

Tutorial for MIMO Communication Systems

27. Januar 2013

Problem 1

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

a) Eigenvalues

$$\begin{aligned} \det(\lambda \mathbf{I} - \mathbf{A}) &\stackrel{!}{=} 0 \quad \rightarrow \begin{vmatrix} \lambda + 1 & 1 \\ 1 & \lambda + 1 \end{vmatrix} \stackrel{!}{=} 0 \\ &\Rightarrow \lambda_1^A = -2, \quad \lambda_2^A = 0 \\ \det(\lambda \mathbf{I} - \mathbf{B}) &\stackrel{!}{=} 0 \quad \rightarrow \begin{vmatrix} \lambda + \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \lambda - \frac{1}{\sqrt{5}} \end{vmatrix} \stackrel{!}{=} 0 \\ &\Rightarrow \lambda_1^B = 1, \quad \lambda_2^B = -1 \end{aligned}$$

b) Determinant & Trace

$$\begin{aligned} |\mathbf{A}| &= \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0, \quad |\mathbf{B}| = -1 \\ \det(\mathbf{A}) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^2(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(-1)^2(a_{21}a_{33} - a_{31}a_{23}) + \dots \\ |\mathbf{AB}| &= |\mathbf{A}| \cdot |\mathbf{B}| \rightarrow \mathbf{AB} = \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix} \rightarrow |\mathbf{AB}| = \frac{1}{5}(3 - 3) = 0 \\ \text{tr}(\mathbf{A}) &= -1 - 1 = -2, \quad \text{tr}(\mathbf{B}) = 0 \\ \lambda_1^A &= -2, \quad \lambda_2^A = 0 \quad \rightarrow \quad \text{tr}(\mathbf{A}) = \sum_i \lambda_i^A \\ \lambda_1^B &= 1, \quad \lambda_2^B = -1 \quad \rightarrow \quad \text{tr}(\mathbf{B}) = \sum_i \lambda_i^B \\ |\mathbf{A}| &= \lambda_1^A \lambda_2^A = \prod_i \lambda_i^A \end{aligned}$$

c)

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}) \quad \forall \mathbf{A}, \mathbf{B}: \text{square matrix } N \times N$$

$$\mathbf{C} = \mathbf{AB} \quad \rightarrow \quad c_{m,n} = \sum_k a_{m,k} b_{k,n} \quad \rightarrow \quad \text{tr}(\mathbf{C}) = \sum_{i=1}^N c_{i,i} = \sum_{i=1}^N \sum_{k=1}^N a_{i,k} b_{k,i}$$

$$\mathbf{D} = \mathbf{BA} \quad \rightarrow \quad d_{m,n} = \sum_k b_{m,k} a_{k,n} \quad \rightarrow \quad \text{tr}(\mathbf{D}) = \sum_{i=1}^N d_{i,i} = \sum_{i=1}^N \sum_{k=1}^N b_{i,k} a_{k,i}$$

d)

①

positive definite matrix : $\forall i, \lambda_i > 0$

positive semidefinite matrix : $\forall i, \lambda_i \geq 0$

negative definite matrix : $\forall i, \lambda_i < 0$

negative semidefinite matrix : $\forall i, \lambda_i \leq 0$

\mathbf{A} : $\lambda_i^A \leq 0 \Rightarrow$ negative semidefinite

\mathbf{B} : indefinite

②

$$\mathbf{X}^H = \mathbf{X} \quad \rightarrow \quad \mathbf{X}^T = \mathbf{X} \quad \rightarrow \quad \mathbf{A}, \mathbf{B}: \text{hermitian}$$

③

$$\text{rank} \left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \right) \rightarrow \text{rank}(\mathbf{A}) = 1$$

$$\text{rank} \left(\frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \right) = \text{rank} \left(\frac{1}{\sqrt{5}} (\mathbf{b}_1 \quad \mathbf{b}_2) \right)$$

$$\beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 = 0 \quad \rightarrow \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \beta_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \beta_2 = -\frac{1}{2}, \quad \beta_2 = -2 \quad \nexists$$

$$\Rightarrow \text{rank}(\mathbf{B}) = 2$$

Important property:

$$\mathbf{X} \text{ has full rank} \quad \rightarrow \quad |\mathbf{X}| \neq 0 \text{ or } \mathbf{X} \text{ is invertible}$$

④

$$|\mathbf{A}| = 0 \quad \Rightarrow \quad \mathbf{A} \text{ nicht invertierbar}$$

$$|\mathbf{B}| = -1 \quad \Rightarrow \quad \mathbf{B} \text{ invertierbar}$$

⑤

\mathbf{X} is unitary $\Leftrightarrow \mathbf{X} \cdot \mathbf{X}^H = \mathbf{I}$, \mathbf{X} : unitary $\rightarrow \mathbf{X}^{-1} = \mathbf{X}^H$

$$\mathbf{A}\mathbf{A}^H = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \Rightarrow \mathbf{A} \text{ is not unitary}$$

$$\mathbf{B}\mathbf{B}^H = \frac{1}{5} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \Rightarrow \mathbf{B} \text{ is unitary}$$

e)

$$|\mathbf{I} + \mathbf{AB}| = |\mathbf{I} + \mathbf{BA}|$$

$$|\mathbf{I} + \mathbf{AB}| = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix} \right| = 2,7789$$

$$|\mathbf{I} + \mathbf{BA}| = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix} \right| = 2,7789$$

■

f)

$$\mathbf{X}^{-1} = \frac{1}{|\mathbf{X}|} \text{adj}(\mathbf{X}), \quad \text{adj}(\mathbf{X}) = \text{Transpose of cofactor matrix}$$

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \frac{1}{\det \mathbf{X}} \begin{pmatrix} (x_{22}x_{33} - x_{32}x_{23}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\rightarrow \mathbf{B}^{-1} = \frac{1}{\det \mathbf{B}} \begin{pmatrix} b_{22} & -b_{21} \\ -b_{21} & b_{11} \end{pmatrix}^T = \frac{1}{-1} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}^T$$

$$\rightarrow \mathbf{B}^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}, \quad |\mathbf{B}^{-1}| = -1 \Rightarrow |\mathbf{B}^{-1}| = \frac{1}{|\mathbf{B}|}$$

■

g)

$$\text{rank}(\mathbf{A}) = 1, \quad \text{rank}(\mathbf{B}) = 2$$

$$\mathbf{AB} = \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightarrow \text{rank}(\mathbf{AB}) = 1$$

$$\text{rank}(\mathbf{AB}) \leq \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}$$

h)

$$\|\bullet\| : \mathbb{C}^{M \times N} \rightarrow \mathbb{R}$$

Axioms for norm function:

