (\lambda x, y) \in T\}. \tag{2.2}$$

The Farrell (1957) radial output efficiency measure $E_{\text{out}} : \mathbb{R}_+^{m+n} \times \mathcal{T} \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ searches for the maximum expansion of an output vector by a scalar θ to the production frontier, i.e.:

$$E_{\text{in}}(x, y, T) = \sup_{\theta}\{\theta > 0 : (x, \theta y) \in T\}. \tag{2.3}$$

The directional distance function is a map $\overrightarrow{D} : \mathbb{R}_+^{m+n} \times \mathbb{R}_+^{m+n} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{\infty, \infty\}$ defined by:

$$\overrightarrow{D}(x, y, h, k, T) = \sup_{\delta \in \mathbb{R}}\{\delta : (x - \delta h, y + \delta k) \in T\}. \tag{2.4}$$

It looks for a simultaneous input and output variation in the direction of a pre-assigned vector $g = (h, k) \in \mathbb{R}_+^{m+n}$ compatible with the technology (see Chambers, Färe and Grosskopf (1996)). The directional distance function is a special case of the shortage function (Luenberger (1992)). It is also closely related to the translation function as developed in Blackorby and Donaldson (1980). Both functions measure the distance in a pre-assigned direction to the boundary of technology.

Färe, Grosskopf and Margaritis (2008: p. 533-534) list a variety of choices for the direction vector. This question on the choice of direction vector has led to a rather substantial amount of literature proposing a variety of directions and also trying to determine some optimal type of direction vector in an endogenous way (see, for instance, Atkinson and Tsionas (2016), Daraio and Simar (2016), Layer et al. (2020) for representative examples). It is clear that the choice of direction vector affects the value of the directional distance function as well as its relative ranking: see, e.g., Kerstens, Mounir and Van de Woestyne (2012) for an empirical illustration.

Finally, the proportional distance function is introduced by Briec (1997). In the following we consider the Hadamard product defined for all $\gamma, z \in \mathbb{R}^d$ by

$$\gamma \odot z = (\gamma_1 z_1, \cdots, \gamma_d z_d).$$

The proportional distance function is the map $D^{\propto} : \mathbb{R}_+^{m+n} \times [0,1]^{m+n} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ defined by

$$D^{\propto}(x, y, \alpha, \beta, T) = \sup_{\delta \in \mathbb{R}} \{\delta : (x - \alpha \odot x, y + \beta \odot y) \in T\}. \tag{2.5}$$

A special case corresponds to the situation where inputs and outputs are equiproportionaly modified. This implies that $\alpha = \mathbb{1}_m$ and $\beta = \mathbb{1}_n$. In such a case, we have:

$$D_T^{\propto}(x, y, T) := D_T^{\propto}(x, y; \mathbb{1}_m, \mathbb{1}_n) = \max \big\{\delta : \big((1-\delta)x, (1+\delta)y\big) \in T\big\}. \tag{2.6}$$

It is generally stated in the literature that this proportional distance function (2.5) is a special case of of the directional distance function (2.4) taking the direction $g = (-\alpha \odot x, \beta \odot y)$. Thus, we have:

$$\overrightarrow{D}(x, y; -\alpha \odot x, \beta \odot y, T) = D^{\propto}(x, y, \alpha, \beta, T). \tag{2.7}$$

However, note that in such a case g is not pre-assigned since it depends on x and y (see Russell and Schworm (2011: p. 146) for details).

In the following we establish that while the directional distance function (2.4) is homogenous of degree 1 under a constant returns to scale assumption, the proportional distance function (2.5) is homogenous of degree 0. The equiproportionate case ($\alpha = \mathbb{1}_m$ and $\beta = \mathbb{1}_n$) is established by Boussemart et al. (2003) who show relationships between the radial and the proportional measures. This confirms that these distance functions are slightly different.

Briec, Dervaux and Leleu (2003: Prop. 1) establish that under a constant returns to scale assumption, the directional distance function is homogeneous of degree 1. Thus, if the technology satisfies a constant returns to scale assumption, then:

$$\overrightarrow{D}(\lambda x, \lambda y, g, T) = \lambda \overrightarrow{D}(x, y, g, T) \quad \forall \lambda \geq 0. \tag{2.8}$$

This result means that proportionally multiplying inputs and outputs by a scalar implies an equivalent proportional multiplication of the directional distance function. It is shown further that this property has some important implications for the Luenberger productivity indicator.

An overview of the axiomatic approach to input efficiency measures is found in Russell and Schworm (2009). A survey of efficiency measures in the graph of technology or in the full $\langle$input, output$\rangle$ space, like the directional and proportional distance functions, is found in Russell and Schworm (2011).

Note that in the remainder of this contribution, we use the simplified notations: $z = (x, y)$, $g = (h, k)$ and $\gamma = (\alpha, \beta)$.

2.3 Weak and Strong Commensurability of Efficiency Measures

This subsection revisits the commensurability condition proposed by Russell (1988: p. 21).[1] In particular, we propose a new distinction between two notions of strong and weak commensurability. This distinctions is necessary since the introductions of efficiency measures

[1] The surveys of Russell and Schworm (2009, 2011) mention the commensurability condition, but provide limited analysis.

depending on some parameters. This is obviously the case of both the directional and proportional distance functions.

We first consider a set of parameters $\Theta \subset \mathbb{R}^d$.

Definition 2.1 *Let $\mathcal{S}$ be a collection of subsets of $\mathbb{R}^d$ and let Θ be a subset of $\mathbb{R}^d$. Let $f : \mathbb{R}^d \times \Theta \times \mathcal{S} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$. We say that f satisfies:*
(a) A strong commensurability condition on S if for all $d \times d$ positive diagonal matrices L we have:

$$f(Lz, \theta, LS) = f(z, \theta, S).$$

(b) A weak commensurability condition on S if for all $d \times d$ positive diagonal matrices L we have:

$$f(Lz, L\theta, LS) = f(z, L\theta, LS).$$

In the first case, one can see that the map f is invariant with respect to any change in the units of measurement and independent of the parameter θ. This definition extends the commensurability condition of Russell (1988) to the broad class of efficiency measures involving additional parameters. This is not true in the second case, where solely the units of measurement of the parameter change.

In the following, we show that the directional distance function satisfies the weak axiom of commensurability, but fails to satisfy the strong axiom. Both the radial efficiency measure and the proportional distance function do satisfy the strong commensurability axiom. It is also shown that the proportional distance function is homogenous of degree 0. Recall that the directional distance function is homogenous of degree 1.

In the next statement, we prove that the strong commensurability axiom implies homogeneity of degree 0 under a constant returns to scale assumption on technology.

Proposition 2.2 *Let $\mathcal{C}$ be the collection of all the conical subsets of $\mathbb{R}^d$. If $f : \mathbb{R}^{m+n} \times \Theta \times \mathcal{C} \longrightarrow \mathbb{R}$ satisfies the strong commensurability condition, then it is homogenous of of degree 0 in its first argument.*

Proof See Appendix A.

Proposition 2.3 *The proportional distance function* (2.5) *satisfies the strong commensurability axiom. The directional distance function* (2.4) *satisfies the weak commensurability axiom.*

Proof See Appendix A.

Proposition 2.2 implies that a map that is not homogenous of degree 0 under a constant returns to scale technology does not satisfy the strong commensurability condition.

Proposition 2.4 *The directional distance function* (2.4) *does not satisfy the strong commensurability axiom under constant returns to scale* (A.5).

Proof See Appendix A.

Proposition 2.5 *If the production technology satisfies a constant returns to scale assumption* (A.5), *then the proportional distance function* (2.5) *is homogenous of degree* 0.

Proof See Appendix A.

In the following, we suggest a slight change in the traditional definition of the directional distance function. Let $g : \mathcal{T} \longrightarrow \mathbb{R}_+^{m+n}$ be a vector valued map defined as: $g : T \mapsto \big(h(T), k(T)\big)$. Let $\mathcal{F}$ be the set of all the maps defined from $\mathcal{T}$ to $\mathbb{R}_+^{m+n}$. The map $\overrightarrow{D}^\sharp$: $\mathbb{R}_+^{m+n} \times \mathcal{F} \times \mathcal{T}$ defined as:

$$\overrightarrow{D}^\sharp(x, y; g, T) = \sup\big\{\delta : \big(x - \delta h(T), y + \delta k(T)\big) \in T\big\} \tag{2.9}$$

is called the adjusted directional distance function. Equivalently, we have:

$$\overrightarrow{D}^{\sharp}(x, y; g, T) = \overrightarrow{D}(x, y; g(T), T) \tag{2.10}$$

Notice that this definition does not involve any fixed parameter: g is just assumed to be a functional defined over $\mathcal{T}$.

