Chapter 5

INTERVAL POLYNOMIALS: KHARITONOV'S THEOREM

In this chapter we present Kharitonov's Theorem on robust Hurwitz stability of interval polynomials, dealing with both the real and complex cases. This elegant result forms the basis for many of the results, to be developed later in the book, on robustness under parametric uncertainty. We develop an important extremal property of the associated Kharitonov polynomials and give an application of this theorem to state feedback stabilization. An extension of Kharitonov's Theorem to nested families of interval polynomials is described. Robust Schur stability of interval polynomials is also discussed and it is shown that robust stability can be ascertained from that of the upper edges.

5.1 INTRODUCTION

We devote this chapter mainly to a result proved in 1978 by V. L. Kharitonov, regarding the Hurwitz stability of a family of *interval polynomials*. This result was so surprising and elegant that it has been the starting point of a renewed interest in robust control theory with an emphasis on *deterministic bounded parameter perturbations*. It is important therefore that control engineers thoroughly understand both result and proof, and this is why we considerably extend our discussion of this subject.

In the next section we first state and prove Kharitonov's Theorem for real polynomials. We emphasize how this theorem generalizes the Hermite-Biehler Interlacing Theorem which is valid for a single polynomial. This has an appealing frequency domain interpretation in terms of *interlacing of frequency bands*. We then give an interpretation of Kharitonov's Theorem based on the evolution of the complex plane image set of the interval polynomial family. Here again the Boundary Crossing Theorem and the monotonic phase increase property of Hurwitz polynomials (Chapter 1) are the key concepts that are needed to establish the Theorem. This proof gives useful geometric insight into the working of the theorem and shows how the result is related to the Vertex Lemma (Chapter 2). We then state the theorem for the case of an interval family of polynomials with complex coefficients. This proof follows quite naturally from the above interlacing point of view. Next, we develop an important *extremal property* of the Kharitonov polynomials. This property establishes that one of the four points represented by the Kharitonov polynomials is the closest to instability over the entire set of uncertain parameters. The latter result is independent of the norm used to measure the distance between polynomials in the coefficient space. We next give an application of the Kharitonov polynomials to robust state feedback stabilization. Following this we establish that Kharitonov's Theorem can be extended to nested families of interval polynomials which are neither interval or polytopic and in fact includes nonlinear dependence on uncertain parameters. In the last section we consider the robust Schur stability of an interval polynomial family. A stability test for this family is derived based on the upper edges which form a subset of all the exposed edges. We illustrate the application of these fundamental results to control systems with examples.

5.2 KHARITONOV'S THEOREM FOR REAL POLYNOMIALS

In this chapter stable will mean Hurwitz stable unless otherwise stated. Of course, we will say that a set of polynomials is stable if and only if each and every element of the set is a Hurwitz polynomial.

Consider now the set $\mathcal{I}(s)$ of real polynomials of degree n of the form

$$\delta(s) = \delta_o + \delta_1 s + \delta_2 s^2 + \delta_3 s^3 + \delta_4 s^4 + \cdots + \delta_n s^n$$

where the coefficients lie within given ranges,

$$\delta_0 \in [x_0, y_0],\ \delta_1 \in [x_1, y_1],\ \cdots,\ \delta_n \in [x_n, y_n].$$

Write

$$\underline{\delta} := [\delta_0,\ \delta_1,\ \cdots,\ \delta_n]$$

and identify a polynomial $\delta(s)$ with its coefficient vector $\underline{\delta}$. Introduce the hyperrectangle or box of coefficients

$$\mathbf{\Delta} := \left\{ \underline{\delta} : \underline{\delta} \in \mathbb{R}^{n+1}, \quad x_i \le \delta_i \le y_i, \quad i = 0, 1, \cdots, n \right\}. \tag{5.1}$$

We assume that the degree remains invariant over the family, so that $0 \notin [x_n, y_n]$. Such a set of polynomials is called a real *interval* family and we loosely refer to $\mathcal{I}(s)$ as an interval polynomial. Kharitonov's Theorem provides a surprisingly simple necessary and sufficient condition for the Hurwitz stability of the entire family.