- 1.) $\|\mathbf{X}\| \geq 0$, $\|\mathbf{X}\| = 0$ if and only if $\mathbf{X} = 0$
- 2.) $\|\alpha\mathbf{X}\| = |\alpha| \|\mathbf{X}\|$
- 3.) $\|\mathbf{X} + \mathbf{Y}\| \leq \|\mathbf{X}\| + \|\mathbf{Y}\|$ (triangular property)
- 4.) $\|\mathbf{XY}\| \leq \|\mathbf{X}\| \cdot \|\mathbf{Y}\|$

i)

$$\|\alpha\mathbf{X}\|_F = \sqrt{\sum_i \sum_j |\alpha\mathbf{X}|^2} = |\alpha| \sqrt{\sum_i \sum_j |\mathbf{X}|^2} = |\alpha| \|\mathbf{X}\|_F \quad \text{Axiom 2}$$

$$\|\mathbf{X} + \mathbf{Y}\|_F^2 \leq \|\mathbf{X}\|_F^2 + \|\mathbf{Y}\|_F^2 \quad \text{Axiom 3}$$

$$\|\mathbf{X} + \mathbf{Y}\|_F^2 = \sum_i \sum_j |x_{ij} + y_{ij}|^2 \quad \text{Axiom 4}$$

$$\leq \sum_i \sum_j |x_{ij}|^2 + \sum_i \sum_j |y_{ij}|^2 = \|\mathbf{X}\|_F^2 + \|\mathbf{Y}\|_F^2$$

$$\|\mathbf{XY}\|_F^2 \leq \|\mathbf{X}\|_F^2 \cdot \|\mathbf{Y}\|_F^2$$

$$\rightarrow \|\mathbf{XY}\|_F^2 = \sum_i \sum_j \mathbf{XY} \leq \sum_i \sum_j |x_{ij}|^2 \cdot \sum_i \sum_j |y_{ij}|^2 = \|\mathbf{X}\|_F^2 \cdot \|\mathbf{Y}\|_F^2$$

■

j)

$$\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$$

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} = [a_{ij} \quad \mathbf{B}] = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & -1 & 2 \\ -2 & 1 & 2 & 1 \\ -1 & 2 & 1 & -2 \\ 2 & 1 & -2 & -1 \end{pmatrix}$$

$$\mathbf{B} \otimes \mathbf{A} = [b_{ij} \quad \mathbf{A}] = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -1 & -2 & 2 \\ -1 & 1 & 2 & -1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$$

■

k)

Hadamard's Inequality: $|\mathbf{A}| \leq \prod_{k=1}^K a_{kk}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -2 & -1 \\ 3 & 1 & 0 \end{pmatrix} = 1 - 2(3) - 1(2 + 6) = -13$$

$$\prod_{k=1}^3 a_{kk} = 0 \rightarrow \det(\mathbf{A}) = -13 \leq \prod_{k=1}^3 a_{kk}$$

$$\tilde{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow |\tilde{\mathbf{A}}| = 1(-2)0 = 0$$

\rightarrow off-diagonal elements should set to zero $\leftrightarrow |\mathbf{A}| = \prod_{k=1}^K a_{kk}$

Problem 2 - Matrix Decompositions

$$\mathbf{A} = \begin{pmatrix} 1 & 0,5 & -0,5 \\ -1 & 1 & -1 \end{pmatrix}$$

a)

$$\text{rank}(\mathbf{A}) : \alpha \begin{pmatrix} 1 \\ 0,5 \\ -0,5 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \stackrel{!}{=} 0$$

we set $\beta = 1$ w/o loss of generality

$$\rightarrow \alpha - 1 = 0 \Rightarrow \alpha = 1$$

$$0,5\alpha + 1 = 0 \Rightarrow \alpha = -2 \quad \nexists$$

$$\Rightarrow \text{rank}(\mathbf{A}) = 2$$

b)

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H \quad \text{svd}(\mathbf{A}) \quad (\text{singular value decomposition})$$

$$\tilde{\mathbf{A}} = \mathbf{A}\mathbf{A}^T = \begin{pmatrix} 1,5 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0 \quad \rightarrow \quad \det \begin{vmatrix} \lambda - 1,5 & 0 \\ 0 & \lambda - 3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda_1 = 1,5, \quad \lambda_2 = 3$$

Eigenvectors $\mathbf{E}\mathbf{v}$:

$$\tilde{\mathbf{A}}\mathbf{E}\mathbf{v} = \lambda\mathbf{E}\mathbf{v}$$

for $\lambda_1 = 1,5$:

$$\rightarrow \begin{pmatrix} 1,5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1,5 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\rightarrow 1,5v_1 = 1,5v_2, \quad 3v_2 = 1,5v_1$$

$$\Rightarrow v_1 = 1, \quad v_2 = 0$$

$$\Rightarrow \mathbf{E}\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $\lambda_2 = 3$:

$$\rightarrow \begin{pmatrix} 1,5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 3 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\rightarrow 1,5u_1 = 3u_2, \quad 3u_2 = 3u_1$$

$$\Rightarrow \mathbf{E}\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H, \quad \mathbf{U} = (\mathbf{E}\mathbf{v}_1 \quad \mathbf{E}\mathbf{v}_2 \quad \dots \quad \mathbf{E}\mathbf{v}_N) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

c)

S_u^2 : Eigenvalue of $\mathbf{X}\mathbf{X}^H$

S_u : singular value of \mathbf{X}

$$\text{tr}(\mathbf{X}\mathbf{X}^H) = \sum_u \lambda_u(\mathbf{X}\mathbf{X}^H) = \sum_u S_u^2$$

d)

QR-decomposition $\rightarrow \mathbf{A} = \mathbf{Q}\mathbf{R}$

with \mathbf{Q} unitary, \mathbf{R} upper triangular matrix

Gram-Schmidt:

$$\mathbf{A} = \begin{bmatrix} 1 & 0,5 & -0,5 \\ -1 & 1 & -1 \end{bmatrix} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$$

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{a}_2 \mathbf{e}_1) \mathbf{e}_1 = \begin{bmatrix} 0,5 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0,5 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right) = \begin{bmatrix} 0,75 \\ 0,75 \end{bmatrix}$$

$$\rightarrow \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_3 = \mathbf{a}_3 - (\mathbf{a}_3 \mathbf{e}_1) \mathbf{e}_1 - (\mathbf{a}_3 \mathbf{e}_2) \mathbf{e}_2 =$$

$$= \begin{bmatrix} -0,5 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} -0,5 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} - \left(\begin{bmatrix} -0,5 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -0,5 \\ -1 \end{bmatrix} + \begin{bmatrix} -0,25 \\ 0,25 \end{bmatrix} + \begin{bmatrix} 0,75 \\ 0,75 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Q} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}_1 \mathbf{e}_1 & \mathbf{u}_2 \mathbf{e}_1 & \mathbf{u}_3 \mathbf{e}_1 \\ 0 & \mathbf{u}_2 \mathbf{e}_2 & \mathbf{u}_3 \mathbf{e}_2 \\ 0 & 0 & \mathbf{u}_3 \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{3}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

e)

$$\mathbf{B} = \begin{bmatrix} -1 & 1 & -1 \\ -0,5 & 0,5 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0,5 & -0,5 \\ -1 & 1 & -1 \end{bmatrix}$$

$\mathbf{B} = \mathbf{Q}\mathbf{L}$ \mathbf{B} is updown and left-right flipped version of \mathbf{A}
 $\mathbf{B} = \text{UD}(\text{LR}(\mathbf{A}))$
 $\mathbf{A} = \mathbf{Q}_A \mathbf{R}_A$ $\rightarrow \mathbf{B} = \text{UD}(\text{LR}(\mathbf{Q}_A \mathbf{R}_A)) = \mathbf{Q}_B \mathbf{L}_B$
 $\mathbf{Q}_B = \text{UD}(\text{LR}(\mathbf{Q}_A))$
 $\mathbf{L}_B = \text{UD}(\text{LR}(\mathbf{R}_A))$

$$\mathbf{Q}_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \mathbf{L}_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ -1,5 & 1,5 & 0 \\ 0,5 & 0,5 & 2 \end{bmatrix}$$

