1.5 HURWITZ STABILITY TEST

We now turn to the problem of left half plane or Hurwitz stability for real polynomials and develop an elementary test procedure for it based on the Interlacing Theorem and therefore on the Boundary Crossing Theorem. This procedure turns out to be equivalent to Routh's well known test.

Let $P(s)$ be a real polynomial of degree n, and assume that all the coefficients of $P(s)$ are positive,

$$P(s) = p_0 + p_1 s + \cdots + p_n s^n, \qquad p_i > 0, \quad \text{for} \quad i = 0, \cdots, n \; .$$

Remember that $P(s)$ can be decomposed into its odd and even parts as

$$P(s) = P^{\text{even}}(s) + P^{\text{odd}}(s).$$

Now, define the polynomial $Q(s)$ of degree $n-1$ by:

$$\begin{aligned} \text{If } n = 2m: \quad Q(s) &= \left[P^{\text{even}}(s) - \frac{p_{2m}}{p_{2m-1}} s P^{\text{odd}}(s) \right] + P^{\text{odd}}(s), \\ \text{If } n = 2m+1: \quad Q(s) &= \left[P^{\text{odd}}(s) - \frac{p_{2m+1}}{p_{2m}} s P^{\text{even}}(s) \right] + P^{\text{even}}(s) \end{aligned} \tag{1.56}$$

that is in general, with $\mu = \frac{p_n}{p_{n-1}}$,

$$Q(s) = p_{n-1}s^{n-1} + (p_{n-2} - \mu p_{n-3})s^{n-2} + p_{n-3}s^{n-3} + (p_{n-4} - \mu p_{n-5})s^{n-4} + \cdots. \tag{1.57}$$

Then the following key result on degree reduction is obtained.

Lemma 1.4 *If $P(s)$ has all its coefficients positive,*

$$P(s) \textit{ is stable } \Longleftrightarrow Q(s) \textit{ is stable.}$$

Proof. Assume, for example that, $n = 2m$, and use the interlacing theorem.

(a) Assume that $P(s) = p_0 + \cdots + p_{2m}s^{2m}$ is stable and therefore satisfies the interlacing theorem. Let

$$0 < \omega_{e,1} < \omega_{o,1} < \omega_{e,2} < \omega_{o,2} < \cdots < \omega_{e,m-1} < \omega_{o,m-1} < \omega_{e,m},$$

be the interlacing roots of $P^e(\omega)$ and $P^o(\omega)$. One can easily check that (1.56) implies that $Q^e(\omega)$ and $Q^o(\omega)$ are given by

$$\begin{aligned} Q^e(\omega) &= P^e(\omega) + \mu\omega^2 P^o(\omega), \quad \mu = \frac{p_{2m}}{p_{2m-1}}, \\ Q^o(\omega) &= P^o(\omega). \end{aligned}$$

From this it is already concluded that $Q^o(\omega)$ has the required number of positive roots, namely the $m-1$ roots of $P^o(\omega)$:

$$\omega_{o,1}, \quad \omega_{o,2}, \quad \cdots, \quad \omega_{o,m-1}.$$

Moreover, due to the form of $Q^e(\omega)$, it can be deduced that,

$$\begin{aligned} Q^e(0) &= P^e(0) > 0, \\ Q^e(\omega_{o,1}) &= P^e(\omega_{o,1}) < 0, \\ &\vdots \\ Q^e(\omega_{o,m-2}) &= P^e(\omega_{o,m-2}), \text{ has the sign of } (-1)^{m-2}, \\ Q^e(\omega_{o,m-1}) &= P^e(\omega_{o,m-1}), \text{ has the sign of } (-1)^{m-1}. \end{aligned}$$

Hence, it is already established that $Q^e(\omega)$ has $m-1$ positive roots $\omega'_{e,1}$, $\omega'_{e,2}$, $\cdots$, $\omega'_{e,m-1}$, that do interlace with the roots of $Q^o(\omega)$. Since moreover $Q^e(\omega)$ is of degree $m-1$ in ω^2, these are the only positive roots it can have. Finally, it has been seen that the sign of $Q^e(\omega)$ at the last root $\omega_{o,m-1}$ of $Q^o(\omega)$ is that of $(-1)^{m-1}$. But the highest coefficient of $Q^e(\omega)$ is nothing but

$$q_{2m-2}(-1)^{m-1}.$$

From this q_{2m-2} must be strictly positive, as $q_{2m-1} = p_{2m-1}$ is, otherwise $Q^e(\omega)$ would again have a change of sign between $\omega_{o,m-1}$ and $+\infty$, which would result in the contradiction of $Q^e(\omega)$ having m positive roots (whereas it is a polynomial of degree only $m-1$ in ω^2). Therefore $Q(s)$ satisfies the interlacing property and is stable if $P(s)$ is.

