Exercise 1

1	2	3	4	5	Exercise
5	5	3	9	3	25 Points

1. With $u_{c_1}(t)$ we denote the the voltage of the high-pass filter capacitor C_1. Then, the dynamics are given by $v_{in}(t) = u_{c_1}(t) + R_1C_1\frac{d}{dt}u_{c_1}(t)$ (**1p.**) and $v_{out}(t) = R_1C_1\frac{d}{dt}u_{c_1}(t)$ (**1p.**). After Applying Laplace transform, we have $v_{in}(s) = u_{c_1}(s) + R_1C_1(su_{c_1}(s) - u_{C_1}(0))$ (**1p.**) and $v_{out}(s) = R_1C_1(su_{c_1}(s) - u_{C_1}(0))$ (**1p.**) yielding to transfer function $G_{\mathrm{HP}}(s) = \frac{sR_1C_1}{sR_1C_1+1}$ (**1p.**).

2. By defining the low-pass filter capacitor voltage as $v_{c_2}(t)$, we have that $v_{in}(t) = R_2C_2\frac{d}{dt}u_{c_2}(t) + u_{c_2}(t)$ and $v_{out}(t) = u_{c_2}(t)$ (**1p.** for both equations). The state of the system is the capacitor voltage $u_{c_2}(t)$, the input of the system is $v_{in}(t)$ and output $v_{out}(t)$ (**1p.** for the state and input and output). Hence $A = -1/(R_2C_2)$, $B = 1/(R_2C_2)$, $C = 1$ and $D = 0$ (**1p.** for all matrices correct). The transfer function can be obtained using $G(s) = C(sI - A)^{-1}B$ (**1p.**), yielding $G_{\mathrm{LP}}(s) = \frac{1}{sR_2C_2+1}$ (**1p.**).

3. The band-pass filter is obtained by connecting the output of low-pass filter to the input of the high-pass filter (**1p.** give a point for connecting high-pass filter output to low-pass filter input). The block diagram is given in Figure 1 (**1p.**). The equivalent transfer function is obtained by $G_{\mathrm{eq}} = G_{\mathrm{HP}}G_{\mathrm{LP}}$ (**1p.**)

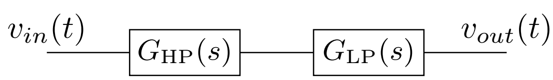

Figure 1: Block diagram

4. First we calculate $R_1C_1 = \frac{1}{5\pi}10^{-3}$ and $R_2C_2 = \frac{1}{10\pi}10^{-3}$ (**1p.** for both expressions). Next, from the equivalent transfer function $G_{\mathrm{eq}}(s)$ we have that the magnitude characteristic is rising with 20dB/dec till $\omega = 5\pi 10^3$rad/s (**1p.**), after which is constant till $\omega = 10\pi 10^3$rad/s (**1p.**) with the amplitude of $20\log 1 = 0$dB (**1p.**), after which is decreasing with 20dB/dec (**1p.**). Since the transfer function has a zero at $\omega = 0$ the phase plot starts from $\pi/2$ (**1p.**). The two poles drop the characteristic for π ending up at $-\pi/2$ (**1p.**). Sketch of the magnitude characteristic is illustrated in Figure 2 a) (**1p.**) and in Figure 2 b) is the sketch of the phase characteristic (**1p.**).

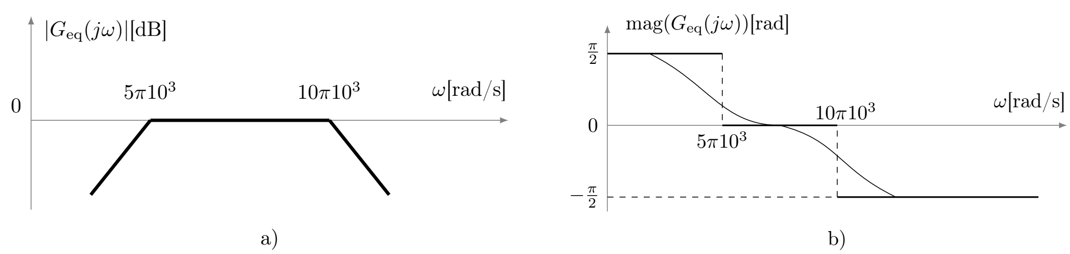

Figure 2: Bode plot sketch: a) magnitude characteristic, b) phase characteristic

5. It can be easily seen that the frequency range that gets passed through the filter is $\omega \in [R_1C_1, R_2C_2] = [5, 10]\pi 10^3$rad/s. Hence $f \in [2.5, 5]$kHz (**1p.**). Since 1kHz$<$ 2.5kHz and 6kHz$>$ 5kHz, those values are not passed through the filter (**1p.**), while 2.5kHz$<$ 3kHz$<$ 5kHz, hence it is passed through the filter (**1p.**).