In the following, it is shown that one can provide a sufficient condition for the strong commensurability of $\overrightarrow{D}^{\sharp}(x, y; g, T)$. This is done by defining a suitable notion of a diagonally homogenous map. A map $g : \mathcal{T} \longrightarrow \mathbb{R}_+^{m+n}$ is diagonally homogenous if for all definite positive diagonal matrix $g(LT) = Lg(T)$. We come now to the following sufficient condition.

Proposition 2.6 *If g is diagonally homogenous, then the adjusted directional distance function is strongly commensurable.*

Proof See Appendix A.

It is not clear that the diagonal homogeneity of g is a necessary condition for strong commensurability. For example, the proportional distance function is strongly commensurable though the direction is not fixed. This condition, however, provides a technical argument to the specification proposed by Chambers, Färe and Grosskopf (1996) in a nonparametric context. Suppose that $A = \{(x_1, y_1), (x_2, y_2), ..., (x_\ell, y_\ell)\}$ is a set of ℓ observed production vectors. From the non-parametric specification proposed by Charnes, Cooper and Rhodes (1978) a constant returns to scale production technology can be reconstructed from the observed data as follows:

$$T_A = \Big\{(x, y) \in \mathbb{R}_+^{m+n} : x \geq \sum_{k\in[\ell]} t_k x_k, y \leq \sum_{k\in[\ell]} t_k y_k, t \in \mathbb{R}_+^{\ell}\Big\}. \tag{2.11}$$

Taking the direction

$$g = \Big(\frac{1}{\ell}\sum_{k\in[\ell]} x_k, \frac{1}{\ell}\sum_{k\in[\ell]} y_k\Big), \tag{2.12}$$

the directional distance function is independent of any change in the units of measurements. This property can be related to Proposition 2.6. Let us denote $\mathcal{P} = \langle \mathbb{R}_+^{m+n} \rangle$ the set of all the finite parts of $\mathbb{R}_+^{m+n}$. Let us consider the collection of production technologies:

$$\mathcal{T}_c = \big\{ T_A : A \in \mathcal{P} \rangle \big\} \tag{2.13}$$

Actually, note that two distinct data sets may yield the same technology. To overcome such a problem, let us introduce the equivalence relation $A \sim A' \iff T_A = T_{A'}$ and let $\tilde{\mathcal{P}} = \mathcal{P} \backslash \sim$ the set of the corresponding equivalence classes, that is the quotient set. Let $\Psi : \mathcal{T}_c \longrightarrow \tilde{\mathcal{P}}$ which associates to any $T \in \mathcal{T}_c$ some $\tilde{A} \in \mathcal{P}$ such that $T_A = T$. By construction, for all positive diagonal matrices L, we have $T_{LA} = LT_A$ and this implies that $\Psi(LT_A) = \Psi(T_{LA}) = L\tilde{A} = L\Psi(T_A)$. It follows that $\Psi(LT) = L\Psi(T)$. Now, let us consider the map $m^\sharp : \tilde{P} \longrightarrow \mathbb{R}_+^{m+n}$ that associates to any equivalence class the arithmetic mean of some arbitrary element of this equivalence class. Namely, $m^\sharp\tilde{(}A) = \frac{1}{|A^\sharp|}\sum_{a \in A^\sharp} a$ where for any $\tilde{A}$, $A^\sharp$ is an arbitrary element of $\tilde{A}$. We retrieve the approach proposed by Chambers, Färe and Grosskopf (1996) and Färe, Grosskopf and Margaritis (2008) by defining the function $g : \mathcal{T}_c \longrightarrow \mathbb{R}_+^n$ as:

$$g(T) = m^\sharp\big(\Psi(T)\big). \tag{2.14}$$

Since $\Psi(LT) = L\Psi(T)$ and $m^\sharp(L\Psi(T)) = Lm^\sharp(\Psi(T))$, we deduce that $g(LT) = Lg(T)$.

Suppose that A is a subset of $\mathbb{R}_{++}^{m+n}$, one could assume that the direction is a generalized mean of the observed production vectors with for all $(i,j) \in [m] \times [n]$

$$h_i = \big(\sum_{k \in \ell} {x_{k,i}}^{\alpha_i} \big)^{\frac{1}{\alpha_i}} \text{ and } k_j = \big(\sum_{k \in \ell} {y_{k,j}}^{\beta_j} \big)^{\frac{1}{\beta_j}} \tag{2.15}$$

and $\alpha_i, \beta_j \neq 0$ for all i, j. For example, if $\alpha_i, \beta_j \longrightarrow \infty$ and $\alpha_i, \beta_j \longrightarrow -\infty$, then we have the limit case:

$$g = \big(\bigvee_{k \in [\ell]} x_k, \bigvee_{k \in [\ell]} y_k \big) \text{ and } g = \big(\bigwedge_{k \in [\ell]} x_k, \bigwedge_{k \in [\ell]} y_k \big), \tag{2.16}$$

where $\vee$ and $\wedge$ are the sup and inf lattice operator, respectively. Note that these results do no contradict Proposition 2.4. In Proposition 2.4 and 2.2, the parameters (direction) are assumed to be independent of T.

3 Productivity Indices and Indicators: Implications of Commensurability

Recently, quite a bit of attention has been devoted to so-called theoretical productivity indices (see Russell (2018)). A theoretical productivity index is defined on the assumption that the technology is known and non-stochastic, but unspecified and thus most often approximated by a nonparametric specification of technology using some form of efficiency measure. The foundational concepts are on the one hand the Malmquist productivity index (Caves, Christensen and Diewert (1982)) and on the other hand the Hicks-Moorsteen productivity index (Bjurek (1996)). While the Malmquist productivity index is fundamentally a measure of the shift of the production frontier, the Hicks-Moorsteen productivity index is a ratio of an aggregate output index over an aggregate input index. Thus, the Malmquist productivity index measures local technical change (i.e., the local shifts in the production frontier), while the Hicks-Moorsteen productivity index has a Total factor Productivity (TFP) interpretation. Kerstens and Van de Woestyne (2014) empirically illustrate that the Malmquist productivity index offers a poor approximation to the Hicks-Moorsteen TFP index in terms of the resulting distributions and that these problems persist under constant as well as under variable returns to scale.

Chambers, Färe and Grosskopf (1996) introduce the Luenberger productivity indicator as a difference-based indicator of directional distance functions (see Chambers (2002)). This generalizes the Malmquist productivity index that is most often either input- or output-oriented in the graph orientation. Briec and Kerstens (2004) define a Luenberger-Hicks-Moorsteen TFP indicator using input- or output-oriented directional distance functions.

Luenberger productivity indicators and Luenberger-Hicks-Moorsteen productivity indicators are also empirically quite different under constant as well as under variable returns to scale (see Kerstens, Shen, and Van de Woestyne (2018)).

Notice that, while these productivity indices and indicators do not require constant returns to scale specifications of technologies, the large majority of empirical applications still imposes such a restrictive assumption.[2]

3.1 Productivity Indices and Indicators: Definitions

At each time period let us denote T_t the production technology at the time period t and suppose that T_t satisfies axioms $(A.1) - (A.4)$. Productivity indices and indicators aim to evaluate productivity changes between discrete time periods and can be decomposed to analyse the origins in the productivity changes.

