Applications of the Barrier Method and Perturbation Analysis for Microwave Resonator Design

Kevin Lie-Atjam

May 2024

Background

The Barrier Method

Constrained optimization is a fundamental problem in various fields, aiming to find the best solution while adhering to certain limitations. There are multiple different ways in which people have tried to optimize different parameters such as the Simplex Method, Simulated Annealing, and genetic algorithms. Imagine minimizing a cost function while staying within the specified parameters. Traditional optimization methods struggle with incorporating these constraints directly.

One recent method that comes to mind that can incorporate these constraints is the Barrier method. The Barrier method can be traced back as early as the 18th century with Euler postulating ideas that mimic this method. The barrier method introduces a special function called a barrier function. This function acts like a fence, pushing the solution away from the boundaries defined by the constraints. As the solution gets closer to violating a constraint, the barrier function's value rapidly increases, making such points undesirable. Instead of directly dealing with the constraints, the barrier method combines the original cost function with a scaled version of the barrier function. This creates a new objective function that considers both minimizing the cost and staying within the allowed region. By minimizing this new objective function, we effectively find a solution that balances the original cost function with staying within the constraints. As the barrier parameter (scaling factor) is gradually reduced, the solution approaches the true optimum while respecting the constraints.

An Example of the Barrier Method Applied

Suppose we want to minimize the function $f(x,y) = x^2 + y^2$ subject to the constraint $x + y \geq 1$. The problem can be formulated as

$$\begin{aligned} \text{minimize} \quad & f(x,y) = x^2 + y^2 \\ \text{subject to} \quad & g(x,y) = x + y - 1 \geq 0 \end{aligned}$$

We now introduce a barrier function to approximate the constraint $g(x,y)$. Typically, a function with a slow growth is used for the barrier method. This function is typically logarithmic to be used in the solution space.

$$B(x,y) = \log(x + y - 1)$$

We also introduce a barrier parameter $\alpha > 0$ and define the modified objective function as

$$\text{minimize} \quad f(x,y) - \alpha B(x,y)$$

This barrier parameter will have a weight that decreases over time such that certain solutions will be penalized. This weight will be decreased gradually such that the solutions will obey the constraints but will get closer towards a feasible yet optimal solution. Our objective function becomes

$$\text{minimize} \quad x^2 + y^2 - \alpha \log(x + y - 1)$$

We can now solve this unconstrained optimization problem using classically known optimization techniques. As we decrease α and solve the modified problems iteratively, the solution will converge to the optimal solution of the original constrained problem.

Perturbation Analysis

Even after optimization, there are certain parameters that we want to know about that could affect the accuracy or convergence of the optimization. This is where we would employ perturbation analysis. The core principle of perturbation analysis hinges on the concept of an unperturbed system. This initial state represents the system under the assumption that all parameters hold their nominal values. The solution to the unperturbed problem serves as a baseline for the system's behavior. Subsequently, a perturbation, ϵ, is introduced. This perturbation signifies a small change in one or more of the system's parameters.

The crux of perturbation analysis lies in analyzing the impact of this ϵ-perturbation on the solution. Tools such as Taylor series expansions, are employed to approximate the solution of the perturbed system. These expansions express the solution as a function of the nominal parameter values and the perturbation ϵ. By analyzing the leading terms of the expansion, we can glean valuable insights into how the solution varies with respect to the parameter changes. Typically, we can view the first order effects of this perturbation to see which factors contribute greatly to small changes.

Perturbation analysis offers plenty of advantages. It facilitates the identification of sensitive parameters within a system. By using the Taylor series expansion, we can determine which parameters significantly influence the solution and warrant closer attention. Additionally, this methodology empowers the estimation of potential errors arising from uncertainties in the input parameters themselves. Perturbation analysis provides a framework to quantify these potential errors based on the level of uncertainty associated with the parameters.

Furthermore, perturbation analysis plays a pivotal role in stability analysis. This subfield investigates how the system's behavior evolves over time in response to perturbations. Through perturbation analysis, we can assess the system's stability. A stable system exhibits resilience, returning to its original state (equilibrium) following a minor perturbation. Conversely, an unstable system deviates significantly from its equilibrium state due to the perturbation.