Gaussian Distribution, Entropy and Capacity

- Data rate
 - Reliability (BER, transmit power)
- $$C = \text{ld}(1 + \text{SNR})$$

Differential Entropy: $h(x) = - \int_{-\infty}^{\infty} f_x(x) \text{ld}(f_x(x)) dx$

$$C = \max(\text{I}(x, y)) = \max(h(y) - h(n))$$

maximized for $h(y)$ Gaussian

noise \rightarrow average power is \rightarrow Gaussian is the most random noise

signal \rightarrow using shaping techniques we get Gaussian distribution

$$C = \max(h(y) - h(n)) = \frac{1}{2} \text{ld}(2\pi e \sigma_y^2) - \frac{1}{2} \text{ld}(2\pi e \sigma_n^2) \stackrel{\sigma_y^2 = \sigma_x^2 + \sigma_n^2}{=} \frac{1}{2} \text{ld}\left(1 + \frac{\sigma_x^2}{\sigma_n^2}\right)$$

$$\text{water-filing: } \frac{P_i}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{\sigma_n^2}{P\|H_i\|^2} & \frac{\|H_i\|^2 P}{\sigma_n^2} > \gamma_0 \\ 0 & \frac{\|H_i\|^2 P}{\sigma_n^2} \leq \gamma_0 \end{cases} \rightarrow \text{channel i is too bad, low SNR}$$

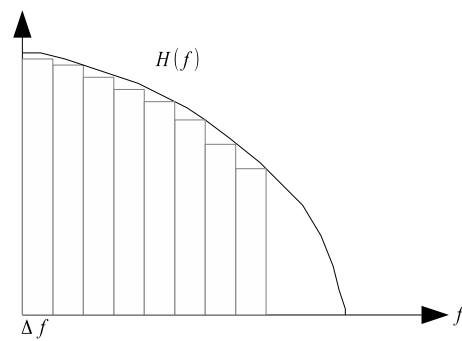
P_i = TX power of channel i

P = total TX power

H_i = gain of channel i

$$\sigma_n^2 = N_0 \cdot B$$

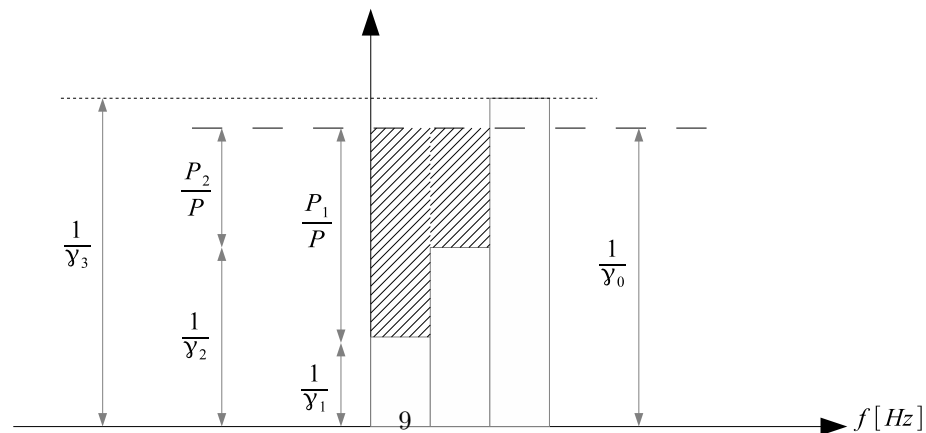
$$\text{Channels} = \begin{cases} \text{AWGN:} & \begin{array}{c} x \xrightarrow{\sigma_x^2} \oplus \xrightarrow{\sigma_y^2} y \\ \downarrow \sigma_n^2 \end{array} & C_T = B \log \left(1 + \frac{\sigma_x^2}{\sigma_n^2} \right) \frac{\text{bit}}{\text{s}} \\ \text{flat fading:} & \begin{array}{c} x \xrightarrow{h} \otimes \xrightarrow{n} \oplus \xrightarrow{\sigma_y^2} y \end{array} & C_T = B \log \left(1 + \frac{h^2 \sigma_x^2}{\sigma_n^2} \right) \\ \text{frequency selective:} & \begin{array}{c} X(f) \xrightarrow{H(f)} \otimes \xrightarrow{N} \oplus \xrightarrow{Y} y \end{array} \end{cases}$$



$H_i(f) = \text{const.} \rightarrow$ Water-filling algorithm

\rightarrow Allocate more power to better (with higher SNR) channels

dB $\begin{cases} \text{comparison: power } P_1, P_2 \rightarrow P_2 \text{ has } 10 \log \frac{P_2}{P_1} \text{ dB more power than } P_1 \\ \text{absolute power: dBm, } P_2 = 10\text{W, } P_1 = 1\text{W} \rightarrow P_2 = 10 \log 10 = 10\text{dB} \end{cases}$



γ_i = SNR of i th channel

P_i = allocated power to the i th channel

$$\frac{P_i}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_i} & \frac{|H_i|^2 P}{\sigma_n^2} = \frac{|H_i|^2 P}{N_0 B} \geq \gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

$\sigma_n^2 = N_0 B$ with noise power σ_n^2 and PSD of noise N_0

H_i = transfer function (channel gain) of i th channel

Constraint: $\sum_{i=1}^3 P_i = P$ total transmit power

$$\sum_{i=1}^3 P_i = P \rightarrow \sum_{i=1}^3 \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) P = P \rightarrow \boxed{\sum_{i=1}^3 \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) P = 1}$$

$$\gamma_1 = \frac{|H_1|^2 P}{N_0 B} = \frac{1 \cdot 10^{-3}}{10^{-11} \cdot 1^7} \rightarrow \gamma_1 = 10 \rightarrow \text{best channel}$$

$$\gamma_2 = \frac{|H_2|^2 P}{N_0 B} = 10 H_2^2 \rightarrow \gamma_2 = 2, 5$$

$$\gamma_3 = \frac{|H_3|^2 P}{N_0 B} = 10 H_3^2 \rightarrow \gamma_3 = 0, 9 \rightarrow \text{worst channel}$$

$$\sum_{i=1}^3 \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) = 1 \Rightarrow \frac{3}{\gamma_0} = 1 + \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} \rightarrow \gamma_0 = 1, 15 \quad (\text{water level})$$

\rightarrow water level check $\rightarrow \gamma_i > \gamma_0 \forall i$

$$\gamma_1 = 10 > \gamma_0 \quad \checkmark$$

$$\gamma_2 = 2, 5 > \gamma_0 \quad \checkmark$$

$$\gamma_3 = 0, 9 < \gamma_0 \quad \nexists \rightarrow \text{no power can be allocated to channel 3}$$

$$\sum_{i=1}^2 \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) = 1 \rightarrow \frac{2}{\gamma_0} = 1 + \frac{1}{10} + \frac{1}{2, 5} \rightarrow \gamma_0 = 1, 33$$

$$\gamma_1 = 10 > \gamma_0 \quad \checkmark$$

$$\gamma_2 = 2, 5 > \gamma_0 \quad \checkmark$$

$$\rightarrow C_T = B \sum_{i=1}^2 \text{ld} \left(1 + \frac{|H_i|^2 P}{N_0 B} \right) = B \sum_{i=1}^2 \text{ld} \left(1 + \frac{\gamma_i}{\gamma_0} - 1 \right) = B \sum_{i=1}^2 \text{ld} \left(\frac{\gamma_i}{\gamma_0} \right)$$