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Exercise 2

1(a)	1(b)	1(c)	2(a)	2(b)	2(c)	2(d)	3(a)	3(b)	3(c)	Exercise
3	3	3	2	4	1	3	2	2	2	25 Points

1. (a) The controllability matrix is $\mathcal{C} = [B \;\; AB]$

$$(\mathbf{1p}) \quad \mathcal{C} = \begin{bmatrix} 1 & -1+ab \\ b & a-b \end{bmatrix},$$

which should be full rank for controllability (**1p**). The characteristic polynominal is

$$p(\mathcal{C}) = (a-b) - (-b+ab^2) = a - ab^2 = a(1-b^2).$$

therefore the system is controllable for $a \neq 0, \;\; b \neq \pm 1$ (**1p**).

(b) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix}.$$

We compute

$$(\mathbf{1p}) \quad \mathcal{O} = \begin{bmatrix} c & 1 \\ a-c & ac-1 \end{bmatrix},$$

which should be full rank for observability (**1p**). The characteristic polynominal is

$$p(\mathcal{C}) = (ac^2 - c) - (a-c) = ac^2 - a = a(c^2-1).$$

therefore the system is controllable for $a \neq 0, \;\; c \neq \pm 1$ (**1p**).

(c) For the given values we have

$$A + LC = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1+\ell_1 \\ 1 & -1+\ell_2. \end{bmatrix}$$

The eigenvalues are given by the characteristic polynomial of $(A + LC - \lambda I)$. Resulting in

$$p(A+LC-\lambda I) = (-1-\lambda)(-1-\lambda+\ell_2) - (1+\ell_1) = \lambda^2 + 2\lambda - \ell_2\lambda - \ell_1 - \ell_2.$$

Writing the characteristic polynomial is (**1p**). The desired poles are at $\lambda_1 = \lambda_2 = -2$ so that we want

$$p(A+LC-\lambda I) = (\lambda+2)^2 = \lambda^2 + 4\lambda + 4 = \lambda^2 + (2-\ell_2)\lambda - (\ell_1+\ell_2).$$

Taking $\ell_1 = -2, \;\; \ell_2 = -2$ gives the desired result. (**2p**, one each gain)

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2. Consider the assignment $a = -0.5$, $b = 0,\ c = 0$ with zero input ($u(t) = 0$).

   (a) We need the eigenvalues of A.

$$A - \lambda I = \begin{bmatrix} -1-\lambda & -0.5 \\ -0.5 & -1-\lambda \end{bmatrix}$$
$$p(A - \lambda I) = (-1-\lambda)^2 - 0.25 = \lambda^2 + 2\lambda + 0.75.$$

   The eigenvalues are $\{-0.5, -1.5\}$ (**1p**), therefore the system with zero input is stable (**1p**).

   (b) The eigenvalues are distinct, so the matrix is diagonalizable (**1p**).
   The columns of W are the eigenvectors of A. The eigenvector for $\lambda_1 = -0.5$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (**1p**), the one for $\lambda_2 = -1.5$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (**1p**).
   Therefore $W = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ (**1p**).

   (c) The A matrix is asymptotically stable, therefore such P always exists (**1p**).

   (d)

$$\begin{aligned} A^T P + PA &= -I \\ WDW^T P + PWDW^T &= -I \quad (A = WDW^T) \\ DW^T P + W^T PWDW^T &= -W^T \quad \text{left multiply } W^T \ (\mathbf{1p}) \\ DW^T PW + +W^T PWD &= -I \quad \text{right multiply } W \ (\mathbf{1p}) \\ D\tilde{P} + \tilde{P}D &= -I \quad \text{definition of } \tilde{P} \ (\mathbf{1p}) \end{aligned}$$

   The last equality shows that $\tilde{P}$ forms the desired Lyapunov function.

