Linear algebra
Exercises

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2022

Contents

1 Week 36

1.1 Which equation are linear in x_1, x_2 and x_3

1.1.1 $x_1 + 5x_2 - \sqrt{2}x^3 = 1$

This is a linear equation

1.1.2 $x_1 = -7x_2 + 3x_3$

This is a linear equation

1.1.3 $x_1^{3/5} - 2x_2 + x_3 = 4$

This is not a linear equation with x_1 having a power

1.2 Convert from matrix form to equation form

$\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$
$2x_1 = 0$
$3x_1 - 4x_2 = 0$
$x_2 = 1$

1.3 Convert from equations to matrix

$-6x_1 - x_2 + 3x_3 = 4$
$5x_2 - x_3 = 1$
$\begin{bmatrix} -6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{bmatrix}$

1.4 Determine if the solution hold in the following system

$(5, 8, 1)$
$x + 2y - 2z = 3$
$3x - y + z = 1$
$-x + 5y - 5z = 5$

$5 + 2(8) - 2(1) = 3$
$19 = 3$
By the first equation the solution does not hold

1.5 Determine if the following matrices are in echoleon form or reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduced echelon form

$$\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Echelon form

2 Week 38

2.1 Show that the determinant og the matix is 0

$$\begin{bmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

Column 2 and 4 are proportional to eachother therefore making the det=0

2.2 Is the following matrix invertible

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix}$$

The determinant is -6 and therefore not zero and therefore invertible

2.3 Find the standard matrix for the transformation defined by the equations

$$\begin{aligned} w_1 &= 7x_1 + 2x_2 - 8x_3 \\ w_2 &= -x_2 + 5x_3 \\ w_3 &= 4x_1 + 7x_2 - x_3 \end{aligned}$$

$$\begin{bmatrix} 7 & 2 & -8 \\ 0 & -1 & 5 \\ 4 & 7 & -1 \end{bmatrix}$$

2.4 Is a linear function a transformation of R

Yes the linear function can be a transform of the space R

2.5 The images of the standard basis vectors for R^3 are given for a linear trasnformation $T : R^3 \to R^3$. Find the standard matrix for the transformation and find $T(x)$

$$T(e_1) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(e_3) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$T(x) = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

2.6 Find the standard matrix A for the linear transformation $T : R^2 \to R^2$ for which

$$T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, T(\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix} = c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}2\\3\end{bmatrix}$$
$$\begin{bmatrix}0\\1\end{bmatrix} = k_1 \begin{bmatrix}1\\1\end{bmatrix} + k_2 \begin{bmatrix}2\\3\end{bmatrix}$$
c variable found by
$$\begin{bmatrix}1 & 2 & 1\\1 & 3 & 0\end{bmatrix}$$
k variable found by
$$\begin{bmatrix}1 & 2 & 0\\1 & 3 & 1\end{bmatrix}$$
Gauss jordian magic!
$$c_1 = 3, c_2 = -1, k_1 = -2, k_2 = 1$$
$$T(\begin{bmatrix}1\\0\end{bmatrix}) = 3T(\begin{bmatrix}1\\1\end{bmatrix}) + (-1)T(\begin{bmatrix}2\\3\end{bmatrix})$$
$$= \begin{bmatrix}3\\-6\end{bmatrix} - \begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}5\\-11\end{bmatrix}$$
$$T(\begin{bmatrix}0\\1\end{bmatrix}) = -2T(\begin{bmatrix}1\\1\end{bmatrix}) + 1T(\begin{bmatrix}2\\3\end{bmatrix})$$
$$= \begin{bmatrix}-2\\4\end{bmatrix} + \begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}-4\\9\end{bmatrix}$$

$$A = \begin{bmatrix}5 & -4\\-11 & 9\end{bmatrix}$$

3 Week 39

3.1 Determine if the set is a vector space

3.1.1 n-tuples of real numbers that have the form $(x, x, ..., x)$ with the standard opeations R^n

- $u, v \in V \to v + u \in V$ True
- $x + v = v + u$ - True
- $u + (v + w) = (u + v) + w$ - True
- $Thezerovectorexists$ - True (0,0,0,0)

- There exists a u in V such $u + (-u) = 0$ - True
- $uk \in V$ where k is a scalar - True
- $k(u + v) = kv + ku$ - True
- $k(mu) = (km)u$ - True
- $1u = u$ - True