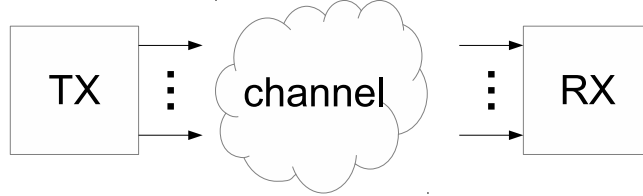
$$C_T = 10^7 \left(\text{ld} \left(\frac{10}{1, 33} \right) + \text{ld} \left(\frac{2, 5}{1, 33} \right) \right) \rightarrow C_T = 38, 2 \text{ Mbits/s}$$

$$C_T = B \text{ld} \left(1 + \frac{S}{N} \right) \quad \text{Shannon formula}$$

Problem 4

Multiplexing gain $G_m = \lim_{\text{SNR} \rightarrow \infty} \frac{R}{\text{ld}(\text{SNR})}$

Diversity gain $C = \text{ld}(1 + \text{SNR}) \Rightarrow C \stackrel{\text{SNR} \rightarrow \infty}{\approx} \text{ld}(\text{SNR})$



Multiplexing gain = number of parallel independent equivalent channels

$\text{svd}(H) \rightarrow$ singular values of H : $\sigma_i^2, i \in \{1, 2, \dots, \text{rank}(H)\}$

$\text{rank}(H) \rightarrow$ number of singular values

$$H_1 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \rightarrow \text{rank}(H_1) = 2$$

Matlab/Octave $\rightarrow \text{svd}(H_1) = \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V}_1^H$

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 2,56 & 0 & 0 \\ 0 & 1,56 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \sigma_1 = 2,56, \sigma_2 = 1,56, \sigma_3 = 0 \rightarrow \text{useless}$$

$$H_2 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \rightarrow \text{rank}(H_2) = 3 \rightarrow \text{full rank}$$

$\text{svd}(H_1) = \mathbf{U}_2 \mathbf{\Sigma}_2 \mathbf{V}_2^H$

$$\mathbf{\Sigma}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \sigma_1 = 2, \sigma_2 = 2, \sigma_3 = 1$$

Problem 5

MIMO channel capacity

short review:

1 - channel matrix is deterministic and known to the transmitter

2 - channel matrix is random and changes during the transmission of a code word

→ ergodic capacity

3 - channel matrix is random and fixed during the transmission of a code word,
but only known to TX

Decompose the MIMO channel to subchannels

$$C_T = B \sum_{i=1}^N \left[\text{ld} \underbrace{\left(\frac{x_i^2}{\sigma_n^2} \mu \right)}_{\text{equivalent SNR of channel } i} \right]$$

B : Bandwidth

x_i : singular value of \mathbf{H}

σ_n^2 : noise power

μ : water level

$$\text{svd}(\mathbf{H}) = \text{svd} \left(\begin{bmatrix} 0,2 & -0,2 & 0,2 \\ 0,1 & 0,4 & -1 \\ 1 & 0,2 & 1 \end{bmatrix} \right) = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^H$$

$$\mathbf{\Sigma} = \begin{bmatrix} 1,61 & 0 & 0 \\ 0 & 0,82 & 0 \\ 0 & 0 & 0,2 \end{bmatrix} \rightarrow \begin{array}{l} x_1 = 1,61 \\ x_2 = 0,82 \\ x_3 = 0,2 \end{array}$$

$$\frac{P}{\sigma_n^2} = 10\text{dB} \rightarrow \frac{P}{\sigma_n^2} = 10$$

$$\gamma_1 = x_1^2 \frac{P}{\sigma_n^2} = 1,61^2 \cdot 10 = 25,9$$

$$\gamma_2 = 0,82^2 \cdot 10 = 7,6$$

$$\gamma_3 = 0,2^2 \cdot 10 = 0,4$$

3 non-zero singular values → number of parallel independent channels = 3

$$\sum_{i=1}^3 \left(\mu - \frac{1}{\gamma_i} \right) = 1$$

$$\rightarrow 3\mu = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} + 1 = 3,7 \rightarrow \mu = 1,23 \text{ water level}$$

$$\text{water level check: } \begin{cases} \frac{1}{\gamma_1} = 0,039 < \mu & \checkmark \\ \frac{1}{\gamma_2} = 0,15 < \mu & \checkmark \\ \frac{1}{\gamma_3} = 2,5 > \mu & \nless \quad \text{3rd channel doesn't have enough SNR} \end{cases}$$

\Rightarrow no power can be allocated to 3rd channel

\Rightarrow allocate power in 1st and 2nd channel

$$\rightarrow \sum_{i=1}^2 \left(\mu - \frac{1}{\gamma_i} \right) = 1$$

$$\rightarrow 2\mu = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + 1 = 1,19 \quad \rightarrow \quad \mu = 0,595 \quad \text{water level}$$

$$\text{water level check: } \begin{cases} \frac{1}{\gamma_1} < \mu & \checkmark \\ \frac{1}{\gamma_2} < \mu & \checkmark \end{cases}$$

$$\rightarrow C_T = B \sum_{i=1}^2 \text{ld}(\mu \gamma_i) = 10^6 \left(\text{ld} \left(\frac{25,9}{0,595} \right) + \text{ld} \left(\frac{6,7}{0,595} \right) \right)$$

$$\rightarrow C_T = 8,937 \text{ Mbit/s}$$

no CSIT:

$$\rightarrow C_T = B \sum_{i=1}^3 \text{ld} \left(1 + x_i^2 \frac{P_i}{\sigma_n^2} \right) = B \sum_{i=1}^3 \text{ld} \left(1 + x_i^2 \frac{P}{3\sigma_n^2} \right)$$

$$\rightarrow C_T = 10^6 (\text{ld}(1 + 8,64) + \text{ld}(1 + 2,24) + \text{ld}(1 + 0,13))$$

$$\rightarrow C_T = 5,14 \text{ Mbit/s}$$

no CSIT at transmitter result in approx. 3,8 Mbit/s data rate degradation

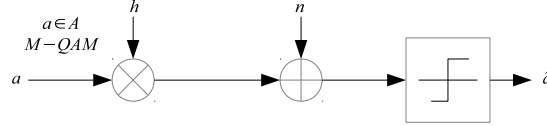
Hier fehlen Aufgabe 6-7???

das folgende geht zu Aufgabe 8 SISO Rayleigh fading channel

ACHTUNG: doppelte Nummerierung: Aufgabe 8 Receiver diversity, MRC, EGC and SC

Problem 8.1¹ - SISO Rayleigh fading channel

$$\overline{\text{SER}} = \mathbb{E}_{|h|^2} \left\{ \text{SER}(|h|^2) \right\}$$



$$\text{SER}(|h|^2) = C_A \text{Q} \left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}} |h|^2 \right)$$

$$C_A = 3 \quad (16\text{-QAM}) \qquad C_A = 2 \quad (4\text{-QAM})$$

$$= C_A \text{Q} \left(\sqrt{\underbrace{\frac{d_{\min}^2}{\text{ld}((M))}}_{d_A^2} \underbrace{\frac{E_s}{N_0}}_{\gamma} |h|^2} \right) = C_A \text{Q} \left(\sqrt{d_A^2 \gamma |h|^2} \right)$$

$$h \sim \mathcal{CN}(0, 1) \quad \rightarrow \quad |h|^2 \text{ is exponentially distributed with variance 1}$$

$$\rightarrow \overline{\text{SER}} = \int_0^\infty \underbrace{f_{|h|^2}(x)}_{\text{pdf of } |h|^2} \text{SER}(x) dx = \int_0^\infty e^{-x} C_A \text{Q} \left(\sqrt{d_A^2 \gamma |h|^2} \right) dx$$