3. Consider the assignment $a = 1,\ b = 0,\ c = 0$.

   (a) We need the eigenvalues of A.

$$A - \lambda I = \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix}$$
$$p(A - \lambda I) = (-1-\lambda)^2 - 1 = \lambda^2 + 2\lambda.$$

   The eigenvalues are $\{0, -2\}$ (**1p**). Therefore the system with zero input is not asymptotically stable due to the pole at zero (**1p**).

   (b) Plugging in the state feedback controller, we have

$$A + BK = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -1+k_1 & 1+k_2 \\ 1 & -1 \end{bmatrix}.$$

   With $k_1 = -1,\ k_2 = -1$ we get

$$A + BK = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix},$$
$$p(A + BK) = (-2-\lambda)(-1-\lambda) - 0 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2),$$

therefore the poles are at $\{-1, -2\}$ (**1p**). Since the real part of the poles are negative, the closed-loop system is asymptotically stable (**1p**).

(c) Following the closed loop form in the previous part, with $k_1 = 2,\ k_2 = -2$ we get

$$A + BK = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$
$$p(A + BK) = (1 - \lambda)(-1 - \lambda) - (-1) = \lambda^2 - 1 + 1 = \lambda^2$$

therefore, we have the poles at $\{0, 0\}$.

Since the real part of the poles are zero, the closed-loop system is not asymptotically stable (**1p**).

Since the closed-loop matrix is non-diagonalizable, the stability in the case of zero real parts cannot be determined from the eigenvalues alone (**1p**).

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Exercise 3

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	Exercise
4	**3**	**3**	**3**	**3**	**4**	**5**	**25 Points**

1. The transfer function of an LTI system is given by:

$$G(s) = C(sI - A)^{-1}B + D,$$

$$\begin{aligned} sI - A &= \begin{bmatrix} s+100 & -1 \\ 0 & s+2 \end{bmatrix} \\ (sI-A)^{-1} &= \frac{1}{(s+100)(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s+100 \end{bmatrix} \\ G(s) &= [(sI-A)^{-1}]_{12} + 0 = \frac{1}{(s+100)(s+2)} \end{aligned}$$

- System (a)

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B + D \\ sI - A &= \begin{bmatrix} s+100 & -1 \\ 0 & s-2 \end{bmatrix} \\ (sI-A)^{-1} &= \frac{1}{(s+100)(s-2)} \begin{bmatrix} s-2 & 1 \\ 0 & s+100 \end{bmatrix} \\ G(s) &= [(sI-A)^{-1}]_{12} = \frac{1}{(s+100)(s-2)} \qquad \textbf{(1 pt.)} \end{aligned}$$

  – $G(0) = \frac{-1}{200} \Rightarrow$ options (ii) and (iv) are valid solutions. **(1 pt.)**
  – $G(j\omega) = \frac{1}{-(200+\omega^2)+98j\omega} \Rightarrow$ there is no value of ω for which $200 + \omega^2 = 0$ **(1 pt.)**. Therefore, it cannot be option (iv) and the right answer is option (ii) **(1 pt.)**.

- System (b)

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B + D \\ sI - A &= \begin{bmatrix} s+100 & -1 \\ 0 & s+2 \end{bmatrix} \\ (sI-A)^{-1} &= \frac{1}{(s+100)(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s+100 \end{bmatrix} \\ G(s) &= 2[(sI-A)^{-1}]_{12} + 0 = \frac{2}{(s+100)(s+2)} \qquad \textbf{(1 pt.)} \end{aligned}$$

  The transfer function has twice the magnitude but the same phase as system (1) **(1 pt.)**. So option (v) is the only valid solution. **(1 pt.)**

- System (c)

$$
\begin{aligned}
G(s) &= C(sI-A)^{-1}B + D \\
sI - A &= \begin{bmatrix} s-100 & -1 \\ 0 & s-2 \end{bmatrix} \\
(sI-A)^{-1} &= \frac{1}{(s-100)(s-2)} \begin{bmatrix} s-2 & 1 \\ 0 & s-100 \end{bmatrix} \\
G(s) &= 2[(sI-A)^{-1}]_{12} + 0 = \frac{1}{(s-100)(s-2)} \qquad \textbf{(1 pt.)}
\end{aligned}
$$

