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Exercise 1

1	2	3	4	5(a)	5(b)	Exercise
5	4	7	3	4	2	25 Points

1. First note that the current flowing in the converter branch is $C\dot{V}$, while the voltage across the inductance is $L\dot{I}$. By applying Kirchhoff's current law to the node right to the inductance, we obtain $I - C\dot{V} + \overline{I}_G - \overline{I}_L = 0$ (**2pt**). Moreover, by applying Kirchhoff's voltage law to the converter's most outer mesh, one obtains $V_{in} - RI - L\dot{I} - V = 0$ (**2pt**). Putting together the two previous equations leads to

$$\begin{bmatrix} \dot{V}(t) \\ \dot{I}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} V(t) \\ I(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_{in}(t) + \begin{bmatrix} (\overline{I}_G - \overline{I}_L)/C \\ 0 \end{bmatrix} \quad (\mathbf{1pt}).$$

2. The set of equilibria are defined as the solutions in x to the set of linear equations $Ax + Bu + d = 0$ (**1pt**). We can write the two equations explicitly as

$$C\dot{V} = I + (\overline{I}_G - \overline{I}_L) = 0$$
$$L\dot{I} = -V - RI + V_{in} = 0\,.$$

The first equation has a unique solution $I = \overline{I}_L - \overline{I}_G$ (**1pt**), and by substituting in the second we also obtain a unique solution $V = V_{in} + R(\overline{I}_G - \overline{I}_L)$ (**1pt**). Hence the equilibrium $\bar{x} = \begin{bmatrix} V_{in} + R(\overline{I}_G - \overline{I}_L) \\ \overline{I}_L - \overline{I}_G \end{bmatrix}$ is unique (**1pt**).

3. Using the hint, we substitute $x = \tilde{x} + \bar{x}$ into the ODE, obtaining the following linear system in perturbed coordinates $\tilde{x}$,

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + \underbrace{A\bar{x} + Bu + d}_{=0} = A\tilde{x}(t) \quad (\mathbf{2pt}).$$

To check for stability, let us first compute the characteristic polynomial of A,

$$\chi_A(\lambda) = \mathrm{DET} \begin{bmatrix} \lambda & -1/C \\ 1/L & \lambda + R/L \end{bmatrix} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} \quad (\mathbf{2pt}).$$

Since $\frac{R}{L}, \frac{1}{LC} > 0$, we can directly infer that the two roots of $\chi_A(\lambda)$ have negative real part, hence the system is asymptotically stable (**2pt**).

Alternative: One can also explicitly compute the two roots of $\chi_A(\lambda)$ as

$$\lambda_{1,2} = \frac{R/L \pm \sqrt{(R/L)^2 - 4/LC}}{2}\,,$$

and note that they have negative real part.

Consequently, $\lim_{t\to\infty} \tilde{x}(t) = 0$, hence $\lim_{t\to\infty} x(t) = \bar{x}$ and the equilibrium $\bar{x}$ is asymptotically stable (**1pt**).

4. Since the equilibrium is asymptotically stable, one has that $\lim_{t\to\infty} V(t) = \overline{V}_{in} + R(\overline{I}_G - \overline{I}_L)$ (**1pt**). Therefore, it suffices to impose $\overline{V}_{in} + R(\overline{I}_G - \overline{I}_L) = V_r$, that is, designing the open-loop controller $\overline{V}_{in} = V_r - R(\overline{I}_G - \overline{I}_L)$ (**2pt**).

5. (a) To verify the inequality we compute

$$\begin{aligned}\dot{E}(V,I) &= CV\dot{V} + LI\dot{I}\\ &= V(I + I_G - I_L) + I(-V - RI + V_{in})\\ &= -YV^2 - RI^2 + V_{in}I + VI_G \quad (\mathbf{2pt})\,.\end{aligned}$$

Therefore, under the controller $V_{in} = -KI$, we obtain

$$\begin{aligned}\dot{E}(V,I) &= -YV^2 - (R+K)I^2 + VI_G\\ &\leq VI_G \quad (\mathbf{2pt})\,.\end{aligned}$$

(b) Systems satisfying such inequalities are called *dissipative systems* because they withhold less energy than they receive. We note that the term $E(V(t), I(t)) = \frac{1}{2}CV(t)^2 + \frac{1}{2}LI(t)^2$ represents the total energy stored in the converter, hence $\frac{d}{dt}E(V(t), I(t))$ is its rate of change. Then, $V(t)I_G(t)$ represents the power supplied from the grid. Under this interpretation, inequality (2) shows that the Buck converter accumulates less energy than provided (**2pt**).