Theorem 5.1 (Kharitonov's Theorem)
Every polynomial in the family $\mathcal{I}(s)$ is Hurwitz if and only if the following four extreme polynomials are Hurwitz:

$$K^1(s) = x_o + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4 + x_5 s^5 + y_6 s^6 + \cdots,$$

$$\begin{aligned}
K^2(s) &= x_o + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4 + y_5 s^5 + y_6 s^6 + \cdots, \\
K^3(s) &= y_o + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4 + x_5 s^5 + x_6 s^6 + \cdots, \\
K^4(s) &= y_o + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4 + y_5 s^5 + x_6 s^6 + \cdots.
\end{aligned} \tag{5.2}$$

The box $\mathbf{\Delta}$ and the vertices corresponding to the Kharitonov polynomials are shown in Figure 5.1. The proof that follows allows for the interpretation of Kharitonov's

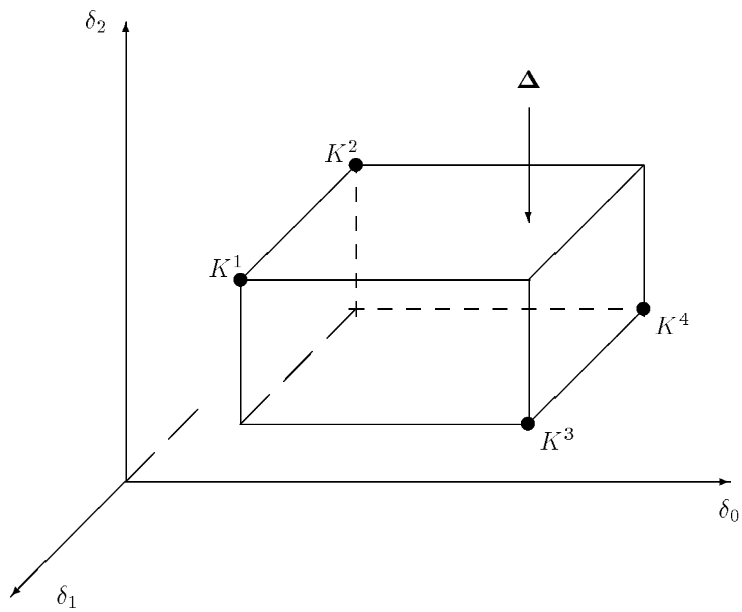

Figure 5.1. The box $\mathbf{\Delta}$ and the four Kharitonov vertices

Theorem as a generalization of the Hermite-Biehler Theorem for Hurwitz polynomials, proved in Theorem 1.7 of Chapter 1. We start by introducing two symmetric lemmas that will lead us naturally to the proof of the theorem.

Lemma 5.1 *Let*

$$\begin{aligned}
P_1(s) &= P^{\text{even}}(s) + P_1^{\text{odd}}(s) \\
P_2(s) &= P^{\text{even}}(s) + P_2^{\text{odd}}(s)
\end{aligned}$$

denote two stable polynomials of the same degree with the same even part $P^{\text{even}}(s)$ *and differing odd parts* $P_1^{\text{odd}}(s)$ *and* $P_2^{\text{odd}}(s)$ *satisfying*

$$P_1^o(\omega) \leq P_2^o(\omega), \qquad \text{for all } \omega \in [0, \infty]. \tag{5.3}$$

Then,

$$P(s) = P^{\text{even}}(s) + P^{\text{odd}}(s)$$

is stable for every polynomial $P(s)$ *with odd part* $P^{\text{odd}}(s)$ *satisfying*

$$P_1^o(\omega) \leq P^o(\omega) \leq P_2^o(\omega), \qquad \text{for all } \ \omega \in [0, \infty]. \tag{5.4}$$

Proof. Since $P_1(s)$ and $P_2(s)$ are stable, $P_1^o(\omega)$ and $P_2^o(\omega)$ both satisfy the interlacing property with $P^e(\omega)$. In particular, $P_1^o(\omega)$ and $P_2^o(\omega)$ are not only of the same degree, but the sign of their highest coefficient is also the same since it is in fact the same as that of the highest coefficient of $P^e(\omega)$. Given this it is easy to see that $P^o(\omega)$ cannot satisfy (5.4) unless it also has this same degree and the same sign for its highest coefficient. Then, the condition in (5.4) forces the roots of $P^o(\omega)$ to interlace with those of $P^e(\omega)$. Therefore, according to the Hermite-Biehler Theorem (Theorem 1.7, Chapter 1), $P^{\text{even}}(s) + P^{\text{odd}}(s)$ is stable. ♣

We remark that Lemma 5.1 as well as its dual, Lemma 5.2 given below, are special cases of the Vertex Lemma, developed in Chapter 2 and follow immediately from it. We illustrate Lemma 5.1 in the example below (see Figure 5.2).