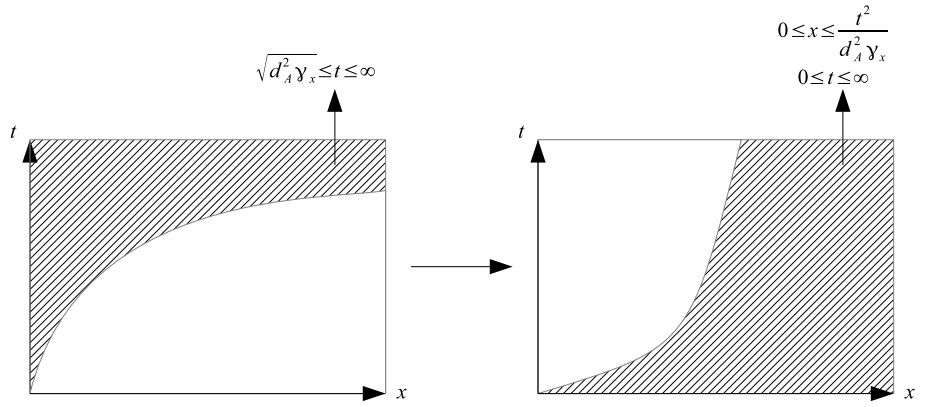
$$\text{Q}(x) = \frac{1}{2\pi} \int_x^\infty e^{-\frac{t^2}{2}} dt$$

$$\rightarrow \overline{\text{SER}} = \int_0^\infty e^{-x} C_A \frac{1}{\sqrt{2\pi}} \int_{\sqrt{d_A^2 \gamma x}}^\infty e^{-\frac{t^2}{2}} dt dx$$

$$\rightarrow \text{change the order of integrals}$$

$$\rightarrow \overline{\text{SER}} = \frac{C_A}{\sqrt{2\pi}} \int_{\sqrt{d_A^2 \gamma x}}^\infty \int_0^\infty e^{-x} e^{-\frac{t^2}{2}} dx dt$$

¹Problem 8 kommt 2mal vor, sind aber unterschiedliche Aufgaben



$$\begin{aligned}
 \overline{\text{SER}} &= \frac{C_A}{\sqrt{2\pi}} \int_0^\infty \int_0^{\frac{t^2}{d_A^2 \gamma}} e^{-x} e^{-\frac{t^2}{2}} dx dt = \frac{C_A}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}} [e^{-x}]_0^{\frac{t^2}{d_A^2 \gamma}} dt = \\
 &= \frac{C_A}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{t^2}{2}} \left(-e^{-\frac{t^2}{d_A^2 \gamma}} + 1 \right) dt = \\
 &= \frac{C_A}{\sqrt{2\pi}} \int_0^\infty \left(e^{-\frac{t^2}{2}} - e^{-\frac{t^2}{2} \left(1 + \frac{2}{d_A^2 \gamma} \right)} \right) dt = \\
 &= \frac{C_A}{\sqrt{2\pi}} \left(\underbrace{\sqrt{2\pi} Q(0)}_{\frac{1}{2}} - \sqrt{\frac{2\pi}{1 + \frac{2}{d_A^2 \gamma}}} \underbrace{Q\left(\sqrt{1 + \frac{2}{d_A^2 \gamma}} 0 \right)}_0 \right) = \\
 &= \frac{C_A}{2} \left(1 - \sqrt{\frac{1}{1 + \frac{2}{d_A^2 \gamma}}} \right) = \boxed{\frac{C_A}{2} \left(1 - \sqrt{\frac{d_A^2 \gamma}{d_A^2 \gamma + 2}} \right)}
 \end{aligned}$$

d)

Asymptotic $\overline{\text{SER}}$ if $\gamma \rightarrow \infty$

$$\overline{\text{SER}} = \frac{C_{\mathcal{A}}}{2} \left(1 - \sqrt{\frac{d_{\mathcal{A}}^2 \gamma}{d_{\mathcal{A}}^2 \gamma + 2}} \right)$$

$C_{\mathcal{A}}$: average number of nearest neighbours in the constellation diagram

$$\text{Taylor series: } f(x) \approx f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) + \dots$$

$$\text{upper bound: } f(x) \approx f(x_0) + (x - x_0) f'(x_0)$$

$$x_0 = 0 \rightarrow f(x) = f(0) + x f'(0)$$

$$x = \frac{1}{d_{\mathcal{A}}^2 \gamma} \text{ if } \gamma \rightarrow \infty \Rightarrow x = 0$$

$$\overline{\text{SER}}(x) = \frac{C_{\mathcal{A}}}{2} \left(1 - \sqrt{\frac{1}{1 + 2x}} \right)$$

$$\overline{\text{SER}}(x) \leq \overline{\text{SER}}(0) + x \overline{\text{SER}}'(0)$$

$$\overline{\text{SER}}(0) = \frac{C_{\mathcal{A}}}{2} \left(1 - \sqrt{\frac{1}{1}} \right) = 0$$

$$\overline{\text{SER}}'(x) = -\frac{C_{\mathcal{A}}}{2} \frac{1}{2\sqrt{\frac{1}{1+2x}}} \frac{-1}{(1+2x)^2} 2 \Rightarrow \overline{\text{SER}}'(0) = -\frac{C_{\mathcal{A}}}{2}$$

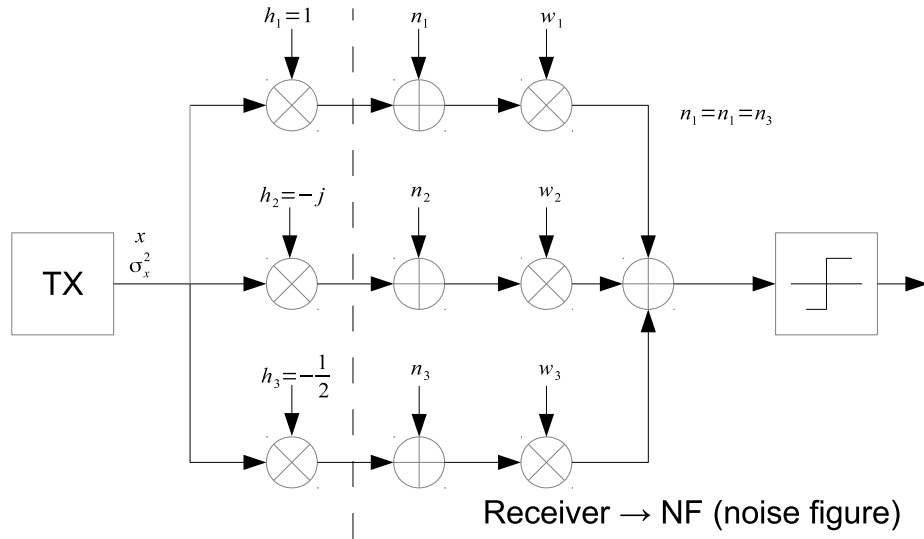
$$\rightarrow \overline{\text{SER}}(x) \leq \frac{C_{\mathcal{A}}}{2} x \rightarrow \boxed{\overline{\text{SER}}(\gamma) \leq \frac{C_{\mathcal{A}}}{2} \frac{1}{d_{\mathcal{A}}^2 \gamma}}$$

$$\text{SER}(|h|^2) = C_{\mathcal{A}} Q\left(\sqrt{d_{\mathcal{A}}^2 \gamma \|h\|^2}\right)$$

$$\text{we could have used Chernoff bound: } Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$$

$$\text{SER}(|h|^2) \leq \frac{C_{\mathcal{A}}}{2} e^{-\frac{d_{\mathcal{A}}^2 \|h\|^2 \gamma}{2}} \rightarrow \dots \rightarrow \boxed{\overline{\text{SER}} \leq C_{\mathcal{A}} \frac{1}{d_{\mathcal{A}}^2 \gamma}}$$

