Statistical Machine Learning

Summer Term 2021, Homework 1

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Total points: 80
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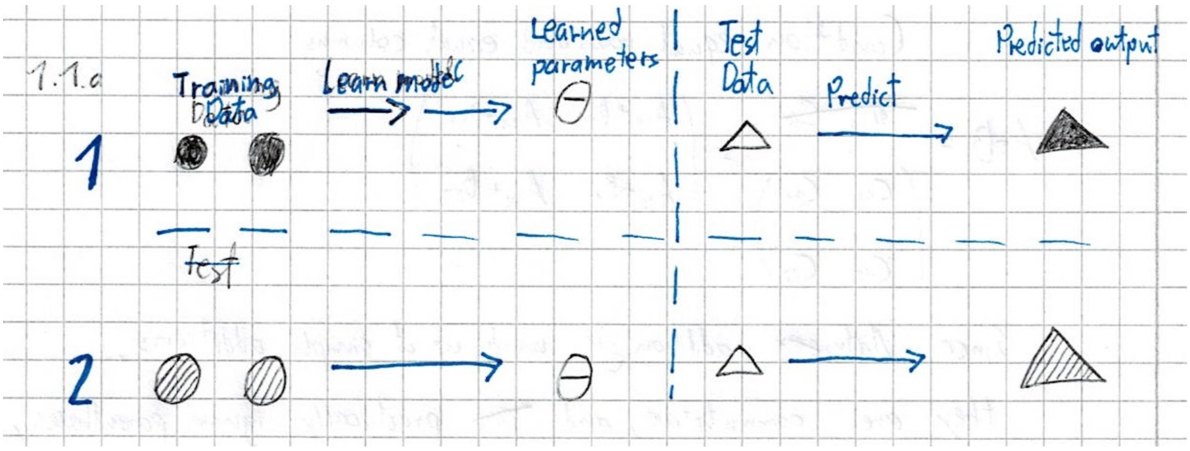
Problem 1.1 Machine Learning Introduction [6 Points]

a) Model Fitting [6 Points]



Model 1 is being trained on filled circles, and as such assumes that the triangle is also to be filled, coming to an incorrect result.
Model 2 is being trained on striped circles, and as such assumes that the triangle is also to be striped, coming to the correct conclusion.

Problem 1.2 Linear Algebra Refresher [20 Points]

a) Matrix Properties [5 Points]

1. Multiplication

$$A*B = \begin{pmatrix} A_{1,1}*B_{1,1}+A_{2,1}*B_{1,2} & A_{1,1}*B_{2,1}+A_{2,1}*B_{2,2} \\ A_{1,2}*B_{1,1}+A_{2,2}*B_{1,2} & A_{1,2}*B_{2,1}+A_{2,2}*B_{2,2} \end{pmatrix}$$

$A*B$ is defined only when Columns of B equal rows of A.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

BUT

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrixmultiplication is not Commutative.
$A(B+C) = AB + AC$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} * \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

Matrixmultiplication is Distributive.

$(A*B)*C = A*(B*C)$

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} * \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

2. Addition Since matrix-addition is made up of simple additions, they are commutative and practically ignore parentheses, so are also distributive and associative.
The condition for any matrix-addition is that the matrices have both equal rows and columns.

$$A_{2,2} + B_{2,2} = \begin{pmatrix} C_{1,1} & C_{2,1} \\ C_{1,2} & C_{2,2} \end{pmatrix} = \begin{pmatrix} A_{1,1}+B_{1,1} & A_{2,1}+B_{2,1} \\ A_{1,2}+B_{1,2} & A_{2,2}+B_{2,2} \end{pmatrix}$$

b) Matrix Inversion [7 Points]

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

...Via Gauß-Jordan. Under the condition that $a = b = d = 0; c = 1$

$$\begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 0 \\ 8 & 3 & 12 \end{pmatrix}$$

Is not invertable, since it's Determinant $Det = 0$.

c) Matrix Pseudoinverse [3 Points]

Left: $A^{\#} * A = \left(A^T * A\right)^{-1} * A^T$
Right: $A * A^{\#} = A * A^T \left(A * A^T\right)^{-1}$
Since A_{2x3} has more rows than columns, the left Moore-Penrose exists.
The equation is: $A^{\#}_{3x2} * A = \left(A^T_{3x2} * A_{2x3}\right)^{-1}_{2x2} * A^T_{3x2}$

d) Basis Transformation [5 Points]

Vector with new Basis $v^* = T^{-1} * v$

1) $T_v = E^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; T_w = B^{-1} = \begin{pmatrix} 2 & -1.5 \\ -1 & 1 \end{pmatrix}$

2) $2 * \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 5 * \begin{bmatrix} -1,5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3,5 \\ 3 \end{bmatrix} = v^*$