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Exercise 2

1	2	3	4	5	6	7	Exercise
4	4	4	3	3	4	3	25 Points

1. The equation $A^TQ + QA = -R$ has a unique positive definite solution if and only if A has eigenvalues with a negative real part (**1p**). A is lower triangular so the eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = \beta$ (**1p**). All eigenvalues are real; they are negative when $\beta < 0$ (**1p**). The stability of the system depends on the eigenvalues of A. As α does not appear in the eigenvalues expressions, α does not affect the system stability (**1p**).

2. The observability matrix is
$$(\mathbf{1p}) \quad \mathcal{Q} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \alpha - 5 & \beta \end{bmatrix},$$
which should be full rank for observability (**1p**). The matrix is full rank when $\alpha - 5 \neq \beta$ (**1p**). Therefore the system is observable for $\alpha \neq \beta + 5$ (**1p**).

3. The controllability matrix is
$$(\mathbf{1p}) \quad \mathcal{P} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & \alpha + \beta \end{bmatrix},$$
which should be full rank for controllability (**1p**). The matrix is full rank when $\alpha + \beta \neq -5$ (**1p**). Therefore the system is controllable for $\alpha \neq -\beta - 5$ (**1p**).

4. If the system is controllable (i.e. for $\alpha \neq -\beta - 5$), it is possible to find an input driving the system from $x(0)$ to $x(t)$ in any given finite time t (**1p**). If the system is uncontrollable (i.e. for $\alpha = -\beta - 5$), the reachable subspace is the Range$(\mathcal{P})$ = Span$\{\begin{bmatrix} 1 & 1 \end{bmatrix}^T\}$ (**1p**). Therefore it is possible to find an input to steer the system from $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ to any $x(t)$ of the form $\begin{bmatrix} a & a \end{bmatrix}^T$, and in particular to $x(1) = \begin{bmatrix} 10 & 10 \end{bmatrix}^T$ (**1p**).

5. We compute the closed loop matrix as
$$A + BK_1 = \begin{bmatrix} -5 - \alpha & -\alpha \\ 0 & \beta - \alpha \end{bmatrix},$$
which is asymptotically stable if its eigenvalues are negative (**1p.**). The matrix is lower triangular so the eigenvalues are $\lambda_1 = -5 - \alpha$ and $\lambda_2 = \beta - \alpha$ (**1p.**). The condition is verified when $\alpha > -5$ and $\alpha > \beta$ (**1p.**).

6. We compute the closed loop system matrix as
$$A + BK_2 = \begin{bmatrix} -5 - \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$
It is a diagonal matrix, which is stable but not asymptotically stable if one of its eigenvalues is zero (**1p.**). The other eigenvalue must be equal to -2 (**1p.**). The eigenvalues are $\lambda_1 = -5 - \alpha$ and $\lambda_2 = \beta$ (**1p.**). The required condition is verified when $\alpha = -5$ and $\beta = -2$ or when $\alpha = -3$ and $\beta = 0$ (**1p.**).

7. With $\alpha < \beta$, the system is observable (it is unobservable only when $\alpha = \beta + 5$) (**1p.**). Thus, there exists an L for which $(A - LC)$ has eigenvalues with negative real part (**2p.**).

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Exercise 3

1	2(a)	2(b)	2(c)	3(a)	3(b)	Exercise
6	**3**	**3**	**3**	**5**	**5**	**25 Points**

1. Since we need to determine the interval of values of k for which the closed-loop is stable, we re-scale the x-axis of the plot in Figure 2 dividing it by k. By doing so, we can treat the point marked with a cross $+$ as $(-1/k, 0)$.

   First, observe that the $L(s)$ does not have unstable poles, and the pole at the origin need not be counted (as in the plot on the left-hand side of Figure 2, the contour used to construct the Nyquist diagram has an indentation to the right and therefore does not encircle the pole in the origin) **(1pt.)**. As a result, the closed-loop $T(s)$ will be stable as long as the Nyquist diagram does not encircle the point $(-1/k, 0)$.

   We distinguish three cases:

   $\mathbf{k = 0}$ In this case $L(s) = 0$ and $T(s) = 0$, so the closed-loop is asymptotically stable. **(1p.)**

   $\mathbf{k > 0}$ In this case we have two clockwise encirclements of $(-1/k, 0)$ if $-1/k > -15.8$, and no encirclements if $-1/k < -15.8$ **(1p.)**. To have asymptotic stability we therefore require

   $$\frac{-1}{k} < -15.8 \iff k < \frac{1}{15.8}. \quad \textbf{(1p.)} \tag{1}$$

   Full points are awarded to students who analyze the case $k \geq 0$ without dealing separately with the case $k = 0$ as long as eq. (1) is derived correctly.

   $\mathbf{k < 0}$ In this case the point $(-1/k, 0)$ belongs to the positive half of the real line, and it is therefore encircled by the Nyquist diagram once in the clockwise direction because of the closure arc (compare Figure 1). We conclude that if $k < 0$ the closed-loop is unstable. **(1p.)**

   We conclude that to have asymptotic stability we require $0 \leq k < 15.8$. **(1p.)**

2. (a) For completeness, we verify that for $\omega = \sqrt{15}\,\mathrm{rad/s}$, we have $\mathrm{Im}(G_2(j\omega)) = 0$. We have:

$$\begin{aligned}
G_2(j\omega) &= \frac{5}{j\omega(j\omega+5)(j\omega+3)} \\
&= \frac{5}{(-\omega^2 + j5\omega)(j\omega+3)} \\
&= \frac{5}{-j\omega^3 - 3\omega^2 - 5\omega^2 + j15\omega} \\
&= \frac{5}{-8\omega^2 + j(-\omega^3 + 15\omega)} \frac{-8\omega^2 - j(-\omega^3 + 15\omega)}{-8\omega^2 - j(-\omega^3 + 15\omega)} \\
&= \frac{-40\omega^2}{\omega^6 + 34\omega^4 + 225\omega^2} + j\frac{5\omega^3 - 75\omega}{w^6 + 34\omega^4 + 225\omega^2} \\
&= \frac{-40}{\omega^4 + 34\omega^2 + 225} + j\frac{5\omega^2 - 75}{w^5 + 34\omega^3 + 225\omega}
\end{aligned}$$

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The imaginary part of $G_2(j\omega)$ is therefore

$$\mathrm{Im}(G_2(j\omega)) = \frac{5\omega^2 - 75}{w^5 + 34\omega^3 + 225\omega} \overset{!}{=} 0 \iff \omega^2 = \frac{75}{5} = 15 \iff \omega = \sqrt{15}.$$

The magnitude of $G_2(j\omega)$ at $\omega = \sqrt{15}\,\mathrm{rad/s}$ is

$$|G_2(j\omega)|_{\omega=\sqrt{15}} = \frac{40}{15^2 + 34 \cdot 15 + 225} = \frac{4}{96} = \frac{1}{24}, \tag{2}$$

and therefore the gain margin is

$$\mathrm{GM} = \frac{1}{|G_2(j\omega)|_{\omega=\sqrt{15}}} = 24. \tag{3}$$

The points are awarded as follows: **(1p.)** for each correct symbolic computation among (2) and (3), **(1p.)** if the numerical value of GM is correct.

(b) The Laplace transform Y_1 of y_1 is given by

$$Y_1(s) = G_2(s) = \frac{5}{s(s+5)(s+3)}.$$

Since Y_1 has only one pole in the origin and the remaining poles are stable, we have that $sY_1(s)$ has no unstable or marginally stable poles and the final value theorem can be applied **(1p.)**. We conclude that

$$\lim_{t\to\infty} y_1(t) = \lim_{s\to 0} sY_1(s) = \lim_{s\to 0} \frac{5}{(s+5)(s+3)} = \frac{1}{3}. \ \textbf{(2p.)}$$

Full points are awarded to students that do not make any statement about the applicability of the final value theorem, as long as the answer states that the limit exists and the value of the limit is correct.

(c) Since the Laplace transform of a step input is $1/s$, the Laplace transform Y_2 of y_2 is given by

$$Y_2(s) = \frac{G_2(s)}{s} = \frac{5}{s^2(s+5)(s+3)}, \ \textbf{(2p.)}$$

and has two poles in the origin. As a result, the function $sY_2(s)$ has one marginally stable pole and the final value theorem cannot be applied **(1p.)**.

3. (a) The Nyquist plot is shown in Figure 1. Points are assigned as follows:
   i. Correct intersections with real axis **(2p. in total, 1p. removed for each mistake)**;
   ii. (Approximately) correct intersections with imaginary axis **(1p. in total)**;
   iii. Correct phase for $\omega \approx 0\,\mathrm{rad/s}$ and for $\omega \to \infty$ **(1p. each)**.

   (b) From the structure of $G_3(s)$, we see that all the poles and the zeros are located at $\omega = 1\,\mathrm{rad/s}$. In the Bode plot, around the frequency $\omega = 1\,\mathrm{rad/s}$, we observe a $-20\,\mathrm{db/dec}$ decrease in the slope of the magnitude, and a $-270°$ decrease in phase **(1p.)**. We proceed with the following observations.