Problem 8.2 - Receiver diversity, MRC, EGC and SC



a)

$$\text{MRC} \rightarrow w_j = h_j^* \quad \forall j \in \{1, \dots, N_R\}$$

$$w_1 = 1$$

$$w_2 = j$$

$$w_3 = -\frac{1}{2}$$

In order to have w_j we need $h_j \rightarrow$ channel estimation

$$\text{EGC} \rightarrow h_n = |h_n| e^{j\phi_n} \Rightarrow w_n = e^{j\phi_n}$$

$$h_1 = 1 \rightarrow w_1 = 1$$

$$h_2 = -j \rightarrow w_2 = j$$

$$h_3 = -\frac{1}{2} \rightarrow w_3 = e^{-j\pi} = -1$$

b)

$$\begin{aligned}
\text{MRC : } \text{SNR}_{\text{total}}^{\text{MRC}} &= \frac{\sigma_x^2 \left| \sum_{n=1}^{N_R} h_n w_n \right|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |w_n|^2} = \frac{\sigma_x^2 \left| \sum_{n=1}^{N_R} |h_n| e^{j\phi_n} |h_n| e^{-j\phi_n} \right|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |h_n|^2} = \\
&= \frac{\sigma_x^2}{\sigma_n^2} \left(\sum_{n=1}^{N_R} |h_n|^2 \right) = \frac{\sigma_x^2}{\sigma_n^2} \left(1 + 1 + \frac{1}{4} \right) \\
&\Rightarrow \boxed{\text{SNR}_{\text{total}}^{\text{MRC}} = \frac{9}{4} \frac{\sigma_x^2}{\sigma_n^2}}
\end{aligned}$$

$$\begin{aligned}
\text{EGC : } \text{SNR}_{\text{total}}^{\text{EGC}} &= \frac{\sigma_x^2 \left| \sum_{n=1}^{N_R} |h_n| e^{j\phi_n} e^{-j\phi_n} \right|^2}{\sigma_n^2 \sum_{n=1}^{N_R} |e^{-j\phi_n}|^2} = \\
&= \frac{\sigma_x^2}{N_R \sigma_n^2} \left(\sum_{n=1}^{N_R} |h_n|^2 \right) = \frac{\sigma_x^2}{3 \sigma_n^2} \left(1 + 1 + \frac{1}{2} \right)^2 \\
&\Rightarrow \boxed{\text{SNR}_{\text{total}}^{\text{EGC}} = \frac{25}{12} \frac{\sigma_x^2}{\sigma_n^2}}
\end{aligned}$$

$$\text{SC : } \text{branch with max. SNR: } \hat{n} = \arg \max_n \text{SNR}_n \rightarrow \text{SNR}_{\text{total}} = \text{SNR}_{\hat{n}}$$

we need SNR estimation algorithm \rightarrow e.g. pilots

$$\begin{aligned}
\text{SNR}_1 &= \frac{\sigma_x^2 |h_1|^2 |w_1|^2}{\sigma_n^2 |w_1|^2} = \frac{\sigma_x^2}{\sigma_n^2} |h_1|^2 = \frac{\sigma_x^2}{\sigma_n^2} \\
\text{SNR}_2 &= \frac{\sigma_x^2}{\sigma_n^2} |h_2|^2 = \frac{\sigma_x^2}{\sigma_n^2} \\
\text{SNR}_3 &= \frac{\sigma_x^2}{\sigma_n^2} |h_3|^2 = \frac{1}{4} \frac{\sigma_x^2}{\sigma_n^2}
\end{aligned}$$

$\text{SNR}_1/\text{SNR}_2$ is the highest SNR, we select first branch

$$\rightarrow \boxed{\text{SNR}_{\text{total}}^{\text{SC}} = \text{SNR}_1 = \frac{\sigma_x^2}{\sigma_n^2}}$$

$$\left. \begin{aligned}
\text{MRC : } \text{SNR}_{\text{total}}^{\text{MRC}} &= \frac{4}{9} \frac{\sigma_x^2}{\sigma_n^2} \\
\text{EGC : } \text{SNR}_{\text{total}}^{\text{EGC}} &= \frac{25}{12} \frac{\sigma_x^2}{\sigma_n^2} \\
\text{SC : } \text{SNR}_{\text{total}}^{\text{SC}} &= \frac{\sigma_x^2}{\sigma_n^2}
\end{aligned} \right\} \boxed{\text{SNR}^{\text{MRC}} \geq \text{SNR}^{\text{EGC}} \geq \text{SNR}^{\text{SC}}}$$

Problem 9 - BPSK error rate in an MRC system over Nakagami-m channel

SNR distribution for Nakagami-m fading

$$\text{pdf: } P_\gamma(x) = \frac{m^m x^{m-1}}{\gamma^{-m} \Gamma(m)} \exp\left(-\frac{mx}{\bar{\gamma}}\right) \begin{cases} m & \text{fading parameter} \\ \Gamma(m) & \text{Gamma function} = (m-1)! \\ \bar{\gamma} & \text{average branch SNR} \end{cases}$$

→ Laplace transform

$$\mathcal{L}(P_\gamma(x)) = M_\gamma(s) \rightarrow \text{moment generating function (mgs)}$$

$$\rightarrow M_\gamma(s) = \left(1 - \frac{s\bar{\gamma}}{m}\right)^{-m} \quad m=1 \Rightarrow \text{Rayleigh}$$

$$\gamma_{\text{total}}^{\text{MRC}} = \sum_{n=1}^{N_R} \gamma_n \quad \gamma_n : \text{branch SNR}$$

Probability density function pdf of total SNR is convolution of branch SNR.