The Malmquist productivity index –introduced by Caves, Christensen and Diewert (1982)– can be based on the radial output measure (2.3). In particular, Caves, Christensen and Diewert (1982) suggest using a geometric mean between a period t Malmquist productivity index $M_t^{\text{out}}(z_t, z_{t+1}, T_t)$:

$$M_t^{\text{out}}(z_t, z_{t+1}, T_t) = \frac{E^{\text{out}}(z_t, T_t)}{E^{\text{out}}(z_{t+1}, T_t)} \tag{3.1}$$

and a period $t+1$ Malmquist productivity index $M_{t+1}^{\text{out}}(z_t, z_{t+1}, T_{t+1})$:

$$M_{t+1}^{\text{out}}(z_t, z_{t+1}, T_{t+1}) = \frac{E^{\text{out}}(z_t, T_{t+1})}{E^{\text{out}}(z_{t+1}, T_{t+1})}. \tag{3.2}$$

Similarly, Färe et al. (1995) define the output-oriented Malmquist productivity index as the

[2]We provide some qualitative evidence for this claim. A Google Scholar search on 2 December 2020 yields about 829 results for the search term "Luenberger productivity indicator". This same search term in conjunction with the search term "constant returns to scale" obtains 375 hits, while this same search term in conjunction with the search term "variable returns to scale" leads to 329 results.

geometric mean of (3.1) and (3.2) as follows:

$$M^{\text{out}}(z_t, z_{t+1}, T_t, T_{t+1}) = \left[\frac{E^{\text{out}}(z_{t+1}, T_t)}{E^{\text{out}}(z_t, T_t)} \frac{E^{\text{out}}(z_{t+1}, T_{t+1})}{E^{\text{out}}(z_t, T_{t+1})} \right]^{1/2}. \tag{3.3}$$

This productivity index allows to analyze productivity changes between different periods and it can be multiplicatively decomposed into efficiency changes (EC) and technological changes (TC):

$$EC = \frac{E^{\text{out}}(x_t, y_t, T_t)}{E^{\text{out}}(x_{t+1}, y_{t+1}, T_{t+1})}, \tag{3.4}$$

and

$$TC = \Big(\frac{E^{\text{out}}(z_{t+1}, T_{t+1})}{E^{\text{out}}(z_{t+1}, T_t)} \frac{E^{\text{out}}(z_t, T_{t+1})}{E^{\text{out}}(z_t, T_t)} \Big)^{\frac{1}{2}}, \tag{3.5}$$

where EC represents the variation in efficiency between two periods and concerns the relative efficiency in the management of input and output quantities over time, while TC captures technological changes (i.e., productivity growth not explained by changes in input and output quantities).

The Luenberger productivity indicator based on the directional distance function (2.4) is defined as follows:

$$\begin{aligned} L(z_t, z_{t+1}, g, T_t, T_{t+1}) = \frac{1}{2} \Big[& \overrightarrow{D}(z_t, g, T_{t+1}) - \overrightarrow{D}(z_{t+1}, g, T_{t+1}) \\ & + \overrightarrow{D}(z_t, g, T_t) - \overrightarrow{D}(z_{t+1}, g, T_t) \Big]. \end{aligned} \tag{3.6}$$

This Luenberger productivity indicator can be additively decomposed into efficiency changes (EC) and technological changes (TC):

$$EC_t = \overrightarrow{D}(z_t, g, T_t) - \overrightarrow{D}(z_{t+1}, g, T_{t+1}) \tag{3.7}$$

and

$$TC_t = \frac{1}{2}\Big[\overrightarrow{D}(z_{t+1}, g, T_{t+1}) - \overrightarrow{D}(z_{t+1}, g, T_t) + \overrightarrow{D}(z_t, g, T_{t+1}) - \overrightarrow{D}(z_t, g, T_t)\Big]. \tag{3.8}$$

where the interpretation follows the one provided for the Malmquist productivity index (3.3).

Paralleling this definition, Boussemart et al. (2003) define a proportional Luenberger indicator based on the proportional directional distance function (2.5) as:

$$\begin{aligned} L^{\propto}(z_t, z_{t+1}, \gamma) = \frac{1}{2}\Big[&D^{\propto}(z_t, \gamma, T_{t+1}) - D^{\propto}(z_{t+1}, \gamma, T_{t+1}) \\ &+ D(z_t, \gamma, T_t) - D^{\propto}(z_{t+1}, \gamma, T_t)\Big]. \end{aligned} \tag{3.9}$$

The decomposition defined in (3.7) and (3.8) is applicable to this proportional case as well.

Note that recently Pastor, Lovell and Aparicio (2020) manage to transgress the distinction between technology and TFP indices outlined above. These authors define a new graph oriented inefficiency measure based on the proportional distance function under constant returns to scale and use it to define a new Malmquist productivity index that has a TFP interpretation.

3.2 Productivity Indices and Indicators: Homogeneity Bias

This subsection analyzes the impact of the commensurability condition on productivity measurement. We define a suitable notion of homogeneity bias for productivity indices and indicators. We also establish a relation between such a notion and the commensurability of the efficiency measure upon which a productivity index or indicator is based.

Definition 3.1 *Let Θ be a subset of $\mathbb{R}^d$. Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$. Let $T_t, T_{t+1} \in \mathcal{T}$. For all, $(z_t, z_{t+1}, \theta) \in T_t \times T_{t+1} \times \Theta$ and all $\lambda > 0$:*

$$B_t(z_t, z_{t+1}, \phi, \theta, \lambda) = \phi(z_t, z_{t+1}, \theta, T_t, T_{t+1}) - \phi(\lambda z_t, z_{t+1}, \theta, T_t, T_{t+1})$$

is called the homogeneity bias of ϕ in period t;

$$B_{t+1}(z_t, z_{t+1}, \phi, \theta, \lambda) = \phi(z_t, z_{t+1}, \theta, T_t, T_{t+1}) - \phi(z_t, \lambda z_{t+1}, \theta, T_t, T_{t+1})$$

is called the homogeneity bias of ϕ in period $t+1$.

The homogeneity bias measures the change of a productivity index or indicator when a firm is proportionally re-scaled at the time periods t and $t+1$. Since productivity is essentially based upon the ratio between the outputs and the inputs involved in the production process, one could expect that a productivity index or indicator should be invariant with respect to such a re-scaling when the technology satisfies a constant returns to scale assumption.

In the case of the Luenberger productivity indicator based on the directional distance function (2.4) the homogeneity bias in t is then defined as:

$$B_t(z_t, z_{t+1}, L, g, \lambda) = L(z_t, z_{t+1}, g, , T_t, T_{t+1}) - L(\lambda z_t, z_{t+1}, g, , T_t, T_{t+1}); \tag{3.10}$$

and the same homogeneity bias at the time period $t+1$ is defined as:

$$B_{t+1}(z_t, z_{t+1}, L, g, \lambda) = L(z_t, z_{t+1}, g, , T_t, T_{t+1}) - L(z_t, \lambda z_{t+1}, g, , T_t, T_{t+1}). \tag{3.11}$$

In the case of the proportional Luenberger productivity indicator based on the proportional directional distance function (2.5) we have the homogeneity bias in t:

$$B_t(z_t, z_{t+1}, L^{\propto}, \gamma) = L^{\propto}(z_t, z_{t+1}, \gamma, T_t, T_{t+1}) - L^{\propto}(\lambda z_t, z_{t+1}, \gamma, T_t, T_{t+1}) \tag{3.12}$$

and the homogeneity bias in $t+1$:

$$B_{t+1}(z_t, z_{t+1}, L^{\propto}, \gamma) = L^{\propto}(z_t, z_{t+1}, \gamma, T_t, T_{t+1}) - L^{\propto}(z_t, \lambda z_{t+1}, \gamma, T_t, T_{t+1}). \tag{3.13}$$

Finally, the output-oriented Malmquist productivity index is independent of any parameter. Hence, for all $\theta \in \mathbb{R}^d$, we have the homogeneity bias in t:

$$B_t(z_t, z_{t+1}, M^{\text{out}}, \theta, \lambda) = M^{\text{out}}(z_t, z_{t+1}, T_t, T_{t+1}) - M^{\text{out}}(\lambda z_t, z_{t+1} T_t, T_{t+1}), \tag{3.14}$$

and the homogeneity bias in $t+1$:

$$B_{t+1}(z_t, z_{t+1}, M^{\text{out}}, \theta, \lambda) = M^{\text{out}}(z_t, z_{t+1}, T_t, T_{t+1}) - M^{\text{out}}(z_t, \lambda z_{t+1} T_t, T_{t+1}). \tag{3.15}$$

The next result shows that given any efficiency measure satisfying the strong commensurability axiom, the corresponding productivity index or indicator has a null homogeneity bias.