3.1.2 The set of matrices in the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

- $u, v \in V \rightarrow v + u \in V$ True
- $x + v = v + u$ - True
- $u + (v + w) = (u + v) + w$ - True
- $Thezerovectorexists$ - True a=0 and b=0
- There exists a u in V such $u + (-u) = 0$ - True
- $uk \in V$ where k is a scalar - True
- $k(u + v) = kv + ku$ - True
- $k(mu) = (km)u$ - True
- $1u = u$ - True

3.2 Use the subspace test to determine if it a subspace of R^3

- $(a, 0, 0)$ - $(a_1, 0, 0) + (a_2, 0, 0) = (a_3, 0, 0)$ and $k(a_1, 0, 0) = (a_2, 0, 0)$, therefore it is a subspace
- $(a, 1, 1)$ - $(a_1, 0, 0) + (a_2, 1, 1 = (a_3, 2, 2$ which is in not in the space and therefore not a subspace
- (a, b, c) where $b = a + c$ - $(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, a_1 + a_2 + c_1 + c_2, c_1 + c_2)$ and $k(a_1, b_1, c_1) = (ka_1, k(a_1 + c_1), c_1)$ which holds true and therefore is asubspace

3.3 Use the subspace test to determine if it a subspace of $F(-\infty, \infty)$

- All functions f in $F(-\infty, \infty)$ for which $f(0) = 0$ - $f_1(0) + f_2(0) = 0$ and $k \cdot f_1(0) = k \cdot 0 = 0$ therfore it is a subspace
- All functions f in $F(-\infty, \infty)$ for which $f(0) = 1$ - $f_1(0) + f_2(0) = 2$ therefore making it not a subspace

3.4 Use the subspace test to determine the a bounded set of sequence of real number is a subspace of the sequence of real number in an infinite tuple

The operations are defined as
$(a_n)_{n \in N} + (b_n)_{n \in N} := (a_n + b_n)_{n \in N}$
$k \cdot (a_n)_{n \in N} := (ka_n)_n$
The sequence is bounded by a C value such $-C \leq a_n \leq C$.
The bounded sequence will be a subspace if we defined $a_n + b_n \leq C$ and $ka_n \leq C$

3.5 Verify that $\mathbb{M}_{m,n}(\mathbb{R})$ is a vector space

- $u, v \in V \rightarrow v + u \in V$ True
- $x + v = v + u$ - True
- $u + (v + w) = (u + v) + w$ - True
- $Thezerovectorexists$ - True everything in the matrix equal 0
- There exists a u in V such $u + (-u) = 0$ - True
- $uk \in V$ where k is a scalar - True
- $k(u + v) = kv + ku$ - True
- $k(mu) = (km)u$ - True in the case that the size of matrices unlike $k_{2,1}, u_{2,2}, m_{2,1}$
- $1u = u$ - True

3.6 Which if the following are linear combination of $u = (0, -2, 2)$ and $v = (1, 3, -1)$

3.6.1 $(2, 2, 2)$

$$\begin{aligned}(2,2,2) &= k_1(0,-2,2) + k_2(1,3,-1)\\ &= (k_1 + k_2, -2k_1 + 3k_2, 2k_1 - k_2)\end{aligned}$$

$$\begin{aligned}k_1 + k_2 &= 2\\ -2k_1 + 3k_2 &= 2\\ 2k_1 - k_2 &= 2\end{aligned}$$

$$\begin{aligned}k_1 &= 0.8\\ k_2 &= 1.2\end{aligned}$$

It can be seen that the third condition is not met and therefore it is not a linear combination

3.6.2 $(0, 4, 5)$

$$\begin{aligned}(0,4,5) &= k_1(0,-2,2) + k_2(1,3,-1)\\ &= (k_1 + k_2, -2k_1 + 3k_2, 2k_1 - k_2)\end{aligned}$$

$$\begin{aligned}k_1 + k_2 &= 0\\ -2k_1 + 3k_2 &= 4\\ 2k_1 - k_2 &= 5\end{aligned}$$

$$\begin{aligned}k_1 &= -0.8\\ k_2 &= 0.8\end{aligned}$$

It can be seen that the second equation is not met and therefore it is not a linear combination

3.6.3 $(0,0,0)$

$$\begin{aligned}(0,4,5) &= k_1(0,-2,2)+k_2(1,3,-1)\\ &= (k_1+k_2,-2k_1+3k_2,2k_1-k_2)\end{aligned}$$

$$\begin{aligned}k_1+k_2 &= 0\\ -2k_1+3k_2 &= 0\\ 2k_1-k_2 &= 0\end{aligned}$$

$$\begin{aligned}k_1 &= 0\\ k_2 &= 0\end{aligned}$$

This is a linear combination and by definition origo will always be a linear combination since a linear transformation does not move it.