  - The transfer function has the same magnitude as system (1) but $\angle G_{(c)}(j\omega) = -\angle G(j\omega)$, where $\angle G(j\omega)$ is the phase as that of system (1). **(1 pt.)**
  - So the correct answer is option (iii). **(1 pt.)**

- System (d)

$$
\begin{aligned}
G(s) &= C(sI-A)^{-1}B + D \\
sI - A &= \begin{bmatrix} s+100 & -1 \\ 0 & s+2 \end{bmatrix} \\
(sI-A)^{-1} &= \frac{1}{(s+100)(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s+100 \end{bmatrix} \\
G(s) &= -[(sI-A)^{-1}]_{12} + 0 = \frac{-1}{(s+100)(s+2)} \qquad \textbf{(1 pt.)}
\end{aligned}
$$

  - The transfer function has the same magnitude as system (1) but $\angle G_{(d)}(j\omega) = \pi + \angle G(j\omega)$, where $\angle G(j\omega)$ is the phase as that of system (1). **(1 pt.)**
  - So the correct answer is option (iv). **(1 pt.)**

2. (a) Let the Laplace transform of $y(t)$ be given by $Y(s)$.

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty y(t)e^{-st}dt.$$

Now, let us perform a variable change $\hat{t} = t-1 \Rightarrow t = \hat{t}+1$ and $d\hat{t} = dt$.

$$
\begin{aligned}
\mathcal{L}\{y(t-1)\} &= \int_0^\infty y(t-1)e^{-st}dt, && (t = \hat{t}+1) \\
&= \int_{-1}^\infty y(\hat{t})e^{-s(\hat{t}+1)}d\hat{t}, && (t = 0 \Rightarrow \hat{t} = -1) \qquad \textbf{(1 pt.)} \\
&= e^{-s}\int_{-1}^\infty y(\hat{t})e^{-s\hat{t}}d\hat{t}, && (y(t) = 0, \quad \forall t < 0) \\
&= e^{-s}\int_0^\infty y(\hat{t})e^{-s\hat{t}}d\hat{t}, \\
&= e^{-s}Y(s). \qquad \textbf{(1 pt.)}
\end{aligned}
$$

Therefore, the expression of the transfer function of the system is,

$$G_2(s) = \frac{\mathcal{L}\{y(t-1)\}}{\mathcal{L}\{y(t)\}} = \frac{e^{-s}Y(s)}{Y(s)} = e^{-s}. \qquad \textbf{(1 pt.)}$$

(b) To plot the Bode plot, we first substitute $s = j\omega$,

$$\begin{aligned} G_2(j\omega) &= e^{-j\omega} \\ |G_2(j\omega)| &= 1 = 0\,dB \\ \angle G_2(j\omega) &= -\omega\ (rad) = -57.3\omega^\circ \qquad \textbf{(1 pt.)} \end{aligned}$$

The bode plot of e^{-s} is shown in Fig. 3. **(1 pt.)** for the correct magnitude plot, **(1 pt.)** for the correct values and **(1 pt.)** for the correct shape.