(b) Conversely assume that $Q(s)$ is stable. Write

$$P(s) = [Q^{\text{even}}(s) + \mu s Q^{\text{odd}}(s)] + Q^{\text{odd}}(s) \quad \mu = \frac{p_{2m}}{p_{2m-1}}.$$

By the same reasoning as in a) it can be seen that $P^o(\omega)$ already has the required number $m-1$ of positive roots, and that $P^e(\omega)$ already has $m-1$ roots in the interval $(0, \omega_{o,m-1})$ that interlace with the roots of $P^o(\omega)$. Moreover the sign of $P^e(\omega)$ at $\omega_{o,m-1}$ is the same as $(-1)^{m-1}$ whereas the term $p_{2m}s^{2m}$ in $P(s)$, makes the sign of $P^e(\omega)$ at $+\infty$ that of $(-1)^m$. Thus $P^e(\omega)$ has an m^{th} positive root,

$$\omega_{e,m} > \omega_{o,m-1},$$

so that $P(s)$ satisfies the interlacing property and is therefore stable.

♣

The above lemma shows how the stability of a polynomial $P(s)$ can be checked by successively reducing its degree as follows.

Algorithm 1.2. (Hurwitz stability for real polynomials)

1) Set $P^{(0)}(s) = P(s)$,

2) Verify that all the coefficients of $P^{(i)}(s)$ are positive,

3) Construct $P^{(i+1)}(s) = Q(s)$ according to (1.57),

4) Go back to 2) until you either find that any 2) is violated ($P(s)$ is not Hurwitz) or until you reach $P^{(n-2)}(s)$ (which is of degree 2) in which case condition 2) is also sufficient ($P(s)$ is Hurwitz).

The reader may verify that this procedure is identical to Routh's test since it generates the Routh table. The proof also shows the known property that for a stable polynomial not only the first column but the entire Routh table must consist only of positive numbers. However the procedure described here does not allow to count the stable and unstable zeroes of the polynomial as can be done with Routh's Theorem.

Example 1.10. Consider a real polynomial of degree 4,

$$P(s) = s^4 + as^3 + bs^2 + cs + d.$$

Following the algorithm above we form the polynomials,

$$\mu = \frac{1}{a}, \text{ and } P^{(1)}(s) = as^3 + \left(b - \frac{c}{a}\right)s^2 + cs + d,$$

and then,

$$\mu = \frac{a^2}{ab-c}, \text{ and } P^{(2)} = \left(b - \frac{c}{a}\right)s^2 + \left(c - \frac{a^2d}{ab-c}\right)s + d.$$

Considering that at each step only the even or the odd part of the polynomial is modified, it is needed to verify the positiveness of the following set of coefficients,

$$\begin{bmatrix} 1 & b & d \\ a & c & \\ b - \dfrac{c}{a} & d & \\ c - \dfrac{a^2d}{ab-c} & & \end{bmatrix}$$

But this is just the Routh table for this polynomial.

Note that a lemma similar to Lemma 1.4 could be derived where the assumption that all the coefficients of $P(s)$ are positive is replaced by the assumption that only the two highest degree coefficients p_{n-1} and p_n are positive. The corresponding algorithm would then exactly lead to checking that the first column of the Routh table is positive. However since the algorithm requires that the entire table be constructed, it is more efficient to check that every new coefficient is positive.

Complex polynomials

A similar algorithm can be derived for checking the Hurwitz stability of complex polynomials. The proof which is very similar to the real case is omitted and a precise description of the algorithm is given below.

Let $P(s)$ be a complex polynomial of degree n,

$$P(s) = (a_0+jb_0)+(a_1+jb_1)s+\cdots+(a_{n-1}+jb_{n-1})s^{n-1}+(a_n+jb_n)s^n, \quad a_n+jb_n \neq 0.$$

Let,

$$T(s) = \frac{1}{a_n + jb_n}P(s).$$

Thus $T(s)$ can be written as,

$$T(s) = (c_0 + jd_0) + (c_1 + jd_1)s + \cdots + (c_{n-1} + jd_{n-1})s^{n-1} + s^n,$$

and notice that,

$$c_{n-1} = \frac{a_{n-1}a_n + b_{n-1}b_n}{a_n^2 + b_n^2}.$$

Assume that $c_{n-1} > 0$, which is a necessary condition for $P(s)$ to be Hurwitz (see Theorem 1.8). As usual write, $T(s) = T_R(s) + T_I(s)$, where

$$\begin{aligned} T_R(s) &= c_0 + jd_1 s + c_2 s^2 + jd_3 s^3 + \cdots \,, \\ T_I(s) &= jd_0 + c_1 s + jd_2 s^2 + c_3 s^3 + \cdots \,. \end{aligned}$$

Now define the polynomial $Q(s)$ of degree $n-1$ by:

$$\begin{aligned} \text{If } n = 2m: \qquad Q(s) &= \left[T_R(s) - \frac{1}{c_{2m-1}} sT_I(s)\right] + T_I(s), \\ \text{If } n = 2m+1: \qquad Q(s) &= \left[T_I(s) - \frac{1}{c_{2m}} sT_R(s)\right] + T_R(s), \end{aligned}$$

that is in general, with $\mu = \frac{1}{c_{n-1}}$,

$$\begin{aligned} Q(s) = {} & [c_{n-1} + j(d_{n-1} - \mu d_{n-2})]s^{n-1} + [(c_{n-2} - \mu c_{n-3}) + jd_{n-2}]s^{n-2} \\ & + [c_{n-3} + j(d_{n-3} - \mu d_{n-4})]s^{n-3} + \cdots. \end{aligned}$$

Now, exactly as in the real case, the following lemma can be proved.

Lemma 1.5 *If $P(s)$ satisfies $a_{n-1}a_n + b_{n-1}b_n > 0$, then*

$$P(s) \text{ is Hurwitz stable} \iff Q(s) \text{ is Hurwitz stable.}$$

The corresponding algorithm is as follows.

Algorithm 1.3. (Hurwitz stability for complex polynomials)

1) Set $P^{(0)}(s) = P(s)$,

2) Verify that the last two coefficients of $P^{(i)}(s)$ satisfy $a_{n-1}^{(i)} a_n^{(i)} + b_{n-1}^{(i)} b_n^{(i)} > 0$,

3) Construct $T^{(i)}(s) = \dfrac{1}{a_n^{(i)} + jb_n^{(i)}} P^{(i)}(s)$,

4) Construct $P^{(i+1)}(s) = Q(s)$ as above,

5) Go back to 2) until you either find that any 2) is violated ($P(s)$ is not Hurwitz) or until you reach $P^{(n-1)}(s)$ (which is of degree 1) in which case condition 2) is also sufficient ($P(s)$ is Hurwitz).

1.6 A FEW COMPLEMENTS

Polynomial functions are analytic in the entire complex plane and thus belong to the so-called class of *entire functions*. It is not straightforward however, to obtain a general version of the Boundary Crossing Theorem that would apply to any family of entire functions. The main reason for this is the fact that a general entire function may have a finite or infinite number of zeros, and the concept of a *degree* is not defined except for polynomials. Similar to Theorem 1.3 for the polynomial case, the following theorem can be considered as a basic result for the analysis of continuous families of entire functions.

Theorem 1.10 *Let A be an open subset of the complex plane C, F a metric space, and f be a complex-valued function continuous in $A \times F$ such that for each α in F, $z \longrightarrow f(z, \alpha)$ is analytic in A. Let also B be an open subset of A whose closure $\overline{B}$ in C is compact and included in A, and let $\alpha_0 \in F$ be such that no zeros of $f(z, \alpha_0)$ belong to the boundary ∂B of B. Then, there exists a neighborhood W of α_0 in F such that,*

1) For all $\alpha \in W$, $f(z, \alpha)$ has no zeros on ∂B.

2) For all $\alpha \in W$, the number (multiplicities included) of zeros of $f(z, \alpha)$ which belong to B is independent of α.

The proof of this theorem is not difficult and is very similar to that of Theorem 1.3. It uses Rouché's Theorem together with the compactness assumption on B. Loosely speaking, the above result states the continuity of the zeros of a parametrized family of analytic functions with respect to the parameter values. However, it only applies to the zeros which are contained in a given compact subset of the complex plane and therefore, in terms of stability, a more detailed analysis is required. In the polynomial case, Theorem 1.3 is used together with a knowledge of the degree of the family to arrive at the Boundary Crossing Theorem. The degree of a polynomial indicates not only the number of its zeros in the complex plane, but also