Convolution in time domain → multiplication in frequency domain.

i.i.d fading → branch SNR independent

$$\rightarrow M_{\gamma_{\text{total}}}(s) = \prod_{n=1}^{N_R} M_{\gamma_n}(s) = (M_\gamma(s))^{N_R}$$

$$\rightarrow M_{\gamma_{\text{total}}}(s) = \left(1 - \frac{s\bar{\gamma}}{m}\right)^{-mN_R}$$

from lecture notes:

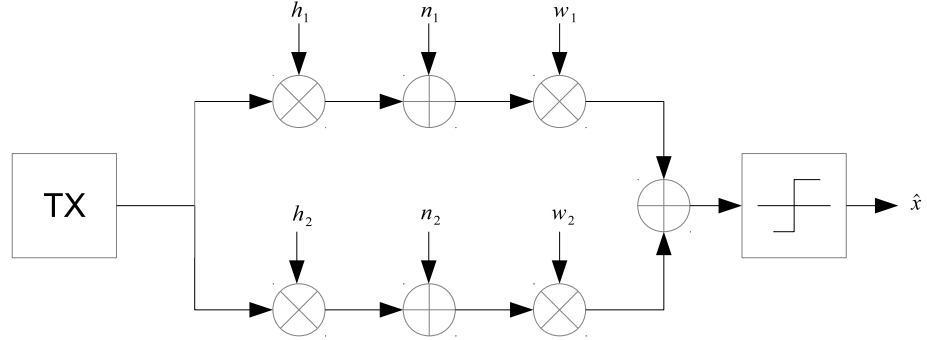
$$\rightarrow \bar{P}_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_{\text{total}}}\left(\frac{b}{2\sin^2 \Theta}\right) d\Theta$$

a, b are parameters depending on modulation scheme

→ for BPSK: $a = 1, b = 2$

$$\rightarrow \boxed{\bar{P}_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \left(1 - \frac{\bar{\gamma}}{m \sin^2 \Theta}\right)^{-mN_R} d\Theta}$$

Problem 10 - MPSK error rate in an MRC system over a mixed Rayleigh, Nakagami-m channel



$$\sigma_{n1}^2 = \sigma_{n2}^2 = \sigma_n^2$$

$$\text{branch SNR: } \bar{\gamma} = \frac{\sigma_x^2}{\sigma_n^2}$$

$$\text{branch 1} \rightarrow \text{Rayleigh} \rightarrow p_{\gamma}(x) = \frac{1}{\bar{\gamma}} \exp\left(-\frac{x}{\bar{\gamma}}\right)$$

$$\text{branch 2} \rightarrow \text{Nakagami-m} \rightarrow p_{\gamma}(x) = \frac{m^m x^{m-1}}{\gamma^{-m} \Gamma(m)} \exp\left(-\frac{mx}{\bar{\gamma}}\right)$$

$$\text{i.i.d fading} \rightarrow M_{\gamma_{\text{total}}}(s) = M_{\gamma_1}(s) M_{\gamma_2}(s)$$

$$\begin{cases} \text{first branch} \rightarrow \text{Rayleigh} \rightarrow M_{\gamma_1}(s) = (1 - s\bar{\gamma})^{-1} \\ \text{second branch} \rightarrow \text{Nakagami-m} \rightarrow M_{\gamma_2}(s) = \left(1 - \frac{s\bar{\gamma}}{m}\right)^{-m} \end{cases}$$

$$\rightarrow M_{\gamma_{\text{total}}} = (1 - s\bar{\gamma})^{-1} \left(1 - s\frac{\bar{\gamma}}{m}\right)^{-m}$$

from lectures notes:

$$\begin{aligned} \rightarrow \bar{P}_e &= \int_0^{\infty} P_e p_{\gamma_{\text{total}}}(x) dx = \\ &= \int_0^{\infty} \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \exp(\cdot) d\Theta p_{\gamma_{\text{total}}}(x) dx = \\ &= \frac{1}{\pi} \int_0^{\frac{(M-1)\pi}{M}} \int_0^{\infty} \exp\left(\frac{-x \sin^2\left(\frac{\pi}{M}\right)}{\sin^2 \Theta}\right) p_{\gamma_{\text{total}}}(x) dx d\Theta = \\ &= \int_0^{\infty} e^{sx} p_{\gamma_{\text{total}}}(x) dx = M_{\gamma_{\text{total}}}(s) \end{aligned}$$

$$\begin{aligned}
& \rightarrow \int_0^\infty \exp\left(\frac{-x \sin^2\left(\frac{\pi}{M}\right)}{\sin^2 \Theta}\right) p_{\gamma_{\text{total}}}(x) dx = M_{\gamma_{\text{total}}}\left(\frac{-\sin^2\left(\frac{\pi}{M}\right)}{\sin^2 \Theta}\right) \\
& \rightarrow \bar{p}_e = \frac{1}{M} \int_0^{\frac{(M-1)\pi}{M}} M_{\gamma_{\text{total}}}\left(\frac{-\sin^2\left(\frac{\pi}{M}\right)}{\sin^2 \Theta}\right) d\Theta \\
& \rightarrow \boxed{\bar{p}_e = \frac{1}{M} \int_0^{\frac{(M-1)\pi}{M}} \left(1 + \frac{\bar{\gamma} \sin^2\left(\frac{\pi}{M}\right)}{\sin^2 \Theta}\right)^{-1} \left(1 + \frac{\bar{\gamma} \sin^2\left(\frac{\pi}{M}\right)}{m \sin^2 \Theta}\right)^{-m} d\Theta} \\
& \bar{\gamma} = 10 \text{ dB} \quad M = 8 \rightarrow 8\text{PSK} \quad m = 3 \quad \Rightarrow \quad \bar{p}_e = 0,0476
\end{aligned}$$

Problem 11 - Transmit and receive diversity

The standard definition of outage probability:

The average rate is lower than a specific value.

Alternative definition:

The error rate exceeds a specific value.

$P_b = 10^{-4}$ & BPSK modulation

$$\rightarrow P_b(\gamma_0) = Q\left(\sqrt{2\gamma_0}\right)$$

$$\rightarrow 10^{-4} = Q\left(\sqrt{2\gamma_0}\right) \rightarrow \gamma_0 = \frac{1}{2} \left(Q^{-1}(10^{-4})\right)^2$$

$$\rightarrow \gamma_0 = 6,9155 \rightarrow \gamma_0 = 8,4 \text{ dB}$$

$$\rightarrow P_{\text{out}} = P_e(\gamma_{\text{total}} < \gamma_0) = P(\gamma_{\text{out}} < 6,9155)$$

from lecture notes:

$$\rightarrow p_{\gamma_{\text{total}}}(x) = \frac{x^{N_R-1} e^{-\frac{\lambda}{\bar{\gamma}}}}{\bar{\gamma}^{-N_R} (N_R - 1)!}$$

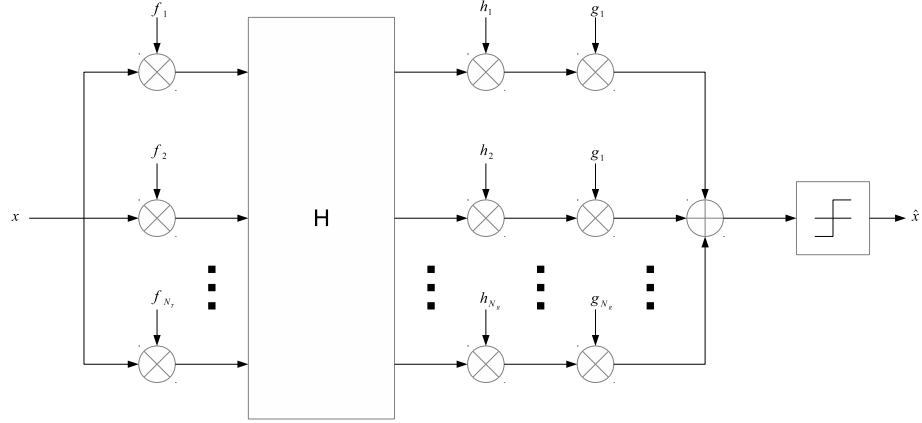
pdf of total SNR from MRC with i.i.d Rayleigh

$$\begin{aligned}
\rightarrow P_{\text{out}} = P_e(\gamma_{\text{total}} < 6,9155) &= \int_0^{6,9155} \frac{x^{N_R-1} e^{-\frac{\lambda}{\bar{\gamma}}}}{\bar{\gamma}^{-N_R} (N_R - 1)!} dx = \\
&= 1 - e^{-\frac{6,9155}{10}} \sum_{k=1}^3 \frac{\left(\frac{6,9155}{10}\right)^{k-1}}{(k-1)!} = 0,033 = 3,3\% \\
\bar{\gamma} &= 10 \quad N_R = 3
\end{aligned}$$

