

Linear algebra

Kristoffer Klokke

2022

Contents

1	Systems of linear Systems of Linear Equations	3
1.1	Gauss-Jordan Elimination	3
1.2	Matrices and Matrix operations	5
1.2.1	Matrix operations	5
1.3	Inverses	6
1.4	Determinants by cofactor expansion	7
1.5	Properties of determinants	8
1.6	Matrix transformation	9
1.7	Projections	10
2	Vector spaces	10
2.1	Subspaces	11

1 Systems of linear Systems of Linear Equations

A system of linear equations are multiple equations containing unknowns which are shared. Ex.

$$2x + y + 3z = 10$$

$$x + y + z = 6$$

$$x + 3y + 2z = 13$$

The same system may be written in form of a matrix

$$\begin{bmatrix} 2 & 1 & 3 & 10 \\ 1 & 1 & 1 & 6 \\ 1 & 3 & 2 & 13 \end{bmatrix}$$

When solving a system there may be

- No solutions - No possible value can be assigned to the variable such that all equations are true (inconsistent)
- One solution - A combination of values can be assigned to make every equation true (consistent)
- Infinite solutions - One or more unknowns may have an infinite amount of possible assignable values (consistent)

By transforming a system of equations to a matrix, the following row operations can be performed:

- Multiply a row through by a nonzero constant.
- Interchange two rows.
- Add a constant times one row to another.

1.1 Gauss-Jordan Elimination

Solving a system of equations in a matrix, can be done such that the matrix has the following requirements

- If the row contains nothing but zeroes the first number should be a 1, called the leading 1.

- If a row is made of nothing but zeroes it should be grouped at the bottom
- In two rows the top row should contain a leading 1 further to the left than the bottom
- Each column which contains a leading 1, every number in the same column below should be 0

This form is called row echelon form.

The solution may also be written as:

$$\{(x_1 = 4, x_2 = 6, x_3 = t, x_4 = v, x_5 = 1) | t \in \mathbb{R}\}$$

Where variables means they can be any possible assignment or a function with given restrictions, called free variables.

In case of the leading 1 column is zero both above and underneath the matrix is in reduced echelon form.

To make a matrix into echelon form the following algorithm can be used:

1. Locate the leftmost column that does not consist entirely of zeros and exchange it to the top
2. Multiply the top row by $\frac{1}{a}$ where a is the leading number in the row
3. subtract top row from every other below row, such the top row is the only non zero value in the column
4. Repeat 2 and 3 but ignore the top row and let the second top row be the top

To reduce the echelon form, from the bottom the bottom row is added to the above rows until the leading 1 is the only in the column. This is repeated until the top is reached.

A homogeneous linear system are systems which all constant (right part of equal) are 0 and the trivial solution (all variables are assigned 0) are a possible solution.

A free variable is the term for a variable which can be assigned multiple values. The number of free variables will be equal to the number of variables minus zero rows.

In a homogeneous linear system if the number of unknowns exceed the number of equation there will be an infinite amount of solutions.

Back substitution is a method taking the echoloe form and starting from the bottom isolating the leading variable and substituting it upwards.

A echoloe form is not unique to a system but a reduced echoloe form is unique and the number of zero rows will be unique.

1.2 Matrices and Matrix operations

A matrix is a rectangular array of numbers called entries.

The size of a matrix is written as rows x columns

A single row matrix is called a row vector and single column is called column vector.

Standard basis is the set of column vectors in a identity matrix.

Variables in matrices are called scalars and unless stated is in the realm of real numbers.

When referring to a number in the matrix A it will be $a_{row\ column}$ and the value can be found by $(A)_{row\ column}$

A matrix scalar matrix can be written as $A = [a_{ij}]_{m \times n}$ the $m \times n$ is optional if the size matters.

1.2.1 Matrix operations

Addition/subtraction - every number in corresponding entries are added/-subtracted.

Addition and subtraction can not be done on two different sized matrices.

Multiplying a matrix with a scalar, is done by multiplying the scalar on every entry.

For multiplying two matrices an entry is found by $a_{ij} = \sum_{t=0}^n b_{i(j+t)} \cdot c_{(i+t)j}$

where n is the size of row size of matrix b and column size of matrix c , if this value does not match the two matrices can not be multiplied.

The resulting product matrix of $a = b \times c$ will have the size rows = c columns and columns = b rows

Partitioning a matrix is the act of splitting it into a smaller valid matrices.