Proposition 3.2 *Let Θ be a subset of $\mathbb{R}^d$. Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$. Let $T_t, T_{t+1} \in \mathcal{T}$ and assume that T_t and T_{t+1} satisfy a constant returns to scale assumption. If ϕ satisfies the strong commensurability condition, then for all $(z_t, z_{t+1}, \theta) \in T_t \times T_{t+1} \times \Theta$ and all $\lambda > 0$,*

$$B_t(z_t, z_{t+1}, \phi, \theta, \lambda) = B_{t+1}(z_t, z_{t+1}, \phi, \theta, \lambda) = 0.$$

Proof See Appendix A.

In the following, let:

$$B_{t,t+1}(z_t, z_{t+1}, \phi) = B_t(z_t, z_{t+1}, \phi) + B_{t+1}(z_t, z_{t+1}, \phi). \tag{3.16}$$

denote the sum of the homogeneity bias in time period t and in time period $t+1$. The next result shows that the homogeneity bias of the proportional Luenberger productivity indicator (3.9) and Malmquist productivity index (3.3) are null, though this is not the case for the Luenberger indicator (3.6) based on the directional distance function for which an explicit form of the bias can be provided.

Corollary 3.3 *Suppose that at each time period T_t and T_{t+1} satisfy $(A.1) - (A.4)$ and a constant returns to scale assumption $(A.5)$. For all $(z_t, z_{t+1}) \in T_t \times T_{t+1}$ we have:*

(a) $B_t(z_t, z_{+1}, M^{\text{out}}, \theta, \lambda) = B_{t+1}(z_t, z_{+1}, M^{\text{out}}, \theta, \lambda) = 0$*;*

(b) $B_t(z_t, z_{+1}, L^{\propto}, \gamma, \lambda) = B_{t+1}(z_t, z_{+1}, L^{\propto}, \alpha, \beta, \lambda) = 0$*;*

(c) We have the identities:

$$B_t(z_t, z_{t+1}, g, \lambda) = \frac{1-\lambda}{2}\big[\overrightarrow{D}_{t+1}(z_t; g) + \overrightarrow{D}_t(z_t; g)\big].$$

$$B_{t+1}(z_t, z_{t+1}, g, \lambda) = \frac{\lambda-1}{2}\big[\overrightarrow{D}(z_{t+1}, g, T_{t+1}) + \overrightarrow{D}(z_{t+1}, g, T_t)\big].$$

and

$$B_{t,t+1}(z_t, z_{t+1}, g, \lambda) = \frac{1-\lambda}{2} L(z_t, z_{t+1}; g, T_t, T_{t+1}).$$

Proof See Appendix A.

Under a constant returns to scale assumption on technology, the Malmquist productivity index and the proportional Luenberger productivity indicator are not affected by a proportional modification of one of the observations. However, this is not true in the case of the Luenberger productivity indicator based on the directional distance function. Remark that Chambers, Färe and Grosskopf (1996: p. 184) in their seminal article do impose a constant returns to scale assumption on technology.

3.3 Translation Homothetic Bias

In this subsection, it is shown that the things are very different when one assume a graph translation homothetic property of the technology. First, notice that it is difficult to define the commensurability axiom from an additive viewpoint. This is due to the fact that the key axioms $(A.1) - (A.4)$ are not preserved via a translation of the technology. However, it is interesting to analyze the impact of the graph translation homotheticity on the structure of the Luenberger productivity indicator (3.6).

We point to the fact that if the technology is graph translation homothetic, then the Luenberger productivity indicator with a fixed direction does not suffer from the shortcomings due to its additive structure. A production technology T is translation homothetic in the direction of g if for all $z \in T$ and all $\delta \in \mathbb{R}$ such that $z + \delta g \in \mathbb{R}_+^{m+n}$, we have $z + \delta g \in T$. It was shown by Briec and Kerstens (2004) that under an assumption of graph translation homotheticity:

$$D(z + \delta g, T) = D(z, T). \tag{3.17}$$

This means that the directional distance function is translation invariant.

Paralleling our earlier definition we define the translation homothetic bias as follows.

Definition 3.4 *Let Θ be a subset of $\mathbb{R}^d$. Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \times \Theta \times \mathcal{T} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$. Let $T_t, T_{t+1} \in \mathcal{T}$. For all, $(z_t, z_{t+1}, \theta) \in T_t \times T_{t+1} \times \Theta$ and all $\lambda > 0$:*

$$TB_t(z_t, z_{t+1}, \phi, \theta, \delta) = \phi(z_t, z_{t+1}, \theta, T_t, T_{t+1}) - \phi(z_t + \delta g, z_{t+1}, \theta, T_t, T_{t+1})$$

is called the translation homothetic bias of ϕ in period t;

$$TB_{t+1}(z_t, z_{t+1}, \phi, \theta, \delta) = \phi(z_t, z_{t+1}, \theta, T_t, T_{t+1}) - \phi(z_t, z_{t+1} + \delta g, \theta, T_t, T_{t+1})$$

is called the translation homothetic bias of ϕ in period $t + 1$.

In the case of the Luenberger productivity indicator (3.6) the translation homothetic bias in t is then defined as:

$$TB_t(z_t, z_{t+1}, L, g, \delta) = L(z_t, z_{t+1}, g, , T_t, T_{t+1}) - L(z_t + \delta g, z_{t+1}, g, T_t, T_{t+1}); \tag{3.18}$$

at the time period $t + 1$

$$TB_{t+1}(z_t, z_{t+1}, L, g, \delta) = L(z_t, z_{t+1}, g, , T_t, T_{t+1}) - L(z_t, z_{t+1} + \delta g, g, T_t, T_{t+1}). \tag{3.19}$$

It follows that if the production technology is graph translation homothetic at both the time periods t and $t+1$, then:

$$TB_t(z_t, z_{t+1}, L, g, \delta) = TB_{t+1}(z_t, z_{t+1}, L, g, \delta) = 0. \tag{3.20}$$

This means that the translation homotheticity bias is zero.

4 Numerical Examples

In the following we compare the output-oriented Malmquist productivity index and the Luenberger productivity indicator. To do so we introduce a numerical example and we show that the Luenberger productivity indicator can yield inconsistent results because of the structure of the directional distance function under a constant returns to scale assumption.

4.1 Output-Oriented Measures

We suppose that the technology is two-dimensional and that $T_0 = \{(x, y) : y \leq x\}$ and $T_1 = \{(x, y) : y \leq 2x\}$, which implies a CRS assumption at each time period. Moreover, we assume that: $z_0 = (x_0, y_0) = (1, \frac{4}{5})$ and $z_1 = (x_1, y_1) = (1, \frac{5}{4})$.

Let us compute the radial output-oriented efficiency measure at each time period:

(i) $E^{\text{out}}(z_1, T_0) = \sup\{\theta : (1, \theta\frac{5}{4}) \in T_0\} = \sup\{\theta : \theta\frac{5}{4} \leq 1\}$. Clearly, we have $\frac{5}{4}\theta^\star = 1$ and $E^{\text{out}}(z_1, T_1) = \theta^\star = \frac{4}{5}$;

(ii) $E^{\text{out}}(z_0, T_0) = \sup\{\theta : \theta\frac{4}{5} \leq 1\}$, hence $E^{\text{out}}(z_0, T_0) = \theta^\star = \frac{5}{4}$;

(iii) $E^{\text{out}}(z_1, T_1) = \sup\{\theta : \theta\frac{5}{4} \leq 2\}$. Clearly, we have $\frac{5}{4}\theta^\star = 2$ and $E^{\text{out}}(z_1, T_1) = \theta^\star = \frac{8}{5}$;

(iv) $E^{\text{out}}(z_0, T_1) = \sup\{\theta : \theta\frac{4}{5} \leq 2\}$, hence $E^{\text{out}}(z_1, T_1) = \theta^\star = \frac{5}{2}$.

Inserting these results leads to the following output-oriented Malmquist productivity index (3.3):

$$M^{\text{out}}(z_0, z_1, T_0, T_1) = \left(\frac{5}{4}.\frac{5}{4}.\frac{5}{8}.\frac{2}{5}\right)^{\frac{1}{2}} = 1.56 \tag{4.1}$$

This result indicates a productivity gain between $t = 0$ and $t = 1$, since indeed the Malmquist productivity index is > 1.