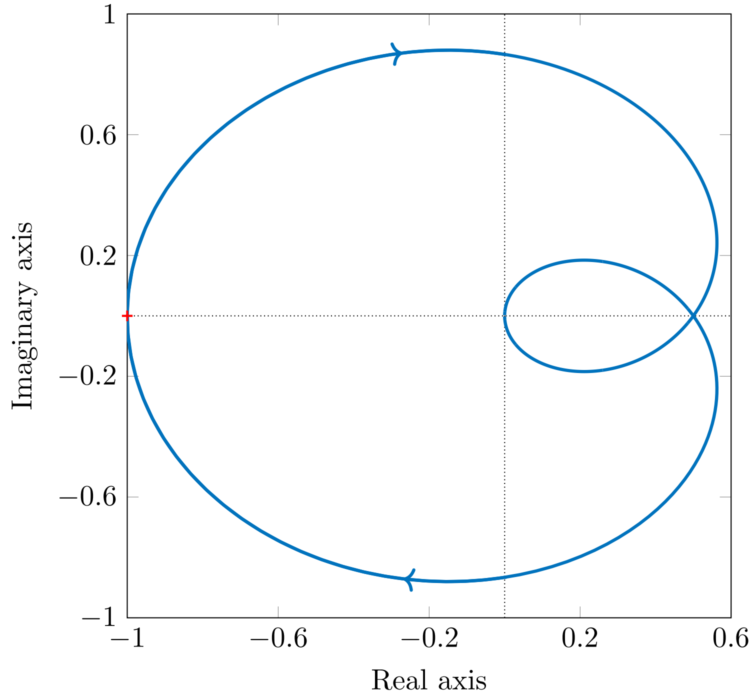

Figure 1: Nyquist plot of $G_3(s)$.

Magnitude First, we observe that the decrease of the slope of the magnitude means that there is an aboundance of poles at $\omega = 1\,\mathrm{rad/s}$ **(1p.)**; specifically, since the decrease is of $-20\,\mathrm{db/dec}$, it means that the number of poles at $\omega = 1\,\mathrm{rad/s}$ exceeds the number of zeros by 1. We conclude that $n_p = n_z + 1$ **(1p.)**.

Phase Around $\omega = 1\,\mathrm{rad/s}$ the phase decreases by a factor of $3 \cdot 90°$. Recall that a phase decrease of $-90°$ is associated to either a stable pole, or to an unstable zero; whereas an increase in phase of $90°$ is associated to either an unstable pole or to a stable zero. We consider the following two cases separately.

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1. $\mathbf{p = -1}$. In this case we require $n_z - n_p = 3 \iff -1 = 3$, which is not possible. **(1p.)**
2. $\mathbf{p = +1}$. In this case we require $n_z + n_p = 3 \iff 2n_z + 1 = 3 \iff n_z = 1$. **(1p.)**

We conclude that the only possibility is to have $p = +1$ and $n_p = 2, n_z = 1$.

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Exercise 4

1	2	3	4	5	Exercise
3	6	7	7	2	25 Points

1. Equilibrium points are obtained by setting system's dynamics to zero. Hence, we have $\dot{x}_1 = 0$, yielding $\hat{x}_2 = 0$. Combining it with $\dot{x}_2 = 0$ we obtain the following equation
$$-2x_1 - 4x_1^3 = -2x_1(1 + 2x_1^2) = 0\,, \quad \textbf{(1 p.)} \tag{4}$$
which is verified for $\hat{x}_1 = 0$ or $\hat{x}_1 = \pm j\frac{1}{\sqrt{2}}$ **(1 p.)**. We can therefore conclude that the system has a unique equilibrium point at $(0, 0)$ **(1 p.)**.