We can use the aspect of perturbation analysis to be applied in specific systems to figure out system stability parameters. The following aforementioned methods can be applied to a case of quantum hardware for quantum computers.

The Readout Resonator

In quantum computers, information is stored in qubits rather than bits in a classical computer. Classical methods of representing bits can be done in multiple ways such as looking at the spin of an electron, the charge on a capacitor, or the current flowing through a transistor. Just like a classical bit, a qubit can also be represented in a multitude of ways. One of the most popular ways of representing these qubits is to utilize a circuit known as a transmission line shunted plasma oscillation qubit or transmon qubit for short. A transmon qubit is a specific type of artificial atom designed for use in quantum computers. Built from superconducting materials, it operates at incredibly cold temperatures and harnesses the quantum properties of electrons. Unlike natural atoms, a transmon qubit encodes its quantum information in the number of Cooper pairs that tunnel across a Josephson junction, a special type of superconductor connection. This design reduces its sensitivity to electrical noise for precise control.

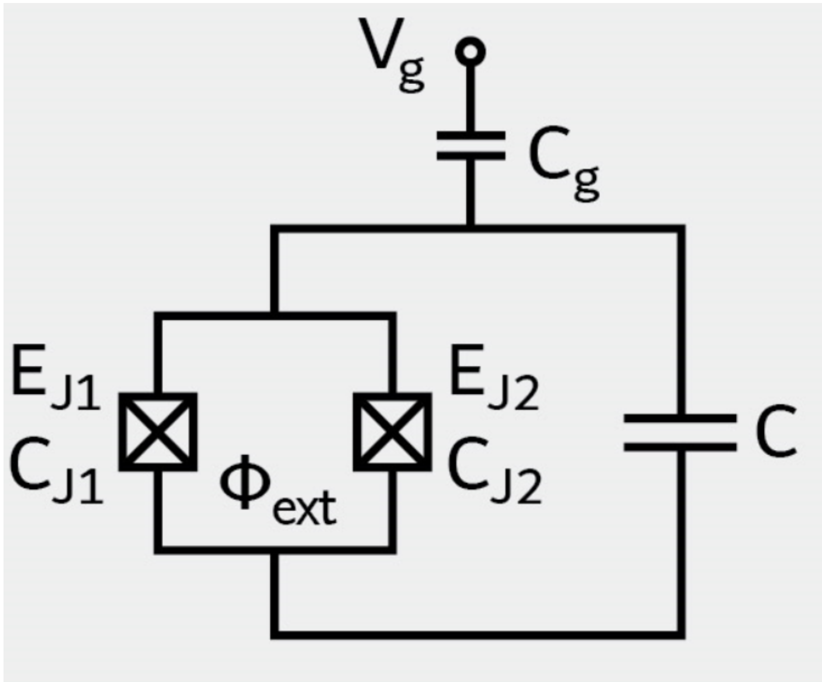

Figure 1: Basic Schematic of A Transmon

One of the key challenges in working with these transmons is reading out and manipulating their state. This is where the readout resonator comes in. It's a separate electrical circuit which is a type of microwave resonator that is carefully designed to interact with the transmon qubit. By precisely coupling them together, the state of the qubit (ground or excited) can influence the resonance frequency of the resonator. By measuring this resonance frequency, we can indirectly determine the qubit's state to be used for quantum algorithms.

Despite the transmon's design to minimize noise sensitivity, it's not entirely immune. While noise can disrupt a qubit's delicate state, it also plays a crucial role in both manipulating and reading out the qubit's information. For instance, applying targeted pulses of specific frequencies can induce transitions between the qubit's ground and excited states. The problem is, these frequencies can easily overlap with unwanted noise frequencies, leading to errors. This is where we could ask how we could improve the readout resonator. Optimizing the readout resonator becomes a balancing act. It needs to be strongly coupled to the qubit for sensitive readout, but weak enough to minimize noise-induced transitions during the process. By carefully designing the resonator's properties, like its shape and size, we can achieve this delicate balance, maximizing readout fidelity while minimizing noise errors. This is where we could use the barrier method to find these optimized parameters. First we need to formulate the objective function and its constraints.