Problem 1.3 Statistics Refreshner [29 Points]

a) Expectation & Variance [8 Points]

1. We can define the expectation by

$$E|f| = \sum_{w \in \Omega} P(w) f(w) \tag{18}$$

Which leads to the variance:

$$var[f] = E\left|f^2\right| - E[f]^2. \tag{19}$$

If we have 2 random variables X,Y and Z = X + Y. Then the expectation is a linear function, since for any 2 points

$$E[Z] = \sum_{w \in \Omega} Z(w) P(w) = \sum_{w \in \Omega} (X(w) + Y(w)) P(w) = E[X] + E[Y] \tag{20}$$

applies. Since the variance of the sum of 2 random variables is

$$var[Z] = E\left[Z^2\right] - E[Z]^2 = var|X| + var[Y] + 2E[XY] - 2E[X]E[Y] isE[XY] \neq E[X]E[Y] \tag{21}$$

$$Andthatisnotalinearoperator. \tag{22}$$

2. Unbiased estimator:

$$\overline{x} = \frac{1}{n} * \sum_{i=1}^{n} x_i \tag{23}$$

$$\overline{xA} = \frac{1}{6} * (1+5+6+3+2+1) = 3 \tag{24}$$

$$\overline{xB} = \frac{1}{6} * (6+1+1+4+1+5) = 3 \tag{25}$$

$$\overline{xC} = \frac{1}{6} * (3+2+3+3+4+3) = 3 \tag{26}$$

unbiased estimator for the variance

$$\overline{\sigma} = \frac{1}{n-1} * \sum_{i=1}^{n} x_i - \overline{x}^2 \tag{27}$$

$$\overline{\sigma A} = \frac{1}{5} * \left((1-3)^2 + (5-3)^2 + (6-3)^2 + (3-3)^2 + (2-3)^2 + (1-3)^2\right) = \frac{22}{5} = 4,4 \tag{28}$$

$$\overline{\sigma B} = \frac{1}{5} * \left((6-3)^2 + (1-3)^2 + (1-3)^2 + (4-3)^2 + (1-3)^2 + (5-3)^2\right) = \frac{26}{5} = 5,2 \tag{29}$$

$$\overline{\sigma C} = \frac{1}{5} * \left((3-3)^2 + (2-3)^2 + (3-3)^2 + (4-3)^2 + (3-3)^2 + (3-3)^2\right) = \frac{2}{5} = 0,4 \tag{30}$$

3.

$$KL : \sum_{x \in X} P(x) ln \frac{P(x)}{Q(x)} \tag{31}$$

$$KL(PA \parallel Q) = \frac{3}{6} * ln\left(\frac{\frac{3}{6}}{\frac{1}{6}}\right) + \frac{1}{6} * ln\left(\frac{\frac{1}{6}}{\frac{1}{6}}\right) + \frac{1}{6} * ln\left(\frac{\frac{1}{6}}{\frac{1}{6}}\right) + \frac{1}{6} * ln\left(\frac{\frac{1}{6}}{\frac{1}{6}}\right) + \frac{1}{6} * ln\left(\frac{\frac{1}{6}}{\frac{1}{6}}\right) = 3 \tag{32}$$

$$KL(PB \parallel Q) = \frac{1}{6} * ln\left(\frac{\frac{1}{6}}{\frac{1}{6}}\right) + \frac{2}{6} * ln\left(\frac{\frac{2}{6}}{\frac{1}{6}}\right) + \frac{1}{6} * ln\left(\frac{\frac{1}{6}}{\frac{1}{6}}\right) + \frac{1}{6} * ln\left(\frac{\frac{1}{6}}{\frac{1}{6}}\right) = 1,52 \tag{33}$$

$$KL(PC \parallel Q) = \frac{4}{6} * ln\left(\frac{\frac{4}{6}}{\frac{1}{6}}\right) + \frac{1}{6} * ln\left(\frac{\frac{1}{6}}{\frac{1}{6}}\right) + \frac{1}{6} * ln\left(\frac{\frac{1}{6}}{\frac{1}{6}}\right) = 2.38 \tag{34}$$

A has the biggest KL divergence, so it is the closest

b) It is a cold world [7 Points]

1.

$$a \in \{0,1\} : if apersonhasbackspin, with 1 = painand 0 = nopain \tag{43}$$

$$b \in \{0,1\} : if apersonhasacold, with 1 = painand 0 = nocold \tag{44}$$

2.