3.7 Explain why the following form linearly dependent sets of vectors

- $u_1 = (-1,2,4)$ and $u_2 = (5,-10,-20)$ in $\mathbb{R}^3$ - Vector u_2 is a scalar multiple of $-5u_1$
- $u_1 = (3,-1), u_2 = (4,5), u_3 = (-4,7)$ in $\mathbb{R}^2$ - A linearly independent system would only need 2 vectors therefore making it dependent

3.8 Determien whether the vectors are linearly independent in R^3

- $(3,8,7),(1,5,3,7),(2,-1,2,6),(4,2,6,4)$ - Each vector is a none scalar of each other such the zero test can only be performed with only 0 coefficients
- $(3,0,-3,6),(0,2,3,1),(0,-2,-2,0),(-2,1,2,1)$ - Each vector is a none scalar and by the zero test shows the only solution is 0 coefficients

3.9 Show that the following set of vectors forms a basis for R^2

$\{(2,1),(3,0)\}$

It can be seen that the determinant of $\begin{bmatrix}2 & 3\\ 1 & 0\end{bmatrix}$ is -3 and therefore not 0 and therefore the rows are linearly independent

3.10 Show that the following polynomials form a basis for P_2

$\{x^2+1, x^{-2}-1, 2x-1\}$

It can be seen that the determinant of $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ is -4 and therefore not 0 and therefore the column vectors span the space

3.11 Find the coordinate vector of v relative to the basis $S = \{v_1, v_2, v_3\}$ for R^3

$$v = (2,-1,3); v_1 = (1,0,0), v_2 = (2,2,0), v_3 = (3,3,3)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 12 \end{bmatrix}$$

3.12 In each part let $T_a : R^3 \to R^3$ be multiplication by A, and let $\{e_1, e_2, e_3\}$ be the standard basis for R^3. Determine whether the set $\{T_A(e_1), T_A(e_2), T_A(e_1),\}$ is linearly independent in R^2

3.12.1 $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{bmatrix}$

Since the determinant of A is not zero means that $\{T_A(e_1), T_A(e_2), T_A(e_1),\}$ is linearly independent

3.12.2 $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

Since the determinant of A is zero means that $\{T_A(e_1), T_A(e_2), T_A(e_1),\}$ is linearly dependent

3.13 In each part, find a basis for the given subspace of R^3, and state its dimension.

3.13.1 The plane $3x - 2y + 5z = 0$

$$
\begin{aligned}
y &= 1.5x + 2.5z \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x \\ 1.5x + 2.5z \\ z \end{bmatrix} \\
&= x \begin{bmatrix} 1 \\ 1.5 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 2.5 \\ 1 \end{bmatrix} \\
\{ &\begin{bmatrix} 1 \\ 1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2.5 \\ 1 \end{bmatrix} \}
\end{aligned}
$$

Since the matrices are non zero and linearly independent they are the basis.

3.13.2 The plane $x - y = 0$

$$
\begin{aligned}
x &= y \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} y \\ y \end{bmatrix} \\
&= y \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\{ &\begin{bmatrix} 1 \\ 1 \end{bmatrix} \}
\end{aligned}
$$

3.13.3 The line $x = 2t, y = -t, z = 4t$

$$\begin{aligned}
y &= -0.5x \\
z &= 2x \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} x \\ -0.5x \\ 2x \end{bmatrix} \\
&= x \begin{bmatrix} 1 \\ -0.5 \\ 2 \end{bmatrix} \\
\{ &\begin{bmatrix} 1 \\ -0.5 \\ 2 \end{bmatrix} \}
\end{aligned}$$

3.13.4 All vectors of the form (a, b, c), where $b = a + c$

$$\begin{aligned}
b &= a + c \\
\begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} a \\ a + c \\ c \end{bmatrix} \\
&= a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
\{ &\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}
\end{aligned}$$

Since the matrices are non zero and linearly independent they are the basis.