Formulating the Optimization

Our main goal is to try to find specific dimensions of a rectangular resonator that correspond to minimizing the noise in the system. Optimizing this resonator is a balancing act - a larger area strengthens the qubit-resonator coupling for better readout, but also increases noise. We can capture this trade-off with the concept of noise flux, the ratio of resonator area to its signal-to-noise ratio (Γ). Minimizing noise flux during design helps achieve the optimal balance between strong coupling and minimal noise for reliable qubit readout.

$$\text{minimize} \quad \frac{A}{\Gamma}$$
$$A \equiv l \cdot w$$

There are a couple of conditions that quantum computer may follow. The first is that we want the qubit to switch between the ground and excited state with a specific frequency which we will denote with ω. The connection between the readout resonator to the qubit is through some type of transmission line. Transmission lines can face an issue of reflections which is when the associated wavelength of an electrical pulse is less than the length of the transmission line. An electric pulse's wavelegth is simply equal to the ratio between the speed of light to the frequency of operation

$$\lambda = \frac{c}{\omega}$$

The key is to make the transmission line the same length as the wavelength or shorter which can be done by evaluating the shorter side of the given transmon qubit

$$\min(l, w) \leq \lambda$$

The next design constraint we might want to follow is defining a maximum we might want a specific dimension to be. This could be due to a multitude of reasons such as manufacturing capabilities, cost, or other factors. We must independently define each factors. The maximum length can be denoted by L and the maximum width can be denoted by W

$$l \leq L$$
$$w \leq W$$

Another factor that is huge in the aspect of quantum computing is decoherence. Decoherence refers to the loss of quantum information due to interactions with the environment, a major enemy of qubit fidelity. By setting a minimum acceptable decoherence time, we can refine the optimization process for the readout resonator. This is where we need to account for the width of the trace that is made for the readout resonator which is defined as

$$\varphi \equiv \frac{\min(l, w) \leq \lambda}{\pi * \max(l, w)}$$

The minimum decoherence time is denoted with T_2, and we can make a constraint with the decoherence

$$\frac{1}{\pi\varphi} \geq T_2$$

Besides decoherence, another crucial factor to consider during resonator optimization is the anharmonicity of the transmon qubit. Anharmonicity refers to the non-linearity of the qubit's energy levels. In simpler terms, the energy difference between the ground and excited states (fundamental frequency) isn't perfectly constant. This anharmonicity is essential for certain quantum operations but can complicate the readout process.

Optimizing the readout resonator based on the qubit's anharmonicity allows for a more targeted approach. By carefully designing the resonator's resonance frequency to specifically interact with the fundamental frequency of the qubit, we can minimize unwanted interactions with higher energy levels. This reduces errors caused by crosstalk between different energy states, leading to more precise and reliable readout of the qubit's actual state. Say we have transistion from the ground state to the first excited state. The required frequency is simply equal to

$$\nu_1 = 2\pi\omega$$

But there can be an excitation from the first excited state to the second excited state with the same required frequency. We can create anharmonicity by taking advantage of the linewidth of the readout resonator

$$\nu_2 = 2\pi(\omega + \varphi)$$

The difference between these two required frequencies is referred to as the anharmonic spacing which has a constraint that is must be greater than a specified minimum Θ

$$\frac{\nu_2 - \nu_1}{\hbar} \geq \Theta$$

Two final key factors to consider are the conductivity and loss tangent of the material used to fabricate the resonator. Conductivity refers to how easily the material conducts electricity. In the context of the resonator, a higher conductivity translates to lower internal losses within the resonator itself. These losses can contribute to noise and ultimately reduce the SNR we discussed earlier. So, ideally, we want a material with high conductivity for the resonator. On the other hand, loss tangent refers to the amount of energy lost in the material as heat when an electric field is applied. A low loss tangent signifies minimal energy dissipation within the resonator. Since this translates to less noise generation, a low loss tangent is also desirable for the resonator material. We can specify parameters for the conductivity and loss tangent. For conductivity, we can say that we have a minimum conductivity of δ with a conductivity per unit area of Ψ. Let's also specify that we have a maximum loss tangent of μ with a loss tangent per unit area of β. We can then formulate our constraints by saying that