Now we suppose that $\lambda = 10$. It follows that we consider the production vector at $t = 1$ defined as:

$$z_1' = 10(x_1, y_1) = (10, \frac{25}{2}).$$

Although in the first and the second case the observation do not use the same level of inputs and outputs, these observations have the same efficiency scores. Thus, the productivity index should yield the same result. This is indeed the case for the Malmquist productivity index, since it is invariant with respect to a proportional change of the second observation.

$$E(z_1', T_0) = \frac{4}{5}, E(z_0, T_0) = \frac{5}{4}, E(z_1', T_1) = \frac{8}{5}, E(z_0, T_1) = \frac{5}{2}$$

Hence, inserting these results we also obtain:

$$M^{\text{out}}(z_0, 10z_1, T_0, T_1) = \Big(\frac{5}{4}.\frac{5}{4}.\frac{5}{2}.\frac{5}{8}\big)^{\frac{1}{2}} = 1.56.$$

Thus, a proportional multiplication of z_1 by 10 does not affect the output-oriented Malmquist productivity index. This is normal because the productivity does not change.

But, for the Luenberger productivity indicator (3.6) such proportional change in input and output quantities does affect the indicator, thereby introducing a bias. Recall that as in the Malmquist productivity index case, the production vectors are $z_0 = (1, \frac{4}{5})$ and $z_1 = (1, \frac{5}{4})$. Let us now consider the Luenberger productivity indicator with the direction of $g = (0, 1)$. This is an output-oriented Luenberger productivity indicator which allows to be compared with the output-oriented Malmquist productivity index:

(i) $\overrightarrow{D}(x_0, y_t, 0, 1, T_1) = \sup\{\delta : (1, \frac{4}{5} + \delta) \in T_1\}$ which implies that $\frac{4}{5} + \delta^\star = 2$ and $\overrightarrow{D}(x_0, y_0, 0, 1, T_1) = \delta^\star = \frac{6}{5}$;

(ii) $\overrightarrow{D}(x_1, y_1, 0, 1, T_1) = \sup\{\delta : (1, \frac{5}{4} + \delta) \in T_1\}$. Hence, $\overrightarrow{D}(x_1, y_1, 0, 1, T_1) = \frac{3}{4}$;

(iii) $\overrightarrow{D}(x_0, y_0, 0, 1, T_0) = \sup\{\delta : (1, \frac{4}{5}+\delta) \in T_t\}$. Hence, $\frac{4}{5}+\delta = 1$ and $\overrightarrow{D}(x_0, y_0, 0, 1, T_0) = \frac{1}{5}$;

(iv) $\overrightarrow{D}(x_1, y_1, 0, 1, T_0) = \sup\{\delta : (1, \frac{5}{4} + \delta) \in T_0\}$. Hence, $\overrightarrow{D}(x_1, y_1, 0, 1, T_0) = -\frac{1}{4}$.

Inserting these results leads to the following output-oriented Luenberger productivity indicator:

$$L^{\text{out}}(z_0, z_1, 0, 1, T_0, T_1) = \frac{1}{2}\left[\frac{6}{5} - \frac{3}{4} + \frac{1}{5} + \frac{1}{4}\right] = \frac{1}{2}.\frac{9}{10} = 0.45 \tag{4.2}$$

Since this indicator is larger than zero, this suggests a productivity gain between periods $t = 0$ and $t = 1$.

Now in the second case, the observation is again characterized by the following conditions: $z_0 = (x_0, y_0) = (1, \frac{4}{5})$ and $z_1' = 10(x_1, y_1) = (10, \frac{25}{2})$.

Again, we compute the output-oriented directional distance function at each time period:
(i) $\overrightarrow{D}(x_0, y_0, 0, 1, T_1) = \sup\{\delta : (1, \frac{4}{5} + \delta) \in T_1\}$ which implies that $\frac{4}{5} + \delta = 2$ and $\delta = \frac{6}{5}$;
(ii) $\overrightarrow{D}(x_1, y_1, 0, 1, T_1) = \sup\{\delta : (10, \frac{25}{2} + \delta) \in T_{t+1}\}$ which implies that $\frac{25}{2} + \delta = 20$ so $\delta = \frac{15}{2}$;
(iii) $\overrightarrow{D}(x_0, y_0, 0, 1, T_0) = \sup\{\delta : (1, \frac{4}{5} + \delta) \in T_t\}$ which implies that $\frac{4}{5} + \delta = 1$ and therefore $\delta = \frac{1}{5}$;
(iv) $\overrightarrow{D}(x_1, y_1, 0, 1, T_0) = \sup\{\delta : (10, \frac{25}{2} + \delta) \in T_t\}$ Thus, $\frac{25}{2} + \delta = 10$ so $\delta = \frac{-5}{2}$.

Collecting again these results leads now to the following output-oriented Luenberger productivity indicator result:

$$L(z_0, 10z_1, g, T_0, T_1) = \frac{1}{2}\left[\frac{6}{5} - \frac{15}{2} + \frac{1}{5} + \frac{5}{2}\right] = \frac{1}{2}.(\frac{-18}{5}) = -1.8 \tag{4.3}$$

Remark that the output-oriented Luenberger productivity indicator is now negative (-1.8) while it was initially positive (0.45). Thus, the Luenberger productivity indicator initially suggests a productivity gain, while it now indicates a productivity loss. However, this is a contradiction: in both cases the observation should have the same productivity. Therefore, the Luenberger productivity indicator is very sensitive to proportional changes in quantities and it does not allow to estimate changes in efficiency.

4.2 Graph-Oriented Measures

The following Figure 1 illustrates the idea behind the homogeneity bias. When a production vector is proportionally expanded the directional distance function is increasing. Hence, the Luenberger productivity indicator may be significantly modified.

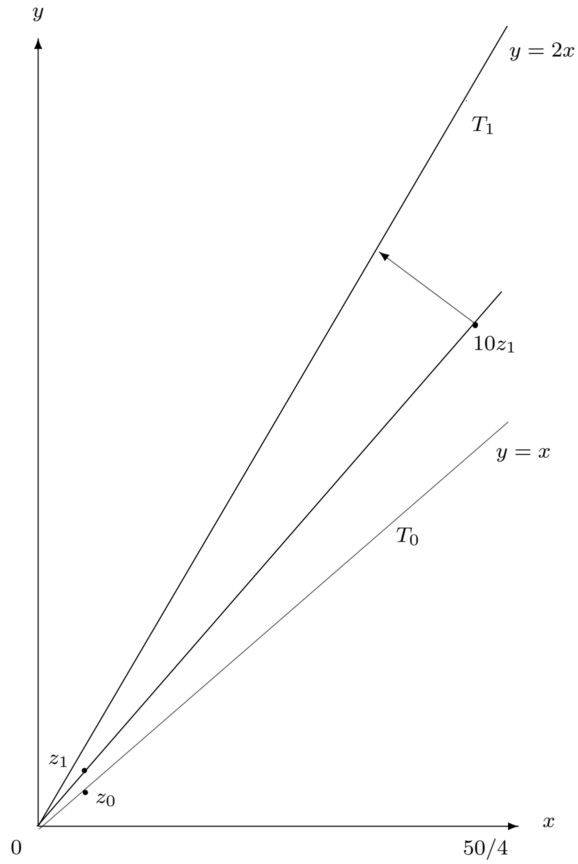

Figure 1: Homogeneity Bias

Consider the production vectors $z_0 = (1, \frac{3}{4})$ and $z_1 = (1, \frac{5}{4})$.

