Seção 1.11

1)

a) Elipsóide $S = \{ (x, y, z) \in \mathbb{R}^3 ; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \}$ menos dois pontos.

b) Hiperbolóide de uma folha

$$S=\{(x,y,z)\in\mathbb{R}^3;\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\}.$$

Como $\operatorname{sen}^2 u+\cos^2 u=1$ e $\cosh^2 v-\operatorname{senh}^2 v=1$, então

$X(u,v)=(a\cosh v\cos u, b\cosh v\operatorname{sen} u, c\operatorname{senh} v)$ é uma parametrização de S.

É regular e injetiva quando $u,v\in[0,2\pi)$, pois

$$\begin{cases} X_v=(a\operatorname{senh} v\cos u, b\operatorname{senh} v\operatorname{sen} u, c\cosh v) \\ X_u=(-a\cosh v\operatorname{sen} u, b\cosh v\cos u, 0)\end{cases}$$

$\Rightarrow |X_u\times X_v|^2=\cosh^2 v\left[(bc\cosh v\cos u)^2+(ac\cosh v\operatorname{sen} u)^2+(ab\operatorname{senh} v)^2\right]\neq 0$ $\quad u,v\in[0,2\pi).$

c) Hiperbolóide de duas folhas

$$S=\{(x,y,z)\in\mathbb{R}^3;\ \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\}.$$

Como $\cosh^2 v-\operatorname{senh}^2 v=1$ e $\operatorname{senh}^2 u-\cosh^2 u=-1$,

então

$X(u,v)=(a\operatorname{senh} v\operatorname{senh} u, b\operatorname{senh} v\cosh u, c\cosh v)$ é uma parametrização de S.

É regular e injetiva quando $u, v \in [0, 2\pi)$, pois

$$\begin{cases} X_u = (a \,\text{senh}\, v \cosh u, b \,\text{senh}\, v \,\text{senh}\, u, 0) \\ X_v = (a \cosh v \,\text{senh}\, u, b \cosh v \cosh u, c \,\text{senh}\, v) \end{cases}$$

$$\Rightarrow |X_u \times X_v|^2 = \text{senh}^2 v \left[(bc \,\text{senh}\, v \,\text{senh}\, u)^2 + (ac \,\text{senh}\, v \cosh u)^2 + (ab \cosh v)^2 \right] \neq 0, \quad u, v \in [0, 2\pi).$$

d) Cone de uma folha menos o vértice

$$S = \{(x, y, z) \in \mathbb{R}^3 \setminus (0,0,0) \,;\, z = \sqrt{x^2 + y^2}\}.$$

Uma parametrização de S é:

$X(u, v) = (u, v, \sqrt{u^2 + v^2})$, que é regular e injetiva quando $u, v \in \mathbb{R} \setminus \{0\}$.

2) Como $\dfrac{(au \,\text{senh}\, v)^2}{a^2} - \dfrac{(bu \cosh v)^2}{b^2} + u^2 = 0$, então $X(u, v)$ descreve o paraboloide hiperbólico menos um ponto.

É uma superfície parametrizada regular, pois é derivável e

$$\begin{cases} X_u = (a \cosh v, b \,\text{senh}\, v, 2u) \\ X_v = (au \,\text{senh}\, v, bu \cosh v, 0) \end{cases}$$

$$\Rightarrow X_u \times X_v = (-2bu^2 \cosh v, 2au^2 \,\text{senh}\, v, -abu) \neq 0, \quad u \in \mathbb{R} \setminus \{0\}, \; v \in \mathbb{R}.$$

3) a) Plano $x=0$. $S=\{(x,y,z)\in\mathbb{R}^3;\ x=0\}$

b) Plano $y=2x$. $S=\{(x,y,z)\in\mathbb{R}^3;\ y=2x\}$

c) $X(u,v)=(\cos u,\ 2\,\mathrm{sen}\,u,\ v)$, $(u,v)\in\mathbb{R}^2$.