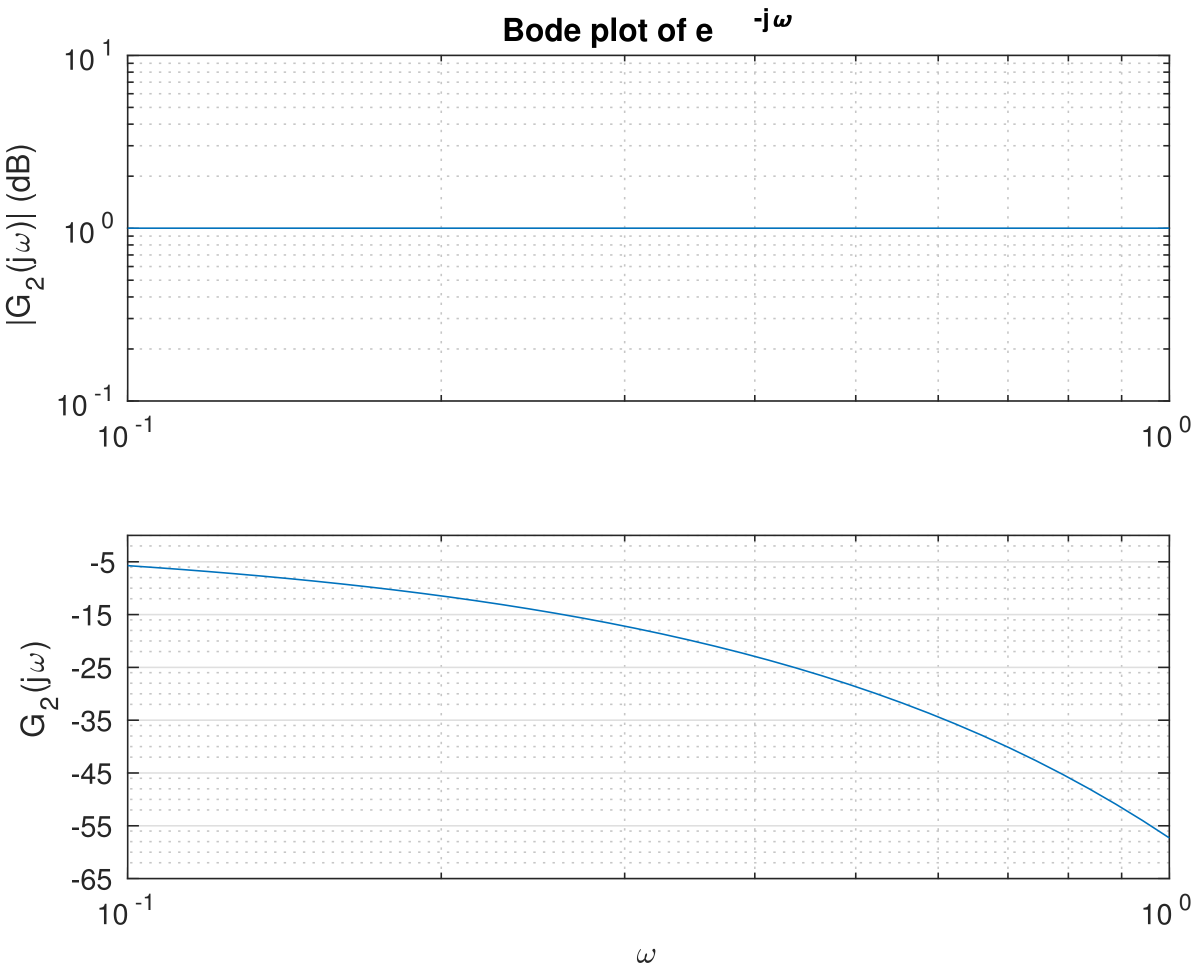

Figure 3: Bode plot of e^{-s}.

(c) To find the phase margin we need to find the ω at which $|G_3(j\omega)| = 1$,

$$
\begin{aligned}
|G_3(j\omega)| &= \frac{|e^{-0.1j\omega}|}{|0.5 + j\omega|} \\
&= \frac{1}{\sqrt{0.25 + \omega^2}} \qquad \textbf{(1 pt.)} \\
\frac{1}{\sqrt{0.25 + \omega^2}} &= 1 \\
0.25 + \omega^2 &= 1 \\
\omega &= 0.5\sqrt{3} \qquad \textbf{(1 pt.)}
\end{aligned}
$$

Next, we compute the phase of $G_3(j\omega)$,

$$
\begin{aligned}
\angle G_3(j\omega) &= -0.1j\omega - \tan^{-1}\left(\frac{\omega}{0.5}\right) \qquad \textbf{(1 pt.)} \\
\angle G_3(j0.5\sqrt{3}) &= 57.3 \times (-0.1 \times 0.5\sqrt{3})^\circ - \tan^{-1}\left(\frac{0.5\sqrt{3}}{0.5}\right)^\circ \\
&= 57.3 \times (\frac{-0.1732}{2})^\circ - \tan^{-1}(\sqrt{3})^\circ \\
&= 57.3 \times (-0.0866)^\circ - 60^\circ \\
&\approx -65^\circ \qquad \textbf{(1 pt.)} \\
\text{Phase Margin} &= 180^\circ - 65^\circ \\
&= 115^\circ \qquad \textbf{(1 pt.)}
\end{aligned}
$$

Exercise 4

1	2	3	4	5	Exercise
3	6	8	6	2	25 Points

1. The standard state space form is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_1 - dx_2^3 \quad \textbf{(1 p.)}\,.\end{aligned}$$

   Equilibrium points are obtained by setting system's dynamics to zero. Hence, we have $\dot{x}_1 = 0$, yielding $\hat{x}_2 = 0$ **(1 p.)**. Combining it with $\dot{x}_2 = 0$ we have $\hat{x}_1 = 0$ **(1 p.)**.