Example 5.1. Let

$$\begin{aligned} P_1(s) &= s^7 + 9s^6 + 31s^5 + 71s^4 + 111s^3 + 109s^2 + 76s + 12 \\ P_2(s) &= s^7 + 9s^6 + 34s^5 + 71s^4 + 111s^3 + 109s^2 + 83s + 12. \end{aligned}$$

Then

$$\begin{aligned} P^{\text{even}}(s) &= 9s^6 + 71s^4 + 109s^2 + 12 \\ P_1^{\text{odd}}(s) &= s^7 + 31s^5 + 111s^3 + 76s \\ P_2^{\text{odd}}(s) &= s^7 + 34s^5 + 111s^3 + 83s. \end{aligned}$$

Figure 5.2 shows that $P^e(\omega)$ and the tube bounded by $P_1^o(\omega)$ and $P_2^o(\omega)$ satisfy the interlacing property.

Thus, we conclude that every polynomial $P(s)$ with odd part $P^{\text{odd}}(s)$ satisfying

$$P_1^o(\omega) \leq P^o(\omega) \leq P_2^o(\omega), \qquad \text{for all } \ \omega \in [0, \infty]$$

is stable. For example, the dotted line shown inside the tube represents

$$P^{\text{odd}}(s) = s^7 + 32s^5 + 111s^3 + 79s.$$

The dual of Lemma 5.1 is:

Lemma 5.2 *Let*

$$\begin{aligned} P_1(s) &= P_1^{\text{even}}(s) + P^{\text{odd}}(s) \\ P_2(s) &= P_2^{\text{even}}(s) + P^{\text{odd}}(s) \end{aligned}$$

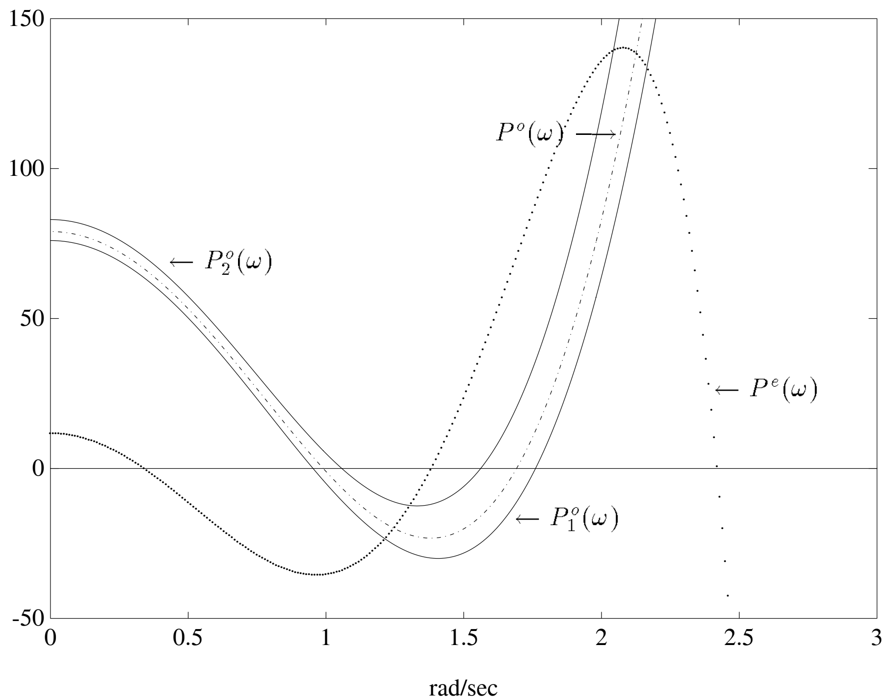

Figure 5.2. $P^e(\omega)$ and $(P_1^o(\omega), P_2^o(\omega))$ (Example 5.1)

denote two stable polynomials of the same degree with the same odd part $P^{\text{odd}}(s)$ and differing even parts $P_1^{\text{even}}(s)$ and $P_2^{\text{even}}(s)$ satisfying

$$P_1^e(\omega) \leq P_2^e(\omega), \qquad \text{for all } \ \omega \in [0, \ \infty]. \tag{5.5}$$