5) $X(u,t)=(t+u,\ t^2+2ut,\ t^3+3ut^2)$

$$\Rightarrow \begin{cases} X_u=(1,\ 2t,\ 3t^2) \\ X_t=(1,\ 2t+2u,\ 3t^2+6ut) \end{cases}$$

$$\Rightarrow |X_u\times X_v|^2 = u^2\left[(6t^2)^2+(6t)^2+(2)^2\right]\neq 0,\quad u\in(0,\infty),\ t\in\mathbb{R}.$$

7) $\alpha'(u)\neq 0$, $f(u)\neq 0$

$$\begin{cases} X_u=(f_u\cos v,\ f_u\,\mathrm{sen}\,v,\ g_u) \\ X_v=(-f(u)\,\mathrm{sen}\,v,\ f(u)\cos v,\ a) \end{cases}$$

$$|X_u\times X_v|^2=(a f_u)^2+f^2(u)\left[f_u^2+g_u^2\right]\neq 0.$$

Seção 2.5

1) $X(u,v)=(u,v,0)$, $(u,v)\in\mathbb{R}^2$

$\bar{X}(u,v)=(u\cos v, u\,\text{sen}\, v, 0)$, $u\in\mathbb{R}\setminus\{0\}$, $v\in\mathbb{R}$

$h(u,v)=(x(u,v), y(u,v)) \Rightarrow X=\bar{X}\circ h$

$\Rightarrow \quad (u,v,0)=(x\cos y, x\,\text{sen}\, y, 0)$

$\Rightarrow \quad x=\dfrac{u}{\cos\left(tg^{-1}\left(\frac{v}{u}\right)\right)}$, $y=tg^{-1}\left(\frac{v}{u}\right)$, $u\in\mathbb{R}\setminus\{0\}$
$v\in\mathbb{R}$.

2) a) $h(\bar{u},\bar{v})=(x(\bar{u},\bar{v}), y(\bar{u},\bar{v}))=(\bar{x},\bar{y})$

$\Rightarrow X\circ h(\bar{u},\bar{v})=(a\bar{x}\cosh\bar{y}, b\bar{x}\,\text{senh}\,\bar{y}, \bar{x}^2)$

$=(a(\bar{u}+\bar{v}), b(\bar{u}-\bar{v}), \bar{u}\bar{v})$

$\Rightarrow \quad \bar{x}=\dfrac{\bar{u}+\bar{v}}{\cosh\left(tgh^{-1}\left(\frac{\bar{u}-\bar{v}}{\bar{u}+\bar{v}}\right)\right)}$, $\bar{y}=tgh^{-1}\left(\dfrac{\bar{u}-\bar{v}}{\bar{u}+\bar{v}}\right)$.

Seção 3.8

1) $X(u,v)=(u,v,f(u,v))$, $f: U\subset\mathbb{R}^2\to\mathbb{R}$ diferenciável.

$$\begin{cases} X_u=(1,0,f_u) \\ X_v=(0,1,f_v)\end{cases} \Rightarrow \begin{array}{l} X_u\times X_v=(-f_u,-f_v,1) \\ |X_u\times X_v|=\sqrt{1+f_u^2+f_v^2}\end{array}$$

$$\Rightarrow N(u,v)=\frac{X_u\times X_v}{|X_u\times X_v|}$$

3) $X(u,v)=(a\cos u, a\,\mathrm{sen}\, u, v)$, $(u,v)\in\mathbb{R}^2$, $a>0$

$$\begin{cases} X_u=(-a\,\mathrm{sen}\, u, a\cos u, 0) \\ X_v=(0,0,1)\end{cases} \Rightarrow \begin{array}{l} X_u\times X_v=(a\cos u, a\,\mathrm{sen}\, u, 0) \\ |X_u\times X_v|=a.\end{array}$$

$$N(u,v)=(\cos u, \mathrm{sen}\, u, 0)$$

Seção 4.8

1)