This can be used if only certain entries are wanted for a matrix multiplication

Linear combinations are an array of matrices (length n) of the same size matrix A and an array of scalars (length n) c computed as $\sum_{i=0}^n A_i \cdot c_i$

Multiplying a vector row and a matrix (with appropriate lengths) can therefore be expressed as an linear combination.

Column row expansion is the act of splitting two matrices being multiplied into rows and columns, and then taking the sum of each product of row and column, and thus getting the multiplied matrix.

Matrix form of linear system is the act of expressing a linear system in the form of matrix A containing all coefficients, matrix X containing a column vector of all unknown variables, and matrix B which is a column vector of all results of the linear system. in this way the linear system can be expressed as $AX = B$

Tranposing a matrix is flipping column and rows such in matrix A the entry a_{ij} is now entry a_{ji} in the matrix A^T .

A trace operation of matrix A is taking the addition of the diagonal line called $tr(A)$, can only be performed on square matrices.

The trace of a square matrix is defined as $tr(A) = a_{11} + a_{22} + \dots + a_{nn}$ or $tr(A) = \sum_{i=1}^n a_{ii}$

For the trace the following is true

- $tr(A^T) = tr(A)$
- $tr(A \pm B) = tr(A) \pm tr(B)$
- $tr(cA) = c \cdot tr(A)$
- $tr(AB) = tr(BA)$

The matrix polynomials are a way of describing multiple equations in matrix form by

If A contains all the coefficients, X contains all variables in a column vector, and B contains all constant the equations are equal to in a column vector.

Then the equations can be described as $A \cdot X = B$

1.3 Inverses

For matrix operations are the following true, if the size allows for it

- Commutative law for matrix addition - $A + B = B + A$
- Associative law for matrix addition - $A + (B + C) = (A + B) + C$
- Associative law for matrix multiplication - $A(BC) = (AB)C$
- Left and Right distributive law - $A(B + C) = AB + AC$

Zero matrices are matrices consisting of only zeros and are denoted by $0_{m \times n}$. Identity matrix I is the matrix containing 1's on the diagonal, and has the property that if multiplied by A the product is A .

The inverse matrix A^{-1} is the matrix B to the square A if $AB = I$, and A is said to be invertible. If B does not exist A is singular.

Only one inverse exists to a matrix.

For two inverse matrices is the following true $B^{-1}A^{-1} = (AB)^{-1}$.

Powers of a matrix are defined as expected, and for $A^0 = I$ and if A is invertible then $A^{-n} = (A^{-1})^n$.

A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

Matrix polynomials are inserting a matrix into a polynomial function.

Transpose is the interchanging of rows in a matrix, and are expressed as $(A^T)^{-1} = (A^{-1})^T$.

The transposed matrix has the following properties:

- $(A^T)^T = A$
- $(A \pm B)^T = A^T \pm B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$

1.4 Determinants by cofactor expansion

Minor of entry a_{ij} is denoted M_{ij} and is defined as the determinant of the submatrix that remains when i row and j column is deleted.

The cofactor C_{ij} of M_{ij} is defined as, if $i + j$ is even then $1^{i+j}M_{ij}$ and $(-1)^{i+j}M_{ij}$ if odd.

The determinant of an $n \times n$ matrix can be found either by a row or column in the following way:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$$

This is via the column and by switching i and j it is by the row.

Therefore making the definition recursive.

If we make the function $f(i, j) = 1 - ((i + j) \% 2) \cdot 2$ such it will be -1 when the sum of i and j is odd and 1 when the sum is even.

$$\det(A) = a_{1j}f(1, j)^{1+j}M_{1j} + a_{2j}f(2, j)^{2+j}M_{2j} + \dots + a_{nj}f(n, j)^{n+j}M_{nj}$$

if we also denote the function $g(A, i, j)$ to be a function which removes row i and j

Then we can make a true recursive definition.

$$\det(A) = a_{1j}f(1, j)^{1+j}\det(g(A, 1, j)) + a_{2j}f(2, j)^{2+j}\det(g(A, 2, j)) + \dots + a_{nj}f(n, j)^{n+j}\det(g(A, n, j))$$

Where for the 1x1 matrix $\det([a_{11}]) = a_{11}$ If A is triangular then $\det(A)$ is equal to the product of the main diagonal.

