On the application of SPH for problems of solid mechanics

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Abstract. We analyze the applicability of the smooth particle hydrodynamics (SPH) to the solution of boundary value problems involving large deformation of solids. The main focus is set on such issues as the reduction of artificial edge effects by implementing corrected kernels and their gradients, accurate and efficient computation of the deformation gradient tensor, evaluation of the internal forces basing on the given stress field. For demonstration purposes hyperelastic body of neo-Hookean type and a visco-elastic body of Maxwell type are considered; the formulation of the Maxwell material is based on the approach of Simo and Miehe (1992). For their implementation, efficient and robust numerical schemes are used. A solution of a test problem is presented. The performance of the implemented algorithms is assessed by checking the preservation of the total energy of the system. The accuracy of the SPH-computations is estimated using nonlinear FEM as a benchmark.

1. Introduction

Smoothed Particle Hydrodynamics (SPH) is a mesh-free simulation method used for boundary value problems of continuum mechanics. In the last decades there has been a large progress in the SPH-simulation of solid dynamics, including both elasticity and inelasticity [6], [8] The advantages of SPH are: in contrast to the FEM, the solution procedure is robust with respect to large distorsions of the body. This make the SPH especially attractive in problems of fracture mechanics and mechanics of multi-phase media. In case of solids, there is a certain similarity between SPH and peridynamics [3]. The drawbacks of SPH are the low accuracy of approximation of discontinuous functions, edge effects, necessity of artificial viscosity to suppress non-physical instabilities. Another disadvantage of SPH lies in a low order of convergence for derived fields like strains and stresses [2]. In [1], different corrected smoothing kernels and corrected kernel gradients are considered to increase the order of approximation of derived variables. More precisely, in [1] a simplified computationally efficient correction of the kernel is considered. Since the simplified correction of the kernel is incomplete, the authors use an additional correction of the kernel gradients. As a byproduct, such a correction allows one to reduce the undesired edge effect. In contrast to [1] and [2], we use a full correction of the smoothing kernel in a combination with a straightforward gradient of the corrected kernel. The well-known $\{\mathbf{F}, \mathbf{p}\}$-approach is implemented here (cf. [2]), whereas the primary discretization quantities are the deformation gradient and the linear momentum. Two different approaches to the computation of the deformation gradient $\mathbf{F}$ can be used. The first one yields the deformation gradient as a straightforward derivative of the displacement with respect to the referential

position vector (see Eq. (8)) and the second one is based on the integration of the evolution equation (see Eq. (9)). The choice between different alternatives depends on the application. The first approach is is free from accumulation of the integration error but the second one is suitable for problems with contact (self-contact). Apart from different methods for $\mathbf{F}$, the SPH-literature contains a big number of ways to compute the vector of internal forces and to stabilize the computations. The goal of the current contribution is to report a suitable combination of various modelling techniques and to demonstrate its applicability to problems of solid mechanics.

Nomenclature

$\boldsymbol{\sigma}$	Cauchy stress tensor
$\boldsymbol{\tau}$	Kirchhoff stress tensor
$\mathbf{1}$	identity tensor
$\mathbf{A}^{\mathrm{D}}$	deviatoric part of a tensor
$\mathbf{B}$	left Cauchy-Green tensor
$\mathbf{C}_i$	inelastic right Cauchy-Green tensor
$\mathbf{F}$	deformation gradient
$\mathbf{v}_b$	particle velocity
$\overline{\mathbf{A}}$	unimodular part of a tensor
ρ	body density
$\mathrm{tr}(\mathbf{A})$	trace of a tensor
$\tilde{\mathbf{T}}$	second Piola-Kirchhoff stress tensor
m	particle mass
U	free-energy
V_b	particle b volume

2. Smoothing kernel

For SPH, it was initially suggested to use the Gaussian kernel [9], but in the follow-up studies the cubic spline [8],[2] was advocated due to a reduced computational effort. In this paper, the following kernel will be taken:

$$W(h,q) = \frac{10}{7\pi h^2}\begin{cases} 1-\frac{3}{2}q^2+\frac{3}{4}q^3, & \text{if } 0\le q\le 1,\\ \frac{1}{4}(2-q)^3, & \text{if } 1\le q\le 2,\\ 0, & \text{else,}\end{cases}$$

where $q = \dfrac{\|\mathbf{x}_a-\mathbf{x}_b\|}{h}$ and $h = \mathrm{k}\cdot\left(\frac{m}{\rho}\right)^{\frac{1}{d}}$. Here, k is a non-dimensional numerical parameter $\mathrm{k}\in[1;1.4]$, d is the dimension of the problems ($d=2$ for plane problems).

2.1. Kernel correction

Liu proposed a modified kernel [1], to ensure that polynomial functions interpolate accurately to a given degree. The corrected kernel $\widetilde{W}_b(\mathbf{x})$ takes the form:

$$\widetilde{W}_b(\mathbf{x}) = W_b(x)\alpha(\mathbf{x})[1+\boldsymbol{\beta}(\mathbf{x})\cdot(\mathbf{x}-\mathbf{x}_b)], \tag{1}$$

$$\boldsymbol{\beta}(\mathbf{x}) = \Big[\sum_b V_b(\mathbf{x}-\mathbf{x}_b)\otimes(\mathbf{x}-\mathbf{x}_b)W_b(\mathbf{x})\Big]^{-1}\sum_b V_b(\mathbf{x}_b-\mathbf{x})W_b(\mathbf{x}), \tag{2}$$

$$\alpha(\mathbf{x}) = \frac{1}{\sum_b V_b[1 + \boldsymbol{\beta}(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_b)]W_b(\mathbf{x})}. \tag{3}$$

The gradient of the corrected kernel $\nabla\widetilde{W}_a(x_b)$ can be approximated by the finite difference

$$(\nabla\widetilde{W}_a(\mathbf{x}_b))_\beta \approx \frac{\widetilde{W}_a(\mathbf{x}_b + dh\ \mathbf{e}_\beta) - \widetilde{W}_a(\mathbf{x}_b)}{dh}, \quad \beta = \overline{1,2}, \quad \mathbf{e}_1 = (1,0), \quad \mathbf{e}_2 = (0,1), \tag{4}$$

where dh is a small real number, $a, b = \overline{1,N}$, N number of particles approximating the body.

3. Smooth Particle Hydrodynamics (SPH) approximation

Following [1], the vector of internal forces $\mathbf{T}$ will be:

$$\mathbf{T}_a = -\sum_b V_b\ \boldsymbol{\sigma}_b\ \nabla\widetilde{W}_{ba}, \tag{5}$$

where a and b are the particle numbers, $\boldsymbol{\sigma}_b$ is the Cauchy (true) stress at the particle b, $\widetilde{W}_{ba} = \widetilde{W}_b(\mathbf{x}_a)$, $\nabla\widetilde{W}_{ba} = \nabla\widetilde{W}_b(\mathbf{x}_a) \neq \nabla\widetilde{W}_{ab}$, V_b is the volume corresponding to particle b. Note that there are alternative expressions for the internal force $\mathbf{T}_a$, cf. equation (59) in [1] and equation (30) in [2].

The velocity gradient at particle a is computed through

$$\mathbf{L}_a = \sum_b V_b\ \mathbf{v}_b \otimes \nabla\widetilde{W}_{ab}. \tag{6}$$

Other ways of computing the velocity gradient can be found in the literature as well (see equation (55) in [1]). The kinetic energy K, the potential energy U, and the total energy E are estimated as

$$K = \frac{1}{2}\ \sum_b \rho V_b \mathbf{v}_b \cdot \mathbf{v}_b, \quad U = \sum_b V_b U_b, \quad E = K + U, \tag{7}$$

where ρV_b is the mass of the particle b; U_b is the potential (free) energy per unit volume in the reference configuration at particle b (see Eq. (12)).

Two different approaches to the computation of the deformation gradient $\mathbf{F}$ can be used. The referential approach employs relation (see [3])

$$\mathbf{F}_a = \sum_b V_b(\mathbf{u}_b - \mathbf{u}_a) \otimes \nabla\widetilde{W}^{(R)}_{ab}\ + \mathbf{1}, \tag{8}$$

where $\mathbf{u}$ is the displacement vector. Its advantage is low computational cost, since the field $\nabla\widetilde{W}^{(R)}_{ab}$ corresponds to the reference configuration and it is calculated just once. Note that instead of using the displacement vector $\mathbf{u}$ one may compute $\mathbf{F}$ as a function of the current position vector $\mathbf{x}$ (see [2]). Still, we prefer using (8) since it provides more accurate results at $t = 0$. However, this referential approach does not account for the change in the topology like the collision/separation and the self-contact. As an alternative, one may calculate the deformation gradient tensor in the current configuration, integrating the evolution equation was presented in [13], and will be

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \quad \Rightarrow \quad \mathbf{F}_a^{n+1} = \exp(\Delta t\ \mathbf{L}_a) \cdot \mathbf{F}_a^n, \quad \Delta t = t_{n+1} - t_n. \tag{9}$$

4. Material models

4.1. Hyperelastic compressible neo-Hookean material

The Jacobian J, the left Cauchy-Green tensor $\mathbf{B}$, and its unimodular part $\overline{\mathbf{B}}$ are defined through

$$J = \det(\mathbf{F}), \quad \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}}, \quad \overline{\mathbf{B}} = J^{-\frac{2}{3}} \cdot \mathbf{B}. \tag{10}$$

In the neo-Hookean material the deviatoric part of the Kirchhoff stress $\boldsymbol{\tau}$ is proportional to the deviatoric part of $\mathbf{B}$. The volumetric part of $\boldsymbol{\tau}$ is a function of J. Here we use the volumetric part proposed in [7]

$$\boldsymbol{\tau} = \mu \overline{\mathbf{B}}^{\mathrm{D}} + \frac{k}{10}(J^5 - J^{-5}) \cdot \mathbf{1}, \quad \boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{\tau}. \tag{11}$$

Here, μ and k are the shear modulus and the bulk modulus, respectively. The corresponding free (potential) energy per unit volume of the reference configuration is given by

$$\psi = \frac{\mu}{2}(\mathrm{tr}\overline{\mathbf{B}} - 3) + \frac{k}{50}(J^5 + J^{-5} - 2). \tag{12}$$

4.2. Maxwell model

Let us recall the formulation of the Maxwell body suggeted by Simo and Miehe in [12]. The model kinematics is based on the multiplicative decomposition of the deformation gradient $\mathbf{F}$ into the elastic part $\mathbf{F}_e$ and the inelastic part $\mathbf{F}_i$; the tensor $\mathbf{F}_i$ gives rise to the inelastic Cauchy-Green tensor $\mathbf{C}_i$:

$$\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_i, \quad \mathbf{C}_i = \mathbf{F}_i^{\mathrm{T}} \cdot \mathbf{F}_i. \tag{13}$$

The tensor $\mathbf{C}_i$ is used to capture the state of the material. Using the same hyperelastic potentials as above, the second Piola-Kirchhoff stress $\tilde{\mathbf{T}}$ is computed by

$$\tilde{\mathbf{T}} = \mu \mathbf{C}^{-1} \cdot (\overline{\mathbf{C}} \cdot \mathbf{C}_i^{-1})^{\mathrm{D}} + \frac{k}{10}(J^5 - J^{-5})\mathbf{C}^{-1}, \quad \overline{\mathbf{C}} = J^{-\frac{2}{3}} \cdot \mathbf{C}. \tag{14}$$

The evolution of the inelastic strain is governed by the ordinary differential equation

$$\dot{\mathbf{C}}_i = \frac{1}{\eta}\big(\mathbf{C}\tilde{\mathbf{T}}\big)^{\mathrm{D}}\mathbf{C}_i = \frac{\mu}{\eta}(\overline{\mathbf{C}} \cdot \mathbf{C}_i^{-1})^{\mathrm{D}}\mathbf{C}_i, \tag{15}$$

where η is a material parameter (Newtonian viscosity). An efficient iteration-free update formula was suggested for this evolution equation in [10]. It reads

$${}^{n+1}\mathbf{C}_i = \overline{{}^{n}\mathbf{C}_i + \frac{\Delta t \mu}{\eta} \cdot {}^{n+1}\overline{\mathbf{C}}}. \tag{16}$$

A more general time-stepping formula was proposed for the Mooney-Rivlin potential in [11]. Finally, the true stress $\boldsymbol{\sigma}$ is computed as

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \tilde{\mathbf{T}} \cdot \mathbf{F}^{T}, \tag{17}$$

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5. Time stepping method

Use explicit integration of the third order Runge Kutta method [4].

Courant condition occurs. Time step $\Delta t = \mathrm{CFL} \cdot \frac{h}{max(c_s, \mathbf{v})}$ [5], where $c_s = \sqrt{(\frac{9 \cdot k \cdot \mu}{3 \cdot k + \mu} + \frac{4/3 \cdot \mu}{\rho})}$ longitudinal wave propagation velocity, CFL number Courant–Friedrichs–Lewy. In this paper, $\mathrm{CFL} = 0.2$

6. Numerical examples

6.1. Trajectory SPH and FEM

Right now the correctness of the calculation algorithm using SPH checks the movement of the square by the trajectory, or more precisely, the movement of its upper right corner.$\mu = 1.25$ $k =$ 68? sph particle 441, 1681. FEM second order approximation elements, 100, implicit euler scheme 400 steps.

6.2. Stress in centre square SPH and FEM

6.3. Energy standard SPH and corrected SPH

6.4. Maxwell Model $\nu = 1, 1/5, 1/25$

7. Conclusion

A working combination of tips was found that allows to obtain acceptable results with medium body deformations. The adjustment method was tested, which consisted in comparisons of SPH and FEM calculations. The method also provides the law of conservation of energy. In the future, it is planned to switch to problems for large deformations, which, in turn, requires the introduction of adoptive smoothing radius. At the moment, the problem of low order of stress convergence stress and strain strain is not solved, but an increase in the approximating particles allows to solve problems. The hyper-model Neo-Hooke trajectory converges to the FEM reference solution. The formulation of the Maxwell material is based on the approach of Simo and Miehe (1992) setting shows results that are consistent with the physical laws ?.

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