$$X(u,v) = (u+v,\ u-v,\ 4uv)$$

$$\bar{X}(\bar{u},\bar{v}) = (\bar{u},\ \bar{v},\ \bar{u}^2 - \bar{v}^2)$$

$$h(\bar{q}) = (x(\bar{u},\bar{v}),\ y(\bar{u},\bar{v})) = (\bar{x},\bar{y})$$

$$\bar{X} = X \circ h(\bar{q}) \Rightarrow (\bar{u},\ \bar{v},\ \bar{u}^2-\bar{v}^2) = (\bar{x}+\bar{y},\ \bar{x}-\bar{y},\ 4\bar{x}\bar{y})$$

$$\Rightarrow \bar{x} = \frac{\bar{u}+\bar{v}}{2},\ \bar{y} = \frac{\bar{u}-\bar{v}}{2}$$

$$\Rightarrow h(\bar{q}) = \left(\frac{\bar{u}+\bar{v}}{2},\ \frac{\bar{u}-\bar{v}}{2}\right)$$

$$\begin{cases} \bar{X}_{\bar{u}} = (1,0,2\bar{u}) \\ \bar{X}_{\bar{v}} = (0,1,-2\bar{v}) \end{cases} \Rightarrow \begin{array}{l} \bar{E} = 1+4\bar{u}^2 \\ \bar{F} = -4\bar{u}\bar{v} \\ \bar{G} = 1+4\bar{v}^2 \end{array}$$

$$\begin{cases} X_u = (1,1,4v) \\ X_v = (1,-1,4u) \end{cases} \Rightarrow \begin{array}{l} E = 2+16v^2 \\ F = 16uv \\ G = 2+16u^2 \end{array}$$

$$\Rightarrow E \circ h(\bar{q}) = 2+8(\bar{u}-\bar{v})^2 \neq \bar{E}$$

$$F \circ h(\bar{q}) = 4(\bar{u}^2-\bar{v}^2) \neq \bar{F}$$

$$G \circ h(\bar{q}) = 2+8(\bar{u}+\bar{v})^2 \neq \bar{G}$$

2) $X(u,v) = (v\cos u, v\,\text{sen}\,u, v)$, $u \in \mathbb{R}$ e $v > 0$

$\alpha(t) = X(\sqrt{2}t, e^t)$, $t \in \mathbb{R}$.

$\alpha(t) = X(u(t), v(t)) \Rightarrow \alpha'(t) = u_t X_u + v_t X_v$

onde:

$$u_t = \sqrt{2}, \quad v_t = e^t$$

$$X_u = (-v\,\text{sen}\,u, v\cos u, 0)$$

$$X_v = (\cos u, \text{sen}\,u, 1).$$

4) $X(u,v) = (u, v, f(u,v))$, $(u,v) \in \mathbb{R}^2$, f é diferenciável

a) $X_u = (1, 0, f_u)$, $X_v = (0, 1, f_v)$

$\langle X_u, X_v \rangle = f_u \cdot f_v$ é ortogonal $\Leftrightarrow f_u \cdot f_v = 0$.

b) $A(X(D)) = \iint_D \sqrt{1 + f_u^2 + f_v^2}\, du\,dv \geq \iint_D du\,dv = A(D)$

$A(X(D)) = A(D) \Leftrightarrow \sqrt{1 + f_u^2 + f_v^2} = 1 \Leftrightarrow f_u^2 + f_v^2 = 0.$

9) Se X e $\bar{X}$ são isométricas, então $E = \bar{E}$, $F = \bar{F}$ e $G = \bar{G}$ $\Rightarrow \sqrt{EG - F^2} = \sqrt{\bar{E}\bar{G} - \bar{F}^2}$

$$\Rightarrow A(X(D)) = A(\bar{X}(D)).$$

8) $X(u,v) = (a\cos u \cos v,\ b\cos u\, \text{sen}\, v,\ c\, \text{sen}\, u)$, $u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$v \in [0, 2\pi)$.

