Tutorial for MIMO Communication Systems

28. Januar 2013

Problem 1

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

a) Eigenvalues

$$\begin{aligned}
\det(\lambda \mathbf{I} - \mathbf{A}) \overset{!}{=} 0 \quad & \rightarrow \begin{vmatrix} \lambda + 1 & 1 \\ 1 & \lambda + 1 \end{vmatrix} \overset{!}{=} 0 \\
& \Rightarrow \lambda_1^A = -2, \quad \lambda_2^A = 0 \\
\det(\lambda \mathbf{I} - \mathbf{B}) \overset{!}{=} 0 \quad & \rightarrow \begin{vmatrix} \lambda + \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \lambda - \frac{1}{\sqrt{5}} \end{vmatrix} \overset{!}{=} 0 \\
& \Rightarrow \lambda_1^B = 1, \quad \lambda_2^B = -1
\end{aligned}$$

b) Determinant & Trace

$$|\mathbf{A}| = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 1 - 1 = 0, \quad |\mathbf{B}| = -1$$

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^2(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(-1)^2(a_{21}a_{33} - a_{31}a_{23}) + \ldots$$

$$|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}| \rightarrow \mathbf{AB} = \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix} \rightarrow |\mathbf{AB}| = \frac{1}{5}(3 - 3) = 0$$

$$\operatorname{tr}(\mathbf{A}) = -1 - 1 = -2, \quad \operatorname{tr}(\mathbf{B}) = 0$$

$$\lambda_1^A = -2, \quad \lambda_2^A = 0 \quad \rightarrow \quad \operatorname{tr}(\mathbf{A}) = \sum_i \lambda_i^A$$

$$\lambda_1^B = 1, \quad \lambda_2^B = -1 \quad \rightarrow \quad \operatorname{tr}(\mathbf{B}) = \sum_i \lambda_i^B$$

$$|\mathbf{A}| = \lambda_1^A \lambda_2^A = \prod_i \lambda_i^A$$

c)

$$\mathrm{tr}\,(\mathbf{AB}) = \mathrm{tr}\,(\mathbf{BA}) \quad \forall \mathbf{A}, \mathbf{B}\text{: square matrix N}\times\text{N}$$

$$\mathbf{C} = \mathbf{AB} \quad \rightarrow \quad c_{m,n} = \sum_k a_{m,k} b_{k,n} \quad \rightarrow \quad \mathrm{tr}\,(\mathbf{C}) = \sum_{i=1}^{N} c_{i,i} = \sum_{i=1}^{N}\sum_{k=1}^{N} a_{i,k} b_{k,i}$$

$$\mathbf{D} = \mathbf{BA} \quad \rightarrow \quad d_{m,n} = \sum_k b_{m,k} a_{k,n} \quad \rightarrow \quad \mathrm{tr}\,(\mathbf{D}) = \sum_{i=1}^{N} d_{i,i} = \sum_{i=1}^{N}\sum_{k=1}^{N} b_{i,k} a_{k,i}$$

d)

1

positive definite matrix : $\forall i,\ \lambda_i > 0$

positive semidefinite matrix : $\forall i,\ \lambda_i \geq 0$

negative definite matrix : $\forall i,\ \lambda_i < 0$

negative semidefinite matrix : $\forall i,\ \lambda_i \leq 0$

$\mathbf{A}$: $\lambda_i^A \leq 0 \quad \Rightarrow$ negative semidefinite

$\mathbf{B}$: indefinite

2

$$\mathbf{X}^H = \mathbf{X} \quad \rightarrow \quad \mathbf{X}^T = \mathbf{X} \quad \rightarrow \quad \mathbf{A}, \mathbf{B}\text{: hermitian}$$

3

$$\mathrm{rank}\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right) \rightarrow \mathrm{rank}\,(\mathbf{A}) = 1$$

$$\mathrm{rank}\left(\frac{1}{\sqrt{5}}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}\right) = \mathrm{rank}\left(\frac{1}{\sqrt{5}}\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{pmatrix}\right)$$

$$\beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 = 0 \quad \rightarrow \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \beta_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \beta_2 = -\frac{1}{2}, \quad \beta_2 = -2 \quad ↯$$

$$\Rightarrow \quad \mathrm{rank}\,(\mathbf{B}) = 2$$

Important property:

$\mathbf{X}$ has full rank $\quad \rightarrow \quad |\mathbf{X}| \neq 0$ or $\mathbf{X}$ is invertible

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$|\mathbf{A}| = 0 \quad \Rightarrow \quad \mathbf{A}$ nicht invertierbar

$|\mathbf{B}| = -1 \quad \Rightarrow \quad \mathbf{B}$ invertierbar

5

$\mathbf{X}$ is unitary $\quad\Leftrightarrow\quad \mathbf{X}\cdot\mathbf{X}^H = \mathbf{T}, \quad \mathbf{X}$: unitary $\rightarrow \mathbf{X}^{-1} = \mathbf{X}^H$

$$\mathbf{A}\mathbf{A}^H = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad \Rightarrow \mathbf{A} \text{ is not unitary}$$

$$\mathbf{B}\mathbf{B}^H = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \qquad \Rightarrow \mathbf{B} \text{ is unitary}$$

e)

$$|\mathbf{I}+\mathbf{A}\mathbf{B}| = |\mathbf{I}+\mathbf{B}\mathbf{A}|$$

$$|\mathbf{I}+\mathbf{A}\mathbf{B}| = \left|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{5}}\begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}\right| = 2,7789$$

$$|\mathbf{I}+\mathbf{B}\mathbf{A}| = \left|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{5}}\begin{pmatrix} 3 & -3 \\ -1 & 1 \end{pmatrix}\right| = 2,7789$$

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f)

$$\mathbf{X}^{-1} = \frac{1}{|\mathbf{X}|}\operatorname{adj}(\mathbf{X}), \quad \operatorname{adj}(\mathbf{X}) = \text{ Transpose of cofactor matrix}$$

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \frac{1}{\det\mathbf{X}}\begin{pmatrix} (x_{22}x_{33} - x_{32}x_{23}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\rightarrow \quad \mathbf{B}^{-1} = \frac{1}{\det\mathbf{B}}\begin{pmatrix} b_{22} & -b_{21} \\ -b_{21} & b_{11} \end{pmatrix}^T = \frac{1}{-1}\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}^T$$

$$\rightarrow \quad \mathbf{B}^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}, \quad \left|\mathbf{B}^{-1}\right| = -1 \quad \Rightarrow \left|\mathbf{B}^{-1}\right| = \frac{1}{|\mathbf{B}|}$$

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g)

$$\operatorname{rank}(\mathbf{A}) = 1, \quad \operatorname{rank}(\mathbf{B}) = 2$$

$$\mathbf{A}\mathbf{B} = \frac{1}{\sqrt{5}}\begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -3 \end{pmatrix} = -3\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \rightarrow \quad \operatorname{rank}(\mathbf{A}\mathbf{B}) = 1$$

$$\operatorname{rank}(\mathbf{A}\mathbf{B}) \leq \min\{\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})\}$$

h)

$$\|\bullet\| : \mathbb{C}^{M\times N} \quad \rightarrow \quad \mathbb{R}$$

Axioms for norm function:

1.) $\|\mathbf{X}\| \geq 0, \quad \|\mathbf{X}\| = 0$ if and only if $\mathbf{X} = 0$

2.) $\|\alpha\mathbf{X}\| = |\alpha|\,\|\mathbf{X}\|$

3.) $\|\mathbf{X}+\mathbf{Y}\| \leq \|\mathbf{X}\| + \|\mathbf{Y}\|$ (triangular property)

4.) $\|\mathbf{XY}\| \leq \|\mathbf{X}\| \cdot \|\mathbf{Y}\|$

i)

$$\|\alpha\mathbf{X}\|_F = \sqrt{\sum_i\sum_j |\alpha\mathbf{X}|^2} = |\alpha| \sqrt{\sum_i\sum_j |\mathbf{X}|^2} = |\alpha|\,\|\mathbf{X}\|_F \qquad \text{Axiom 2}$$

$$\|\mathbf{X}+\mathbf{Y}\|_F^2 \leq \|\mathbf{X}\|_F^2 + \|\mathbf{Y}\|_F^2 \qquad \text{Axiom 3}$$

$$\|\mathbf{X}+\mathbf{Y}\|_F^2 = \sum_i\sum_j |x_{ij}+y_{ij}|^2 \qquad \text{Axiom 4}$$

$$\leq \sum_i\sum_j |x_{ij}|^2 + \sum_i\sum_j |y_{ij}|^2 = \|\mathbf{X}\|_F^2 + \|\mathbf{Y}_F^2$$

$$\|\mathbf{XY}\|_F^2 \leq \|\mathbf{X}\|_F^2 \cdot \|\mathbf{Y}\|_F^2$$

$$\rightarrow \|\mathbf{XY}\|_F^2 = \sum_i\sum_j \mathbf{XY} \leq \sum_i\sum_j |x_{ij}|^2 \cdot \sum_i\sum_j |y_{ij}|^2 = \|\mathbf{X}\|_F^2 \cdot \|\mathbf{Y}\|_F^2$$

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j)

$$\mathbf{A}\otimes\mathbf{B} \neq \mathbf{B}\otimes\mathbf{A}$$

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} a_{ij} & \mathbf{B} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 & -1 & 2 \\ -2 & 1 & 2 & 1 \\ -1 & 2 & 1 & -2 \\ 2 & 1 & -2 & -1 \end{pmatrix}$$

$$\mathbf{B}\otimes\mathbf{A} = \begin{bmatrix} b_{ij} & \mathbf{A} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -1 & -2 & 2 \\ -1 & 1 & 2 & -1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{A}\otimes\mathbf{B} \neq \mathbf{B}\otimes\mathbf{A}$$

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k)

Hadamard's Inequality: $|\mathbf{A}| \leq \prod_{k=1}^{K} a_{kk}$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -2 & -1 \\ 3 & 1 & 0 \end{pmatrix} = 1 - 2\,(3) - 1\,(2+6) = -13$$

$$\prod_{k=1}^{3} a_{kk} = 0 \quad \rightarrow \quad \det(\mathbf{A}) = -13 \leq \prod_{k=1}^{3} a_{kk}$$

$$\tilde{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \left|\tilde{\mathbf{A}}\right| = 1\,(-2)\,0 = 0$$

$$\rightarrow \text{off-diagonal elements should set to zero} \leftrightarrow |\mathbf{A}| = \prod_{k=1}^{K} a_{kk}$$

Problem 2 - Matrix Decompositions

$$\mathbf{A} = \begin{pmatrix} 1 & 0,5 & -0,5 \\ -1 & 1 & -1 \end{pmatrix}$$

a)

$$\text{rank}\,(\mathbf{A}): \quad \alpha \begin{pmatrix} 1 \\ 0,5 \\ -0,5 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \stackrel{!}{=} 0$$

we set $\beta = 1$ w/o loss of generality

$\rightarrow \quad \alpha - 1 = 0 \quad \Rightarrow \quad \alpha = 1$

$\quad 0,5\alpha + 1 = 0 \quad \Rightarrow \quad \alpha = -2 \quad \lightning$

$\Rightarrow \text{rank}\,(\mathbf{A}) = 2$

b)

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H \qquad \text{svd}\,(\mathbf{A}) \quad \text{(singular value decomposition)}$$

$$\tilde{\mathbf{A}} = \mathbf{A}\mathbf{A}^T = \begin{pmatrix} 1,5 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0 \quad \rightarrow \quad \det \left| \begin{pmatrix} \lambda - 1,5 & 0 \\ 0 & \lambda - 3 \end{pmatrix} \right| = 0$$

$$\Rightarrow \quad \lambda_1 = 1,5, \quad \lambda_2 = 3$$

Eigenvectors $\mathbf{Ev}$:

$$\tilde{\mathbf{A}}\mathbf{Ev} = \lambda \mathbf{Ev}$$

for $\lambda_1 = 1,5$:

$$\rightarrow \quad \begin{pmatrix} 1,5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1,5 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\rightarrow 1,5v_1 = 1,5v_2, \quad 3v_2 = 1,5v_1$$

$$\Rightarrow v_1 = 1, \quad v_2 = 0$$

$$\Rightarrow \mathbf{Ev}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $\lambda_2 = 3$:

$$\rightarrow \quad \begin{pmatrix} 1,5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 3 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\rightarrow 1,5u_1 = 3u_2, \quad 3u_2 = 3u_1$$

$$\Rightarrow \mathbf{Ev}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H, \quad \mathbf{U} = \begin{pmatrix} \mathbf{Ev_1} & \mathbf{Ev_2} & \dots & \mathbf{Ev_N} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

c)

S_u^2 : Eigenvalue of $\mathbf{X}\mathbf{X}^H$

S_u : singular value of $\mathbf{X}$

$$\text{tr}\left(\mathbf{X}\mathbf{X}^H\right) = \sum_u \lambda_u \left(\mathbf{X}\mathbf{X}^H\right) = \sum_u S_u^2$$

d)

QR-decomposition $\quad \rightarrow \quad \mathbf{A} = \mathbf{QR}$

with $\mathbf{Q}$ unitary, $\mathbf{R}$ upper triangular matrix

Gram-Schmidt:

$$\mathbf{A} = \begin{bmatrix} 1 & 0,5 & -0,5 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow \quad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{a}_2 \mathbf{e}_1)\, \mathbf{e}_1 = \begin{bmatrix} 0,5 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0,5 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right) = \begin{bmatrix} 0,75 \\ 0,75 \end{bmatrix}$$

$$\rightarrow \quad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_3 = \mathbf{a}_3 - (\mathbf{a}_3 \mathbf{e}_1)\, \mathbf{e}_1 - (\mathbf{a}_3 \mathbf{e}_2)\, \mathbf{e}_2 =$$

$$= \begin{bmatrix} -0,5 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} -0,5 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} - \left(\begin{bmatrix} -0,5 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -0,5 \\ -1 \end{bmatrix} + \begin{bmatrix} -0,25 \\ 0,25 \end{bmatrix} + \begin{bmatrix} 0,75 \\ 0,75 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad \mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{-\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}_1\mathbf{e}_1 & \mathbf{u}_2\mathbf{e}_1 & \mathbf{u}_3\mathbf{e}_1 \\ 0 & \mathbf{u}_2\mathbf{e}_2 & \mathbf{u}_3\mathbf{e}_2 \\ 0 & 0 & \mathbf{u}_3\mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{3}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

e)

$$\mathbf{B} = \begin{bmatrix} -1 & 1 & -1 \\ -0,5 & 0,5 & 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0,5 & -0,5 \\ -1 & 1 & -1 \end{bmatrix}$$

$\mathbf{B} = \mathbf{QL}$ — $\mathbf{B}$ is updown and left-right flipped version of $\mathbf{A}$

$\mathbf{B} = \mathrm{UD}\left(\mathrm{LR}\left(\mathbf{A}\right)\right)$

$\mathbf{A} = \mathbf{Q}_A \mathbf{R}_A$ — $\rightarrow \mathbf{B} = \mathrm{UD}\left(\mathrm{LR}\left(\mathbf{Q}_A \mathbf{R}_A\right)\right) = \mathbf{Q}_B \mathbf{L}_B$

$\mathbf{Q}_B = \mathrm{UD}\left(\mathrm{LR}\left(\mathbf{Q}_A\right)\right)$

$\mathbf{L}_B = \mathrm{UD}\left(\mathrm{LR}\left(\mathbf{R}_A\right)\right)$

$$\mathbf{Q}_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{L}_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ -1,5 & 1,5 & 0 \\ 0,5 & 0,5 & 2 \end{bmatrix}$$

Gaussian Distribution, Entropy and Capacity

- Data rate
- Reliability (BER, transmit power)

$C = \mathrm{ld}\left(1 + \mathrm{SNR}\right)$

Differential Entropy: $h\left(x\right) = -\int\limits_{-\infty}^{\infty} f_x\left(x\right) \mathrm{ld}\left(f_x\left(x\right)\right) \mathrm{d}x$

$C = \max\left(\mathrm{I}\left(x, y\right)\right) = \max\left(h\left(y\right) - h\left(n\right)\right)$

maximized for $h\left(y\right)$ Gaussian

noise $\rightarrow$ average power is .... $\rightarrow$ Gaussian is the most random noise

signal $\rightarrow$ using shaping techniques we get Gaussian distribution

$$C = \max\left(h\left(y\right) - h\left(n\right)\right) = \frac{1}{2}\mathrm{ld}\left(2\pi e \sigma_y^2\right) - \frac{1}{2}\mathrm{ld}\left(2\pi e \sigma_n^2\right) \overset{\sigma_y^2 = \sigma_x^2 + \sigma_n^2}{=} \frac{1}{2}\mathrm{ld}\left(1 + \frac{\sigma_x^2}{\sigma_n^2}\right)$$

water-filing: $\dfrac{P_i}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{\sigma_n^2}{P\|H_i\|^2} & \frac{\|H_i\|^2 P}{\sigma_n^2} > \gamma_0 \\ 0 & \frac{\|H_i\|^2 P}{\sigma_n^2} \leq \gamma_0 \quad \rightarrow \quad \text{channel i is too bad, low SNR} \end{cases}$

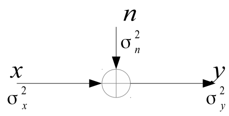
P_i = TX power of channel i

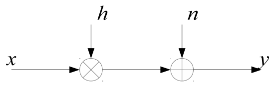
P = total TX power

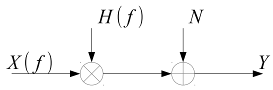
H_i = gain of channel i

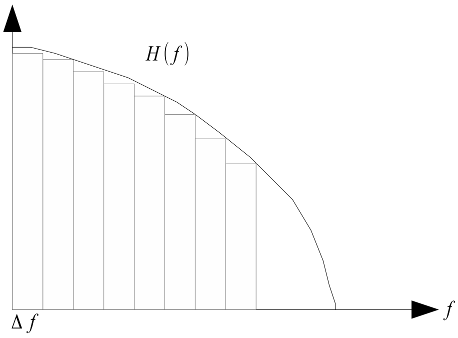
$\sigma_n^2 = N_0 \cdot B$

Channels =

- AWGN: 