Let us compute the Luenberger productivity indicator based on the proportional distance function (3.9) as introduced by Boussemart et al. (2003). We consider the case where $\alpha = \mathbb{1}_m$ and $\beta = \mathbb{1}_n$. At each time periods t, s we have

$$D^{\propto}(x_t, y_t, T_s) = \max_{\delta}\{\delta : ((1-\delta)x_t, (1+\delta)y_t) \in T_s\} \tag{4.4}$$

Under a constant returns to scale assumption, we have the relation:

$$D^{\propto}(x_t, y_t, T_s) = \frac{E^{\text{out}}(x_t, y_t, T_s) - 1}{E^{\text{out}}(x_t, y_t, T_s) + 1} \tag{4.5}$$

Boussemart et al. (2003) define the Luenberger productivity indicator based on the proportional distance function as follows:

$$\begin{aligned} L^{\propto}(x_t, y_t, x_{t+1}, y_{t+1}, T_t, T_t + 1) = \frac{1}{2}[&D^{\propto}(x_t, y_t, T_t) - D^{\propto}(x_{t+1}, y_{t+1}, T_t) \\ &+ D^{\propto}(x_t, y_t, T_{t+1}) - D^{\propto}(x_{t+1}, y_{t+1}, T_{t+1})] \end{aligned} \tag{4.6}$$

Since the proportional distance function is homogenous of degree 0, we obviously have for all $\lambda > 0$:

$$L^{\propto}(x_t, y_t, x_{t+1}, y_{t+1}, T_t, T_{t+1}) = L^{\propto}(x_t, y_t, \lambda x_{t+1}, \lambda y_{t+1}, T_t, T_{t+1}). \tag{4.7}$$

Moreover, from Boussemart et al. (2003), we also have under a constant returns to scale assumption, the second order approximation:

$$L^{\propto}(x_t, y_t, x_{t+1}, y_{t+1}, T_t, T_{t+1}) \approx \frac{1}{2} \ln \Big(M^{\text{out}}(x_t, y_t, x_{t+1}, y_{t+1}, T_t, T_{t+1}) \Big) \tag{4.8}$$

Assuming that $z_0 = (1, \frac{4}{5})$, $z_1 = (1, \frac{5}{4})$, one can compute the proportional distance functions at each time period as follows:

(i) $D^{\propto}(x_0, y_0, T_1) = \max\{\delta : (1 - \delta, \frac{4}{5} + \frac{4}{5}\delta) \in T_1\}$. Hence, we should have $\frac{4}{5} + \frac{4}{5}\delta = 2(1 - \delta)$ and $\delta = \frac{3}{7}$;

(ii) $D^{\propto}(x_1, y_1, T_1) = \max\{\delta : (1 - \delta, \frac{5}{4} + \frac{5}{4}\delta) \in T_1\}$ so $\frac{5}{4} + \frac{5}{4}\delta = 2(1 - \delta)$ and $\delta = \frac{3}{13}$;

(iii) $D^{\propto}(x_0, y_0, T_0) = \max\{\delta : (1 - \delta, \frac{4}{5} + \frac{4}{5}\delta) \in T_0\}$. Thus, $\frac{4}{5} + \frac{4}{5}\delta = 1 - \delta$ and $\delta = \frac{1}{7}$;

(iv)$D^{\propto}(x_1, y_1, T_0) = \max\{\delta : (1 - \delta, \frac{5}{4} + \delta) \in T_0\}$. Hence, we deduce $\delta = -\frac{1}{9}$

Inserting these results yields the following proportional Luenberger productivity indicator:

$$L^{\propto}(z_0, z_1, T_0, T_1) = \frac{1}{2}\Big[\frac{5}{11} - \frac{3}{13} + \frac{1}{7} + \frac{1}{9}\Big] = 0.238 \tag{4.9}$$

Suppose now that $z_1 = (10, \frac{25}{2})$, since the proportional distance function is homogenous of degree 0, we have:

$$L^{\propto}(z_0, z_1, T_0, T_1) = L^{\propto}(z_0, 10z_1) = 0.238 \tag{4.10}$$

Therefore, the productivity change is the same. The results are parallel to those obtained using the output-oriented Malmquist productivity index.

Let us now compute the Luenberger productivity indicator based on the directional distance function (3.6) as follows:

(*i*) $\overrightarrow{D}(x_0, y_0, 1, 1, T_1) = \sup\{\delta : (1-\delta, \frac{3}{4}+\delta) \in T_1\}$. Thus so $\frac{3}{4}+\delta = 2(1-\delta)$ and $\delta = \frac{5}{12}$;

(*ii*) $\overrightarrow{D}(x_1, y_1, 1, 1, T_1) = \sup\{\delta : (1-\delta, \frac{5}{4}+\delta) \in T_1\}$ thus $\delta = \frac{3}{12}$;

(*iii*) $\overrightarrow{D}(x_0, y_0, 1, 1, T_0) = \sup\{\delta : (1-\delta, \frac{3}{4}+\delta) \in T_0\}$ so $\frac{3}{4}+\delta = 1-\delta$ and $\delta = \frac{1}{8}$;

(*iv*) $\overrightarrow{D}(x_1, y_1, 1, 1, T_0) = \sup\{\delta : (1-\delta, \frac{5}{4}+\delta) \in T_0\}$, thus $\delta = -\frac{1}{8}$.

Inserting these results into the Luenberger productivity indicator yields:

$$L(z_0, z_1, g, T_0, T_1) = \frac{1}{2}\Big[\frac{5}{12} - \frac{3}{12} + \frac{1}{8} + \frac{1}{8}\Big] = \frac{1}{2}(\frac{5}{12}) = 0.21 \tag{4.11}$$

Thus, this indicator being larger than > 0 suggests a productivity gain between periods $t = 0$ and $t = 1$.

Now in the second case the production vectors become $z_0 = (x_0, y_0) = (1, \frac{3}{4})$ and $z_1' = 10(x_1, y_1) = (10, \frac{25}{2})$.

The directional distance functions in each time period are now:

(*i*) $\overrightarrow{D}(x_0, y_0, h, k, T_1) = \frac{5}{12}$;

(*ii*) $\overrightarrow{D}(x_1, y_1, h, k, T_1) = \sup\{\delta : (10-\delta, \frac{25}{2}+\delta) \in T^1\}$ so $\frac{25}{2}+\delta = 2(10-\delta)$ and $\delta = \frac{15}{6}$;

(*iii*) $\overrightarrow{D}(x_0, y_0, h, k, T_0) = \frac{1}{8}$;

(*iv*) $\overrightarrow{D}(x_1, y_1, h, k, T_0) = \sup\{\delta : (10-\delta, \frac{25}{2}+\delta) \in T^0\}$ so $\delta = -\frac{5}{4}$.

Collecting these results leads to the following Luenberger productivity indicator:

$$L(z_0, 10z_1, g, T_0, T_1) = \frac{1}{2}\Big[\frac{5}{12} - \frac{15}{6} + \frac{1}{8} + \frac{5}{4}\Big] = \frac{1}{2}(-\frac{17}{24}) = -0.35 \tag{4.12}$$

Since the indicator is now negative, it suggests a productivity loss between periods $t = 0$ and $t = 1$.

Again, one can remark contradictory results between these two cases. The Luenberger productivity indicator based on the directional distance function fails to measure productivity changes properly. This is due to the homogeneity degree of the directional distance function.

These numerical results are summarized in Table 1.

Table 1: Malmquist Index and Luenberger Indicator: Numerical Examples

	Case 1	Productivity	Case 2	Productivity
Output case	$z_t = (1, \frac{4}{5})$ $z_{t+1} = (1, \frac{5}{4})$		$z_t = (1, \frac{4}{5})$ $z_{t+1} = (10, \frac{50}{4})$	
Malmquist	$M^o = 1.56 > 1$	+	$M^o = 1.56 > 1$	+
Luenberger	$L = 0.45 > 0$	+	$L = -1.8 < 0$	−
Graph case	$z_t = (1, \frac{3}{4})$ $z_{t+1} = (1, \frac{5}{4})$		$z_t = (1, \frac{3}{4})$ $z_{t+1} = (1, \frac{50}{4})$	
Proportional	$L^{\propto} = 0.238 > 0$	+	$L^{\propto} = 0.105 > 0$	+
Luenberger	$L = 0.21 > 0$	+	$L = -0.35 < 0$	−

5 Empirical Illustration

As an empirical illustration, we propose to focus on the schooling productivity of European countries using the PISA-OECD and Eurostat data. Indeed, PISA (Programme for International Student Assessment) is an OECD program that aims to evaluate the performances of educational systems of OECD member countries. Since 2000 and every three years, surveys are conducted to evaluate 15-year-olds' ability to use their reading, mathematics, and science knowledge in 36 OECD member countries and partner countries. In parallel, Eurostat collects and harmonizes published data from national statistics institutes of European Union countries for various themes like education.