3.14 Find a standard basis vector for R^3 that can be added to the set, to produce a bassi for R^3 / Enlarge to vector set to a basis

3.14.1 $v_1 = (-1, 2, 3), v_2 = (1, -2, -2)$

$$\begin{aligned}
det(\begin{bmatrix} -1 & 2 & 3 \\ 1 & -2 & -2 \\ a & b & c \end{bmatrix}) &= -1 \cdot det(\begin{bmatrix} 2 & 3 \\ b & c \end{bmatrix}) - 2 \cdot det(\begin{bmatrix} 1 & -2 \\ a & c \end{bmatrix} + 3 \cdot det(\begin{bmatrix} 1 & -2 \\ a & b \end{bmatrix}) \\
&= -(2c - 3b) - 2(1c - (-2a)) + 3(ab - (-2a)) \\
&= 2c + 3b - 2c - 4a + 3ab - 2a \\
&= 3b - 6a + 3ab \\
0 &\neq 3b - 6a + 3ab
\end{aligned}$$

Therefore the missing vector in the form (a, b, c), c can be any value and a and b must values such the last statement holds true. Otherwise the determinant will be 0 and the vectors would be linearly dependent.

3.14.2 $v_1 = (1, -1, 0), v_2 = (3, 1, -2)$

$$\begin{aligned}
det(\begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \\ a & b & c \end{bmatrix}) &= 1 \cdot det(\begin{bmatrix} 1 & -2 \\ b & c \end{bmatrix}) - (-1) \cdot det(\begin{bmatrix} 3 & -2 \\ a & c \end{bmatrix} + 0 \cdot det(\begin{bmatrix} 3 & 1 \\ a & b \end{bmatrix}) \\
&= 1(1c - 2b) + 1(3c - (-2a)) \\
&= c - 2b + 3c + 2a \\
0 &\neq 2a + 2b + 2c \\
2a &\neq -2b - 2c \\
a &\neq -b - c
\end{aligned}$$

Therefor in the missing vector in the form (a, b, c) it must hold true that a is not equal to $-b - c$. Otherwise the determinant will be 0 and the vectors would be linearly dependent.

3.15 Find a bassi for the subspace of R^3 that is spanned by the vectors $v_1 = (1, 0, 0), v_2 = (1, 0, 1), v_3 = (2, 0, 1), v_4 = (3, 3, 3, 4)$

By setting it up in a matrix and using gauss elimination the basis vectors can be found $(1, 0, 0), (0, 0, 1)$

3.16 Consider the bases $B = \{u_1, u_2\}$ and $B' = \{u_1', u_2'\}$ for R^2 NOT DONE

$$u_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, u_1' = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, u_2' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

3.16.1 Find the transmission matrix from B to B'

[new basis — old basis]

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 3 & -1 & 2 & 1 \end{bmatrix}$$

Reducing gives

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & -5 \end{bmatrix}$$

Therefore making the transistion matrix

$$\begin{bmatrix} 0 & -1 \\ -2 & -5 \end{bmatrix}$$

3.16.2 Find the transmission matrix from B to B'

[new basis — old basis]

$$\begin{bmatrix} 2 & 4 & 1 & -1 \\ 2 & 1 & 3 & -1 \end{bmatrix}$$

Reducing gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Therefore making the transistion matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

3.16.3 Calculate the vector in both spaces

$$w = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$[w]_b = B \cdot w$$
$$[w]_b = 3\begin{bmatrix}2\\2\end{bmatrix} - 5\begin{bmatrix}4\\-1\end{bmatrix}$$
$$[w]_b = \begin{bmatrix}-14\\11\end{bmatrix}$$
$$[w]_{b'} = \begin{bmatrix}0 & -1\\-2 & 5\end{bmatrix} \cdot [w]_b$$
$$[w]_{b'} = -14\begin{bmatrix}0\\-2\end{bmatrix} + 11\begin{bmatrix}-1\\5\end{bmatrix}$$
$$[w]_{b'} = \begin{bmatrix}-11\\37\end{bmatrix}$$
$$[w]_{b'} = 3\begin{bmatrix}1\\3\end{bmatrix} - 5\begin{bmatrix}-1\\-1\end{bmatrix}$$
$$[w]_{b'} = \begin{bmatrix}-2\\4\end{bmatrix}$$

3.17 Let V be the space spanned by $f_1 = \sin x$ and $f_2 = \cos x$

3.17.1 Show that $g_1 = 2\sin x + \cos x$ and $g_2 = 3\cos x$ form a basis for V

It can be seen that since one consist

3.17.2 Find the transition matrix from $B = \{f_1, f_2\}$ to $B' = \{g_1, g_2\}$

$$x_1 \cdot (2\sin x + \cos x) + x_3 \cdot 3\cos x = \sin x$$
$$x_2 \cdot (2\sin x + \cos x) + x_4 \cdot 3\cos x = \cos x$$
$$x_1 = \frac{1}{2}$$
$$x_2 = -\frac{1}{6}$$
$$x_3 = 0$$
$$x_4 = \frac{1}{3}$$
$$\begin{bmatrix}\frac{1}{2} & 0\\-\frac{1}{6} & \frac{1}{3}\end{bmatrix}$$