It can also be noted that determination of the zero matrix is zero

And for the transpose $\det(A) = \det(A^T)$

To make life easier row operation can be done such less operations are needed to find the determinant.

When performing row operation the following is the for the A matrix with size $n \times n$

- If B is the result of a row or column in A being multiplied by a constant k then $\det(B) = k \cdot \det(A)$
- if B is the result of two rows or two columns being interchanged then $\det(B) = -\det(A)$
- if B is the result of one row being multiplied and added to another row then $\det(B) = \det(A)$

For the special case where $A = I_n$ and E is the result of an row operation on A the following applies

- If E is the result of multiplying a row with the constant k which is not 0 then $\det(E) = k$
- If E is the result of interchanging two rows then $\det(E) = -1$
- If E is the result of adding a constant to a row and adding to another then $\det(E) = 1$

If A has a row/column which is proportional to another then $\det(A) = 0$

Reducing a matrix the echeloen, in doing so instead of reducing the a row by $1/\text{the leading value}$, the leading value is taken to the side, then the product of every value taken to the side will be equal to the determinant.

1.5 Properties of determinants

If A is a $n \times n$ matrix and C_{ij} is the cofactor of A_{ij} , then the matrix is called matrix of cofactors from A

The transpose of the matrix of cofactors from A is called adjoint of A

- $\det(kA) = k^n \det(A)$
- $\det(C) = \det(A) + \det(B)$ If A and B only differ in one row/column and C is equal to A and the the differ row/column sum
- $\det(AB) = \det(A)\det(B)$ if both A and B are $n \times n$ matrices
- A is only invertible if $\det(A) \neq 0$
- If A is invertible then $\det(A^{-1}) = \frac{1}{\det(A)}$
- If A is invertible then $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

In the system of linear equation $Ax = b$ then the solutions can be found by

$$x_n = \frac{\det(A_n)}{\det(A)}$$

1.6 Matrix transformation

The matrix transformation is function $T_A(x) = A \cdot x$, which transforms the axis linearly, such origo stays in the same place.

The function then takes a vector of length n and multiply it / transforms it with a matrix of size $n \times n$

The transformation which has the follwing proberties:

- $T_A(0) = 0$ Origin stays in place
- $T_A(k \cdot x) = k \cdot T_A(x)$
- $T_A(x + y) = T_A(x) + T_A(y)$
- $T_A(x - y) = T_A(x) - T_A(y)$

To think of the matrix A which the transformation use, it can be though of in the coordination system x,y,z A would then be a 3x3 matrix.

Each column then represent the standard basis for the axis.

Such for the Cartesian (standard) coordination system, the matrix would be a 3x3 identity matrix.

Generally speaking the identity matrix is also called to standard matrix

The transformation may also be composited together like $(T_B \cdot T_A)(x) = T_B(T_A(X))$

Likewise multiplication the transformation is not commutative $T_A \cdot T_B \neq T_B \cdot T_A$

The exception to this is rotation and reflection transformations.

If A is invertible then $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is also invertible.

This is written as $T_A^{-1} = T_{A^{-1}}$

A linear transformation is when $T : \mathbb{R}^N \rightarrow \mathbb{R}^M$ satisfies the following linear conditions

- $T(k \cdot x) = k \cdot T(x)$
- $T(x + y) = T(x) + T(y)$

1.7 Projections

A projection is a transformation to a lower dimension system.

An orthogonal projection is a projection on a line, plane or the appropriate amount of dimensions.

2 Vector spaces

If V defines all objects which are non empty then

- $u, v \in V \rightarrow v + u \in V$
- $u + v = v + u$
- $u + (v + w) = (u + v) + w$
- The zero vector exist called 0 with the property $v + 0 = v$
- For each u in V exists a $-u$ such $u + (-u) = 0$
- If k is a scalar and u is in V then uk is in V
- $k(u + v) = ku + kv$
- $k(mu) = (km)u$
- $1u = u$

If an object satisfies all these axioms then we can classify a set of objects as being in the vector space.

Therefore \mathbb{R} is in the vector space, \mathbb{R}^n being the vector of real number length n is in the vector space.

Likewise $\mathbb{R}^{n \cdot m}$ matrix is in the vector space and even functions like the polynomial functions are in the vector space.

2.1 Subspaces

A subspace W of V is a subspace if u and v are in W and $u + v$ is in W .
And if k is a scalar and u is in W then ku is in W