$$P(a=1 \mid b=1) = 0,25 \tag{45}$$

$$P(b=1) = 0,04 \tag{46}$$

$$P(a=1 \mid b=0) = 0,1 \tag{47}$$

3. Rule of Bayes

$$P(b=1 \mid a=1) = \frac{P(a=1 \mid b=1) P(b=1)}{P(b=1)} \tag{48}$$

$$\frac{P(a=1 \mid b=1) P(b=1)}{P(a=1 \mid b=1) P(b=1) + P(a=1 \mid b=0) P(b=0)} \tag{49}$$

$$Werteeinsetzen : \frac{0,25 * 0,04}{0,25 * 0,04 + 0,10 * (1 - 0,04)} = \frac{5}{53} \approx 0,094 \tag{50}$$

c) Cure the virus [14 Points]

1. Markov Chain

$$S_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{62}$$

$$S_1 = \begin{pmatrix} 0,42 \\ 0,58 \end{pmatrix} \tag{63}$$

$$P = \begin{pmatrix} 0,42 & 0,974 \\ 0,58 & 0,026 \end{pmatrix} \tag{64}$$

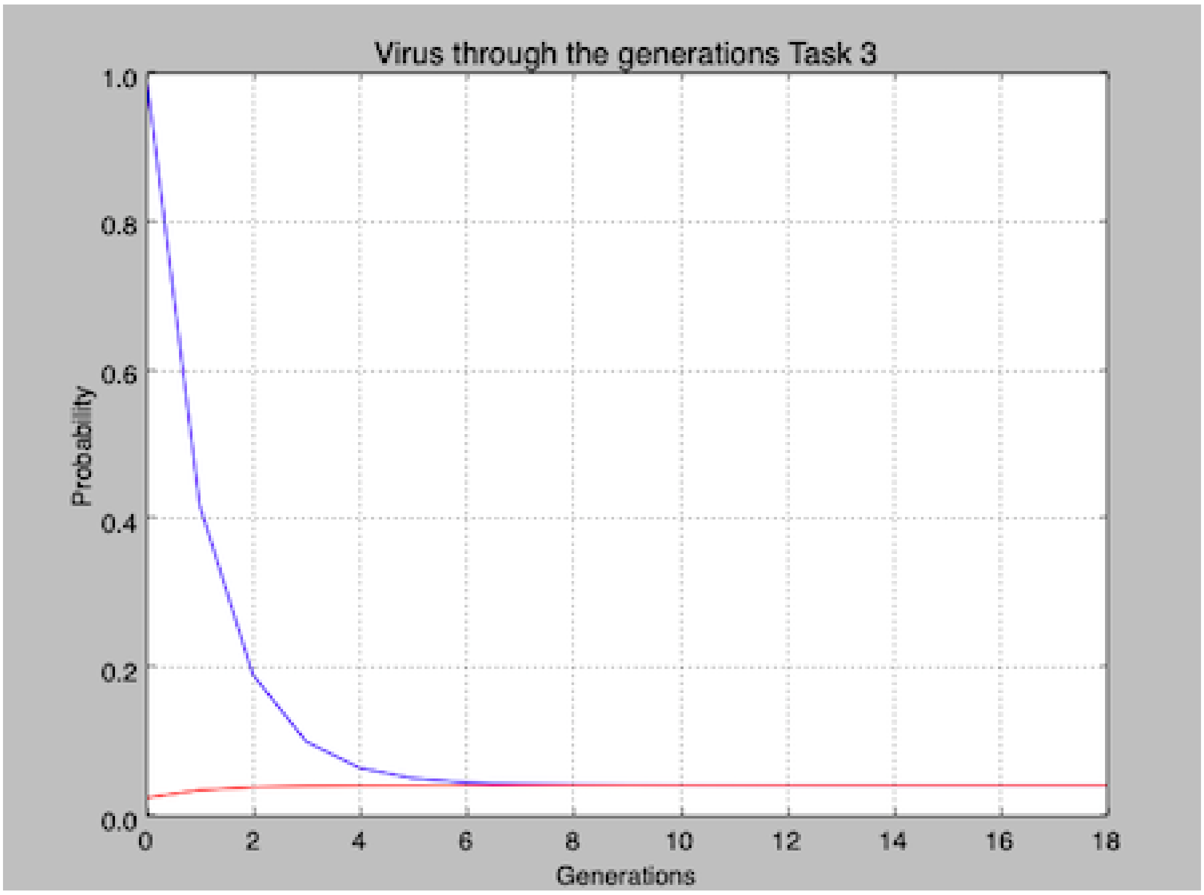
$$S_1 = P * S_0 \leftrightarrow \begin{pmatrix} 0,42 \\ 0,58 \end{pmatrix} = \begin{pmatrix} 0,42 & 0,974 \\ 0,58 & 0,026 \end{pmatrix} * \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{65}$$

$$S_2 = P * S_1 \leftrightarrow \begin{pmatrix} 0,741 \\ 0,394 \end{pmatrix} = \begin{pmatrix} 0,42 & 0,974 \\ 0,58 & 0,026 \end{pmatrix} * \begin{pmatrix} 0,42 \\ 0,58 \end{pmatrix} \tag{66}$$

