Tutorial for MIMO Communication Systems

25. Januar 2013

Problem 1

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

a) Eigenvalues

$$\det\left(\lambda\mathbf{I} - \mathbf{A}\right) \overset{!}{=} 0 \quad \rightarrow \quad \begin{vmatrix} \lambda+1 & 1 \\ 1 & \lambda+1 \end{vmatrix} \overset{!}{=} 0$$

$$\Rightarrow \lambda_1^A = -2, \quad \lambda_2^A = 0$$

$$\det\left(\lambda\mathbf{I} - \mathbf{B}\right) \overset{!}{=} 0 \quad \rightarrow \quad \begin{vmatrix} \lambda+\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \lambda-\frac{1}{\sqrt{5}} \end{vmatrix} \overset{!}{=} 0$$

$$\Rightarrow \lambda_1^B = 1, \quad \lambda_2^B = -1$$

b) Determinant & Trace

$$|\mathbf{A}| = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0, \quad |\mathbf{B}| = -1$$

$$\det\left(\mathbf{A}\right) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\left(-1\right)^2\left(a_{22}a_{33} - a_{32}a_{23}\right) + a_{12}\left(-1\right)^2\left(a_{21}a_{33} - a_{31}a_{23}\right) + \dots$$

$$|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}| \rightarrow \mathbf{AB} = \frac{1}{\sqrt{5}}\begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix} \rightarrow |\mathbf{AB}| = \frac{1}{5}\left(3 - 3\right) = 0$$

$$\operatorname{tr}\left(\mathbf{A}\right) = -1 - 1 = -2, \quad \operatorname{tr}\left(\mathbf{B}\right) = 0$$

$$\lambda_1^A = -2, \quad \lambda_2^A = 0 \quad \rightarrow \quad \operatorname{tr}\left(\mathbf{A}\right) = \sum_i \lambda_i^A$$

$$\lambda_1^B = 1, \quad \lambda_2^B = -1 \quad \rightarrow \quad \operatorname{tr}\left(\mathbf{B}\right) = \sum_i \lambda_i^B$$

$$|\mathbf{A}| = \lambda_1^A \lambda_2^A = \prod_i \lambda_i^A$$

c)

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}) \quad \forall \mathbf{A}, \mathbf{B}: \text{square matrix N}\times\text{N}$$

$$\mathbf{C} = \mathbf{AB} \quad \rightarrow \quad c_{m,n} = \sum_{k} a_{m,k}b_{k,n} \quad \rightarrow \quad \text{tr}(\mathbf{C}) = \sum_{i=1}^{N} c_{i,i} = \sum_{i=1}^{N}\sum_{k=1}^{N} a_{i,k}b_{k,i}$$

$$\mathbf{D} = \mathbf{BA} \quad \rightarrow \quad d_{m,n} = \sum_{k} b_{m,k}a_{k,n} \quad \rightarrow \quad \text{tr}(\mathbf{D}) = \sum_{i=1}^{N} d_{i,i} = \sum_{i=1}^{N}\sum_{k=1}^{N} b_{i,k}a_{k,i}$$

d)

①

positive definite matrix :	$\forall i, \lambda_i > 0$
positive semidefinite matrix :	$\forall i, \lambda_i \geq 0$
negative definite matrix :	$\forall i, \lambda_i < 0$
negative semidefinite matrix :	$\forall i, \lambda_i \leq 0$

$\mathbf{A}: \quad \lambda_i^A \leq 0 \quad \Rightarrow \quad$ negative semidefinite

$\mathbf{B}: \quad$ indefinite

②

$$\mathbf{X}^H = \mathbf{X} \quad \rightarrow \quad \mathbf{X}^T = \mathbf{X} \quad \rightarrow \quad \mathbf{A}, \mathbf{B}: \text{hermitian}$$

