Institute for Operations Research
ETH Zurich HG G21-23
math.ethz.ch/ifor

Exam

Introduction to Mathematical Optimization

18.08.2021

Student ID:	

General Remarks:

- ☺ Place your student identity card on the table.
- ☺ Turn off all your electronic devices, including cell phones and smart watches, and put them away. Do not wear them on your body or in your pockets.
- ☺ No extra material is allowed, except for a subject-neutral dictionary.
- ☺ Use a pen to write your solutions. Pencils and red pens are not allowed.
- ☺ Write your solutions on the provided paper. Always clearly indicate which problem you are solving. Do not use your own paper, we will hand out more if needed.
- ☺ For all problems, provide a complete solution in English, including all explanations and implications in a mathematically clear and well-structured way. Please hand in a readable and clean solution. Cross out any invalid solution attempts.
- ☺ Auxiliary statements that you use to solve a problem must be proved. Unless you are explicitly asked to prove them, results known from the lectures can be used without a proof. In each such case, state the result or its name.
- ☺ Ask any questions that you might have immediately and during the exam.
- ☺ You have 2 hours to solve the exam.

Good luck!

Problem	1	2	3	4	5	Total
Points	/ 10	/ 10	/ 10	/ 10	/ 10	/ 50

Problem 1 Job Assignment (10 points)

Imagine that you are in charge of distributing tasks in a small company. The company has n employees, among which m are managers, and there are t jobs on the agenda, among which s are marked as important. For every employee i and job j, you know the amount $d_{ij} \in \mathbb{R}_{\geq 0}$ of dissatisfaction that will be caused by assigning i to work on j. If an employee is not assigned a task, then their dissatisfaction is 0 by default.

Your task is to find an assignment of jobs to employees that satisfies the following criteria.

- Every job can be assigned to at most one employee.
- Every employee can be assigned at most one job.
- At least k jobs are assigned.
- At least ℓ important jobs are assigned to managers.
- The total dissatisfaction of the employees is minimized.

Write an ILP that finds an optimal assignment with the described properties.

Problem 2 Production Planning (10 points)

Suppose you were hired to optimize production at a chemical plant specializing in nitrogen compounds. The production chain consists of three steps:

- Produce one unit of ammonia NH_3.
- Produce one unit of nitric acid HNO_3 from one unit of ammonia NH_3.
- Produce one unit of ammonium nitrate NH_4NO_3 from one unit of ammonia NH_3 and one unit of nitric acid HNO_3.

The table below contains sale prices and production costs of each of the three compounds.

Compund	**Sale price, CHF/ton**	**Production cost, CHF/ton**
NH_3	275	125
HNO_3	350	75
NH_4NO_3	500	50

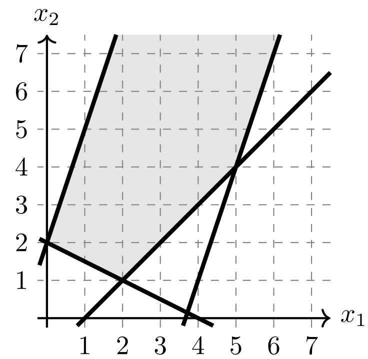
For example, to produce one ton of ammonium nitrate NH_4NO_3, one needs to first produce two tons of ammonia NH_3, then convert one of them into one ton of nitric acid HNO_3, and finally combine the remaining ton of ammonia NH_3 with one ton of nitric acid HNO_3. Here the total production costs are CHF 375, so the profit after selling the compound is CHF 125.

The chemical plant has a monthly budget of CHF 5 000 000 and can ship at most 20 000 tons of final products each month. Write an LP that finds the amount of each substance to be produced monthly so that the monthly profit (i.e., total revenue from sales minus total production costs) is maximized.

Problem 3 Simplex Method and Duality (10 points)

Consider the following LP whose feasible region depicted in gray in the figure below.

$$\begin{array}{lrcrcr} \max & 5x_1 & - & 2x_2 & & \\ \text{s.t.} & 3x_1 & - & x_2 & \geq & -2 \\ & x_1 & + & 2x_2 & \geq & 4 \\ & x_1 & - & x_2 & \leq & 1 \\ & 3x_1 & - & x_2 & \leq & 11 \end{array}$$



(a) Bring this LP into canonical form. Then write down the corresponding standard form LP. (1 pt)
Hint: The canonical form can be written using the same variables x_1 and x_2. Use the picture to argue why this is the case.

(b) Write down the auxiliary tableau for phase I of the simplex method corresponding to the origin $(x_1, x_2) = (0, 0)$. Circle the element pivoting on which makes the tableau feasible. (1 pt)
Note: The row corresponding to the original objective is optional. You do **not** need to do the pivot.

(c) At the end of phase I of the simplex method, one may arrive at the following tableau:

y_1	y_2	y_3	y_4	x_1	x_2	x_0	1
0	0	0	0	0	0	1	0
0	1	0	0	-6	0	-1	-4
1	$\frac{1}{2}$	0	0	$-\frac{7}{2}$	0	$-\frac{3}{2}$	0
0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	2
0	$-\frac{1}{2}$	1	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	3
0	$-\frac{1}{2}$	0	1	$\frac{7}{2}$	0	$-\frac{1}{2}$	13

The original objective was kept along during phase I in the second row of the auxiliary tableau. Explain how to convert the given tableau into a feasible tableau for the original LP and write down the resulting tableau. (1 pt)

(d) Obtain an optimal tableau for the original LP by performing phase II of the simplex method starting from the following tableau: (2 pts)

y_1	y_2	y_3	y_4	x_1	x_2	1
0	-1	4	0	0	0	8
1	$-\frac{2}{3}$	$\frac{7}{3}$	0	0	0	7
0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	1	1
0	$-\frac{1}{3}$	$\frac{2}{3}$	0	1	0	2
0	$\frac{2}{3}$	$-\frac{7}{3}$	1	0	0	6

(e) Write down the dual LP. (1 pts)

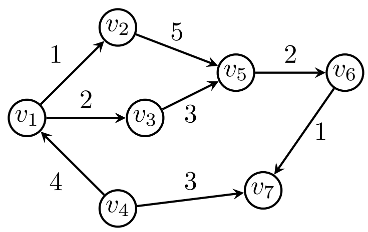
(f) Given that $(x_1, x_2) = (5, 4)$ is an optimal solution of the original LP, find an optimal solution of the dual LP using the complementary slackness conditions. (2 pts)

(g) Suppose the constraint $3x_1 - x_2 \leq 11$ of the original LP is changed to $3x_1 - x_2 \geq 11$. Is the dual of the modified LP feasible? Justify your answer. (2 pts)
Hint: You may use geometric intuition for the modified primal and refer to the picture in your solution.

Problem 4 Combinatorial Optimization (10 points)

Let $G = (V, A)$ be the directed graph shown in the figure below with arc lengths written next to them.

(a) Run Dijkstra's algorithm to determine all shortest path lengths from v_1 to all the nodes in the graph. For this purpose complete the table given below. After filling in the table, write the distance from v_1 to every other vertex, along with a corresponding shortest path (provided one exists). (5 pts)



current node		v_1	v_2	v_3	v_4	v_5	v_6	v_7
initialization	distance from v_1: visited (yes/no):							
	distance from v_1: visited (yes/no):							
	distance from v_1: visited (yes/no):							
	distance from v_1: visited (yes/no):							
	distance from v_1: visited (yes/no):							
	distance from v_1: visited (yes/no):							
	distance from v_1: visited (yes/no):							
	distance from v_1: visited (yes/no):							

- Distance and a shortest path from v_1 to v_2:
- Distance and a shortest path from v_1 to v_3:
- Distance and a shortest path from v_1 to v_4:
- Distance and a shortest path from v_1 to v_5:
- Distance and a shortest path from v_1 to v_6:
- Distance and a shortest path from v_1 to v_7:

(b) Suppose we modify the original graph G by reversing the direction of the arc (v_4, v_1) while keeping its length the same. How do the shortest path lengths from v_1 to every other vertex change compared to the original graph? (3 pts)

(c) Suppose we modify the original graph G by replacing the length of the arc (v_2, v_5) with a parameter $\ell \in \mathbb{R}_{\geq 0}$. Find the shortest path length from v_1 to v_7 for every value of the parameter ℓ. (2 pts)

Problem 5 Theory (10 x 1 = 10 points)

For each of the following statements, state if it is **true or false** and provide a short **justification**. No points are given for unjustified, or wrongly justified answers.

(a) There is an LP with a finite optimum and an unbounded feasible region.

☐ **TRUE** ☐ **FALSE**

(b) The set $A := \{(x, y) \in \mathbb{R}^2 \mid y \leq |x| + x\}$ is the feasible region of a linear program.

☐ **TRUE** ☐ **FALSE**

(c) Let $P := \{x \in \mathbb{R}^2 \mid Ax \leq b\}$ and $Q := \{x \in \mathbb{R}^2 \,|\, Cx \leq d\}$ be two convex polyhedra. Then

$$(P \setminus Q) \cup (Q \setminus P) = \{x \in \mathbb{R}^2 \mid (x \in P \text{ and } x \notin Q) \text{ or } (x \in Q \text{ and } x \notin P)\}$$

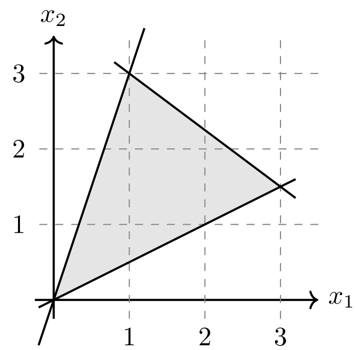
is a polyhedron, i.e., can be written in terms of finitely many linear constraints.

☐ **TRUE** ☐ **FALSE**

(d) Consider the linear program $\{\max c^T x \mid Ax \leq b\}$ for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Let x_1, $x_2 \in \mathbb{R}^n$ be distinct points, i.e., $x_1 \neq x_2$. It can happen that the set of optimal solutions is exactly $\{x_1, x_2\}$.

☐ **TRUE** ☐ **FALSE**

(e) Consider the linear programs $P : \{\max\ c_1x + c_2y \mid y - 3x \leq 0,\ 2y - x \geq 0,\ 4y + 3x \leq 15,\ x, y \in \mathbb{R}_{\geq 0}\}$ and $P_{\text{int}} : \{\max\ c_1x + c_2y \mid y - 3x \leq 0,\ 2y - x \geq 0,\ 4y + 3x \leq 15,\ x, y \in \mathbb{Z}_{\geq 0}\}$, for some $c_1, c_2 \in \mathbb{Z}$. The feasible region of P and P_{int} is depicted below.



Then for every $c_1, c_2 \in \mathbb{Z}$ the optimal value of P is equal to the optimal value of P_{int}.

☐ **TRUE** ☐ **FALSE**

(f) Consider the following long tableau.

z	x_1	x_2	x_3	x_4	1
1	-1	0	8	0	0
0	5	0	5	0	10
0	-3	0	-3	1	1
0	4	1	2	0	7

Performing a pivot on the circled element leads to a feasible tableau.

☐ **TRUE** ☐ **FALSE**

(g) Let $f(n) = 2^n$ and $g(n) = n^{128}$. Then $f(n) = O(g(n))$.

☐ **TRUE** ☐ **FALSE**

(h) Let G be a connected graph with $n \geq 1$ vertices and n edges. Then G has exactly one cycle.

☐ **TRUE** ☐ **FALSE**

(i) There exists a graph with an odd number of vertices such that every vertex has odd degree.

☐ **TRUE** ☐ **FALSE**

(j) Consider a directed graph $G = (V, A)$ with arc lengths $\ell : A \to \mathbb{Z}_{\geq 0}$. Let v, w, u be three distinct nodes in V. Let p denote the length of a shortest v-w path and let q denote the length of a shortest w-u path. Suppose that there is a shortest v-u path that contains w. Then the length of a shortest v-u path is necessarily $p + q$.

☐ **TRUE** ☐ **FALSE**