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Question 1

a) $\lim_{n\to\infty} \left| \frac{a_n+1}{a_n} \right| = \lim_{n\to\infty} \frac{(-1)^{n+1} \cdot x^{n+1}/(n+1)^2}{(-1)^n \cdot x^n / n^2}$

$= \lim_{n\to\infty} \left| \frac{-x \cdot n^2}{(n+1)^2} \right| = |x|$

For convergence $|x| < 1$

$\Rightarrow$ Radius of convergence is $R=1$ and the point of convergence is $(-1, 1)$.

b) Given the series $\sum_{n=1}^{\infty} \frac{e^n \cdot x^n}{n}$

Applying the concep of the ratio test as in part (A);

$\lim_{n\to\infty} \left| \frac{a_m+1}{a_n} \right| = \lim_{n\to\infty} \left| \frac{e^{n+1} \cdot x^{n+1}/(n+1)}{e^n \cdot x^n / n} \right|$

$= \lim_{n\to\infty} \left| \frac{e \cdot x \cdot n}{n+1} \right| = e \cdot |x|$

For convergence, $e \cdot |x| < 1$ $|x| < 1/e$. Therefor Radius of convergence is $R = 1/e$ and the internal of convergence is $(-\frac{1}{e}, \frac{1}{e})$.

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Question2

Given $f(x) = \frac{x}{1-4x^2}$, using the geometric series

formula:

$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$, where $|u| < 1$

In this case, $u = 4x^2$ so $|u| < 1$ where

$|4x^2| < 1$ which simplifies to $|x| < \frac{1}{2}$

We use the geometric series formular to represent $\frac{1}{1-4x^2}$ as: $\frac{1}{1-4x^2} = \sum_{n=0}^{\infty} (4x^2)^n$

$\Rightarrow F(x) = x \cdot \sum_{n=0}^{\infty} (4x^2)^n$

$= \sum_{n=0}^{\infty} x \cdot (4x^2)^n = \sum_{n=0}^{\infty} 4^n x^{2n+1}$

Thus the power series representation for $f(x)$ is $\sum_{n=0}^{\infty} 4^n x^{2n+1}$, and the internal of convergence is $-\frac{1}{2} < x < \frac{1}{2}$

Question 3

given function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ differentiating with respect to 'x'

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!}$$

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} \quad \Rightarrow f'(x) = f(x).$$

Therefore, the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Satifies the differential equation hence $f(x)$ is the solution.

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Question 4

a) Sum of the series $\sum_{n=1}^{\infty} nx^{n-1}$

The geometric series sum is $\sum_{n=1}^{\infty} x^n = \frac{1}{x-x}$, $\forall |x|<1$
When differentiating both side with respect to x:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\frac{1}{1-x}$$

Then the left side becomes:

$$\sum_{n=1}^{\infty} nx^{n-1}$$

The right side is:

$$\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x^2)}$$ Thus $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{1-x^2}$ $\forall |x|<1$

b) Sum of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$

We know from the previous part (a) that

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \forall |x|<1$$

Substitute $= x = \frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{x}{(1-x)^2} \Big|_{x=\frac{1}{2}}$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = \frac{\frac{1}{2}}{(\frac{1}{2})^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = \frac{1}{2} \times 4 = 2.$$

$$\therefore \sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

c) Sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

We know from a) that:

$$\sum_{n=1}^{\infty} n^2 x^n = x \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right), \quad \forall\ |x| < 1$$

Differentiate $\frac{x}{(1-x^2)}$: let $f(x) = \frac{x}{1-x^2}$

Then $f'(x) = \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right)$

Using the quotient Rule:

$$f'(x) = \frac{(1-x)^2 \cdot 1 - x \cdot 2(1-x)(-1)}{(1-x)^4}$$

Numerator: $(1-x)^2 + 2x(1-x) = 1-2x-x^2+2x-2x^2$
$= 1-x^2$

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Thus: $f'(x) = \dfrac{1-x^2}{(1-x)^4}$

Substituting back into the formula:

$$\sum_{n=1}^{\infty} n^2 x^n = x \cdot \frac{1-x^2}{(1-x)^4} = \frac{x(1-x^2)}{(1-x)^4}$$

Subsitituting $x = \frac{1}{2}$:

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{x(1-x^2)}{(1-x)^4} \Bigg|_{x=\frac{1}{2}}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^2\right)}{\left(1-\frac{1}{2}\right)^4} = \frac{\frac{1}{2}\left(1-\frac{1}{4}\right)}{\left(\frac{1}{2}\right)^4} = \frac{\frac{3}{8}}{\frac{1}{16}}$$

$$= \frac{3}{8} \times \frac{16}{1} = 6$$

$$\therefore \sum_{n=1}^{\infty} n^2 x^n = 6.$$