Then,

$$P(s) = P^{\text{even}}(s) + P^{\text{odd}}(s)$$

is stable for every polynomial $P(s)$ with even part $P^{\text{even}}(s)$ satisfying

$$P_1^e(\omega) \leq P^e(\omega) \leq P_2^e(\omega), \qquad \text{for all } \ \omega \in [0, \ \infty]. \tag{5.6}$$

We are now ready to prove Kharitonov's Theorem.

Proof of Kharitonov's Theorem The Kharitonov polynomials repeated below, for convenience are four specific vertices of the box $\mathbf{\Delta}$:

$$K^1(s) = x_o + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4 + x_5 s^5 + y_6 s^6 + \cdots,$$
$$K^2(s) = x_o + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4 + y_5 s^5 + y_6 s^6 + \cdots,$$

$$\begin{aligned} K^3(s) &= y_o + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4 + x_5 s^5 + x_6 s^6 + \cdots, \\ K^4(s) &= y_o + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4 + y_5 s^5 + x_6 s^6 + \cdots. \end{aligned} \tag{5.7}$$

These polynomials are built from two different even parts $K_{\max}^{\text{even}}(s)$ and $K_{\min}^{\text{even}}(s)$ and two different odd parts $K_{\max}^{\text{odd}}(s)$ and $K_{\min}^{\text{odd}}(s)$ defined below:

$$\begin{aligned} K_{\max}^{\text{even}}(s) &:= y_o + x_2 s^2 + y_4 s^4 + x_6 s^6 + y_8 s^8 + \cdots, \\ K_{\min}^{\text{even}}(s) &:= x_o + y_2 s^2 + x_4 s^4 + y_6 s^6 + x_8 s^8 + \cdots, \end{aligned}$$

and

$$\begin{aligned} K_{\max}^{\text{odd}}(s) &:= y_1 s + x_3 s^3 + y_5 s^5 + x_7 s^7 + y_9 s^9 + \cdots, \\ K_{\min}^{\text{odd}}(s) &:= x_1 s + y_3 s^3 + x_5 s^5 + y_7 s^7 + x_9 s^9 + \cdots. \end{aligned}$$

The Kharitonov polynomials in (5.2) or (5.7) can be rewritten as:

$$\begin{aligned} K^1(s) &= K_{\min}^{\text{even}}(s) + K_{\min}^{\text{odd}}(s), \\ K^2(s) &= K_{\min}^{\text{even}}(s) + K_{\max}^{\text{odd}}(s), \\ K^3(s) &= K_{\max}^{\text{even}}(s) + K_{\min}^{\text{odd}}(s), \\ K^4(s) &= K_{\max}^{\text{even}}(s) + K_{\max}^{\text{odd}}(s). \end{aligned} \tag{5.8}$$

The motivation for the subscripts "max" and "min" is as follows. Let $\delta(s)$ be an arbitrary polynomial with its coefficients lying in the box $\mathbf{\Delta}$ and let $\delta^{\text{even}}(s)$ be its even part. Then

$$\begin{aligned} K_{\max}^{e}(\omega) &= y_0 - x_2\omega^2 + y_4\omega^4 - x_6\omega^6 + y_8\omega^8 + \cdots, \\ \delta^e(\omega) &= \delta_0 - \delta_2\omega^2 + \delta_4\omega^4 - \delta_6\omega^6 + \delta_8\omega^8 + \cdots, \\ K_{\min}^{e}(\omega) &= x_0 - y_2\omega^2 + x_4\omega^4 - y_6\omega^6 + x_8\omega^8 + \cdots, \end{aligned}$$

so that

$$K_{\max}^{e}(\omega) - \delta^e(\omega) = (y_0 - \delta_0) + (\delta_2 - x_2)\omega^2 + (y_4 - \delta_4)w^4 + (\delta_6 - x_6)\omega^6 + \cdots,$$

and

$$\delta^e(\omega) - K_{\min}^{e}(\omega) = (\delta_0 - x_0) + (y_2 - \delta_2)\omega^2 + (\delta_4 - x_4)\omega^4 + (y_6 - \delta_6)\omega^6 + \cdots.$$

