Linear algebra

Kristoffer Klokker

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1 Systems of linear Systems of Linear Equations

A system of linear equations are multiple equations containing unknows which are shared. Ex.

$$\begin{aligned} 2x + y + 3z &= 10 \\ x + y + z &= 6 \\ x + 3y + 2z &= 13 \end{aligned}$$

The same system may be wirtten in form of a matrix

$$\begin{bmatrix} 2 & 1 & 3 & 10 \\ 1 & 1 & 1 & 6 \\ 1 & 3 & 2 & 13 \end{bmatrix}$$

When solving a system their may be

- No solutions - No possible value can be assigned to the variable such all equation are true (inconsistent)
- One solution - A combination of values can be assigned to make every equation true (consistent)
- Infinite solutions - One or more unknows may have an infinite amount of possible assignable values (consistent)

By transforming a system of equations to a matrice, the following row operations can be performed:

- Multiply a row through by a nonzero constant.
- Interchange two rows.
- Add a constant times one row to another.

1.1 Gauss-Jordan Elimination

Solving a system of equation in a matrice, can be done such the matrice has the following requirements

- If the row contain nothing but zeroes the first number should be a 1, called the leading 1.

- If a row is made of nothing but zeroes it should be grouped at the bottom
- In two rows the top row should contain a leading 1 further to the left than the bottom
- Each column which contains a leading 1, every number in the same column below should be 0

This form is called row echelon form.
The solution may also be written as:

$$\{(x_1 = 4, x_2 = 6, x_3 = t, x_4 = v, x_5 = 1)|t \in \mathbb{R}\}$$

Where variables means they can be any possible asignment or a function with given restrictions, called free variables.
In case of the leading 1 column is zero both above and underneath the matrice is in reduced echlon form.

To make a matrice into echoleon form the following algorithm can be used:

1. Locate the leftmost column that does not consist entruely of zeros and exhange it to the top
2. Multiply the top row by $\frac{1}{a}$ where a is the leading number in the row
3. subtract top row from every other below row, such the top row is the only non zero value in the column
4. Repeat 2 and 3 but ignore the top row and let the second top row be the top

To reduce the echloen form, from the bottom the bottom row is added to the above rows until the leading 1 is the only in the column. This is repeated until the top is reached.
A homogeneous linear system are systems which all constant (right part of equal) are 0 and the trivial solution (all variables are assigned 0) are a possible solution.
A free variable is the term for a variable which can be assigned multiple values. The number of free variables wil lbe equal to the number of variables minus zero rows.
In a homogenous linear system if the number of unknows exceed the number of equation there will be an infinite amount of solutions.

Back substitution is a method taking the echoloen form and starting from the bottom isolating the leading variable and substituting it upwards.
A echoloen form is not unique to a system but a reduced echoloen form is unique and the number of zero rows will be unique.

1.2 Matrices and Matrix operations

A matrix is a rectangular array of numbers called entries.
The size of a matrix is written as rows x columns
A single row matrix is alled a row vector and single column is called column vector.
Variables in matrices are called scalars and unless stated is in the realm of real numbers.
When refering to a number in the matrix A it will be $a_{row\ column}$ and the value can be found by $(A)_{row\ column}$
A matrix scalar matrix can be written as $A = [a_{ij}]_{m\times n}$ the $m \times n$ is optional if the size matters.

1.2.1 Matrix operations

Addition/subtraction - every number in corresponding entries are added/-subtracted.
Addtion and subtraction can not be done on two different sized matrices.

Multiplying a matrix with a scalar, is done by multiplying the scalar on every entry.
For multiplying two matrices an entry is found by $a_{ij} = \sum\limits_{t=0}^{n} b_{i(j+t)} \cdot c_{(i+t)j}$ where n is the size of row size of matrix b and column size of matrix c, if this value does not match the two matrices can not be multiplied.
The reuslting product matrix of $a = b \times c$ will have the size rows = c columns and columns = b rows

Partioning a matrix is the act of splitting it into a smaller valid matrices. This can be used if only certain entries are wanted for a matrix multiplication
Linear combinations are an array of matrices (length n) of the same size matrix A and an array of scalars (length n) c computed as $\sum\limits_{i=o}^{n} A_i \cdot c_i$
Multiplying a vector row and a matrix (with approprate lengths) can therefore be expressed as an linear combination.
Column row expansion is the act of splitting two matrices being multiplied

into rows and columns, and then taking the sum of each product of row and column, and thus getting the muiltiplied matrix.

Matrix form of linear system is the act of expressing a linear system in the form of matrix A containing all coefficients, matrix X containing a column vector of all unknown variables, and matrix B which is a column vector of all results of the linear system. in this way the linear system can be expressed as $AX = B$
Tranposing a matrix is flipping column and rows such in matrix A the entry a_{ij} is now entry a_{ji} in the matrix A^T.
A trace operation of matrix A is taking the addition of the diagonal line called $tr(A)$, can only be performed on square matrices.
The trace of a square matrix is defined as $tr(A) = a_{11} + a_{22} + ... + a_{nn}$ or $tr(A) = \sum_{i=0}^{n} a_{ii}$
For the trace the following is true

- $tr(A^T) = tr(A)$
- $tr(A \pm B) = tr(A) \pm tr(B)$
- $tr(cA) = c \cdot tr(A)$
- $tr(AB) = tr(BA)$

The matrix polynomials are a way of describing multiple equations in matrix form by
If A contains all the coefficients, X contains all variables in a column vector, and B contains all constant the equations are equal to in a column vector.
Then the equations can be described as $A \cdot X = B$

1.3 Inverses

For matrix operations are the following true, if the size allows for it

- Commutative law for matrix addition - $A + B = B + A$
- Associative law for matrix addition - $A + (B + C) = (A + B) + C$
- Associative law for matrix multiplication - $A(BC) = (AB)C$
- Left and Right distributive law - $A(B + C) = AB + AC$

Zero matrices are matrices consisting of only zeros and are denoted by $0_{m\times n}$
Identity matrix I is the matrix containing 1's on the diagonal, and has the proberty that if multiplies by A the product is A.
The inverse matrix A^{-1} is the matrix B to the square A if $AB = I$, and A is said to be invertible. If B does not exist A is singular
Only one inverse exist to a matrix.
For two inverse matrices is the following true $B^{-1}A^{-1} = (AB)^{-1}$.
Powers if a matrix is defined as expected, and for $A^0 = I$ and if A is invertible then $A^{-n} = (A^{-1})^n$
A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$
Matrix pylonomials are inserting a matrix into a polynomial function.
Transpose if the interchangning of rows in a matrix, and are expressed as $(A^T)^{-1} = (A^{-1})^T$.
The transposed matrix has the follwoing properties:

- $(A^T)^T = A$
- $(A \pm B)^T = A^T \pm B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$