Problem 12 - Linear and decision-feedback MI-MO detection

MIMO-MRC

$$\sigma_{n_1}^2 = \sigma_{n_2}^2 = \dots = \sigma_{N_R}^2 = \sigma_n^2$$



$$f = (f_1, f_2, \dots, f_{N_R})^T; \quad g = (g_1^*, g_2^*, \dots, g_{N_R}^*)$$

$$\rightarrow \text{Nomenclature: } \begin{cases} f^H f = 1 = \|f\|^2 \\ g^H g = 1 = \|g\|^2 \end{cases}$$

singular value decomposition: $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$

$$\mathbf{\Sigma} = \text{diag}(\xi_1^2, \xi_2^2, \dots, \xi_N^2)$$

$$\rightarrow \text{sorted in descending order} \rightarrow \xi_1^2 = \xi_{max}^2$$

$$f = \mathbf{V}f' \rightarrow \mathbf{V}^H f = \underbrace{\mathbf{V}^H \mathbf{V}}_{\mathbf{I}} f' = f' \rightarrow \|f'\|^2 = \|\mathbf{V}^H f\|^2 \stackrel{\mathbf{V}: \text{unitary}}{=} \|f\|^2 = 1$$

in a similar way:

$$\rightarrow g = u \cdot g'$$

$$\rightarrow \|g\|^2 = \|g'\|^2 = 1$$

$$\text{SNR} = \frac{\sigma_x^2 \left| \overbrace{\text{total transfer function}}^{h_{total}} \right|^2}{\sigma_n^2 \underbrace{\|g\|^2}_{\rightarrow=1}} = \frac{\sigma_x^2 |g^H H f|}{\sigma_n^2} = \frac{\sigma_x^2 |g'^H \mathbf{U}^H \mathbf{U} \mathbf{\Sigma} \mathbf{V}^H \mathbf{V} f'|}{\sigma_n^2}$$

$$\begin{aligned}
& \rightarrow \text{SNR} = \frac{\sigma_x^2}{\sigma_n^2} |g'^H \Sigma f'| \\
g' &= (1 \ 0 \ \dots \ 0)'; \quad f' = (1 \ 0 \ \dots \ 0)' \\
& \rightarrow \text{to maximize SNR: } \xi_1 = \xi_{\max} \\
\text{SNR}_{\max} &= \frac{\sigma_x^2}{\sigma_n^2} \xi_1^2 \\
\xi_k^2 &\text{ are also eigenvalues of } HH^H \\
\|H\|_F^2 &= \text{tr}(HH^H) = \sum_{k=1}^N \xi_k^2 \quad N = \min\{N_T, N_R\}
\end{aligned}$$

Frobenius norm

$$\xi_1^2 = \xi_{\max}^2 \quad \rightarrow \quad \underbrace{\frac{1}{N} \sum_{k=1}^N \xi_k^2}_{\text{average}} \leq \xi_{\max}^2 \leq \sum_{k=1}^N \xi_k^2$$

$$\|H\|_F^2 = \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} |h_{ij}|^2, \quad h_{ij} \sim \mathcal{CN}(0, 1)$$

$\rightarrow \|H\|_F^2$ is the sum of squared complex Gaussian RVs

$\rightarrow \chi^2$ -square (χ^2 -distribution)

$$f_{\|H\|_F^2}(x) = \frac{1}{\Gamma(N_T N_R)} x^{N_T N_R - 1} e^{-x}$$

pdf of the channel

$$\overbrace{f_{\frac{1}{N} \|H\|_F^2}}^{\text{pdf of the channel}} = \frac{1}{\Gamma(N_T N_R)} (Nx)^{N_T N_R - 1} e^{-Nx} = \frac{N^{N_T N_R} x^{N_T N_R - 1} e^{-Nx}}{\Gamma(N_T N_R)}$$

$$\begin{aligned}
\overline{\text{SER}} &\geq \int_0^\infty f_{\frac{1}{N} \|H\|_F^2} \text{SER}(x) dx = \int_0^\infty \frac{N^{N_T N_R} x^{N_T N_R - 1} e^{-Nx}}{\Gamma(N_T N_R)} \frac{C_A}{2} e^{-\frac{d_A^2}{2} x \gamma} dx = \\
&= \frac{N^{N_T N_R} C_A}{2 \Gamma(N_T N_R)} \int_0^\infty x^{N_T N_R - 1} e^{-\left(N + \frac{d_A^2}{2} \gamma\right) x} dx = \frac{N^{N_T N_R} C_A}{2 \Gamma(N_T N_R)} \frac{\Gamma(N_T N_R)}{\left(N + \frac{d_A^2}{2} \gamma\right)^{N_T N_R}}
\end{aligned}$$

$$\rightarrow \overline{\text{SER}} \geq \frac{C_A N^{N_T N_R}}{2 \left(N + \frac{d_A^2}{2} \underbrace{\gamma}_{\text{SNR}} \right)^{N_T N_R}}$$

$$\text{SNR} \rightarrow \infty \Rightarrow \text{SER} \geq \frac{C_A}{2 \left(\frac{d_A^2}{2N} \gamma \right)^{N_T N_R}}$$

$$\begin{aligned}
\frac{\sum_{k=1}^N \xi_k^2}{N} &\leq \xi_{max}^2 \leq \sum_{k=1}^N \xi_k^2 \quad \rightarrow \quad \overline{\text{SER}} \leq \frac{C_{\mathcal{A}}}{2 \left(\frac{D_{\mathcal{A}}^2}{2} \gamma \right)^{N_T N_R}} \\
&\rightarrow \quad \frac{C_{\mathcal{A}}}{2 \left(\frac{D_{\mathcal{A}}^2}{2} \gamma \right)^{N_T N_R}} \leq \overline{\text{SER}} \leq \frac{C_{\mathcal{A}}}{2 \left(\frac{D_{\mathcal{A}}^2}{2} \gamma \right)^{N_T N_R}} \\
\text{we have: } \frac{\dots}{\dots \gamma^{N_T N_R}} &\rightarrow \quad \text{Diversity gain} = N_T N_R
\end{aligned}$$

$$\text{equalization} \quad \left\{ \begin{array}{l} \text{at the receiver} \\ \text{(detection)} \end{array} \right\} \left\{ \begin{array}{l} \text{linear} \left\{ \begin{array}{l} \text{Zero-Forcing ZF} \\ \text{Minimum Mean Square Error MMSE} \end{array} \right. \\ \text{decision feedback} \left\{ \begin{array}{l} \text{ZF} \\ \text{MMSE} \end{array} \right. \end{array} \right. \\
\left\{ \begin{array}{l} \text{at the transmitter} \\ \text{(precoding)} \end{array} \right\} \left\{ \begin{array}{l} \text{linear} \\ \text{with feedback} \end{array} \right.
\end{array} \right.$$

Linear equalizer: