Exercise 1

1	2	3	4	5	Exercise
6	6	1	6	6	25 Points

1. Based on Newton's law, the equations of motions are the following:

$$m\,\ddot{z}(t) + d\,\dot{z}(t) = F(t).$$

The circuit equation is the following:

$$L\,\frac{di(t)}{dt} + R\,i(t) = v_a(t) - v_c(t).$$

Combining the equations gives:

$$m\,\ddot{z}(t) + d\,\dot{z}(t) = \ell\,i(t)$$

$$\Rightarrow \ddot{z}(t) = -\frac{d}{m}\dot{z}(t) + \frac{\ell}{m}\,i(t) \qquad \textbf{[1pt]}$$

and

$$L\,\frac{di(t)}{dt} + R\,i(t) = v_a(t) - \ell\,\dot{z}(t)$$

$$\Rightarrow \frac{di(t)}{dt} = -\frac{\ell}{L}\,\dot{z}(t) - \frac{R}{L}\,i(t) + v_a(t). \qquad \textbf{[1pt]}$$

Input and output and state are $u(t) = v_a(t)$ and $y(t) = z(t)$, $x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \\ i(t) \end{bmatrix}$.

The state-space dynamics are the following

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{d}{m} & \frac{\ell}{m} \\ 0 & -\frac{\ell}{L} & \frac{R}{L} \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} x(t) + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u(t)$$

[1pt] for each = **[4pts]**

Click to BUY NOW! PDF-XChange Editor www.pdf-xchange.com

2. The observability matrix is

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{d}{m} & \frac{\ell}{m} \end{bmatrix}. \qquad \textbf{[1pt]}$$

Yes, the system is observable **[1pt]**. Check full rank condition with $d, m, \ell > 0$, two ways **[1pt]** :

(1) Argue for linear independence of rows: Clearly none of the vectors are a linear combination of the other ones.

(2) Show that $\det(Q) = 1\,(1\,\frac{\ell}{m}) = \frac{\ell}{m} \neq 0$. (lower triangular matrix, can read off determinant).

The controllability matrix is

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\ell}{m} \\ 0 & \frac{\ell}{m} & -\frac{d\ell}{m^2} + \frac{\ell R}{mL} \\ 1 & \frac{R}{L} & -\frac{\ell^2}{Lm} + \frac{R^2}{L^2} \end{bmatrix}. \qquad \textbf{[1pt]}$$

Yes, it is controllable **[1pt]**. Check full rank condition **[1pt]**, two ways:

(1) Argue for linear independence of rows: Clearly none of the vectors are a linear combination of the other ones.

(2) Show that $\det(P) = \frac{\ell}{m}(0\,(-\frac{R}{L}) - 1\,\frac{\ell}{m}) = -\frac{\ell^2}{m^2} \neq 0$.

3. Setting the dynamics to zero under the constant input v_0 gives the following

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{d}{m} & \frac{\ell}{m} \\ 0 & -\frac{\ell}{L} & \frac{R}{L} \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} 0,$$

which directly gives

$$\hat{x} = \begin{bmatrix} \hat{z} \\ 0 \\ 0 \end{bmatrix} \text{ for any constant } \hat{z} \in \mathbb{R} \quad \textbf{[1pt]}.$$

4. The state matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{d}{m} & \frac{\ell}{m} \\ 0 & -\frac{\ell}{L} & \frac{R}{L} \end{bmatrix}.$$

The characteristic polynomial is: $\lambda^3 + \lambda^2 \left(\frac{dL-Rm}{mL}\right) + \lambda \left(\frac{\ell^2-dR}{mL}\right) = 0$ **[2pts]**. Thus, $\lambda_1 = 0$ **[1pts]** and the second order polynomial $\lambda^2 + \lambda \left(\frac{dL-Rm}{mL}\right) + \left(\frac{\ell^2-dR}{mL}\right) = 0$. The system is stable if $dL > Rm$ and $\ell^2 > dR$ **[1pt]**.

For no set of parameter values the system is asymptotically stable as one of the eigenvalues is zero, i.e., $\lambda_1 = 0$ **[2pts]**.

5. The closed-loop system matrix is the following

$$(A + BK) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{d}{m} & \frac{\ell}{m} \\ k_1 & 0 & \frac{R}{L} + k_3 \end{bmatrix}.$$

The determinant can be computed as:

$$\begin{aligned} \det(\lambda I - (A + BK)) &= \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda + \frac{d}{m} & -\frac{\ell}{m} \\ -k_1 & 0 & \lambda - \frac{R}{L} - k_3 \end{bmatrix} \quad \textbf{[1pt]} \\ &= \lambda[(\lambda + \frac{d}{m})(\lambda - \frac{R}{L} - k_3)] - k_1 \frac{\ell}{m} \\ &= \lambda^3 + \underbrace{(\frac{d}{m} - \frac{R}{L} - k_3)}_{a_2} \lambda^2 + \underbrace{\frac{d}{m}(-\frac{R}{L} - k_3)}_{a_1} \lambda \underbrace{- k_1 \frac{\ell}{m}}_{a_0} \end{aligned}$$

[1pt] (for the correct polynomial). The following conditions need to hold:

1. $a_2, a_1, a_0 > 0$
   (i) $a_2 = (\frac{d}{m} - \frac{R}{L} - k_3) > 0 \quad \Rightarrow \frac{d}{m} - \frac{R}{L} > k_3$
   (ii) $a_1 = \frac{d}{m}(-\frac{R}{L} - k_3) > 0 \quad \Rightarrow -\frac{R}{L} > k_3$
   (iii) $a_0 = -k_1 \frac{\ell}{m} > 0 \quad \Rightarrow k_1 < 0$
   (ii) and (iii) need to hold **[2pts]**
2. $a_2 a_1 > a_0 : (\frac{d}{m} - \frac{R}{L} - k_3)(-\frac{R}{L} - k_3) > -k_1 \frac{\ell}{m}$ **[2pts]**.

(accept any rearranged form).

Click to BUY NOW! PDF-XChange Editor www.pdf-xchange.com

Exercise 2

1	2	3	4	5	Exercise
4	4	7	5	5	25 Points

1. The equation $A^T P + PA = -I$ has a unique positive definite solution if and only if A has negative eigenvalues. (**1p**) We need the eigenvalues of A:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -1-\lambda & 4 \\ a & -2-\lambda \end{vmatrix} \\ &= (-1-\lambda)(-2-\lambda) - 4a \\ &= \lambda^2 + 3\lambda + 2 - 4a \quad (\mathbf{1p}) \end{aligned}$$

   No point if the last equation is found through wrong derivation. We need all coefficients with the same sign for negative eigenvalues (Finding the eigenvalues directly is also fine). The condition holds for

$$a < \frac{1}{2}. \quad (\mathbf{1p})$$

   Negative eigenvalues means A is asymptotically stable, therefore ZIR of the system tends to 0 as $t \to \infty$. (**1p**)

2. The controllability matrix is

$$(\mathbf{1p}) \quad \mathcal{P} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 4b \\ b & -2b \end{bmatrix},$$

   which should be full rank for controllability (**1p**). The determinant is

$$\mathrm{Det}(\mathcal{P}) = -4b^2 \quad (\mathbf{1p}).$$

   Therefore the system is controllable for all a and c as long as $b \neq 0$ (**1p**).

3. For $(a, b, c) = (3, 1, 1)$ we have the closed loop system matrix

$$\begin{aligned} \tilde{A} = A + BK &= \begin{bmatrix} -1 & 4 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & k \end{bmatrix} \\ &= \begin{bmatrix} -1 & 4 \\ 3 & -2+k \end{bmatrix}. \quad (\mathbf{1p}) \end{aligned}$$

   We compute the eigenvalues of the closed-loop matrix as

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -1-\lambda & 4 \\ 3 & -2+k-\lambda \end{vmatrix} \\ &= (-1-\lambda)(-2+k-\lambda) - 12 \\ &= \lambda^2 - k\lambda + 3\lambda + 2 - k - 12 \\ &= \lambda^2 + (3-k)\lambda - 10 - k. \quad (\mathbf{1p}) \end{aligned}$$

PDF-XChange Editor
Click to BUY NOW!
www.pdf-xchange.com

We would like to have $\lambda_2 = \alpha\lambda_1$

$$0 = (\lambda - \lambda_1)(\lambda - \alpha\lambda_1) = \lambda^2 - (1+\alpha)\lambda_1\lambda + \alpha\lambda_1^2 \quad (\mathbf{1p})$$

Therefore we need

$$\begin{aligned} 3 - k &= -(1+\alpha)\lambda_1 \\ -10 - k &= \alpha\lambda_1^2, \end{aligned}$$

which, by substituting to eliminate k, results in

$$0 = \alpha\lambda_1^2 + (1+\alpha)\lambda_1 + 13 \quad (\mathbf{1p}).$$

Noting that we need k to be a real number, we need the above equation to have a real solution with $\lambda_1 < 0$. This is possible only if the discriminant of the above equation is positive (**1p**).

$$\Delta = (1+\alpha)^2 - 4(13\alpha) = \alpha^2 - 50\alpha + 1.$$

The roots of Δ are

$$roots(\Delta) = \frac{50 \pm \sqrt{50^2 - 4}}{2} = 25 \pm \frac{\sqrt{2496}}{2} \approx 25 \pm 24.98 \quad (\mathbf{1p}),$$

where we used the approximation given in the hint. Therefore λ_1, and consequently k, has real solutions for $\alpha \in (0, 0.02] \cup [49.98, \infty)$ (**1p**).

4. Given $\alpha = 0.02$, from the previous solution we have that

$$\begin{aligned} 0 &= \alpha\lambda_1^2 + (1+\alpha)\lambda_1 + 13 \\ &= 0.02\lambda_1^2 + (1.02)\lambda_1 + 13 \quad (\mathbf{1p}) \\ &= \lambda_1^2 + 51\lambda_1 + 650 \\ &= (s_1 + 25)(s_2 + 26) \quad (\mathbf{1p}), \end{aligned}$$

where s_1, s_2 denote the roots of the polynomial. For faster response we choose the root $\lambda_1 = -26$ (**1p**, accept also if stated that more negative roots lead to faster response without explicit calculation) and using the equation in previous solution to get

$$3 - k = -(1+\alpha)\lambda_1 \quad (\mathbf{1p}) \implies k = 3 + \lambda_1(1+\alpha) = 3 - 26 - \frac{52}{100} = -23.52 \quad (\mathbf{1p}).$$

5. We need to check the observability of the system. The observability matrix is

$$(\mathbf{1p}) \quad Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} c & 1 \\ a - c & 4c - 2 \end{bmatrix},$$

which should be full rank for observability (**1p**). The determinant is

$$\text{Det}(Q) = 4c^2 - c - a.$$

Using $(a, b, c) = (3, 1, 1)$, we have $\text{Det}(Q) = 0$ (**1p**), thus the system is not observable (**1p**). Therefore, we cannot build an observer to estimate the state from measurements (**1p**).

Click to BUY NOW! PDF-XChange Editor www.pdf-xchange.com

Exercise 3

1(a)	1(b)	2	Exercise
5	10	10	25 Points

1. (a) For the system in Figure 2 it holds that $(R(s) - Y(s)F(s))H(s) = Y(s)$ (**2p.**). Hence, $G(s) = \frac{Y(s)}{R(s)} = \frac{H(s)}{1+H(s)F(s)}$ (**1p.**). Substituting values for $H(s)$ and $F(s)$ we obtain $G(s) = \frac{s+2}{s^2+as+1}$ (**2p.**, 1 for numerator and 1 for denominator).

   (b) Plot 1 corresponds to $a = 2$ (**1p.**). For $a = 2$, the system has two real poles in the left-half plane (**1p.**).

   Plot 2 corresponds to $a = 0$ (**1p.**). For $a = 0$, the system has conjugate-complex poles with zero real part (**1p.**), its step-response is the sinusoidal signal (**1p.**).

   Plot 3 corresponds to $a = 0.5$ (**1p.**). For $a = 0.5$, the system has a pair of conjugate-complex poles with real parts in the left-half plane (**1p.**). Hence, its step-response corresponds to the dumped oscillator (**1p.**).

   Plot 4 corresponds to $a = -2$ (**1p.**). For $a = -2$, the system has two real poles in the right-half plane (**1p.**).

2. (a) For $a = -2$, the system is unstable (**1p.**) and its poles are given by $s_{1,2} = \frac{-a\pm\sqrt{a^2-4}}{2}$ (**1p.**). Hence, there are two poles at 1(**1p.**). The number of poles P with positive real part is $P = 2$ (**2p.**), and the number of zeros Z with positive real part is $Z = 0$ (**1p.**). Finally, from Nyquist criterion we have that the system (with $Z = 0$) is stable if and only if $N = -P$ (**1p.**), where N is the number of encirclement of $(-1/K, 0)$ (**1p.**). The number of counter clockwise encirclements is $N = -2$ when $-0.5 < -1/K < 0$ (**1p.**). Hence the system is stable for $K > 2$ (**1p.**).

PDF-XChange Editor Click to BUY NOW! www.pdf-xchange.com

Exercise 4

1	2	3	4	5	Exercise
6	1	5	8	5	25 Points

1. For the equilibrium points it holds that $\dot{x}_1(t) = 0$ and $\dot{x}_2(t) = 0$ (**1 p.**). Hence from $\dot{x}_1(t) = x_1(t)x_2(t) = 0$ we have either $\hat{x}_1 = 0$ or $\hat{x}_2 = 0$ (**1 p.**). For $\hat{x}_1 = 0$, $\hat{x}_2 = \frac{a}{k}$ (**1 p.**), and when $\hat{x}_2 = 0$, $\hat{x}_1 = \pm\sqrt{a}$ (**1 p.**). Consequently, for $a > 0$ the equilibrium points are $\hat{x} = (\hat{x}_1, \hat{x}_2) \in \{(0, \frac{a}{k}), (\pm\sqrt{a}, 0)\}$. Hence, the system has three equilibrium points (**1 p.**). If $a < 0$, the system has only one real equilibrium point $\hat{x} = (\hat{x}_1, \hat{x}_2) = (0, \frac{a}{k})$ (**1 p.**).

2. From $\dot{x}_1 = 0$ we have $\hat{x}_1 = 0$ or $\hat{x}_2 = 0$. When $\hat{x}_1 = 0$ if $\dot{x}_2 = 0$ then $\hat{x}_2 = 0$. Additionally, when $\hat{x}_2 = 0$ from $\dot{x}_2 = 0$ we have $\hat{x}_1 = 0$. (**Give a point only if it is explicitly shown that $(0, 0)$ is the only equilibrium. Plugging in values in $\dot{x}_1$ and $\dot{x}_2$ and showing that they are zero does not show that the equilibrium is unique.**).

3. Indirect Lyapunov method requires linearizaton around the equilibrium point $\hat{x} = (0, 0)$. By linearizing the system we have $\dot{x}_\delta(t) = Ax_\delta(t)$ where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{|x=\hat{x}} = \underbrace{\begin{bmatrix} x_2 & x_1 \\ -2x_1 & -k \end{bmatrix}_{|x=\hat{x}}}_{(\mathbf{1\ p.})} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -k \end{bmatrix}}_{(\mathbf{1\ p.})} \tag{1}$$

The eigenvalues of the characteristic polynomial $p(s) = \det(sI_2 - A) = s(s + k)$ are $s = 0$ and $s = -k$ (**1 p.**). For $k < 0$ the system is unstable (**1 p.**) while for $k \geq 0$ the indirect Lyapunov method is inconclusive (**1 p.**).

4. For the open set $S = \mathbb{R}^2$ and the Lyapunov function $V = \frac{1}{2}(x_1^2 + x_2^2)$ we have

   - $V(\hat{x}) = 0$ trivially holds for $\hat{x} = (0, 0)$ (**1 p.**),
   - $V(x) > 0$ for $x \in \mathbb{R}^2 \setminus (0, 0)$ trivially holds since $V(x)$ is quadratic (**1 p.**),
   - $\frac{\mathrm{d}}{\mathrm{d}t}V = x_1\dot{x}_2 + x_2\dot{x_2} = x_1^2x_2 - kx_2^2 - x_2x_1^2 = \underbrace{-kx_2^2}_{(\mathbf{1\ p.})} \leq 0$ for $k > 0$, $x \in S$ (**1 p.**).

   Consequently, according to the Lyapunov direct method, the system is stable (**1 p.**).

   Next, we note that $\frac{\mathrm{d}}{\mathrm{d}t}V = 0$ for $x = (x_1, 0)$ where $x_1 \in \mathbb{R}$ (**1 p.**). Hence the condition $\frac{\mathrm{d}}{\mathrm{d}t}V < 0$ for all $x \in S \setminus (0, 0)$ is not satisfied (**1 p.**) and we cannot show asymptotic stability using the direct Lyapunov method (**1 p.**).

5. Using Part 3, we recall that $\dot{V}(t) \leq 0$ for all $x \in S_K$. Hence, S_k is invariant and $\bar{S} = \{x \in S \mid \frac{\mathrm{d}}{\mathrm{d}t}V = 0\} = \{x \in S \mid x_1 \in \mathbb{R},\ x_2 = 0\}$ (**1 p.**).

   Next, the largest invariant set M, contained in $\bar{S}$ needs to be quantified. The set $\bar{S}$ is a horizontal line at $x_2 = 0$. From the system's trajectories, if $x_2 = 0$ and $x_1 \neq 0$, the system diverges from the horizontal line $x_2 = 0$ (**1 p**). Consequently, the system stays on the line $x_2 = 0$ only when $x_1 = x_2 = 0$ and $M = \{(0, 0)\}$ is the largest invariant set contained in $\bar{S}$ (**1 p**). Hence, by LaSalle's theorem all trajectories that start in S_K converge to the equilibrium $\hat{x} = (0, 0)$.

Since the level-set $K > 0$ can be chosen arbitrarily, when $||x(t)|| \to \infty \implies V(x) \to \infty$ (**1 p**). Finally, according to the La'Salle's theorem the origin is globally asymptotically stable equilibrium (**1 p**).