2. We compute the Jacobian of the dynamics and we then evaluate it at the equilibrium point
$$\begin{aligned} A(\hat{x}) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\hat{x}) & \frac{\partial f_1}{\partial x_2}(\hat{x}) \\ \frac{\partial f_2}{\partial x_1}(\hat{x}) & \frac{\partial f_2}{\partial x_2}(\hat{x}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -2 - 12\hat{x}_1^2 & -2a \end{bmatrix}. \end{aligned} \tag{5}$$
We plug $\hat{x} = (0, 0)$, which leads to
$$A((0,0)) = \quad = \begin{bmatrix} 0 & 1 \\ -2 & -2a \end{bmatrix} \quad \textbf{(1 p.)}\,. \tag{6}$$
We compute the eigenvalues of $A((0, 0))$ as follow
$$\begin{aligned} \det(\lambda I - A((0,0))) &= \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 2a \end{bmatrix} \\ &= \lambda^2 + 2a\lambda + 2 \overset{!}{=} 0 \iff \lambda_{1,2} = -a \pm \sqrt{a^2 - 2} \quad \textbf{(1 p.)}\,. \end{aligned} \tag{7}$$
Since $a \in [-\sqrt{2}, \sqrt{2}]$, $2 - a^2 \geq 0$ and we have $\lambda_{1,2} = -a \pm j\sqrt{2 - a^2}$ **(1 p.)**. When $a = 0$, the Lyapunov's linearization method is inconclusive since $\mathrm{Re}\{\lambda_i\} = 0$ for $i = 1, 2$ **(1 p.)**. When $a \in (0, \sqrt{2}]$, we can conclude that $(0, 0)$ is locally asymptotically stable since $\mathrm{Re}\{\lambda_i\} < 0$ for $i = 1, 2$ **(1 p.)**, and when $[-\sqrt{2}, 0)$ we can conclude that $(0, 0)$ is unstable since $\mathrm{Re}\{\lambda_i\} > 0$ for $i = 1, 2$ **(1 p.)**.

3. The derivative of Lyapunov function $V(x) = 4x_1^2 + 2x_2^2 + 4x_1^4$ is
$$\begin{aligned} \dot{V}(x) &= \nabla V(x) f(x) \\ &= \begin{bmatrix} 8x_1 + 16x_1^3 & 4x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -2x_1 - 2ax_2 - 4x_1^3 \end{bmatrix} \; \textbf{(1 p.)} \\ &= 8x_1x_2 + 16x_1^3x_2 - 8x_1x_2 - 8ax_2^2 - 16x_1^3x_2 \\ &= -8ax_2^2 \; \textbf{(1 p.)}\,. \end{aligned}$$
On the given open set $S = \mathbb{R}^2$ we have

(a) $V(\hat{x}) = 0$ **(1 p.)**

(b) $V(x) > 0$ for all $x \in S \setminus \{\hat{x}\}$ **(1 p.)**

(c) $\dot{V}(x) \leq 0$ for all $x \in S$ since $a \geq 0$ and $x_2^4 \geq 0$ for all $x_2 \in \mathbb{R}$ **(1 p.)**.

Hence, according to the Lyapunov direct method, the equilibrium $\hat{x}$ is stable.

No, the asymptotic stability cannot be concluded via the direct Lyapunov method because $\dot{V}(x) = 0$ for all $x_1, x_2 \in \mathbb{R}$ when $a = 0$ **(1 p.)**, and for all $(x_1, 0)$ with $x_1 \in \mathbb{R}$ when $a > 0$ **(1 p.)**. Hence $\frac{d}{dt}V(x) < 0$ for all $x \in S \setminus \{\hat{x}\}$ does not hold.

4. When $a > 0$, then $\dot{V}(x(t)) = 0$ for $(x_1, 0)$ with $x_1 \in \mathbb{R}$. Therefore

$$\bar{S} = \{x \in S, \quad x_2 = 0\} \subseteq \mathbb{R}^2 \; \textbf{(1 p.)}.$$

To derive the largest invariant set in $\bar{S}$ we must have a look at the system equations **(1 p.)**. Indeed, from the system equations we see that in order for the trajectories to be confined to the line where $x_2 = 0$ we need $x_1 = 0$ or $x_1 = \pm j\frac{1}{\sqrt{2}}$, or the system would diverge from the $x_2 = 0$ line **(1 p.)**. Therefore, the equilibrium $\hat{x} = (0, 0)$ is the largest invariant set in $\bar{S}$ **(1 p.)** since $\bar{S} \subseteq \mathbb{R}^2$, which excludes the other two values **(1 p.)**.

Since S is compact and invariant and $\dot{V}(x(t)) \leq 0$ for all $x \in S$, we can invoke LaSalle's theorem. According to LaSalle's theorem, all trajectories starting in S tend to $M = \{(0, 0)\}$ for $t \to \infty$. We can therefore conclude that $\hat{x}$ is locally asymptotically stable **(1 p.)**. In addition, since $\epsilon > 0$ can be chosen to be arbitrarily large, we can conclude that $\hat{x}$ is globally asymptotically stable **(1 p.)**.

5. By looking at the trajectories in Fig. 6 we can conclude that when $a = 0$ the equilibrium $\hat{x}$ is stable but not asymptotically stable **(1 p.)** since the trajectories will not converge to it but remain close **(1 p.)**.