$C_T = B \,\mathrm{ld}\left(1 + \frac{\sigma_x^2}{\sigma_n^2}\right) \frac{\text{bit}}{\text{s}}$
- flat fading: 

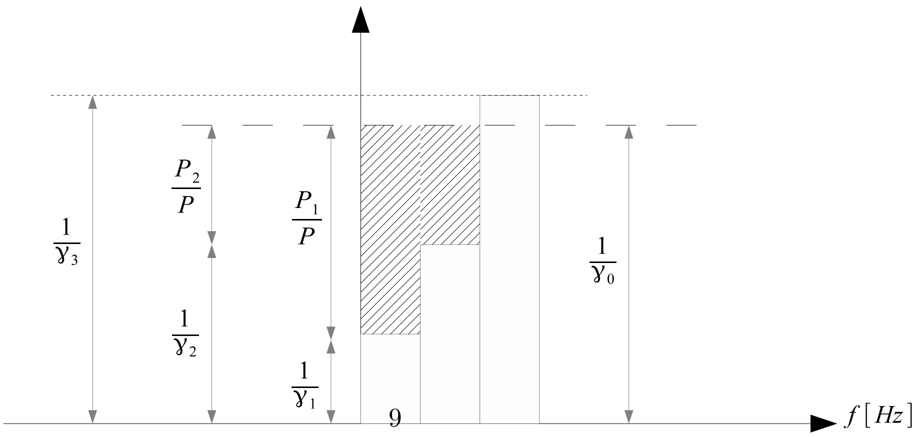
$C_T = B \,\mathrm{ld}\left(1 + \frac{h^2\sigma_x^2}{\sigma_n^2}\right)$
- frequency selective: 



$H_i(f) = \text{const.} \quad \rightarrow \quad$ Water-filling algorithm

$\rightarrow \quad$ Allocate more power to better (with highter SNR) channels

dB $\begin{cases} \text{comparison: power } P_1,\, P_2 \quad \rightarrow \quad P_2 \text{ has } 10\log\frac{P_2}{P_1}\text{dB more power than } P_1 \\ \text{absolute power: dBm, } P_2 = 10\text{W}, \quad P_1 = 1\text{W} \quad \rightarrow \quad P_2 = 10\log 10 = 10\text{dB} \end{cases}$



$\gamma_i = \text{SNR of } i\text{th channel}$

$P_i = \text{allocated power to the } i\text{th channel}$

$$\frac{P_i}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_i} & \frac{|H_i|^2 P}{\sigma_n^2} = \frac{|H_i|^2 P}{N_0 B} \geq \gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

$\sigma_n^2 = N_0 B \quad \text{with noise power } \sigma_n^2 \text{ and PSD of noise } N_0$

$H_i = \text{transfer function (channel gain) of } i\text{th channel}$

$$\text{Constraint:} \quad \sum_{i=1}^{3} P_i = P \quad \text{total transmit power}$$

$$\sum_{i=1}^{3} P_i = P \quad \rightarrow \quad \sum_{i=1}^{3} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) P = P \quad \rightarrow \quad \boxed{\sum_{i=1}^{3} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) P = 1}$$

$$\gamma_1 = \frac{|H_1|^2 P}{N_0 B} = \frac{1 \cdot 10^{-3}}{10^{-11} \cdot 1^7} \quad \rightarrow \quad \gamma_1 = 10 \quad \rightarrow \quad \text{best channel}$$

$$\gamma_2 = \frac{|H_2|^2 P}{N_0 B} = 10 H_2^2 \quad \rightarrow \quad \gamma_2 = 2,5$$

$$\gamma_3 = \frac{|H_3|^2 P}{N_0 B} = 10 H_3^2 \quad \rightarrow \quad \gamma_3 = 0,9 \quad \rightarrow \quad \text{worst channel}$$

$$\sum_{i=1}^{3} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) = 1 \quad \Rightarrow \quad \frac{3}{\gamma_0} = 1 + \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} \quad \rightarrow \quad \gamma_0 = 1,15 \quad \text{(water level)}$$

$$\rightarrow \text{water level check} \quad \rightarrow \quad \gamma_i > \gamma_0 \forall i$$

$\gamma_1 = 10 > \gamma_0 \quad \checkmark$

$\gamma_2 = 2,5 > \gamma_0 \quad \checkmark$

$\gamma_3 = 0,9 < \gamma_0 \quad \lightning \quad \rightarrow \quad \text{no power can be allocated to channel 3}$

$$\sum_{i=1}^{2} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) = 1 \quad \rightarrow \quad \frac{2}{\gamma_0} = 1 + \frac{1}{10} + \frac{1}{2,5} \quad \rightarrow \quad \gamma_0 = 1,33$$

$\gamma_1 = 10 > \gamma_0 \quad \checkmark$

$\gamma_2 = 2,5 > \gamma_0 \quad \checkmark$

$$\rightarrow \quad C_\mathrm{T} = B \sum_{i=1}^{2} \mathrm{ld} \left(1 + \frac{|H_1|^i P}{N_0 B} \right) = B \sum_{i=1}^{2} \mathrm{ld} \left(1 + \frac{\gamma_i}{\gamma_0} - 1 \right) = B \sum_{i=1}^{2} \mathrm{ld} \left(\frac{\gamma_i}{\gamma_0} \right)$$

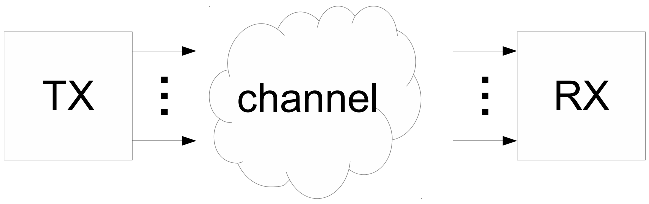
$$C_\mathrm{T} = 10^7 \left(\mathrm{ld} \left(\frac{10}{1,33} \right) + \mathrm{ld} \left(\frac{2,5}{1,33} \right) \right) \quad \rightarrow \quad C_\mathrm{T} = 38,2 \text{ Mbits/s}$$

$$C_\mathrm{T} = B \,\mathrm{ld} \left(1 + \frac{S}{N} \right) \quad \text{Shannon formula}$$

Problem 4

Multiplexing gain $\quad G_{\mathrm{m}} = \lim\limits_{\mathrm{SNR}\to\infty} \frac{R}{\mathrm{ld}\,(\mathrm{SNR})}$

Diversity gain $\quad C = \mathrm{ld}\,(1+\mathrm{SNR}) \quad \Rightarrow \quad C \overset{\mathrm{SNR}\to\infty}{\approx} \mathrm{ld}\,(\mathrm{SNR})$



Multiplexing gain = number of parallel independent equivalent channels

svd (H) $\quad\rightarrow\quad$ singular values of H: σ_i^2, $i \in \{1, 2, \ldots, \mathrm{rank}\,(H)\}$

rank (H) $\quad\rightarrow\quad$ number of singular values

$$H_1 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \quad\rightarrow\quad \mathrm{rank}\,(H_1) = 2$$

Matlab/Octave $\quad\rightarrow\quad$ $\mathrm{svd}\,(H_1) = \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V}_1^H$

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 2,56 & 0 & 0 \\ 0 & 1,56 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad\rightarrow\quad \sigma_1 = 2,56,\ \sigma_2 = 1,56,\ \sigma_3 = 0 \rightarrow \text{useless}$$

$$H_2 = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad\rightarrow\quad \mathrm{rank}\,(H_2) = 3 \quad\rightarrow\quad \text{full rank}$$

$\mathrm{svd}\,(H_1) = \mathbf{U}_2 \mathbf{\Sigma}_2 \mathbf{V}_2^H$

$$\mathbf{\Sigma}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad\rightarrow\quad \sigma_1 = 2,\ \sigma_2 = 2,\ \sigma_3 = 1$$

Problem 5

MIMO channel capacity

short review:

1 - channel matrix is deterministic and known to the transmitter

2 - channel matrix is random and changes during the transmission of a code word
→ ergodic capacity

3 - channel matrix is random and fixed during the transmission of a code word, but only known to TX

Decompose the MIMO channel to subchannels

$$C_{\mathrm{T}} = B \sum_{i=1}^{N} \left[\mathrm{ld} \underbrace{\left(\frac{x_i^2}{\sigma_n^2} \mu \right)}_{\substack{\text{equivalent SNR} \\ \text{of channel } i}} \right]$$