$$\begin{cases} X_u = (-a\,\text{sen}\, u \cos v,\ -b\,\text{sen}\, u\, \text{sen}\, v,\ c\cos u) \\ X_v = (-a\cos u\, \text{sen}\, v,\ b\cos u \cos v,\ 0) \end{cases}$$

$E(u,v) = |X_u|^2 = (a\,\text{sen}\, u \cos v)^2 + (b\,\text{sen}\, u\, \text{sen}\, v)^2 + (c\cos u)^2$

$F(u,v) = \langle X_u, X_v \rangle = \text{sen}\, u \cos u\, \text{sen}\, v \cos v\, (a^2 - b^2)$

$G(u,v) = |X_v|^2 = \cos^2 u\, (a^2 \text{sen}^2 v + b^2 \cos^2 v)$

$$A(X(D)) = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{EG - F^2}\, du\, dv.$$

7) $X(u,v) = (u\cos v,\ u\,\text{sen}\, v,\ av + b)$

$\alpha(t) = X(u(t), v(t))$

$\Rightarrow \begin{cases} X_u = (\cos v, \text{sen}\, v, 0) \\ X_v = (-u\,\text{sen}\, v, u\cos v, a) \end{cases} \Rightarrow E(u,v) = 1,\ F(u,v) = 0$
$G(u,v) = u^2 + a^2$

Seja $w = \alpha'(t) = u_t X_u + v_t X_v$

$\Rightarrow I_q(w) = (u_t)^2 E + 2u_t \cdot v_t F + (v_t)^2 G$

$\Rightarrow I_q(w) = (u_t)^2 + (v_t)^2 (u^2 + a^2)$

5) $X(u,v) = (\operatorname{sen} v \cos u, \operatorname{sen} v \operatorname{sen} u, \cos v)$, $u \in \mathbb{R}$, $v \in (0,\pi)$

$\alpha(t) = X(u(t), v(t))$, $u(t) = \log \cot\left(\frac{\pi}{4} - \frac{t}{2}\right)$,

$v(t) = \frac{\pi}{2} - t$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\alpha'(t) = u_t X_u + v_t X_v$

$$\begin{cases} X_u = (-\operatorname{sen} v \operatorname{sen} u, \operatorname{sen} v \cos u, 0) \\ X_v = (\cos v \cos u, \cos v \operatorname{sen} u, -\operatorname{sen} v) \end{cases} \Rightarrow \begin{array}{l} E = \operatorname{sen}^2 v \\ F = 0, \ G = 1 \end{array}$$

$$u_t = \frac{1}{\cot\left(\frac{\pi}{4} - \frac{t}{2}\right)} \cdot \left(-\csc^2\left(\frac{\pi}{4} - \frac{t}{2}\right)\right) \cdot \left(-\frac{1}{2}\right), \quad v(t) = -1$$

$$\Rightarrow u_t = \frac{1}{2 \operatorname{sen}\theta \cos\theta}, \quad \theta = \frac{\pi}{4} - \frac{t}{2}$$

$$\Rightarrow I_q(w) = (u_t)^2 E + 2 u_t v_t F + (v_t)^2 G$$

$$= \frac{1}{4 \operatorname{sen}^2\theta \cos^2\theta} \cdot \operatorname{sen}^2 v - 1$$

$\operatorname{sen}(2\theta) = 2 \operatorname{sen}\theta \cos\theta$ e $2\theta = v$

$\Rightarrow \operatorname{sen}^2 v = 4 \operatorname{sen}^2\theta \cos^2\theta$

$\Rightarrow I_q(w) = 0$

11)

a) Se p_1 e p_2 estão, localmente, em um plano, então o ínfimo dos comprimentos das curvas que ligam p_1 a p_2 é uma reta, ou seja,

$|p_1 - p_2| = d(p_1, p_2)$.