Therefore,

$$K_{\min}^{e}(\omega) \le \delta^e(\omega) \le K_{\max}^{e}(\omega), \qquad \text{for all } \ \omega \in [0, \infty]. \tag{5.9}$$

Similarly, if $\delta^{\text{odd}}(s)$ denotes the odd part of $\delta(s)$, and $\delta^{\text{odd}}(j\omega) = j\omega\delta^o(\omega)$ it can be verified that

$$K_{\min}^{o}(\omega) \le \delta^o(\omega) \le K_{max}^{o}(\omega), \qquad \text{for all } \ \omega \in [0, \infty]. \tag{5.10}$$

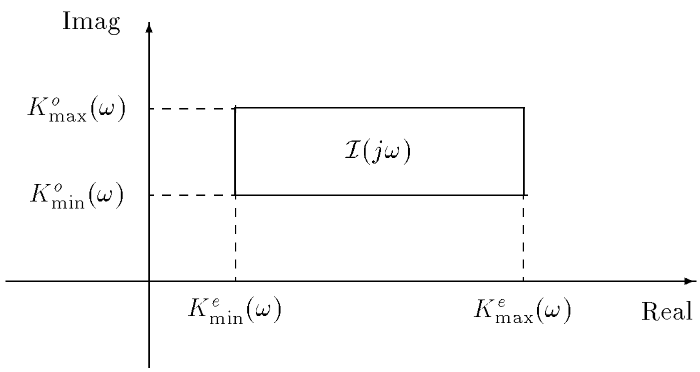

Figure 5.3. Axis parallel rectangle $\mathcal{I}(j\omega)$

Thus $\delta(j\omega)$ lies in an axis parallel rectangle $\mathcal{I}(j\omega)$ as shown in Figure 5.3.

To proceed with the proof of Kharitonov's Theorem we note that necessity of the condition is trivial since if all the polynomials with coefficients in the box $\mathbf{\Delta}$ are stable, it is clear that the Kharitonov polynomials must also be stable since their coefficients lie in $\mathbf{\Delta}$. For the converse, assume that the Kharitonov polynomials are stable, and let $\delta(s) = \delta^{\text{even}}(s) + \delta^{\text{odd}}(s)$ be an *arbitrary* polynomial belonging to the family $\mathcal{I}(s)$, with even part $\delta^{\text{even}}(s)$ and odd part $\delta^{\text{odd}}(s)$.

We conclude, from Lemma 5.1 applied to the stable polynomials $K^3(s)$ and $K^4(s)$ in (5.8), that

$$K^{\text{even}}_{\max}(s) + \delta^{\text{odd}}(s) \quad \text{is stable.} \tag{5.11}$$

Similarly, from Lemma 5.1 applied to the stable polynomials $K^1(s)$ and $K^2(s)$ in (5.8) we conclude that

$$K^{\text{even}}_{\min}(s) + \delta^{\text{odd}}(s) \quad \text{is stable.} \tag{5.12}$$

Now, since (5.9) holds, we can apply Lemma 5.2 to the two stable polynomials

$$K^{\text{even}}_{\max}(s) + \delta^{\text{odd}}(s) \quad \text{and} \quad K^{\text{even}}_{\min}(s) + \delta^{\text{odd}}(s)$$

to conclude that

$$\delta^{\text{even}}(s) + \delta^{\text{odd}}(s) = \delta(s) \quad \text{is stable.}$$

Since $\delta(s)$ was an arbitrary polynomial of $\mathcal{I}(s)$ we conclude that the entire family of polynomials $\mathcal{I}(s)$ is stable and this concludes the proof of the theorem. ♣

Remark 5.1. The Kharitonov polynomials can also be written with the highest order coefficient as the first term:

$$\hat{K}^1(s) = x_n s^n + y_{n-1}s^{n-1} + y_{n-2}s^{n-2} + x_{n-3}s^{n-3} + x_{n-4}s^{n-4} + \cdots,$$

$$
\begin{aligned}
\hat{K}^2(s) &= x_n s^n + x_{n-1}s^{n-1} + y_{n-2}s^{n-2} + y_{n-3}s^{n-3} + x_{n-4}s^{n-4} + \cdots, \\
\hat{K}^3(s) &= y_n s^n + x_{n-1}s^{n-1} + x_{n-2}s^{n-2} + y_{n-3}s^{n-3} + y_{n-4}s^{n-4} + \cdots, \quad (5.13) \\
\hat{K}^4(s) &= y_n s^n + y_{n-1}s^{n-1} + x_{n-2}s^{n-2} + x_{n-3}s^{n-3} + y_{n-4}s^{n-4} + \cdots.
\end{aligned}
$$