B : Bandwidth

x_i : singular value of H

σ_n^2 : noise power

μ : water level

$$\mathrm{svd}\,(\mathbf{H}) = \mathrm{svd}\left(\begin{bmatrix} 0,2 & -0,2 & 0,2 \\ 0,1 & 0,4 & -1 \\ 1 & 0,2 & 1 \end{bmatrix} \right) = \mathbf{U\Sigma V}^H$$

$$\mathbf{\Sigma} = \begin{bmatrix} 1,61 & 0 & 0 \\ 0 & 0,82 & 0 \\ 0 & 0 & 0,2 \end{bmatrix} \quad \rightarrow \quad \begin{aligned} x_1 &= 1,61 \\ x_2 &= 0,82 \\ x_3 &= 0,2 \end{aligned}$$

$$\frac{P}{\sigma_n^2} = 10\mathrm{dB} \quad \rightarrow \quad \frac{P}{\sigma_n^2} = 10$$

$$\gamma_1 = x^2 \frac{P}{\sigma_n^2} = 1,61^2 \cdot 10 = 25,9$$

$$\gamma_2 = 0,82^2 \cdot 10 = 7,6$$

$$\gamma_3 = 0,2^2 \cdot 10 = 0,4$$

3 non-zero singular values → number of parallel independent channels = 3

$$\sum_{i=1}^{3} \left(\mu - \frac{1}{\gamma_i} \right) = 1$$

$$\rightarrow \quad 3\mu = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} + 1 = 3,7 \quad \rightarrow \quad \mu = 1,23 \quad \text{water level}$$

water level check: $\begin{cases} \frac{1}{\gamma_1} = 0,039 < \mu & \checkmark \\ \frac{1}{\gamma_2} = 0,15 < \mu & \checkmark \\ \frac{1}{\gamma_3} = 2,5 > \mu & \lightning \quad \text{3rd channel doesn't have enough SNR} \end{cases}$

$\Rightarrow$ no power can be allocated to 3rd channel

$\Rightarrow$ allocate power in 1st and 2nd channel

$$\rightarrow \sum_{i=1}^{2} \left(\mu - \frac{1}{\gamma_i} \right) = 1$$

$$\rightarrow 2\mu = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + 1 = 1,19 \quad \rightarrow \quad \mu = 0,595 \quad \text{water level}$$

water level check: $\begin{cases} \frac{1}{\gamma_1} < \mu & \checkmark \\ \frac{1}{\gamma_2} < \mu & \checkmark \end{cases}$

$$\rightarrow C_{\mathrm{T}} = B \sum_{i=1}^{2} \mathrm{ld}\left(\mu \gamma_i\right) = 10^6 \left(\mathrm{ld}\left(\frac{25,9}{0,595}\right) + \mathrm{ld}\left(\frac{6,7}{0,595}\right) \right)$$

$\rightarrow C_{\mathrm{T}} = 8,937$ Mbit/s

no CSIT:

$$\rightarrow C_{\mathrm{T}} = B \sum_{i=1}^{3} \mathrm{ld}\left(1 + x_i^2 \frac{P_i}{\sigma_n^2}\right) = B \sum_{i=1}^{3} \mathrm{ld}\left(1 + x_i^2 \frac{P}{3\sigma_n^2}\right)$$

$$\rightarrow C_{\mathrm{T}} = 10^6 \left(\mathrm{ld}\left(1 + 8,64\right) + \mathrm{ld}\left(1 + 2,24\right) + \mathrm{ld}\left(1 + 0,13\right)\right)$$

$\rightarrow C_{\mathrm{T}} = 5,14$ Mbit/s

no CSIT at transmitter result in approx. $3,8$ Mbit/s data rate degradation

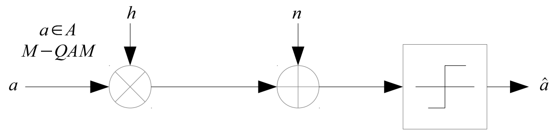
Hier fehlen Aufgabe 6-7???

das folgende gehrt zu Aufgabe 8 SISO Rayleigh fading channel

ACHTUNG: doppelte Nummerierung: Aufgabe 8 Receiver diversity, MRC, EGC and SC

Problem 8.1[1] - SISO Rayleigh fading channel

$$\overline{\mathrm{SER}} = \mathbb{E}_{|h|^2}\left\{\mathrm{SER}\left(|h|^2\right)\right\}$$



$$\mathrm{SER}\left(|h|^2\right) = C_{\mathcal{A}}\mathrm{Q}\left(\sqrt{d_{\min}^2 \frac{E_b}{N_0}\,|h|^2}\right)$$

$$C_{\mathcal{A}} = 3 \quad \text{(16-QAM)} \qquad\qquad C_{\mathcal{A}} = 2 \quad \text{(4-QAM)}$$

$$= C_{\mathcal{A}}\mathrm{Q}\left(\sqrt{\underbrace{\frac{d_{\min}^2}{\mathrm{ld}((M)}}_{d_{\mathcal{A}^2}}\underbrace{\frac{E_s}{N_0}}_{\gamma}\,|h|^2}\right) = C_{\mathcal{A}}\mathrm{Q}\left(\sqrt{d_{\mathcal{A}}^2\gamma\,|h|^2}\right)$$

$h \sim CN(0,1) \quad \rightarrow \quad |h|^2$ is exponentially distributed with variance 1

$$\rightarrow \overline{\mathrm{SER}} = \int_0^\infty \underbrace{f_{|h|^2}(x)}_{\text{pdf of } |h|^2}\,\mathrm{SER}(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-x} C_{\mathcal{A}}\mathrm{Q}\left(\sqrt{d_{\mathcal{A}}^2\gamma\,|h|^2}\right)\mathrm{d}x$$

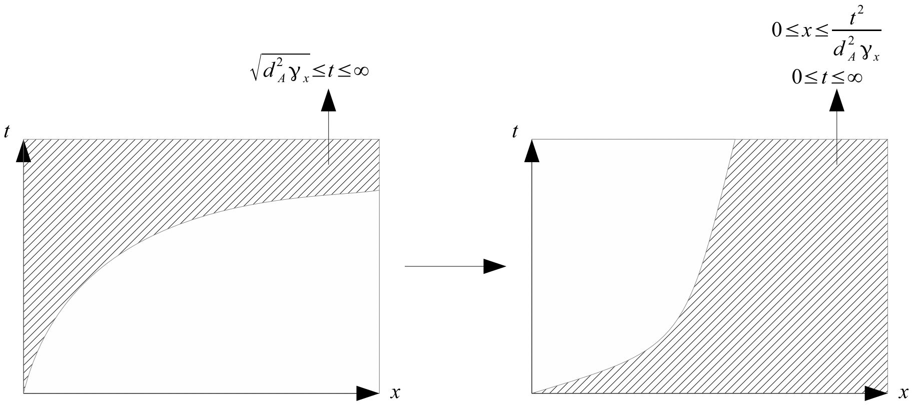
$$\mathrm{Q}(x) = \frac{1}{2\pi}\int_x^\infty e^{-\frac{t^2}{2}}\,\mathrm{d}t$$

$$\rightarrow \quad \overline{\mathrm{SER}} = \int_0^\infty e^{-x} C_{\mathcal{A}} \frac{1}{\sqrt{2\pi}} \int_{\sqrt{d_{\mathcal{A}}^2\gamma_x}}^\infty e^{-\frac{t^2}{2}}\,\mathrm{d}t\,\mathrm{d}x$$

$\rightarrow$ change the order of integrals

$$\rightarrow \quad \overline{\mathrm{SER}} = \frac{C_{\mathcal{A}}}{\sqrt{2\pi}} \int_{\sqrt{d_{\mathcal{A}}^2\gamma_x}}^\infty \int_0^\infty e^{-x} e^{-\frac{t^2}{2}}\,\mathrm{d}x\,\mathrm{d}t$$

[1] Problem 8 kommt 2mal vor, sind aber unterschiedliche Aufgaben



$$\begin{aligned}
\overline{\mathrm{SER}} &= \frac{C_{\mathcal{A}}}{\sqrt{2\pi}} \int_0^{\infty} \int_0^{\frac{t^2}{d_{\mathcal{A}}^2\gamma}} e^{-x} e^{-\frac{t^2}{2}} \,\mathrm{d}x \,\mathrm{d}t = \frac{C_{\mathcal{A}}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{t^2}{2}} \left[e^{-x}\right]_0^{\frac{t^2}{d_{\mathcal{A}}^2\gamma}} \mathrm{d}t = \\
&= \frac{C_{\mathcal{A}}}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{t^2}{2}} \left(-e^{-\frac{t^2}{d_{\mathcal{A}}^2\gamma}} + 1\right) \mathrm{d}t = \\
&= \frac{C_{\mathcal{A}}}{\sqrt{2\pi}} \int_0^{\infty} \left(e^{-\frac{t^2}{2}} - e^{-\frac{t^2}{2}\left(1+\frac{2}{d_{\mathcal{A}}^2\gamma}\right)}\right) \mathrm{d}t = \\
&= \frac{C_{\mathcal{A}}}{\sqrt{2\pi}} \left(\sqrt{2\pi}\,\underbrace{\mathrm{Q}\left(0\right)}_{\frac{1}{2}} - \sqrt{\frac{2\pi}{1+\frac{2}{d_{\mathcal{A}}^2\gamma}}}\,\underbrace{\mathrm{Q}\left(\sqrt{1+\frac{2}{d_{\mathcal{A}}^2\gamma}}0\right)}_{0}\right) = \\
&= \frac{C_{\mathcal{A}}}{2} \left(1 - \sqrt{\frac{1}{1+\frac{2}{d_{\mathcal{A}}^2\gamma}}}\right) = \boxed{\frac{C_{\mathcal{A}}}{2} \left(1 - \sqrt{\frac{d_{\mathcal{A}}^2\gamma}{d_{\mathcal{A}}^2\gamma + 2}}\right)}
\end{aligned}$$

d)

Asymptotic $\overline{\text{SER}}$ if $\gamma \to \infty$

$$\overline{\text{SER}} = \frac{C_{\mathcal{A}}}{2}\left(1 - \sqrt{\frac{d_{\mathcal{A}}^2\gamma}{d_{\mathcal{A}}^2\gamma + 2}}\right)$$

$C_{\mathcal{A}}$: average number of nearest neighbours in the constellation diagram

Taylor series: $f(x) \approx f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) + \dots$

uppper bound: $f(x) \approx f(x_0) + (x - x_0) f'(x_0)$

$x_0 = 0 \quad \to \quad f(x) = f(0) + x f'(0)$

$x = \frac{1}{d_{\mathcal{A}}^2\gamma}$ if $\gamma \to \infty \quad \Rightarrow \quad x = 0$

$$\overline{\text{SER}}(x) = \frac{C_{\mathcal{A}}}{2}\left(1 - \sqrt{\frac{1}{1 + 2x}}\right)$$

$$\overline{\text{SER}}(x) \leq \overline{\text{SER}}(0) + x\text{SER}'(0)$$

$$\overline{\text{SER}}(0) = \frac{C_{\mathcal{A}}}{2}\left(1 - \sqrt{\frac{1}{1}}\right) = 0$$

$$\overline{\text{SER}'}(x) = -\frac{C_{\mathcal{A}}}{2} \frac{1}{2\sqrt{\frac{1}{1+2x}}} \frac{-1}{(1+2x)^2} 2 \quad \Rightarrow \quad \overline{\text{SER}'}(0) = -\frac{C_{\mathcal{A}}}{2}$$

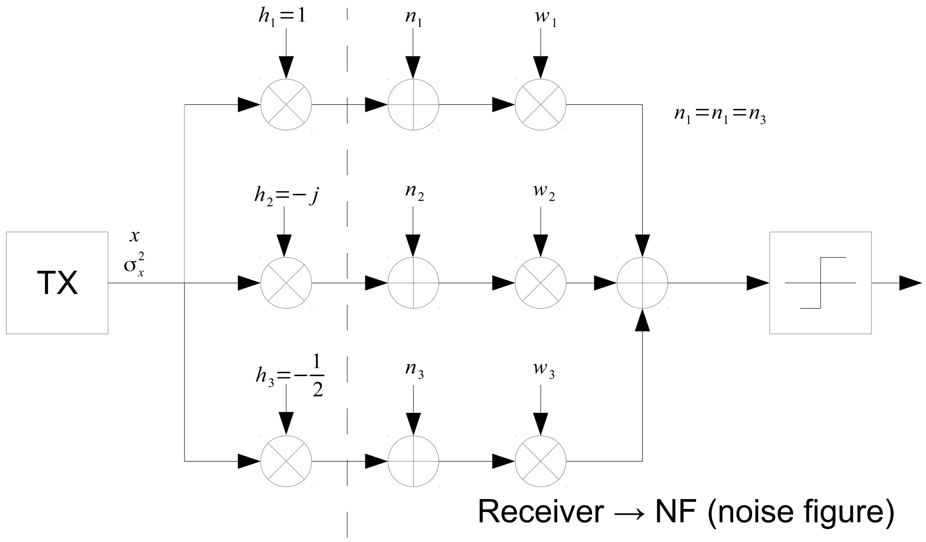
$$\to \quad \overline{\text{SER}}(x) \leq \frac{C_{\mathcal{A}}}{2} x \quad \to \quad \boxed{\overline{\text{SER}}(\gamma) \leq \frac{C_{\mathcal{A}}}{2} \frac{1}{d_{\mathcal{A}}^2\gamma}}$$

$$\text{SER}\left(|h|^2\right) = C_{\mathcal{A}} \text{Q}\left(\sqrt{d_{\mathcal{A}}^2\gamma \|h\|^2}\right)$$

we could have used Chernoff bound: $\text{Q}(x) \leq \frac{1}{2}\text{e}^{-\frac{x^2}{2}}$

$$\text{SER}\left(|h|^2\right) \leq \frac{C_{\mathcal{A}}}{2} e^{-\frac{d_{\mathcal{A}}^2\|h\|^2\gamma}{2}} \quad \to \quad \dots \quad \to \quad \boxed{\overline{\text{SER}} \leq C_{\mathcal{A}} \frac{1}{d_{\mathcal{A}}^2\gamma}}$$

Problem 8.2 - Receiver diversity, MRC, EGC and SC



a)

$$
\begin{aligned}
\text{MRC} \quad &\rightarrow \quad w_j = h_j^* \quad \forall j \in \{1, \ldots, N_R\} \\
& \quad\quad w_1 = 1 \\
& \quad\quad w_2 = j \\
& \quad\quad w_3 = -\frac{1}{2} \\
& \quad\quad \text{In order to have } w_j \text{ we need } h_j \rightarrow \text{channel estimation} \\
\text{EGC} \quad &\rightarrow \quad h_n = |h_n| \, e^{j\phi_n} \quad \Rightarrow \quad w_n = e^{j\phi_n} \\
& \quad\quad h_1 = 1 \quad \rightarrow \quad w_1 = 1 \\
& \quad\quad h_2 = -j \quad \rightarrow \quad w_2 = j \\
& \quad\quad h_3 = -\frac{1}{2} \quad \rightarrow \quad w_3 = e^{-j\pi} = -1
\end{aligned}
$$

b)

$$\text{MRC}: \quad \text{SNR}_{\text{total}}^{\text{MRC}} = \frac{\sigma_x^2 \left| \sum\limits_{n=1}^{N_R} h_n w_n \right|^2}{\sigma_n^2 \sum\limits_{n=1}^{N_R} |w_n|^2} = \frac{\sigma_x^2 \left| \sum\limits_{n=1}^{N_R} |h_n| \, e^{j\phi_n} \, |h_n| \, e^{-j\phi_n} \right|^2}{\sigma_n^2 \sum\limits_{n=1}^{N_R} |h_n|^2} =$$

$$= \frac{\sigma_x^2}{\sigma_n^2} \left(\sum_{n=1}^{N_R} |h_n|^2 \right) = \frac{\sigma_x^2}{\sigma_n^2} \left(1 + 1 + \frac{1}{4} \right)$$

$$\Rightarrow \boxed{\text{SNR}_{\text{total}}^{\text{MRC}} = \frac{9}{4} \frac{\sigma_x^2}{\sigma_n^2}}$$

$$\text{EGC}: \quad \text{SNR}_{\text{total}}^{\text{EGC}} = \frac{\sigma_x^2 \left| \sum\limits_{n=1}^{N_R} |h_n| \, e^{j\phi_n} e^{-j\phi_n} \right|^2}{\sigma_n^2 \sum\limits_{n=1}^{N_R} \left| e^{-j\phi_n} \right|^2} =$$

$$= \frac{\sigma_x^2}{N_R \sigma_n^2} \left(\sum_{n=1}^{N_R} |h_n|^2 \right) = \frac{\sigma_x^2}{3\sigma_n^2} \left(1 + 1 + \frac{1}{2} \right)^2$$

$$\Rightarrow \boxed{\text{SNR}_{\text{total}}^{\text{EGC}} = \frac{25}{12} \frac{\sigma_x^2}{\sigma_n^2}}$$

SC : branch with max. SNR: $\hat{n} = \arg\max\limits_{n} \text{SNR}_n \quad \rightarrow \quad \text{SNR}_{\text{total}} = \text{SNR}_{\hat{n}}$

we need SNR estimation algorithm $\rightarrow$ e.g. pilots

$$\text{SNR}_1 = \frac{\sigma_x^2 \, |h_1|^2 \, |w_1|^2}{\sigma_n^2 \, |w_1|^2} = \frac{\sigma_x^2}{\sigma_n^2} \, |h_1|^2 = \frac{\sigma_x^2}{\sigma_n^2}$$

$$\text{SNR}_2 = \frac{\sigma_x^2}{\sigma_n^2} \, |h_2|^2 = \frac{\sigma_x^2}{\sigma_n^2}$$

$$\text{SNR}_3 = \frac{\sigma_x^2}{\sigma_n^2} \, |h_3|^2 = \frac{1}{4} \frac{\sigma_x^2}{\sigma_n^2}$$

$\text{SNR}_1/\text{SNR}_2$ is the highest SNR, we select first branch

$$\rightarrow \quad \boxed{\text{SNR}_{\text{total}}^{\text{SC}} = \text{SNR}_1 = \frac{\sigma_x^2}{\sigma_n^2}}$$

$$\left.\begin{array}{ll} \text{MRC}: & \text{SNR}_{\text{total}}^{\text{MRC}} = \dfrac{4}{9} \dfrac{\sigma_x^2}{\sigma_n^2} \\[2ex] \text{EGC}: & \text{SNR}_{\text{total}}^{\text{EGC}} = \dfrac{25}{12} \dfrac{\sigma_x^2}{\sigma_n^2} \\[2ex] \text{SC}: & \text{SNR}_{\text{total}}^{\text{SC}} = \dfrac{\sigma_x^2}{\sigma_n^2} \end{array}\right\} \quad \boxed{\text{SNR}^{MRC} \geq \text{SNR}^{EGC} \geq \text{SNR}^{SC}}$$

Problem 9 - BPSK error rate in an MRC system over Nakagami-m channel

SNR distribution for Nakagami-m fading

$$\text{pdf}: \quad P_\gamma(x) = \frac{m^m x^{m-1}}{\gamma^{-m}\Gamma(m)} \exp\left(-\frac{mx}{\bar{\gamma}}\right) \begin{cases} m & \text{fading parameter} \\ \Gamma(m) & \text{Gamma function } = (m-1)! \\ \bar{\gamma} & \text{average branch SNR} \end{cases}$$

$\rightarrow$ Laplace transform

$\mathcal{L}(P_\gamma(x)) = M_\gamma(s) \quad \rightarrow \quad$ moment generating function (mgs)

$$\rightarrow M_\gamma(s) = \left(1 - \frac{s\bar{\gamma}}{m}\right)^{-m} \qquad m = 1 \quad \Rightarrow \quad \text{Rayleigh}$$

$$\gamma_{\text{total}}^{\text{MRC}} = \sum_{n=1}^{N_R} \gamma_n \qquad \gamma_n: \text{ branch SNR}$$

Probability density function pdf of total SNR is convolution of branch SNR.
Convolution in time domain $\rightarrow$ multiplication in frequency domain.
i.i.d fading $\rightarrow$ branch SNR independent

$$\rightarrow M_{\gamma_{\text{total}}}(s) = \prod_{n=1}^{N_R} M_{\gamma_n}(s) = (M_\gamma(s))^{N_R}$$

$$\rightarrow M_{\gamma_{\text{total}}}(s) = \left(1 - \frac{s\bar{\gamma}}{m}\right)^{-mN_R}$$

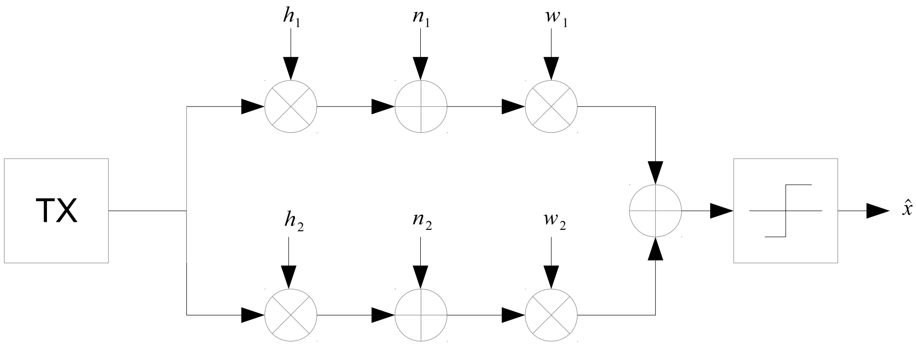
from lecture notes:

$$\rightarrow \bar{P}_e = \frac{a}{\pi} \int_0^{\frac{\pi}{2}} M_{\gamma_{\text{total}}}\left(\frac{b}{2\sin^2\Theta}\right) \mathrm{d}\Theta$$

a, b are parameters depending on modulation scheme
$\rightarrow$ for BPSK: $a = 1$, $b = 2$

$$\rightarrow \quad \boxed{\bar{P}_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \left(1 - \frac{\bar{\gamma}}{m\sin^2\Theta}\right)^{-mN_R} \mathrm{d}\Theta}$$

Problem 10 - MPSK error rate in an MRC system over a mixed Rayleigh, Nakagami-m channel



$\sigma_{n1}^2 = \sigma_{n2}^2 = \sigma_{n1}^2$

branch SNR: $\bar{\gamma} = \dfrac{\sigma_x^2}{\sigma_n^2}$

branch 1 $\quad\rightarrow\quad$ Rayleigh $\quad\rightarrow\quad$ $p_\gamma(x) = \dfrac{1}{\bar{\gamma}} \exp\left(\dfrac{x}{\bar{\gamma}}\right)$

branch 2 $\quad\rightarrow\quad$ Nakagami-m $\quad\rightarrow\quad$ $p_\gamma(x) = \dfrac{m^m x^{m-1}}{\gamma^{-m}\Gamma(m)} \exp\left(-\dfrac{mx}{\bar{\gamma}}\right)$

i.i.d fading $\quad\rightarrow\quad$ $M_{\gamma_{\text{total}}}(s) = M_{\gamma_1}(s)\, M_{\gamma_2}(s)$

$$\begin{cases} \text{first branch} \quad\rightarrow\quad \text{Rayleigh} \quad\rightarrow\quad M_{\gamma_1}(s) = (1 - s\bar{\gamma})^{-1} \\ \text{second branch} \quad\rightarrow\quad \text{Nakagami-m} \quad\rightarrow\quad M_{\gamma_2}(s) = \left(1 - \frac{s\bar{\gamma}}{m}\right)^{-m} \end{cases}$$

$$\rightarrow \quad M_{\gamma_{\text{total}}} = (1 - s\bar{\gamma})^{-1} \left(1 - s\frac{\bar{\gamma}}{m}\right)^{-m}$$

from lectures notes:

$$\rightarrow \bar{P}_e = \int_0^\infty P_e p_{\gamma_{\text{total}}}(x)\, \mathrm{d}x =$$

$$= \int\limits_0^\infty \frac{1}{\pi} \int\limits_0^{\frac{(M-1)\pi}{M}} \exp()\, \mathrm{d}\Theta\, p_{\gamma_{\text{total}}}(x)\, \mathrm{d}x =$$

$$= \frac{1}{\pi} \int\limits_0^{\frac{(M-1)\pi}{M}} \int\limits_0^\infty \exp\left(\frac{-x \sin^2\left(\frac{\pi}{M}\right)}{\sin^2\Theta}\right) p_{\gamma_{\text{total}}}(x)\, \mathrm{d}x\, \mathrm{d}\Theta =$$

$$= \int\limits_0^\infty e^{sx} p_{\gamma_{\text{total}}}(x)\, \mathrm{d}x = M_{\gamma_{\text{total}}}(s)$$

$$\rightarrow \int_0^{\infty} \exp\left(\frac{-x\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\Theta}\right) p_{\gamma_{\text{total}}}(x)\,\mathrm{d}x = M_{\gamma_{\text{total}}}\left(\frac{-\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\Theta}\right)$$

$$\rightarrow \bar{p}_e = \frac{1}{M}\int_0^{\frac{(M-1)\pi}{M}} M_{\gamma_{\text{total}}}\left(\frac{-\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\Theta}\right)\mathrm{d}\Theta$$

$$\rightarrow \boxed{\bar{p}_e = \frac{1}{M}\int_0^{\frac{(M-1)\pi}{M}} \left(1+\frac{\bar{\gamma}\sin^2\left(\frac{\pi}{M}\right)}{\sin^2\Theta}\right)^{-1}\left(1+\frac{\bar{\gamma}\sin^2\left(\frac{\pi}{M}\right)}{m\sin^2\Theta}\right)^{-m}\mathrm{d}\Theta}$$

$\bar{\gamma} = 10\,\mathrm{dB} \qquad M = 8 \rightarrow 8\mathrm{PSK} \qquad m = 3 \qquad \Rightarrow \quad \bar{p}_e = 0,0476$

Problem 11 - Transmit and receive diversity

The standard definition of outage probability:

The average rate is lower than a specific value.

Alternative definition:

The error rate exceeds a specific value.

$P_b = 10^{-4} \quad \& \quad$ BPSK modulation

$$\rightarrow \quad P_b(\gamma_0) = \mathrm{Q}\left(\sqrt{2\gamma_0}\right)$$

$$\rightarrow \quad 10^{-4} = \mathrm{Q}\left(\sqrt{2\gamma_0}\right) \quad \rightarrow \quad \gamma_0 = \frac{1}{2}\left(\mathrm{Q}_{???}\left(10^{-4}\right)\right)^2$$

$$\rightarrow \quad \gamma_0 = 6,9155 \quad \rightarrow \quad \gamma_0 = 8,4\,\mathrm{dB}$$

$$\rightarrow \quad P_{out} = P_e\left(\gamma_{total} < \gamma_0\right) = P\left(\gamma_{out} < 6,9155\right)$$

from lecture notes:

$$\rightarrow \quad p_{\gamma_{total}}(x) = \frac{x^{N_R-1}e^{-\frac{\lambda}{\bar{\gamma}}}}{\gamma^{-N_R}\left(N_R-1\right)!}$$

pdf of total SNR from MRC with i.i.d Rayleigh

$$\rightarrow P_{out} = P_e\left(\gamma_{total} < 6,9155\right) = \int_0^{6,9155} \frac{x^{N_R-1}e^{-\frac{\lambda}{\bar{\gamma}}}}{\gamma^{-N_R}\left(N_R-1\right)!}\mathrm{d}x =$$

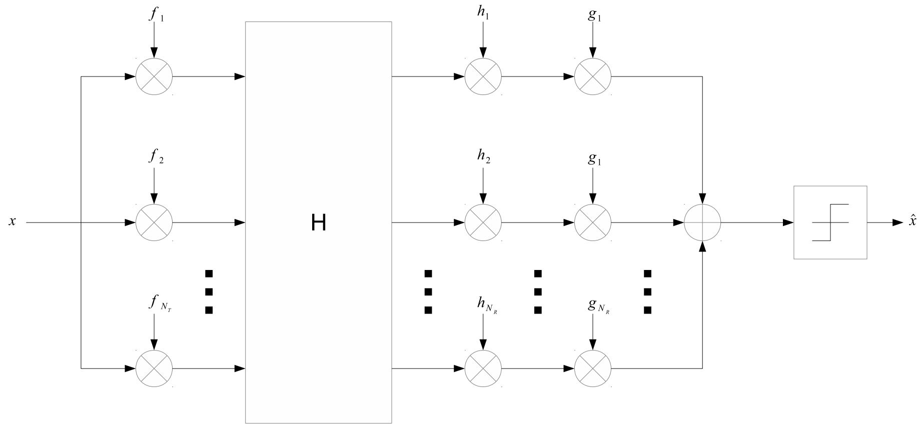
$$= 1 - e^{-\frac{6,9155}{10}}\sum_{k=1}^{3}\frac{\left(\frac{6,9155}{10}\right)^{k-1}}{(k-1)!} = 0,033 = 3,3\%$$

$\bar{\gamma} = 10 \qquad N_R = 3$

Problem 12 - Linear and decision-feedback MIMO detection

<u>MIMO-MRC</u>

$\sigma_{n_1}^2 = \sigma_{n_2}^2 = \ldots = \sigma_{N_R}^2 = \sigma_n^2$



$f = (f_1, f_2, \ldots, f_{N_R})^T$; $\qquad g = \left(g_1^*, g_2^*, \ldots, g_{N_R}^*\right)$

$\rightarrow$ Nomenclature: $\begin{cases} f^H f = 1 = \|f\|^2 \\ g^H g = 1 = \|g\|^2 \end{cases}$

singular value decomposition: $\mathbf{H} = \mathbf{U\Sigma V}^H$

$\mathbf{\Sigma} = \text{diag}\left(\xi_1^2, \xi_2^2, \ldots, \xi_N^2\right)$

$\rightarrow$ sorted in descending order $\rightarrow$ $\xi_1^2 = \xi_{max}^2$

$$f = \mathbf{V} f' \quad \rightarrow \quad \mathbf{V}^H f = \underbrace{\mathbf{V}^H \mathbf{V}}_{\mathbf{I}} f' = f' \quad \rightarrow \quad \|f'\|^2 = \|\mathbf{V}^H f\|^2 \overset{\mathbf{V}:unitary}{=} \|f\|^2 = 1$$

in a similar way:

$\rightarrow$ $g = u \cdot g'$

$\rightarrow$ $\|g\|^2 = \|g'\|^2 = 1$

$$\text{SNR} = \frac{\sigma_x^2 \left| \overbrace{h_{total}}^{\substack{\text{total transfer}\\ \text{function}}} \right|^2}{\sigma_n^2 \underbrace{\|g\|^2}_{\rightarrow = 1}} = \frac{\sigma_x^2 \left| g^H H f \right|}{\sigma_n^2} = \frac{\sigma_x^2 \left| g'^H \mathbf{U}^H \mathbf{U \Sigma V}^H \mathbf{V} f' \right|}{\sigma_n^2}$$

$$\rightarrow \quad \mathrm{SNR} = \frac{\sigma_x^2}{\sigma_n^2} \left| g'^H \mathbf{\Sigma} f' \right|$$

$$g' = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}'; \quad f' = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}'$$

$\rightarrow$ to maximize SNR: $\xi_1 = \xi_{max}$

$$\mathrm{SNR}_{\max} = \frac{\sigma_x^2}{\sigma_n^2} \xi_1^2$$

ξ_k^2 are also eigenvalues of HH^H

$$\|H\|_F^2 = \operatorname{tr}\left(HH^H\right) = \sum_{k=1}^{N} \xi_k^2 \qquad N = \min\{N_T, N_R\}$$

Frobenius norm

$$\xi_1^2 = \xi_{max}^2 \quad \rightarrow \quad \underbrace{\frac{1}{N} \sum_{k=1}^{N} \xi_k^2}_{\text{average}} \leq \xi_{max}^2 \leq \sum_{k=1}^{N} \xi_k^2$$

$$\|H\|_F^2 = \sum_{i=1}^{N_T} \sum_{j=1}^{N_R} \left| h_{ij} \right|^2, \quad h_{ij} \sim \mathcal{CN}\left(0, 1\right)$$

$\rightarrow$ $\|H\|_F^2$ is the sum of squared complex Gaussian RVs

$\rightarrow$ χ^2-square (χ^2-distribution

$$f_{\|H\|_F^2}\left(x\right) = \frac{1}{\Gamma\left(N_T N_R\right)} x^{N_T N_R - 1} e^{-x}$$

$$\overbrace{f_{\frac{1}{N}\|H\|_F^2}}^{\text{pdf of the channel}} = \frac{1}{\Gamma\left(N_T N_R\right)} \left(Nx\right)^{N_T N_R - 1} e^{-Nx} = \frac{N^{N_T N_R} x^{N_T N_R - 1} e^{-Nx}}{\Gamma\left(N_T N_R\right)}$$

$$\overline{\mathrm{SER}} \geq \int_0^\infty f_{\frac{1}{N}\|H\|_F^2} \mathrm{SER}\left(x\right) \mathrm{d}x = \int_0^\infty \frac{N^{N_T N_R} x^{N_T N_R - 1} e^{-Nx}}{\Gamma\left(N_T N_R\right)} \frac{C_{\mathcal{A}}}{2} e^{-\frac{d_{\mathcal{A}}^2}{2} x \gamma} \mathrm{d}x =$$

$$= \frac{N^{N_T N_R} C_{\mathcal{A}}}{2\Gamma\left(N_T N_R\right)} \int_0^\infty x^{N_T N_R - 1} e^{-\left(N + \frac{d_{\mathcal{A}}^2}{2}\gamma\right)x} \mathrm{d}x = \frac{N^{N_T N_R} C_{\mathcal{A}}}{2\Gamma\left(N_T N_R\right)} \frac{\Gamma\left(N_T N_R\right)}{\left(N + \frac{d_{\mathcal{A}}^2}{2}\gamma\right)^{N_T N_R}}$$

$$\rightarrow \quad \overline{\mathrm{SER}} \geq \frac{C_{\mathcal{A}} N^{N_T N_R}}{2\left(N + \frac{d_{\mathcal{A}^2}}{2} \underbrace{\gamma}_{\mathrm{SNR}}\right)^{N_T N_R}}$$

$$\mathrm{SNR} \quad \rightarrow \quad \infty \quad \Rightarrow \quad \mathrm{SER} \geq \frac{C_{\mathcal{A}}}{2\left(\frac{d_{\mathcal{A}}^2}{2N}\gamma\right)^{N_T N_R}}$$

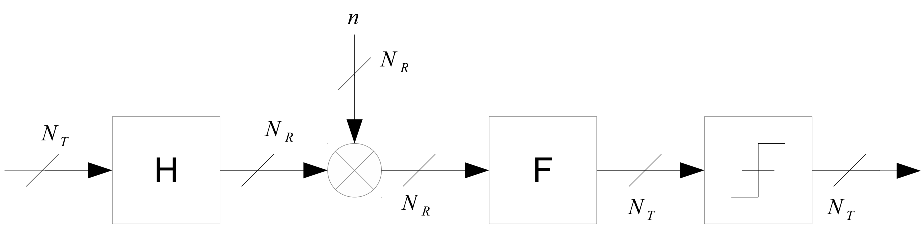
$$\frac{\sum_{k=1}^{N}\xi_k^2}{N} \le \xi_{max}^2 \le \sum_{k=1}^{N}\xi_k^2 \quad \rightarrow \quad \overline{\text{SER}} \le \frac{C_{\mathcal{A}}}{2\left(\frac{D_{\mathcal{A}}^2}{2}\gamma\right)^{N_T N_R}}$$

$$\rightarrow \quad \frac{C_{\mathcal{A}}}{2\left(\frac{D_{\mathcal{A}}^2}{2N}\gamma\right)^{N_T N_R}} \le \overline{\text{SER}} \le \frac{C_{\mathcal{A}}}{2\left(\frac{D_{\mathcal{A}}^2}{2}\gamma\right)^{N_T N_R}}$$

we have: $\dfrac{\ldots}{\ldots\gamma^{N_T N_R}} \quad \rightarrow \quad$ Diversity gain $= N_T N_R$

$$\text{equalization} \begin{cases} \underset{\text{(detection)}}{\text{at the receiver}} \begin{cases} \text{linear} \begin{cases} \text{Zero-Forcing ZF} \\ \text{Minimum Mean Square Error MMSE} \end{cases} \\ \text{decision feedback} \begin{cases} \text{ZF} \\ \text{MMSE} \end{cases} \end{cases} \\ \underset{\text{(precoding)}}{\text{at the transmitter}} \begin{cases} \text{linear} \\ \text{with feedback} \end{cases} \end{cases}$$

Linear equalizer:



$\mathbf{H} \in \mathbb{C}^{N_R \times N_T}$; $\quad \mathbf{F} \in \mathbb{C}^{N_T \times N_R}$

$$\mathbf{H} = \begin{pmatrix} 0,2 & -0,2 \\ 0,1 & 0,4 \\ 1 & 0,2 \end{pmatrix}, \qquad \frac{\sigma_x^2}{\sigma_n^2} = 10$$

$N_R > N_T \quad \rightarrow \quad \mathbf{F} = \left(\mathbf{H}^H\mathbf{H}\right)^{-1} \cdot \mathbf{H}^H \quad \rightarrow$ left Moore-Penrose pseudo-inverse

received signal: $r = \mathbf{FH}x + \mathbf{F}n \quad \rightarrow$ drawback: noise amplification

$r = \mathbf{FH}x + \mathbf{F}n = \underbrace{\left(\mathbf{H}^H\mathbf{H}\right)^{-1}\mathbf{H}^H\mathbf{H}}_{\mathbf{I}}x + \mathbf{F}n = x + \mathbf{F}n \quad \rightarrow$ no interference

$\rightarrow$ error signal: $e = \mathbf{F}n \quad \rightarrow \quad \phi_{ee} = ee^H = \mathbf{F}\underbrace{nn^H}_{\sigma_n^2\mathbf{I}}\mathbf{F}^H$

$\rightarrow \quad nn^H = \sigma_n^2\mathbf{I}$ white noise

$$\rightarrow \quad \phi_{ee}^{ZF} = \sigma_n^2 \mathbf{F}\mathbf{F}^H \quad \rightarrow \quad \phi_{ee}^{ZF} = \sigma_n^2 \begin{pmatrix} 1,13 & -0,94 \\ -0,94 & 4,95 \end{pmatrix}$$

$\sigma_e^2 = \operatorname{tr}\left(\phi_{ee}^{ZF}\right) = \sigma_n^2 \left(1,13 + 4,95\right)$

$\Rightarrow \sigma_e^2 = \sigma_n^2 \cdot 6,1$

Now determine $\mathbf{F}$ such that the error-variance is minimal!

$$\rightarrow \quad \text{MMSE} \quad \rightarrow \quad \mathbf{F} = \left(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\sigma_x^2}\mathbf{I}\right)^{-1} \underbrace{\mathbf{H}^H}_{\substack{\text{matched}\\\text{filter}}} \; ; \qquad \frac{\sigma_x^2}{\sigma_n^2} = 10$$

$\Rightarrow$ for low SNR: MMSE

$\Rightarrow$ for high SNR: ZF

$$\rightarrow \mathbf{F} = \begin{pmatrix} 0,31 & -0,13 & 0,85 \\ -0,77 & 1,25 & 0,086 \end{pmatrix}$$

$\rightarrow$ from lecture notes:

$$\rightarrow \quad \phi_{ee} = \sigma_n^2 \left(\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\sigma_x^2}\mathbf{I}\right)^{-1} \quad \rightarrow \quad \phi_{ee}^{MMSE} = \sigma_n^2 \begin{pmatrix} 0,97 & -0,57 \\ -6,57 & 3,28 \end{pmatrix}$$

$\rightarrow \quad$ total error variance $=\ \operatorname{tr}\left(\phi_{ee}\right)$

$\Rightarrow \quad \sigma_e^2 = 4,25\sigma_n^2$

End-to-end channel: $\mathbf{K} = \mathbf{F}\mathbf{H} = \begin{pmatrix} 0,9 & 0,06 \\ 0,06 & 0,67 \end{pmatrix}$

residual interference in off-diagonal elements

diagonal elements should be 1, but are not $\Rightarrow$ biased!

$\rightarrow$ solution: $\quad \rightarrow \quad$ remove bias by multiplying with $\mathbf{C} = \begin{pmatrix} \frac{1}{0,9} & 0 \\ 0 & \frac{1}{0,67} \end{pmatrix}$

$$\rightarrow \mathbf{C} = \begin{pmatrix} 1,11 & 0 \\ 0 & 1,49 \end{pmatrix}$$

$$r' = \mathbf{C}r = \begin{pmatrix} 1,11 & 0 \\ 0 & 1,49 \end{pmatrix} \begin{pmatrix} 0,9 & 0,06 \\ 0,06 & 0,67 \end{pmatrix} x + \begin{pmatrix} 1,11 & 0 \\ 0 & 1,49 \end{pmatrix} \mathbf{F}n =$$

$$= \begin{pmatrix} 1 & 0,07 \\ 0,09 & 1 \end{pmatrix} x + \begin{pmatrix} 1,11 & 0 \\ 0 & 1,49 \end{pmatrix} \mathbf{F}n$$

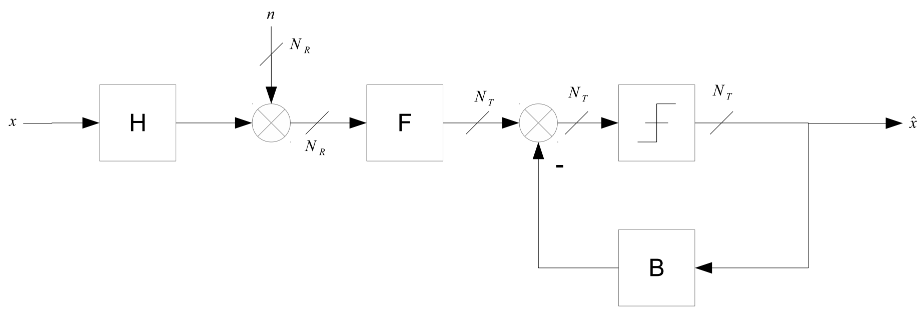
from lecture notes:

$$\phi_{e'e'} = \sigma_x^2 \left(\mathbf{I} + \left(\mathbf{C} - \mathbf{I}\right)\mathbf{K}^H\mathbf{C}^H - \mathbf{C}\mathbf{K}\right); \qquad \frac{\sigma_x^2}{\sigma_n^2} = 10$$

$$\Rightarrow \quad \sigma_x^2 \begin{pmatrix} 0,11 & -0,05 \\ -0,05 & 0,49 \end{pmatrix} = \sigma_n^2 \begin{pmatrix} 1,1 & -0,5 \\ -0,5 & 4,5 \end{pmatrix}$$

$$\rightarrow \quad \sigma_e'^2 = \operatorname{tr}\left(\phi_{e'e'}\right) = 6\sigma_n^2$$

Decision feedback:



$\mathbf{H} \in \mathbb{C}^{N_R \times N_T}$; $\qquad \mathbf{F} \in \mathbb{C}^{N_T \times N_R}$

drawback: propagation of error when errors predictions are not accurate

$[\mathbf{L}, \mathbf{D}] =??? \left(\mathbf{H}^H\mathbf{H}\right)$ Cholesky factorization

$$\mathbf{H}^H\mathbf{H} = \begin{pmatrix} 1,03 & 0,2 \\ 0,2 & 0,24 \end{pmatrix} \quad \rightarrow \quad \mathbf{L} = \begin{pmatrix} 1 & 0 \\ 0,19 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1,03 & 0 \\ 0 & 0,2 \end{pmatrix}$$

$$\text{feedforward filter:} \quad \mathbf{F} = \mathbf{D}^{-1}\mathbf{C}^{-H}\mathbf{H}^H = \begin{pmatrix} 0,23 & 0,02 & 0,96 \\ -0,99 & 1,98 & 0,99 \end{pmatrix}$$

$$\text{feedback filter:} \quad \mathbf{B} = \mathbf{C} - \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0,19 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0,19 & 0 \end{pmatrix}$$

MMSE-DFE:

$$\mathbf{LDL}^H = \mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\sigma_x^2}\mathbf{I} = \begin{pmatrix} 1,15 & 0,2 \\ 0,2 & 0,34 \end{pmatrix} \quad \rightarrow \quad (\mathbf{L}, \mathbf{D}) \rightarrow \left(\mathbf{H}^H + \frac{\sigma_n^2}{\sigma_x^2}\mathbf{I}\right)$$

$$\rightarrow \quad \mathbf{L} = \begin{pmatrix} 1 & 0 \\ 0,17 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1,15 & 0 \\ 0 & 0,31 \end{pmatrix}$$

$$\rightarrow \text{Feedforward filter: } \mathbf{F} = \mathbf{C}\left|\mathbf{H}^H\mathbf{H} + \frac{\sigma_n^2}{\sigma_x^2}\mathbf{I}\right|^{-1}\mathbf{H}^H = \begin{pmatrix} 0,19 & 0,21 & 1,13 \\ 0,005 & 0,15 & 0,48 \end{pmatrix}$$

$$\rightarrow \text{Feedback filter: } \mathbf{B} = \mathbf{L} - \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0,1739 & 0 \end{pmatrix}$$

$$\rightarrow \text{Error covariance matrix: } \phi_{ee} = \sigma_n^2\mathbf{D}^{-1} = \sigma_n^2\begin{pmatrix} 0,87 & 0 \\ 0 & 3,28 \end{pmatrix}$$

$$\rightarrow \quad \sigma_e^2 = \operatorname{tr}(\phi_{ee}) = 4,15\sigma_n^2$$