3.17.3 Find the transition matrix from $B' = \{g_1, g_2\}$ to $B = \{f_1, f_2\}$

$$x_1 \cdot (\sin x) + x_3 \cdot \cos x = 2\sin x + \cos x$$
$$x_2 \cdot (\sin x) + x_4 \cdot \cos x = 3\cos x$$
$$x_1 = 2$$
$$x_2 = 0$$
$$x_3 = 1$$
$$x_4 = 3$$
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

3.17.4 Find h in the two basis

$$h = \begin{bmatrix} 2\sin x \\ -5\cos x \end{bmatrix}$$
$$[h]_B = \begin{bmatrix} \sin x \\ \cos x \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = h$$
$$x_1 = 2$$
$$x_2 = -5$$
$$[h]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$
$$[h]_{B'} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \end{bmatrix} [h]_{B'} = \begin{bmatrix} 2 \cdot \frac{1}{2} - 5 \cdot 0 \\ 2 \cdot (-\frac{1}{6}) - 5 \cdot \frac{1}{3} \end{bmatrix}$$
$$[h]_{B'} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$[h]_{B'} = \begin{bmatrix} 2\sin x + \cos x \\ 3\cos x \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = h$$
$$x_1 = 1$$
$$x_2 = -2$$

3.18 Which of these are linear transistions and what is their kernel

- $T(A) = A^2$ - Not linear
- $T(A) = tr(A)$ - all matrices in the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$

- $T(A) = A + A^T$ - all matrices in the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

3.19 Let $T : P_2 \to P_3$ be the linear transformaiton defined by $T(p(x)) = xp(x)$. which are kernels to T

- x^2 false
- 0 true
- $x + 1$ false
- $-x$ false

3.20 Consider the basis $S = \{v_1, v_2\}$ for R^2 where $v_1 = (1, 1)$ and $v_2 = (1, 0)$ and let T be a linear transformation, find $T(5, -3)$

$$\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$
$$x_1 + x_3 = 1$$
$$x_2 + x_4 = -2$$
$$x_1 = -4$$
$$x_2 = 1$$
$$x_3 = 1 - (-4) = 5$$
$$x_4 = -2 - 1 = -3$$
$$\begin{bmatrix} -4 & 5 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -35 \\ 14 \end{bmatrix}$$

3.21 Are the following operator one-to-one

- Orthogonal projection onto x axis in R^2 - false
- The reflection abouth the y axis in R^2 - true
- The reflection about the line $y = x$ in R^2 - true

3.22 Are the following transformation one-to-one, determine using kernel

- $T : R^2 \to R^2$ where $T(x, y) = (y, x)$ - only kernel is $\{(0, 0)\}$ therefore it is one-to-one
- $T : R^3 \to R^2$ where $T(x, y, z) = (x + y + z, x - y - z)$ - only kernel is $\{(0, 0)\}$ therefore it is one-to-one

3.23 Is multiplication by the matrix one-to-one?

$$\begin{bmatrix} 1 & 2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1x_1 - 2x_2 = 0$$
$$2x_1 - 4x_2 = 0$$
$$-3x_1 + 6x_2 = 0$$
$$x_1 = 2x_2$$

It can be seen as long that $x_1 = 2x_2$ the result will be zero therefore the kernel is not $\{0\}$ therefore making it not one-to-one

3.24 Dertermine if the transformation is one-to-one

- $T : V \to W$; nullity$(T) = 0$ - T is one-to-one
- $T : V \to W$ rank$(T) = dim(V)$ - T is one-to-one
- $T : V \to W$ $dim(W) < dim(V)$ - T is not one-to-one

4 Week 40

4.1 Determine if the transformation is isomorphic

$$c_0 + c_1 x \to (c_0 - c_1, c_1)$$

From P_1 to R^2
$ker(T) : \{c_0 = 0, c_1 = 0\}$ therefore making it one-to-one, and onto since every vector is obtainable, making it isomorphic

4.2 Determine if the transformation is isomorphic

$$a + bx + cx^2 + dx^3 \to \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

from P_3 to M_{22}.
Isomorphic

4.3 Find isomorphism between the spaces

4.3.1 All 3×3 symmetric matrices and R^6

$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \to (x_1, x_2, x_3, x_4, x_5, x_6)$

4.3.2 All 2×2 matrices and R^4 in two ways

$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$
(x_1, x_2, x_3, x_4) (x_2, x_3, x_4, x_1)

4.4 Is the transformation isomorphic?

- $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$ the determinant is -3 therefore it is linear independent and isomorphic
- $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$ the determinant is 0 therefore linear dependent and not isomorphic

4.5 Let $T : P_2 \to P_1$ be the linear transformation $T(a_1 + a_1 x + a_2 x^2) = (a_0 + a_1) - (2a_1 + 3a_2)x$

4.5.1 Find the matrix for the linear transformation T relatie to the standard bases $B = \{1, x, x^2\}$ and $B' = \{1, x\}$ for P_2 and P_1

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$b_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(b_1) = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(b_2) = 1 - 2x = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$T(b_3) = -3x = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$[T(b_1)]_{B'} = b_1' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[T(b_2)]_{B'} = b_1' - 2b_2' = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$[T(b_3)]_{B'} = -3b_2' = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$[T]_{B',B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix}$$

4.5.2 Validate the matrix $[T]_{B',B}$ satisfies for every vector $x = c_0 + c_1x + c_2x^2$ in P_2

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix} [c_0, c_1, c_2] = \begin{bmatrix} c_0 + c_2x \\ -2c_2 - 3c_3 \end{bmatrix}$$

4.6 Let $T : R^2 \to R^2$ be the linear operator defined by $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$ and let $B = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\}$ be the basis

4.6.1 Find $[T]_B$

$$T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$T(\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}$$

4.6.2 Verify $[T]_B$

$$[T]_B[x]_B = [T(x)]_B$$

$$\begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

4.7 Let $T : R^2 \to R^3$ be $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 + 2x_2 \\ -x_1 \\ 0 \end{bmatrix}$

4.7.1 Find the matrix $[T]_{B',B}$ relative to bases $B' = \{v_1, v_2, v_3\}$ and $B = \{u_1, u_2\}$

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$T(u_1) = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$$

$$T(u_2) = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}$$

$$[T(u_1)]_B = \begin{bmatrix} 0 \\ -0.5 \\ \frac{8}{3} \end{bmatrix}$$

$$[T(u_2)]_B = \begin{bmatrix} 0 \\ 1 \\ \frac{4}{3} \end{bmatrix}$$

$$[T]_{B',B} = \begin{bmatrix} 0 & 0 \\ -0.5 & 1 \\ \frac{8}{3} & \frac{4}{3} \end{bmatrix}$$

4.8 Let $v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, and let $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ be the matrix for $T : R^2 \to R^2$ relative to the basis $B = \{v_1, v_2\}$

4.8.1 Find $[T(v_1)]_B$ and $[T(v_2)]_B$

$$[T(v_1)]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix} [T(v_2)]_B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}\}$$

4.8.2 Find $T(v_1)$ and $T(v_2)$

4.9 Let $D : P_2 \to P_2$ be the differentiation operator $D(P) = p'(x)$

4.9.1 Find the matrix for D relative to the basis $B = \{p_1, p_2, p_3\}$ for P_2 in which $p_1 = 1, p_2 = x, p_3 = 2 - 3x + 8x^2$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{bmatrix}$$

4.9.2 Use the matrix to compute $D(6 - 6x + 23x^2)$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ -6 \\ 23 \end{bmatrix} = \begin{bmatrix} 63 \\ 368 \\ 0 \end{bmatrix}$$

$$63 + 368x$$

4.10 **Let** $T : R^2 \to R^2$ **be a linear operator and let** B **and** B' **be bases for** R^2 **for which** $[T]_B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ **and** $P_{B\to B'} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$**. Find the matrix for** T **relative to the basis** B'

$$P^{-1} = P_{B'\to B} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$T_{B'} = P_{B\to B'} T_B P_{B'\to B}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 8 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$$

4.11 $T : P_1 \to P_1$ **is defined by** $T(a_0+a_1x) = -a_0+(a_0+a_1)x$ B **is the standard basis for** P_1 **and** $B' = \{q_1, q_2\}$ **where** $q_1 = x+1, q_2 = x-1$**. Find T relative to the basis** B

$$T_B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B'^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$B'^{-1} T_B B' = \begin{bmatrix} 0.5 & 0.5 \\ 1.5 & -0.5 \end{bmatrix}$$

4.12 Check that x is an eigenvector of A and find the eigenvalue

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$
$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$Ax = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\lambda = -1$$

4.13 Check that x is an eigenvector of A and find the eigenvalue

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$
$$x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$
$$Ax = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix}$$
$$\lambda = 5$$

4.14 Find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix

4.14.1 $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}) = 0$$
$$det(\begin{bmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{bmatrix}) = 0$$
$$(\lambda - 1)(\lambda - 3) - (-4 \cdot -2) = 0$$
$$\lambda^2 - \lambda - 3\lambda + 3 - 8 = 0$$
$$\lambda^2 - 4\lambda - 5 = 0$$
$$\lambda = (5, -1)$$

$$\begin{bmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 - x_2 = 0$$
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 + 2x_2 = 0$$
$$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

4.14.2 $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

$$\begin{aligned} det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}) = 0 \\ det(\begin{bmatrix} \lambda + 2 & 7 \\ -1 & \lambda - 2 \end{bmatrix}) = 0 \\ (\lambda - 2)(\lambda - 2) - (7 \cdot -1) = 0 \\ \lambda^2 + 3 = 0 \end{aligned}$$

No real solutions

4.14.3 $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$det(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}) = 0$$

$$det(\begin{bmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{bmatrix}) = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

$$x = (1, 2, 3)$$

$$x = 1$$

$$\begin{bmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 & -1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0 \land x_3 = 0$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x = 2$$

$$\begin{bmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 0.5x_3 = 0 \land x_2 - x_3 = 0$$

$$\begin{bmatrix} 0.5 \\ 1 \\ 1 \end{bmatrix}$$

$$x = 3$$

$$\begin{bmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$

$$x_2 - x_3 = 0$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

4.15 Find the characteristic equation by inspection

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$$

Lower triangle therefore

$$(x - 3)(x - 7)(x - 1) = 0$$

4.16 Find eigenvalues and basis, for $T(x, y) = (x+4y, 2x+3y)$

$$\begin{bmatrix} x & 4y \\ 2x & 3y \end{bmatrix}$$

$$det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}) = 0$$

$$det(\begin{bmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{bmatrix}) = 0$$

$$x^2 - 4x - 5 = 0$$

$$x = (5, -1)$$

$$x = 5$$

$$\begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2 = 0$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x = -1$$

$$\begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

4.17 Show that A and B are not similar matrices

4.17.1 $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$

$det(A) = -1$
$det(B) = -2$
Therefore since they do not equal they can not be similar

4.18 Find the matrix P that diagonalizes $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$det(\begin{bmatrix} \lambda-2 & 0 & 2 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix}) = 0$$
$$(x-2)(x-3)^2 = 0$$
$$x = (2,3)$$

$$x = 2$$
$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$x_2 = 0$$
$$x_3 = 0$$
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x = 3$$
$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$x_1 + 2x_3 = 0$$
$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4.19 Find the geometric and algebraic multiplicity of each eigenvalue of $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$, and if it diagonalizable

$$det(\begin{bmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{bmatrix}) = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

$$x = (1, 2, 3)$$

$$x = 1$$

$$alg : 1$$

$$\begin{bmatrix} 2 & -4 & 2 \\ 3 & -3 & 0 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$

$$x_2 - x_3 = 0$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

geo : 1 since the whole space can be made with 1 vector

$$x = 2$$

$$alg : 1$$

$$\begin{bmatrix} 3 & -4 & 2 \\ 3 & -2 & 0 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - \frac{2}{3}x_3 = 0$$

$$x_2 - x_3 = 0$$

$$\begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}$$

geo : 1 since the whole space can be made with 1 vector

$$x = 3$$

$$alg : 1$$

$$\begin{bmatrix} 4 & -4 & 2 \\ 3 & -1 & 0 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 0.25x_3 = 0$$

$$x_2 - 0.75x_3 = 0$$

$$\begin{bmatrix} 0.25 \\ 0.75 \\ 1 \end{bmatrix}$$

geo : 1 since the whole space can be made with 1 vector

4.20 Compute $\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}^{10}$

$$det(\begin{bmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{bmatrix}) = 0$$
$$x^2 + x - 6 = 0$$
$$x = (2, -3)$$

$$x = 2$$
$$\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 - 1.5x_2 = 0$$
$$\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

$$x = -3$$
$$\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 + x_2 = 0$$
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.5 & 1 \\ 1 & -1 \end{bmatrix}$$
$$P^{-1} = \begin{bmatrix} 0.4 & 0.4 \\ 0.4 & -0.6 \end{bmatrix}$$
$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$
$$A^{10} = PD^{10}p^{-1}$$
$$A^{10} = \begin{bmatrix} 24234 & -34815 \\ -23210 & 35839 \end{bmatrix}$$

5 Week 41

5.1 Compute the standard inner product of $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

$$\begin{aligned} &< U, V > \\ &3 \cdot (-1) + (-2) \cdot 3 + 4 \cdot 1 + 8 \cdot 1 \\ &< U, V >= -3 - 6 + 4 + 8 = 3 \end{aligned}$$