Remark 5.2. The assumption regarding invariant degree of the interval family can be relaxed. In this case some additional polynomials need to be tested for stability. This is dealt with in Exercise 5.13.

Remark 5.3. The assumption inherent in Kharitonov's Theorem that the coefficients perturb independently is crucial to the working of the theorem. In the examples below we have constructed some control problems where this assumption is satisfied. Obviously in many real world problems this assumption would fail to hold, since the characteristic polynomial coefficients would perturb interdependently through other primary parameters. However even in these cases Kharitonov's Theorem can give useful and computationally simple answers by overbounding the actual perturbations by an axis parallel box $\mathbf{\Delta}$ in coefficient space.

Remark 5.4. As remarked above Kharitonov's Theorem would give conservative results when the characteristic polynomial coefficients perturb interdependently. The Edge Theorem and the Generalized Kharitonov Theorem described in Chapters 6 and 7 respectively were developed precisely to deal nonconservatively with such dependencies.

Example 5.2. Consider the problem of checking the robust stability of the feedback system shown in Figure 5.4.

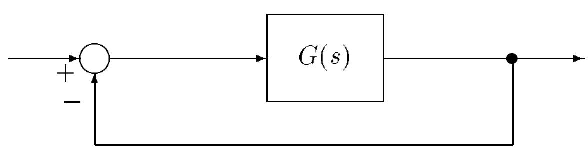

Figure 5.4. Feedback system (Example 5.2)

The plant transfer function is

$$G(s) = \frac{\delta_1 s + \delta_0}{s^2(\delta_4 s^2 + \delta_3 s^3 + \delta_2)}$$

with coefficients being bounded as

$$\delta_4 \in [x_4, y_4], \quad \delta_3 \in [x_3, y_3], \quad \delta_2 \in [x_2, y_2], \quad \delta_1 \in [x_1, y_1], \quad \delta_0 \in [x_0, y_0].$$

The characteristic polynomial of the family is written as

$$\delta(s) = \delta_4 s^4 + \delta_3 s^3 + \delta_2 s^2 + \delta_1 s + \delta_0.$$

The associated even and odd polynomials for Kharitonov's test are as follows:

$$K_{\min}^{\text{even}}(s) = x_0 + y_2 s^2 + x_4 s^4, \qquad K_{\max}^{\text{even}}(s) = y_0 + x_2 s^2 + y_4 s^4,$$
$$K_{\min}^{\text{odd}}(s) = x_1 s + y_3 s^3, \qquad K_{\max}^{\text{odd}}(s) = y_1 s + x_3 s^3.$$

The Kharitonov polynomials are:

$$K^1(s) = x_0 + x_1 s + y_2 s^2 + y_3 s^3 + x_4 s^4, \qquad K^2(s) = x_0 + y_1 s + y_2 s^2 + x_3 s^3 + x_4 s^4,$$
$$K^3(s) = y_0 + x_1 s + x_2 s^2 + y_3 s^3 + y_4 s^4, \qquad K^4(s) = y_0 + y_1 s + x_2 s^2 + x_3 s^3 + y_4 s^4.$$

The problem of checking the Hurwitz stability of the family therefore is reduced to that of checking the Hurwitz stability of these four polynomials. This in turn reduces to checking that the coefficients have the same sign (positive, say; otherwise multiply $\delta(s)$ by -1) and that the following inequalities hold:

$$\begin{array}{lll}
K^1(s) \quad \text{Hurwitz} : y_2 y_3 > x_1 x_4, & & x_1 y_2 y_3 > x_1^2 x_4 + y_3^2 x_0, \\
K^2(s) \quad \text{Hurwitz} : y_2 x_3 > y_1 x_4, & & y_1 y_2 x_3 > y_1^2 x_4 + x_3^2 x_0, \\
K^3(s) \quad \text{Hurwitz} : x_2 y_3 > x_1 y_4, & & x_1 x_2 y_3 > x_1^2 y_4 + y_3^2 y_0, \\
K^4(s) \quad \text{Hurwitz} : x_2 x_3 > y_1 y_4, & & y_1 x_2 x_3 > y_1^2 y_4 + x_3^2 y_0.
\end{array}$$

Example 5.3. Consider the control system shown in Figure 5.5.