$$\begin{gathered} A\Psi \geq \delta \\ A\beta \leq \mu \end{gathered}$$

Now we have all of our system constraints we can use towards our objective function

$$\begin{gathered} \text{minimize} \quad \frac{A}{\Gamma} \\ \min(l, w) \leq \lambda \\ l \leq L \\ w \leq W \\ \frac{\nu_2 - \nu_1}{\hbar} \geq \Theta \\ \frac{1}{\pi\varphi} \geq T_2 \\ A\Psi \geq \delta \\ A\beta \leq \mu \end{gathered}$$

To translate this into a barrier method problem, what we must do is rewrite the constraints to be greater than or equal to zero

$$
\begin{gathered}
\text{minimize} \quad \frac{A}{\Gamma} \\
\lambda - \min(l, w) \geq 0 \\
L - l \geq 0 \\
W - w \geq 0 \\
\frac{\nu_2 - \nu_1}{\hbar} - \Theta \geq 0 \\
\frac{1}{\pi\varphi} - T_2 \geq 0 \\
A\Psi - \delta \geq 0 \\
\mu - A\beta \geq 0
\end{gathered}
$$

Next we will construct our barrier functions by taking the log on the left hand side

$$
\begin{gathered}
B_1(l, w) = \log(\lambda - \min(l, w)) \\
B_2(l) = \log(L - l) \\
B_3(w) = \log(W - w) \\
B_4(l, w) = \log\left(\frac{\nu_2 - \nu_1}{\hbar} - \Theta\right) \\
B_5(l, w) = \log\left(\frac{1}{\pi\varphi} - T_2\right) \\
B_6(l, w) = \log(A\Psi - \delta) \\
B_7(l, w) = \log(\mu - A\beta)
\end{gathered}
$$

We now have 7 barrier functions, but we will multiply some barrier functions to others. This is because some parameters are geometry dependent features such as the length and width limit or the wavelength restriction, but other parameters depend on the system requirement such as material considerations or operation modes. We can construct a new objective function by multiplying the composite logarithmic to the barrier term

$$
\begin{gathered}
\text{minimize} \quad \frac{A}{\Gamma} - \alpha(B) \\
B = B_1(l, w)B_2(l)B_3(w) + B_4(l, w)B_5(l, w)B_6(l, w)B_7(l, w) \\
\text{minimize} \quad \frac{A}{\Gamma} - \alpha(B_1(l, w)B_2(l)B_3(w) + B_4(l, w)B_5(l, w)B_6(l, w)B_7(l, w))
\end{gathered}
$$

We will not expand the expression any further, but this formulation is the unconstrained function that we can use iteratively to find solutions that are feasible in the optimization. For this project gradient descent was utilized with the unconstrained function to be used in the optimization. The results to the optimization can be viewed in the presentation in the repository.

Perturbation Analysis on The System

The perturbation analysis can be completed by computing the specific parameter values obtained from each of the constraints. The value obtained from each of the constraints can be thought of as a contributing factor to a vector. The higher the value of the constraint, the more the vector points in that direction. We can then take the magnitude of that vector to make a directional unit vector which will take the system to converge to a different aspect of the system. A small perturbation can be applied with respect to the direction of this vector to see if the parameters are stable within a reasonable boundary. Arguably, the parameters do not falter too much from a first order perturbation, but they do falter much more in a perturbation of the initial conditions/dimensions of the system. This ties in to a major weakness of the barrier method. Despite its strengths, the barrier method can be challenged by initial conditions. The method often requires an initial solution that adheres to all the constraints, which can be difficult to find for complex problems. Even with a feasible starting point, its location can significantly impact how many iterations are needed to reach the optimal solution, potentially slowing down the optimization process.

Beyond these considerations, the biggest factor for manufacturers, understandably so, is the resonator's size. While a larger area might improve coupling, it also increases noise and footprint. Manufacturers must carefully consider this trade-off during optimization, using simulations to identify the ideal size that balances strong coupling, minimal noise, and a manufacturable design. This ensures both reliable qubit readout and efficient use of space in their quantum circuits.