2. We compute the Jacobian of the dynamics and we evaluate it at the equilibrium $\hat{x}$

$$\begin{aligned}A &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\hat{x}) & \frac{\partial f_1}{\partial x_2}(\hat{x}) \\ \frac{\partial f_2}{\partial x_1}(\hat{x}) & \frac{\partial f_2}{\partial x_2}(\hat{x}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -k & -3d\hat{x}_2^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \quad \textbf{(2 p.)}\,.\end{aligned} \tag{1}$$

   We compute the eigenvalues of the A as follow

$$\begin{aligned}\det(\lambda I - A) &= \det \begin{bmatrix} \lambda & -1 \\ k & \lambda \end{bmatrix} \\ &= \lambda^2 + k \overset{!}{=} 0 \iff \lambda_{1,2} = \pm i\sqrt{k} \quad \textbf{(2 p.)}\,.\end{aligned} \tag{2}$$

   The Lyapunov's linearization method is inconclusive since there are imaginary eigenvalues **(2 p.)**.

3. The derivative of Lyapunov function $V(x) = \frac{1}{2}\left(kx_1^2 + x_2^2\right)$ is

$$\begin{aligned}\dot{V}(x) &= \nabla V(x) f(x) \\ &= \begin{bmatrix} kx_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -kx_1 - dx_2^3 \end{bmatrix} \quad \textbf{(1 p.)} \\ &= kx_1x_2 - kx_1x_2 - dx_2^4 \\ &= -dx_2^4 \ \textbf{(1 p.)}\,.\end{aligned}$$

   On the given open set $S = \mathbb{R}^2$ we have

   (a) $V(\hat{x}) = 0$ **(1 p.)**
   (b) $V(x) > 0$ for all $x \in S \setminus \{\hat{x}\}$ **(1 p.)**

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(c) $\dot{V}(x) \leq 0$ for all $x \in S$ since $d > 0$ and $x_2^4 \geq 0$ for all $x_2 \in \mathbb{R}$ **(1 p.)**.

Hence, according to the Lyapunov direct method, the equilibrium $\hat{x}$ is stable **(1 p.)**.

No, the asymptotic stability cannot be concluded via the direct Lyapunov method because $\dot{V}(x) = 0$ for $(x_1, 0)$ with $x_1 \in \mathbb{R}$ **(1 p.)**. Hence $\frac{d}{dt}V(x) < 0$ for all $x \in S\backslash\{\hat{x}\}$ does not hold **(1 p.)**.

4. $\dot{V}(x(t)) = 0$ for $(x_1, 0)$ with $x_1 \in \mathbb{R}$. Therefore

$$\bar{S} = \left\{ x \in \mathbb{R}^2 \,\middle|\, x_1 \in [-\ell_1, \ell_1],\, x_2 = 0 \right\} .$$

To derive the largest invariant set in $\bar{S}$ we must have a look at the system equations. Indeed, from the system equations we see that in order for the trajectories to be confined to the line where $x_2 = 0$ we need $x_1 = 0$, or the system would diverge from the $x_2 = 0$ line **(2 p.)**. Therefore, the equilibrium $\hat{x} = (0, 0)$ is the largest invariant set in $\bar{S}$ **(1 p.)**.

Since S is bounded and closed and $\dot{V}(x(t)) \leq 0$ for all $x \in S$, we can invoke LaSalle's theorem **(1 p.)**. According to LaSalle's theorem, all trajectories starting in S tend to $\hat{x}$ for $t \to \infty$. We can therefore conclude that $\hat{x}$ is locally asymptotically stable **(1 p.)**. In addition, since $\|x(t)\| \to \infty \implies V(x(t)) \to \infty$, following the hint we can conclude that $\hat{x}$ is globally asymptotically stable **(1 p.)**.

5. By looking at the phase plane plot in Fig. 6 we can conclude that the equilibrium $\hat{x}$ is unstable **(1 p.)**. From the phase portrait we also see that from any initial state, the system will enter a limit cycle, which is marked in red in the figure **(1 p.)**.