③

$$\text{rank}\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right) \rightarrow \text{rank}(\mathbf{A}) = 1$$

$$\text{rank}\left(\frac{1}{\sqrt{5}}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}\right) = \text{rank}\left(\frac{1}{\sqrt{5}}\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{pmatrix}\right)$$

$$\beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2 = 0 \quad \rightarrow \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \beta_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \beta_2 = -\frac{1}{2}, \quad \beta_2 = -2 \quad \text{\textcolor{}{↯}}$$

$$\Rightarrow \quad \text{rank}(\mathbf{B}) = 2$$

Important property:

$\mathbf{X}$ has full rank $\quad \rightarrow \quad |\mathbf{X}| \neq 0$ or $\mathbf{X}$ is invertible

④

$|\mathbf{A}| = 0 \quad \Rightarrow \quad \mathbf{A}$ nicht invertierbar

$|\mathbf{B}| = -1 \quad \Rightarrow \quad \mathbf{B}$ invertierbar

2

$\boxed{5}$

$\mathbf{X}$ is unitary $\quad\Leftrightarrow\quad \mathbf{X}\cdot\mathbf{X}^H=\mathbf{T}, \quad \mathbf{X}:$ unitary $\rightarrow \mathbf{X}^{-1}=\mathbf{X}^H$

$\mathbf{A}\mathbf{A}^H = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ $\qquad\qquad \Rightarrow \mathbf{A}$ is not unitary

$\mathbf{B}\mathbf{B}^H = \dfrac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ $\qquad \Rightarrow \mathbf{B}$ is unitary

e)

$|\mathbf{I}+\mathbf{A}\mathbf{B}| = |\mathbf{I}+\mathbf{B}\mathbf{A}|$

$|\mathbf{I}+\mathbf{A}\mathbf{B}| = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dfrac{1}{\sqrt{5}}\begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix} \right| = 2,7789$

$|\mathbf{I}+\mathbf{B}\mathbf{A}| = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dfrac{1}{\sqrt{5}}\begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix} \right| = 2,7789$

$\blacksquare$

f)

$\mathbf{X}^{-1} = \dfrac{1}{|\mathbf{X}|}\operatorname{adj}(\mathbf{X}), \quad \operatorname{adj}(\mathbf{X}) = $ Transpose of cofactor matrix

$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \dfrac{1}{\det\mathbf{X}}\begin{pmatrix} (x_{22}x_{33}-x_{32}x_{23}) & \cdots & \cdots \\ & \cdots & \cdots & \cdots \\ & \cdots & & \cdots & \cdots \end{pmatrix}$

$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$\rightarrow \quad \mathbf{B}^{-1} = \dfrac{1}{\det\mathbf{B}}\begin{pmatrix} b_{22} & -b_{21} \\ -b_{21} & b_{11} \end{pmatrix}^T = \dfrac{1}{-1}\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}^T$

$\rightarrow \quad \mathbf{B}^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}, \quad |\mathbf{B}^{-1}| = -1 \quad \Rightarrow |\mathbf{B}^{-1}| = \dfrac{1}{|\mathbf{B}|}$

$\blacksquare$

g)

$\operatorname{rank}(\mathbf{A}) = 1, \quad \operatorname{rank}(\mathbf{B}) = 2$

$\mathbf{A}\mathbf{B} = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -3 \end{pmatrix} = -3\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \rightarrow \quad \operatorname{rank}(\mathbf{A}\mathbf{B}) = 1$

$\operatorname{rank}(\mathbf{A}\mathbf{B}) \leq \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}$

h)

$$\| \bullet \| : \mathbb{C}^{M \times N} \quad \rightarrow \quad \mathbb{R}$$

Axioms for norm function:
1.) $\|\mathbf{X}\| \geq 0, \quad \|\mathbf{X}\| = 0$ if and only if $\mathbf{X} = 0$
2.) $\|\alpha\mathbf{X}\| = |\alpha|\,\|\mathbf{X}\|$
3.) $\|\mathbf{X} + \mathbf{Y}\| \leq \|\mathbf{X}\| + \|\mathbf{Y}\|$ $\qquad\qquad$ (triangular property)
4.) $\|\mathbf{XY}\| \leq \|\mathbf{X}\| \cdot \|\mathbf{Y}\|$

i)

$$\|\alpha\mathbf{X}\|_F = \sqrt{\sum_i \sum_j |\alpha\mathbf{X}|^2} = |\alpha|\sqrt{\sum_i \sum_j |\mathbf{X}|^2} = |\alpha|\,\|\mathbf{X}\|_F \qquad\qquad \text{Axiom 2}$$

$$\|\mathbf{X} + \mathbf{Y}\|_F^2 \leq \|\mathbf{X}\|_F^2 + \|\mathbf{Y}\|_F^2 \qquad\qquad \text{Axiom 3}$$

$$\|\mathbf{X} + \mathbf{Y}\|_F^2 = \sum_i \sum_j |x_{ij} + y_{ij}|^2 \qquad\qquad \text{Axiom 4}$$

$$\leq \sum_i \sum_j |x_{ij}|^2 + \sum_i \sum_j |y_{ij}|^2 = \|\mathbf{X}\|_F^2 + \|\mathbf{Y}_F^2$$

$$\|\mathbf{XY}\|_F^2 \leq \|\mathbf{X}\|_F^2 \cdot \|\mathbf{Y}\|_F^2$$

$$\rightarrow \|\mathbf{XY}\|_F^2 = \sum_i \sum_j \mathbf{XY} \leq \sum_i \sum_j |x_{ij}|^2 \cdot \sum_i \sum_j |y_{ij}|^2 = \|\mathbf{X}\|_F^2 \cdot \|\mathbf{Y}\|_F^2$$

$\blacksquare$

j)

$$\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$$

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{ij} & \mathbf{B} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 & -1 & 2 \\ -2 & 1 & 2 & 1 \\ -1 & 2 & 1 & -2 \\ 2 & 1 & -2 & -1 \end{pmatrix}$$

$$\mathbf{B} \otimes \mathbf{A} = \begin{bmatrix} b_{ij} & \mathbf{A} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -1 & -2 & 2 \\ -1 & 1 & 2 & -1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$$

$\blacksquare$

k)

Hadamard's Inequality: $|\mathbf{A}| \leq \prod_{k=1}^{K} a_{kk}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -2 & -1 \\ 3 & 1 & 0 \end{pmatrix} = 1 - 2\,(3) - 1\,(2+6) = -13$$

$$\prod_{k=1}^{3} a_{kk} = 0 \quad \rightarrow \quad \det(\mathbf{A}) = -13 \leq \prod_{k=1}^{3} a_{kk}$$

$$\tilde{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \left|\tilde{\mathbf{A}}\right| = 1\,(-2)\,0 = 0$$

$\rightarrow$ off-diagonal elements should set to zero $\leftrightarrow |\mathbf{A}| = \prod_{k=1}^{K} a_{kk}$

Problem 2

$$\mathbf{A} = \begin{pmatrix} 1 & 0,5 & -0,5 \\ -1 & 1 & -1 \end{pmatrix}$$

a)

$$\mathrm{rank}\,(\mathbf{A}): \quad \alpha \begin{pmatrix} 1 \\ 0,5 \\ -0,5 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \overset{!}{=} 0$$

we set $\beta = 1$ w/o loss of generality

$\rightarrow \quad \alpha - 1 = 0 \quad \Rightarrow \quad \alpha = 1$

$\quad 0,5\alpha + 1 = 0 \quad \Rightarrow \quad \alpha = -2 \quad \lightning$

$\Rightarrow \mathrm{rank}\,(\mathbf{A}) = 2$

5

b)

$$\mathbf{A} = \mathbf{U\Sigma V}^H \qquad \text{svd}\,(\mathbf{A}) \quad \text{(singular value decomposition)}$$

$$\tilde{\mathbf{A}} = \mathbf{AA}^T = \begin{pmatrix} 1,5 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\det\,(\lambda\mathbf{I} - \mathbf{A}) = 0 \quad \rightarrow \quad \det\left|\begin{pmatrix} \lambda - 1,5 & 0 \\ 0 & \lambda - 3 \end{pmatrix}\right| = 0$$

$$\Rightarrow \quad \lambda_1 = 1,5, \quad \lambda_2 = 3$$

Eigenvectors $\mathbf{Ev}$:

$$\tilde{\mathbf{A}}\mathbf{Ev} = \lambda\mathbf{Ev}$$

for $\lambda_1 = 1,5$:

$$\rightarrow \quad \begin{pmatrix} 1,5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1,5\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\rightarrow 1,5v_1 = 1,5v_2, \quad 3v_2 = 1,5v_1$$

$$\Rightarrow v_1 = 1, \quad v_2 = 0$$

$$\Rightarrow \mathbf{Ev}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $\lambda_2 = 3$:

$$\rightarrow \quad \begin{pmatrix} 1,5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 3\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\rightarrow 1,5u_1 = 3u_2, \quad 3u_2 = 3u_1$$

$$\Rightarrow \mathbf{Ev}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{U\Sigma V}^H, \quad \mathbf{U} = \begin{pmatrix} \mathbf{Ev_1} & \mathbf{Ev_2} & \dots & \mathbf{Ev_N} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

c)

$$S_u^2 : \text{ Eigenvalue of } \mathbf{XX}^H$$

$$S_u : \text{ singular value of } \mathbf{X}$$

$$\text{tr}\left(\mathbf{XX}^H\right) = \sum_u \lambda_u\left(\mathbf{XX}^H\right) = \sum_u S_u^2$$

d)

QR-decomposition $\rightarrow$ $\mathbf{A} = \mathbf{Q}\mathbf{R}$

with $\mathbf{Q}$ unitary, $\mathbf{R}$ upper triangular matrix

Gram-Schmidt:

$$\mathbf{A} = \begin{bmatrix} 1 & 0,5 & -0,5 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \quad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{a}_2\mathbf{e}_1)\,\mathbf{e}_1 = \begin{bmatrix} 0,5 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0,5 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right) = \begin{bmatrix} 0,75 \\ 0,75 \end{bmatrix}$$

$$\rightarrow \quad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_3 = \mathbf{a}_3 - (\mathbf{a}_3\mathbf{e}_1)\,\mathbf{e}_1 - (\mathbf{a}_3\mathbf{e}_2)\,\mathbf{e}_2 =$$

$$= \begin{bmatrix} -0,5 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} -0,5 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} - \left(\begin{bmatrix} -0,5 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -0,5 \\ -1 \end{bmatrix} + \begin{bmatrix} -0,25 \\ 0,25 \end{bmatrix} + \begin{bmatrix} 0,75 \\ 0,75 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad \mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{-\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}_1\mathbf{e}_1 & \mathbf{u}_2\mathbf{e}_1 & \mathbf{u}_3\mathbf{e}_1 \\ 0 & \mathbf{u}_2\mathbf{e}_2 & \mathbf{u}_3\mathbf{e}_2 \\ 0 & 0 & \mathbf{u}_3\mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{3}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

e)

$$\mathbf{B} = \begin{bmatrix} -1 & 1 & -1 \\ -0,5 & 0,5 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0,5 & -0,5 \\ -1 & 1 & -1 \end{bmatrix}$$

$\mathbf{B} = \mathbf{Q}\mathbf{L}$ $\mathbf{B}$ is updown and left-right flipped version of $\mathbf{A}$

$$\mathbf{B} = \mathrm{UD}\left(\mathrm{LR}\left(\mathbf{A}\right)\right)$$

$\mathbf{A} = \mathbf{Q}_A \mathbf{R}_A$ $\rightarrow \mathbf{B} = \mathrm{UD}\left(\mathrm{LR}\left(\mathbf{Q}_A \mathbf{R}_A\right)\right) = \mathbf{Q}_B \mathbf{L}_B$

$$\mathbf{Q}_B = \mathrm{UD}\left(\mathrm{LR}\left(\mathbf{Q}_A\right)\right)$$

$$\mathbf{L}_B = \mathrm{UD}\left(\mathrm{LR}\left(\mathbf{R}_A\right)\right)$$

$$\mathbf{Q}_B = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \mathbf{L}_B = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ -1,5 & 1,5 & 0 \\ 0,5 & 0,5 & 2 \end{bmatrix}$$

Gaussian Distribution, Entropy and Capacity

- Data rate
- Reliability (BER, transmit power)

$$C = \mathrm{ld}\left(1 + \mathrm{SNR}\right)$$

Differential Entropy: $h\left(x\right) = -\int\limits_{-\infty}^{\infty} f_x\left(x\right)\mathrm{ld}\left(f_x\left(x\right)\right)\mathrm{d}x$

$C = \max\left(\mathrm{I}\left(x,y\right)\right) = \max\left(h\left(y\right) - h\left(n\right)\right)$

       maximized for $h\left(y\right)$ Gaussian

noise $\rightarrow$ average power is .... $\rightarrow$ Gaussian is the most random noise

signal $\rightarrow$ using shaping techniques we get Gaussian distribution

$$C = \max\left(h\left(y\right) - h\left(n\right)\right) = \frac{1}{2}\mathrm{ld}\left(2\pi e \sigma_y^2\right) - \frac{1}{2}\mathrm{ld}\left(2\pi e \sigma_n^2\right) \overset{\sigma_y^2 = \sigma_x^2 + \sigma_n^2}{=} \frac{1}{2}\mathrm{ld}\left(1 + \frac{\sigma_x^2}{\sigma_n^2}\right)$$

water-filing: $\dfrac{P_i}{P} = \begin{cases} \dfrac{1}{\gamma_0} - \dfrac{\sigma_n^2}{P\|H_i\|^2} & \dfrac{\|H_i\|^2 P}{\sigma_n^2} > \gamma_0 \\ 0 & \dfrac{\|H_i\|^2 P}{\sigma_n^2} \leq \gamma_0 \quad \rightarrow \quad \text{channel i is too bad, low SNR} \end{cases}$
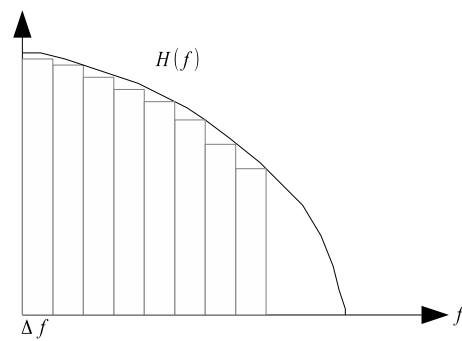
$P_i = \text{TX power of channel i}$

$P = \text{total TX power}$

$H_i = \text{gain of channel i}$

$\sigma_n^2 = N_0 \cdot B$

$\text{Channels} = \begin{cases} \text{AWGN:} \\ \\ \text{flat fading:} \\ \\ \text{frequency selective:} \end{cases}$

AWGN: $C_T = B \operatorname{ld}\left(1 + \dfrac{\sigma_x^2}{\sigma_n^2}\right) \dfrac{\text{bit}}{\text{s}}$

flat fading: $C_T = B \operatorname{ld}\left(1 + \dfrac{h^2 \sigma_x^2}{\sigma_n^2}\right)$

$H_i(f) = \text{const.} \quad \rightarrow \quad \text{Water-filling algorithm}$

$\rightarrow \quad \text{Allocate more power to better (with highter SNR) channels}$

$\text{dB} \begin{cases} \text{comparison: power } P_1, P_2 \quad \rightarrow \quad P_2 \text{ has } 10\log\frac{P_2}{P_1}\text{dB more power than } P_1 \\ \text{absolute power: dBm, } P_2 = 10\text{W}, \quad P_1 = 1\text{W} \quad \rightarrow \quad P_2 = 10\log 10 = 10\text{dB} \end{cases}$
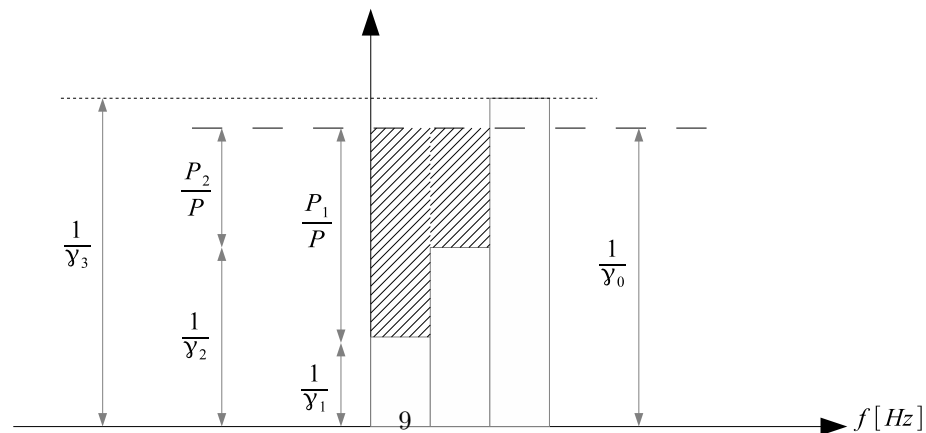
γ_i = SNR of ith channel

P_i = allocated power to the ith channel

$$\frac{P_i}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_i} & \frac{|H_i|^2 P}{\sigma_n^2} = \frac{|H_i|^2 P}{N_0 B} \geq \gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

$\sigma_n^2 = N_0 B$ with noise power σ_n^2 and PSD of noise N_0

H_i = transfer function (channel gain) of ith channel

Constraint: $\sum_{i=1}^{3} P_i = P$ total transmit power

$$\sum_{i=1}^{3} P_i = P \quad \rightarrow \quad \sum_{i=1}^{3} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) P = P \quad \rightarrow \quad \boxed{\sum_{i=1}^{3} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) P = 1}$$

$$\gamma_1 = \frac{|H_1|^2 P}{N_0 B} = \frac{1 \cdot 10^{-3}}{10^{-11} \cdot 17} \quad \rightarrow \quad \gamma_1 = 10 \quad \rightarrow \quad \text{best channel}$$

$$\gamma_2 = \frac{|H_2|^2 P}{N_0 B} = 10 H_2^2 \quad \rightarrow \quad \gamma_2 = 2,5$$

$$\gamma_3 = \frac{|H_3|^2 P}{N_0 B} = 10 H_3^2 \quad \rightarrow \quad \gamma_3 = 0,9 \quad \rightarrow \quad \text{worst channel}$$

$$\sum_{i=1}^{3} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) = 1 \quad \Rightarrow \quad \frac{3}{\gamma_0} = 1 + \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} \quad \rightarrow \quad \gamma_0 = 1,15 \quad \text{(water level)}$$

$$\rightarrow \text{water level check} \quad \rightarrow \quad \gamma_i > \gamma_0 \forall i$$

$\gamma_1 = 10 > \gamma_0$ ✓

$\gamma_2 = 2,5 > \gamma_0$ ✓

$\gamma_3 = 0,9 < \gamma_0$ ✗ $\rightarrow$ no power can be allocated to channel 3

$$\sum_{i=1}^{2} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) = 1 \quad \rightarrow \quad \frac{2}{\gamma_0} = 1 + \frac{1}{10} + \frac{1}{2,5} \quad \rightarrow \quad \gamma_0 = 1,33$$

$\gamma_1 = 10 > \gamma_0$ ✓

$\gamma_2 = 2,5 > \gamma_0$ ✓

$$\rightarrow \quad C_T = B \sum_{i=1}^{2} \mathrm{ld} \left(1 + \frac{|H_1|^i P}{N_0 B} \right) = B \sum_{i=1}^{2} \mathrm{ld} \left(1 + \frac{\gamma_i}{\gamma_0} - 1 \right) = B \sum_{i=1}^{2} \mathrm{ld} \left(\frac{\gamma_i}{\gamma_0} \right)$$
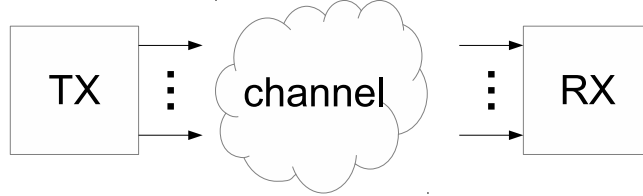
$$C_T = 10^7 \left(\mathrm{ld} \left(\frac{10}{1,33} \right) + \mathrm{ld} \left(\frac{2,5}{1,33} \right) \right) \quad \rightarrow \quad C_T = 38,2 \text{ Mbits/s}$$

$$C_T = B \, \mathrm{ld} \left(1 + \frac{S}{N} \right) \quad \text{Shannon formula}$$

Problem 4

Multiplexing gain $\qquad G_{\mathrm{m}} = \lim\limits_{\mathrm{SNR}\to\infty} \dfrac{R}{\mathrm{ld}\,(\mathrm{SNR})}$

Diversity gain $\qquad\qquad C = \mathrm{ld}\,(1 + \mathrm{SNR}) \quad\Rightarrow\quad C \overset{\mathrm{SNR}\to\infty}{\approx} \mathrm{ld}\,(\mathrm{SNR})$

Multiplexing gain = number of parallel independent equivalent channels

$\mathrm{svd}\,(H) \quad\to\quad$ singular values of H: σ_i^2, $i \in \{1, 2, \ldots, \mathrm{rank}\,(H)\}$

$\mathrm{rank}\,(H) \quad\to\quad$ number of singular values

$H_1 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \quad\to\quad \mathrm{rank}\,(H_1) = 2$

Matlab/Octave $\quad\to\quad \mathrm{svd}\,(H_1) = \mathbf{U}_1 \boldsymbol{\Sigma}_1 \mathbf{V}_1^H$

$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 2,56 & 0 & 0 \\ 0 & 1,56 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad\to\quad \sigma_1 = 2,56,\ \sigma_2 = 1,56,\ \sigma_3 = 0 \to \text{useless}$

$H_2 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad\to\quad \mathrm{rank}\,(H_2) = 3 \quad\to\quad \text{full rank}$

$\mathrm{svd}\,(H_1) = \mathbf{U}_2 \boldsymbol{\Sigma}_2 \mathbf{V}_2^H$

$\boldsymbol{\Sigma}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad\to\quad \sigma_1 = 2,\ \sigma_2 = 2,\ \sigma_3 = 1$

Problem 5

MIMO channel capacity

short review:

1 - channel matrix is deterministic and known to the transmitter

2 - channel matrix is random and changes during the transmission of a code word

    $\rightarrow$ ergodic capacity

3 - channel matrix is random and fixed during the transmission of a code word, but only known to TX

Decompose the MIMO channel to subchannels

$$C_{\mathrm{T}} = B \sum_{i=1}^{N} \left[\mathrm{ld} \quad \underbrace{\left(\frac{x_i^2}{\sigma_n^2} \mu \right)}_{\substack{\text{equivalent SNR} \\ \text{of channel } i}} \right]$$

$B :$ Bandwidth

$x_i :$ singular value of H

$\sigma_n^2 :$ noise power

$\mu :$ water level

$$\mathrm{svd}\,(\mathbf{H}) = \mathrm{svd}\left(\begin{bmatrix} 0,2 & -0,2 & 0,2 \\ 0,1 & 0,4 & -1 \\ 1 & 0,2 & 1 \end{bmatrix} \right) = \mathbf{U\Sigma V}^H$$

$$\mathbf{\Sigma} = \begin{bmatrix} 1,61 & 0 & 0 \\ 0 & 0,82 & 0 \\ 0 & 0 & 0,2 \end{bmatrix} \quad \rightarrow \quad \begin{array}{l} x_1 = 1,61 \\ x_2 = 0,82 \\ x_3 = 0,2 \end{array}$$

$$\frac{P}{\sigma_n^2} = 10\mathrm{dB} \quad \rightarrow \quad \frac{P}{\sigma_n^2} = 10$$

$$\gamma_1 = x^2 \frac{P}{\sigma_n^2} = 1,61^2 \cdot 10 = 25,9$$

$$\gamma_2 = 0,82^2 \cdot 10 = 7,6$$

$$\gamma_3 = 0,2^2 \cdot 10 = 0,4$$

    3 non-zero singular values $\rightarrow$ number of parallel independent channels $= 3$

$$\sum_{i=1}^{3} \left(\mu - \frac{1}{\gamma_i} \right) = 1$$

$$\rightarrow \quad 3\mu = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} + 1 = 3,7 \quad \rightarrow \quad \mu = 1,23 \quad \text{water level}$$

water level check: $\begin{cases} \frac{1}{\gamma_1} = 0,039 < \mu \quad \checkmark \\ \frac{1}{\gamma_2} = 0,15 < \mu \quad \checkmark \\ \frac{1}{\gamma_3} = 2,5 > \mu \quad \text{\scriptsize ǂ} \quad \text{3rd channel doesn't have enough SNR} \end{cases}$

$\Rightarrow$ no power can be allocated to 3rd channel

$\Rightarrow$ allocate power in 1st and 2nd channel

$$\rightarrow \sum_{i=1}^{2} \left(\mu - \frac{1}{\gamma_i} \right) = 1$$

$$\rightarrow 2\mu = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + 1 = 1,19 \quad \rightarrow \quad \mu = 0,595 \quad \text{water level}$$

water level check: $\begin{cases} \frac{1}{\gamma_1} < \mu \quad \checkmark \\ \frac{1}{\gamma_2} < \mu \quad \checkmark \end{cases}$

$$\rightarrow C_\text{T} = B \sum_{i=1}^{2} \text{ld}\,(\mu\gamma_i) = 10^6 \left(\text{ld}\left(\frac{25,9}{0,595} \right) + \text{ld}\left(\frac{6,7}{0,595} \right) \right)$$

$$\rightarrow C_\text{T} = 8,937 \text{ Mbit/s}$$

no CSIT:

$$\rightarrow C_\text{T} = B \sum_{i=1}^{3} \text{ld}\left(1 + x_i^2 \frac{P_i}{\sigma_n^2} \right) = B \sum_{i=1}^{3} \text{ld}\left(1 + x_i^2 \frac{P}{3\sigma_n^2} \right)$$

$$\rightarrow C_\text{T} = 10^6 \left(\text{ld}\,(1 + 8,64) + \text{ld}\,(1 + 2,24) + \text{ld}\,(1 + 0,13) \right)$$

$$\rightarrow C_\text{T} = 5,14 \text{ Mbit/s}$$

no CSIT at transmitter result in approx. $3,8$ Mbit/s data rate degradation

Hier fehlen Aufgabe 6-7???

das folgende gehrt zu Aufgabe 8 SISO Rayleigh fading channel

ACHTUNG: doppelte Nummerierung: Aufgabe 8 Receiver diversity, MRC, EGC and SC