To analyze schooling productivity, we consider as outputs the PISA reading scores, mathematics scores, and science scores in 2018 and 2009 of 15-year-olds' pupils to measure schooling productivity over almost one decade. Following Agasisti, Munda and Hippe (2019), as inputs we select three types of resources: student/teacher ratio, government expenditure per student, and total public expenditure on education as percent of GDP. Furthermore, we distinguish those inputs for primary and secondary education levels and consider those resources during the schooling of pupils, i.e., for primary education in 2003 and 2012 so theoretically when pupils are 9-year-olds' and for secondary education in 2007 and 2016 so theoretically when pupils are 13-years-olds'. The reader can consult Table 2 for more details on these data. A sample of 21 European Union countries is collected. The original data can be found in Table B.1 in Appendix B.

We compute on these data four productivity indices and indicators: (i) the output-oriented Malmquist index (3.3), (ii) the input-oriented Luenberger indicator based on the proportional distance function (3.9), (iii) the input-oriented Luenberger indicator based on directional distance function (3.6) with input direction: (0.01, 0.01, 1000, 1000, 0.1, 0.1), and (iv) the input-oriented Luenberger indicator based on directional distance function (3.6) with as input direction the means in the sample (0.073, 0.096, 4609.34, 6211.84, 1.254, 2.039). The results and the rankings obtained for each index and indicator are presented in Table 3. In the top row, these four productivity indices and indicators are labeled "Malmquist", "Luenberger Prop.", "Luenberger Dir." and "Luenberger Mean", respectively.

Note that in this empirical illustration we opt for input-oriented Luenberger productivity indicators rather than graph-oriented ones. This methodological choice avoids any complications due to infeasibilities (see Briec and Kerstens (2009a)) and due to the need for positivity constraints on the projection of the outputs (see Briec and Kerstens (2009b)).

Table 2: Description of Inputs and Outputs

Variable	Label	Time Period 0	Time Period 1
Output 1	Reading scores	2009	2018
Output 2	Mathematic scores	2009	2018
Output 3	Science scores	2009	2018
Input 1	student/teacher ratio (inverse) for primary education	2003 (except: Estonia 2001)	2012 (except: Greece 2013)
Input 2	student/teacher ratio (inverse) for secondary education	2007	2016 (except: Norway 2017)
Input 3	Government expenditure per student (based on FTE) for primary education (PPS)	2003 (except: Estonia 2005; Greece 2005; Hungary 2004)	2012 (except: Belgium 2011; Norway 2011)
Input 4	Government expenditure per student (based on FTE) for secondary education (PPS)	2007 (except: Hungary 2006)	2016
Input 5	Total public expenditure on primary, lower and upper secondary education as % of GDP for primary education	2003	2012 (except: Slovakia 2011)
Input 6	Total public expenditure on primary, lower and upper secondary education as % of GDP for secondary education	2007 (except: Greece 2005)	2016

Our results show similar sign interpretation and ranking for the Malmquist productivity index and for the proportional Luenberger indicator. But, for the Luenberger indicator based on the directional distance function, the results are different. Indeed, the ranking is seriously modified. Some countries are better ranked with the directional Luenberger indicator (Czechia (+8); Lithuania (+4), Poland (+4)), while some other countries are worse ranked (Norway (-7), Portugal (-6), Austria (-4), Slovenia (-4)). We also notice that the sign interpretation of the productivity indices and indicators is even inverted for Austria. Indeed, the Malmquist index and the proportional Luenberger indicator highlight that Austria has increased its schooling productivity between 2009 and 2018 by 3.8 %, whereas the directional Luenberger indicator reveals a productivity decrease for this same period of time. Finally, using inputs means as direction for the directional Luenberger indicator somewhat limits this issue. This confirms the idea that the choice of a direction as the mean of the observed data also yields relevant results. Therefore, the strong commensurability, inherited from the diagonal homogeneity of the direction, has a significant impact on the evaluation of productivity changes as shown in Proposition 2.2. The results indeed become closer to the Malmquist productivity index and the proportional Luenberger productivity indicator. This confirms that the choice of the direction as an arithmetic means of the observed production vectors yields more relevant results.

Table 3: Productivity Scores and Ranking

Country	Malmquist	Rank	Luenberger Prop.	Rank	Luenberger Dir.	Rank	Luenberger Mean	Rank
Italy	1,117	1	0,055	1	1,114	1	0,059	1
Sweden	1,111	2	0,052	2	0,492	3	0,055	2
Estonia	1,081	3	0,039	3	0,675	2	0,037	3
Austria	1,039	4	0,019	4	-0,278	8	0,027	4
Portugal	1,001	5	0,000	5	-0,419	11	-0,006	6
Netherlands	0,998	6	-0,001	6	-0,045	4	-0,003	5
UK	0,977	7	-0,012	7	-0,127	5	-0,011	7
France	0,959	8	-0,021	9	-0,213	7	-0,018	8
Norway	0,959	9	-0,020	8	-0,721	15	-0,024	10
Hungary	0,934	10	-0,034	10	-0,378	9	-0,020	9
Germany	0,932	11	-0,035	11	-0,399	10	-0,032	11
Greece	0,923	12	-0,040	12	-0,503	12	-0,033	12
Belgium	0,914	13	-0,043	13	-0,696	14	-0,053	14
Czechia	0,889	14	-0,058	14	-0,174	6	-0,034	13
Slovenia	0,875	15	-0,067	15	-1,270	20	-0,054	15
Latvia	0,845	16	-0,084	16	-0,836	17	-0,066	17
Poland	0,828	17	-0,093	17	-0,528	13	-0,062	16
Slovakia	0,786	18	-0,119	18	-0,958	18	-0,080	18
Finland	0,781	19	-0,123	19	-1,551	21	-0,130	21
Lithuania	0,715	20	-0,162	20	-0,754	16	-0,093	19
Bulgaria	0,686	21	-0,181	21	-1,053	19	-0,099	20
Average	0,921		-0,044		-0,411		-0,031	

6 Conclusion

This paper has verified in detail some examples in which it is shown that the Luenberger productivity indicator based upon the directional distance function may not be a relevant productivity indicator. We have refined the notion of commensurability and have shown that it plays a crucial role. A distance function that is not strongly commensurable is not homogenous of degree 0 under a constant returns to scale assumption. Therefore, it may yield wrong evaluations to measure productivity. The simplest alternative to avoid these problems is to employ the Luenberger productivity indicator based upon the proportional distance function.

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Appendices: Supplementary Material

A Proofs of Propositions and Corollaries

Proof of Proposition 2.2:

Let us consider the positive diagonal matrix L of dimension $n+m$ whose components are all identical and equal to $\lambda > 0$. Since the map f satisfies the strong commensurability we have for all $z \in \mathbb{R}^d$ and all $C \in \mathcal{C}$, $f(Lz, \theta, LC) = f(z, \theta, C) = f(\lambda z, \theta, \lambda C)$. Since C is a cone, $\lambda C = C$. Hence $f(z, \theta, C) = f(\lambda z, \theta, C)$ which proves the homogeneity of degree 0 in the first argument.□

Proof of Proposition 2.3:

These results are established in Briec (1997: p. 103) and in Briec, Dervaux and Leleu (2003: p. 249-250), respectively. □

Proof of Proposition 2.4:

Let us consider the $m+n$-dimensional diagonal matrix L whose components are all identical and equal to $\lambda > 0$. Assume that the technology satisfies a constant returns to scale assumption $(A.5)$. In such a case $LT = T$. However, since the directional distance function is homogenous of degree 1, hence it is not homogenous of degree 0. From Proposition 2.2 it follows that it does not satisfy the strong commensurability condition. □

Proof of Proposition 2.5:

A production technology satisfying $(A.1) - (A.5)$ is a cone. Since the proportional distance function (2.5) satisfies the commensurability condition, the result is immediate from Proposition 2.2. □

Proof of Proposition 2.6:

Suppose that g is diagonally homogenous. Let L be a definite positive diagonal matrix. Suppose that

$$L = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

where M and N are respectively two $m \times m$ and $n \times n$ diagonal matrices. By hypothesis, we have $h(LT) = Mh(T)$ and $k(LT) = Nk(T)$. We have

$$\begin{aligned} \overrightarrow{D}^{\sharp}\big(Lz; g, L(T)\big) &= \sup\big\{\delta : \big(Mx - \delta h(LT), Ny + \delta k(LT)\big) \in LT\big\} \\ &= \sup\big\{\delta : \big(Mx - \delta Mh(T), Ny + \delta Nk(T)\big) \in LT\big\} \\ &= \overrightarrow{D}\big(Lz; Lg(T), L(T)\big). \end{aligned}$$