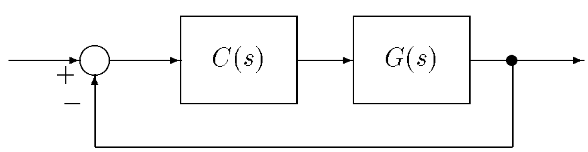

Figure 5.5. Feedback system with controller (Example 5.3)

The plant is described by the rational transfer function $G(s)$ with numerator and denominator coefficients varying independently in prescribed intervals. We refer to such a family of transfer functions $\mathbf{G}(s)$ as an *interval plant*. In the present example we take

$$\mathbf{G}(s) := \left\{ G(s) = \frac{n_2 s^2 + n_1 s + n_0}{s^3 + d_2 s^2 + d_1 s + d_0} : \begin{array}{l} n_0 \in [1, 2.5],\ n_1 \in [1, 6], \qquad n_2 \in [1, 7], \\ d_2 \in [-1, 1],\ d_1 \in [-0.5, 1.5],\ d_0 \in [1, 1.5] \end{array} \right\}.$$

The controller is a constant gain, $C(s) = k$ that is to be adjusted, if possible, to robustly stabilize the closed loop system. More precisely we are interested in determining the range of values of the gain $k \in [-\infty, +\infty]$ for which the closed loop system is robustly stable, i.e. stable for all $G(s) \in \mathbf{G}(s)$.

The characteristic polynomial of the closed loop system is:

$$\delta(k, s) = s^3 + \underbrace{(d_2 + kn_2)}_{\delta_2(k)} s^2 + \underbrace{(d_1 + kn_1)}_{\delta_1(k)} s + \underbrace{(d_0 + kn_0)}_{\delta_0(k)}.$$

Since the parameters d_i, n_j, $i = 0, 1, 2$, $j = 0, 1, 2$ vary independently it follows that for each fixed k, $\delta(k, s)$ is an interval polynomial. Using the bounds given to describe the family $\mathbf{G}(s)$ we get the following coefficient bounds for positive k:

$$\begin{aligned} \delta_2(k) &\in [-1 + k, 1 + 7k], \\ \delta_1(k) &\in [-0.5 + k, 1.5 + 6k], \\ \delta_0(k) &\in [-1 + k, 1.5 + 2.5k]. \end{aligned}$$

Since the leading coefficient is $+1$ the remaining coefficients must be all positive for the polynomial to be Hurwitz. This leads to the constraints:

(a) $-1 + k > 0, \quad -0.5 + k > 0, \quad 1 + k > 0.$

From Kharitonov's Theorem applied to third order interval polynomials it can be easily shown that to ascertain the Hurwitz stability of the entire family it suffices to check in addition to positivity of the coefficients, the Hurwitz stability of only the third Kharitonov polynomial $K_3(s)$. In this example we therefore have that the entire system is Hurwitz if and only if in addition to the above constraints (a) we have:

$$(-0.5 + k)(-1 + k) > 1.5 + 2.5k.$$

From this it follows that the closed loop system is robustly stabilized if and only if

$$k \in (2 + \sqrt{5}, +\infty].$$

To complete our treatment of this important result we give Kharitonov's Theorem for polynomials with complex coefficients in the next section.

5.3 KHARITONOV'S THEOREM FOR COMPLEX POLYNOMIALS

Consider the set $\mathcal{I}^*(s)$ of all complex polynomials of the form,

$$\delta(s) = (\alpha_0 + j\beta_0) + (\alpha_1 + j\beta_1)s + \cdots + (\alpha_n + j\beta_n)s^n \tag{5.14}$$

with

$$\alpha_0 \in [x_0, y_0], \quad \alpha_1 \in [x_1, y_1], \quad \cdots, \quad \alpha_n \in [x_n, y_n] \tag{5.15}$$