5.2 Compute the standard inner product of $p = -2 + x + 3x^2$, $q = 4 - 7x^2$

$$\begin{aligned} &< p, q > \\ &-2 \cdot 4 + 1 \cdot 0 + 3 \cdot (-7) \\ &< p, q >= -8 + 0 - 21 = -29 \end{aligned}$$

5.3 Determine whether the vectors are orthogonal with repsect to the euclidean inner product

- $u = (-1, 3, 2), v = (4, 2, -1)$ - $< u, v >= -4 + 6 - 2 = 0$
- $u = (-2, -2, -2), v = (1, 1, 1)$ - $< u, v >= -2 - 2 - 2 = -4$
- $u = (a, b), v = (-b, a)$ - $< u, v >= -ab + ab = 0$

Only the last and the first vectors are orthogonal relative given the inner product is 0

5.4 Determine if the vectors are orthogonal and/or orthonormal respect to the inner product

- $(0, 1)$, $(2, 0)$ - $<>= 0$
- $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ - $<>= 0$
- $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}))$ - $<>= -1$

The two first are orthogonal relative and the third has an orthonormal basis

5.5 Transform the basis $\{(1,1,1),(-1,1,0),(1,2,1)\}$ into an orthonormal basis

$$u_1 = (1,1,1)$$
$$u_2 = (-1,1,0) - \frac{<(-1,1,0),(1,1,1)>}{|(1,1,1)|^2}(1,1,1)$$
$$u_2 = (-1,1,0) - \frac{0}{\sqrt{3}^2}(1,1,1)$$
$$u_2 = (-1,1,0) - (0,0,0) = (-1,1,0)$$
$$u_3 = (1,2,1) - \frac{<(1,2,1),(1,1,1)>}{|(1,1,1)|^2}(1,1,1) - \frac{<(1,2,1),(-1,1,0)>}{|(-1,1,0)|^2}(-1,1,0)$$
$$u_3 = (1,2,1) - \frac{4}{\sqrt{3}^2}(1,1,1) - \frac{1}{\sqrt{2}^2}(-1,1,0)$$
$$u_3 = (1,2,1) - (\frac{4}{3},\frac{4}{3},\frac{4}{3}) - (-0.5,0.5,0)$$
$$u_3 = (\frac{1}{6},\frac{1}{6},-\frac{1}{3})$$
$$u_1 = \frac{1}{\sqrt{3}}(1,1,1)$$
$$u_2 = \frac{1}{\sqrt{2}}(-1,1,0)$$
$$u_3 = \frac{\sqrt{6}}{6}(\frac{1}{6},\frac{1}{6},-\frac{1}{3})$$

5.6 Determine if the matrix is orthogonal, and if so find the inverse

- $a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ - $a^T = a^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $a = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ - $a^T = a^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

5.7 For the matrix $A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

5.7.1 Verify that A is orthogonally diagonalisable

It can be seen that A is symmetric therefore it is orthogonally diagonalizable

5.7.2 Find the orthogonal matrix P such that $P^{-1}AP$ is diagonal

$$det(\begin{bmatrix} \lambda - \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \lambda - \frac{3}{2} \end{bmatrix}) = 0$$
$$x^2 - 3x + 2 = 0$$
$$x = (1, 2)$$

$$x = 1$$
$$(\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 - x_2 = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x = 2$$
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 + x_2 = 0$$
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$P^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$$
$$D = P^T AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

5.7.3 Calculate A^{10}

$$P^{-1}D^{10}P = \begin{bmatrix} 512.5 & -511.5 \\ -511.5 & 512.5 \end{bmatrix}$$

5.8 Find the characteristic equation, and the dimensions of the eigenspaces by inspections

5.8.1 $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$det(\begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 4 \end{bmatrix}) = 0$$
$$x^2 - 5x = 0$$
$$x = (0, 5)$$
$$dim(0) = 1$$
$$dim(5) = 1$$

5.8.2 $\begin{bmatrix} \lambda - 1 & 4 & -2 \\ 4 & \lambda - 1 & 2 \\ -2 & 2 & \lambda + 2 \end{bmatrix}$

$$det \begin{bmatrix} \lambda - 1 & 4 & -2 \\ 4 & \lambda - 1 & 2 \\ -2 & 2 & \lambda + 2 \end{bmatrix}) = 0$$
$$x = (6, -3, -3)$$
$$dim(6) = 3$$
$$dim(-3) = 2$$