Since the directional distance distance function is weakly commensurable, we have:

$$\overrightarrow{D}\big(Lz; Lg(T), L(T)\big) = \overrightarrow{D}(z; g(T), T).$$

It follows that:

$$\overrightarrow{D}^{\sharp}\big(Lz, g, L(T)\big) = \overrightarrow{D}^{\sharp}(z, g, T).$$

which proves the strong commensurability of $\overrightarrow{D}^{\sharp}$. $\square$

Proof of Proposition 3.2:

Let $N = \begin{pmatrix} \lambda I_d & 0 \\ 0 & I_d \end{pmatrix}$ be a $d \times d$ be a positive diagonal matrix where I_d is the d-dimensional identity matrix and $\lambda > 0$. If ϕ is strongly commensurable then

$$\phi(\lambda I_d I_d z_t, z_{t+1}, \theta, \lambda T_t, T_{t+1}) = \phi(\lambda z_t, z_{t+1}, \theta, \lambda T_t, T_{t+1}) = \phi(z_t, z_{t+1}, \theta, \lambda T_t, T_{t+1}).$$

Since T_t satisfies a constant returns to scale assumption, we have $T_t = \lambda T_t$. It follows that

$B_t(z_t, z_{t+1}, \phi, \theta, \lambda) = 0$. The proof is similar in period $t+1$. □

Proof of Corollary 3.3:

(a) and (b) are immediate since the radial output measure (2.3) and the proportional distance function (2.5) satisfy the strong commensurability condition (Proposition 2.3). At the time period t, we have

$$\begin{aligned}
&B_t(z_t, z_{t+1}, L, g, \lambda)\\
&\quad = \frac{1}{2}\big[\overrightarrow{D}(z_t, g, T_{t+1}) - \overrightarrow{D}(z_{t+1}, g, T_{t+1}) + \overrightarrow{D}(z_t, g, T_t) - \overrightarrow{D}(z_{t+1}, g, T_t)\\
&\quad - \lambda\overrightarrow{D}(z_t, g, T_{t+1}) + \overrightarrow{D}(z_{t+1}, g, T_{t+1}) - \lambda\overrightarrow{D}(z{,}g, T_t) + \overrightarrow{D}(z_{t+1}, g, T_t)\big].
\end{aligned}$$

This implies that the bias is:

$$B_t(z_t, z_{t+1}, g, \lambda) = \frac{1-\lambda}{2}\big[\overrightarrow{D}(z_t, g, T_{t+1}) + \overrightarrow{D}(z_t, g, T_t)\big].$$

At the time period $t+1$ we have

$$\begin{aligned}
&B_{t+1}(z_t, z_{t+1}, g, \lambda)\\
&\quad = \frac{1}{2}\big[\overrightarrow{D}(z_t, g, T_{t+1}) - \overrightarrow{D}(z_{t+1}, g, T_{t+1}) + \overrightarrow{D}(z_t, g, T_t) - \overrightarrow{D}(z_{t+1}, g, T_t)\\
&\quad - \overrightarrow{D}(z_t, g_{t+1}) + \lambda\overrightarrow{D}(z_{t+1}, g, T_{t+1}) - \overrightarrow{D}(z_t, g, T_t) + \lambda\overrightarrow{D}(z_{t+1}, g, T_t)\big].
\end{aligned}$$

It follows that:

$$B_{t+1}(z_t, z_{t+1}, g, \lambda) = \frac{\lambda-1}{2}\big[\overrightarrow{D}(z_{t+1}, g, T_{t+1}) + \overrightarrow{D}(z_{t+1}, g, T_t)\big].$$

The last statement is immediate. □

B Empirical Data

To allow for replication of our empirical results in the main body of the text, we provide the small data set that we have employed in developing our empirical illustration.

Table B.1: Data on Inputs and Outputs

Country	Year	Outp 1	Outp 2	Outp 3	Inp 1	Inp 2	Inp 3	Inp 4	Inp 5	Inp 6
Belgium	2009	506	515	507	0,076	0,109	5217,9	7477,3	1,45	2,58
Belgium	2018	493	508	499	0,080	0,111	6981,2	9543,5	1,56	2,69
Bulgaria	2009	429	428	439	0,058	0,083	1223,5	1825,1	0,78	1,73
Bulgaria	2018	420	436	424	0,057	0,078	2315,5	3609,6	0,69	1,51
Czechia	2009	478	493	500	0,055	0,081	1983,1	4442	0,68	1,96
Czechia	2018	490	499	497	0,053	0,083	3447	5978,2	0,75	1,62
Germany	2009	497	513	520	0,053	0,066	4034,1	6590,8	0,67	2,25
Germany	2018	498	500	503	0,063	0,076	5700	7855,7	0,63	2,13
Estonia	2009	501	512	528	0,068	0,088	2679,6	4122,3	1,39	2,19
Estonia	2018	523	523	530	0,076	0,099	4432,8	4848,6	1,27	1,44
Greece	2009	483	466	470	0,083	0,130	2905,8	4788,8	1,01	1,45
Greece	2018	457	451	452	0,105	0,130	3641,8	4662,2	1,1	1,42
France	2009	496	497	498	0,052	0,070	4280,6	7896,7	1,19	2,56
France	2018	493	495	493	0,053	0,069	4982,7	7290,7	1,13	2,39
Italy	2009	486	483	489	0,092	0,105	5884,7	6500	1,21	1,97
Italy	2018	476	487	468	0,083	0,092	5622,3	6271	0,98	1,74
Latvia	2009	484	482	494	0,063	0,105	2038,4	3501,5	0,96	2,14
Latvia	2018	479	496	487	0,091	0,128	5316,4	4645,4	1,86	1,65
Lituania	2009	468	477	491	0,082	0,125	1414,4	2920,2	0,76	2,4
Lituania	2018	476	481	482	0,099	0,137	3671,8	3985,1	0,74	1,68
Hungary	2009	494	490	503	0,094	0,098	3129,2	3387,7	1	2,33
Hungary	2018	476	481	481	0,093	0,097	3270,9	4004	0,75	2,17
Netherlands	2009	508	526	522	0,063	0,064	5011,5	8507,1	1,48	2,16
Netherlands	2018	485	519	503	0,048	0,062	6073,8	8988	1,4	2,21
Austria	2009	470	496	494	0,069	0,097	6103,5	8809,8	1,09	2,49
Austria	2018	484	499	490	0,083	0,116	7494,6	11681,9	0,88	2,2
Poland	2009	500	495	508	0,084	0,081	2379,6	2801,4	1,79	1,89
Poland	2018	512	516	511	0,091	0,104	4652,5	4899,9	1,5	1,47
Portugal	2009	489	487	493	0,088	0,127	3583,5	5469	1,63	2,02
Portugal	2018	492	492	492	0,084	0,102	4185,1	6992,4	1,44	2,11
Slovenia	2009	483	501	512	0,078	0,105	5563,5	4891,7	2,56	1,16
Slovenia	2018	495	509	507	0,063	0,164	6525,5	6925,8	1,6	1,66
Slovakia	2009	477	497	490	0,052	0,072	1734,2	2690,7	0,64	1,69
Slovakia	2018	458	486	464	0,060	0,081	3721,1	4408,8	0,77	1,69
Finland	2009	536	541	554	0,060	0,101	4322,8	6484,8	1,39	2,51
Finland	2018	520	507	522	0,074	0,111	6324,7	10893,9	1,37	2,55
Sweden	2009	497	494	495	0,081	0,087	6124,1	7561,6	1,99	2,58
Sweden	2018	506	502	499	0,085	0,081	7945,1	8906,5	1,75	2,15
UK	2009	494	492	514	0,050	0,060	4819,4	7443	1,33	2,4
UK	2018	504	502	505	0,047	0,068	6635,4	6664,2	1,83	2,05
Norway	2009	503	498	500	0,085	0,098	6847,1	9798,8	2,01	2,33
Norway	2018	499	501	490	0,097	0,106	9371,6	9931,6	1,66	2,31