2.
```
def markovChain(s0, p, g):
    s = s0.copy()
    result = np.zeros(g+1)
    result[0]=s[0]
    for i in range(1, g+1):
        s = s.dot(p)
        result[i]=s[0]
    return result

s0 = np.array([1, 0])
s1 = np.array([0.026, 0.974])
p = np.array([[0.42, 0.58],[0.026, 0.974]])
n = np.arange(0, 19, 1)

gen18 = markovChain(s0, p, 18)
gen18prog = markovChain(s1, p, 18)
```



3. After 6 timesteps does the ratios stop to change significantly. Stable probability:

$$P = \begin{pmatrix} 0,42 & 0,974 \\ 0,58 & 0,026 \end{pmatrix} \tag{67}$$

$$\overline{X} = \begin{pmatrix} A \\ B \end{pmatrix} \tag{68}$$

$$P * \overline{X} = \overline{X} \leftrightarrow \begin{pmatrix} 0,42 & 0,974 \\ 0,58 & 0,026 \end{pmatrix} * \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \tag{69}$$

$$0,42A + 0,5B = A \Rightarrow 0,5B = A - 0,42A \Rightarrow B = \frac{25}{29} = 0,86 \tag{70}$$

$$0,58A + 0,026B = B \tag{71}$$

$$A + B = 1 \Rightarrow A + 0,86 = 1 \Rightarrow A = 0,14 \tag{72}$$

We can see the probability converge to our solution.

Problem 1.4 Information Theory [5 Points]

a) Entropy [5 Points]

1.
$$-0,04 * \log_2 0,04 - 0,22 * \log_2 0,22 - 0,67 * \log_2 0,67 - 0,07 * \log_2 0,07 = 1,3219 \tag{75}$$

An average of 1 bit can be transmitted.

2.
$$H = ln(4) = 1,386 \approx 2 \tag{76}$$

Maximum of 2 bits per symbol can be transmitted using a set of four symbols. The distribution over the symbols requires that at least the maximum is as great as that of all other members.

Problem 1.5 Bayesian Decision Theory [20 Points]

a) Optimal Boundary [4 Points]

1. Bayesian decision theory is a fundamental statistical approach to the problem of pattern classification based probabilities.
2. The goal is to decide which class an example x most likely belongs to. This is done by comparing the class posterior probabilities $p(C_i \mid x)$, which can be calculated by Bayes' theorem:

$$p(C_i \mid x) = \frac{p(x \mid C_i)\, p(C_i)}{p(x)} \propto p(x \mid C_i)\, p(C_i)$$

3. The decision boundary of two classes C_1 and C_2 is given by $p(C_1 \mid x) = p(C_2 \mid x)$, where C_1 is chosen over C_2 if $p(C_1 \mid x) > p(C_2 \mid x)$.

b) Decision Boundaries [8 Points]

Given that the propabilites and variances of the two classes are equal, the decision boundary should only be influenced by the two means, sitting centered between them.(x represents the boundary)

$$(x-\mu_1)^2 = (x-\mu_2)^2\, x = \frac{(x-\mu_1)^2 = (x-\mu_2)}{Q*(\mu_1-\mu_2)} \tag{80}$$

$$If\,\mu_1 = \mu_2, NodecisionBoundary \tag{81}$$

$$else : x = \frac{\mu_1+\mu_2}{2} \tag{82}$$

c) Different Misclassification Cost [8 Points]

Given that wrongly identifying a case of C_2 as C_1 is more costly than the other way around, the Decision Boundary has to be moved towards C_1, causing samples to be more often identified as C_2
$\mu_1 > 0; \mu_1 = 2*\mu_2; \delta_1 = \delta_2; p(C_1) = p(C_2)$

$4\left(2\pi*\delta_1^2\right)^{\frac{-1}{2}} * \exp\left(-\frac{(x-\mu_1)^2}{2\delta_1^2}\right) * p(C_1) = \left(2\pi*\delta_2^2\right)^{\frac{-1}{2}} * \exp\left(-\frac{(x-\mu_2)^2}{2\delta_2^2}\right) * p(C_2)$

$\log(4) + \frac{(x-\mu_2)^2}{2\delta_2^2} = \frac{(x-\mu_2)^2}{2\delta_2^2}$

$2\mu_2^2 * x - 3\mu_2^2 = -\log(4) * \delta_2^2$

$x = \frac{3\mu_2^2 - \log(4)*2\delta_2}{2\mu_2^2}$