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15 December 2022

# Tutorial on Hypothesis Test

The purpose of this exercise is to design a statistical test to discover a signal process by counting events in a detector. Suppose the detector can for each event measure a quantity $x$ with $0 \leq x \leq 1$, for which probability density functions (pdfs) are for signal ($s$) and background ($b$),

$$f(x|s) = 3(1-x)^2 , \quad (1)$$

$$f(x|b) = 3x^2 . \quad (2)$$

**1(a)** Suppose for each event we test the hypothesis that it is background. We reject this hypothesis if the observed value of $x$ is less than a specified cut value $x_{\rm cut}$. Find the value of $x_{\rm cut}$ such that the probability to reject the background hypothesis (i.e., accept as signal) if it is background is $\alpha = 0.05$. (The value $\alpha$ is the *size* or significance level of the test.)

**1(b)** For the value of $x_{\rm cut}$ that you find, what is the probability to reject the background hypothesis (i.e., accept as a candidate signal event) with $x < x_{\rm cut}$ given that it is signal. (This is the *power* of the test of the background hypothesis with respect to the signal alternative or equivalently the signal efficiency.)

**1(c)** Suppose that the expected number of background events is $b_{\rm tot} = 100$ and for a given signal model one expects $s_{\rm tot} = 10$ signal events. Find the expected numbers of events $s$ and $b$ of signal and background events that will satisfy $x < x_{\rm cut}$ using the value of $x_{\rm cut} = 0.1$.

**1(d)** Assuming the numbers from 1(c), the prior probabilities for an event to be signal or background are

$$\pi_s = \frac{s_{\rm tot}}{s_{\rm tot} + b_{\rm tot}} = 0.09 , \quad (3)$$

$$\pi_b = \frac{b_{\rm tot}}{s_{\rm tot} + b_{\rm tot}} = 0.91 . \quad (4)$$

Based on these values, what is the probability for an event to be signal given that one finds $x < x_{\rm cut}$. (Recall Bayes' theorem or consult `arXiv:1307.2487`.)

**1(e)** Now suppose we do the experiment and observe $n_{\rm obs}$ events in the search region $x < x_{\rm cut}$. We now want to test the hypothesis that $s = 0$ (the background-only hypothesis or "$b$"), against the alternative that signal is present with $s \neq 0$ (the "$s + b$" hypothesis).

The actual number of events $n$ found in the experiment with $x < x_{\rm cut}$ can be modeled as following a Poisson distribution with a mean value of $s + b$. That is, the probability to find $n$ events is

$$P(n|s,b) = \frac{(s+b)^n}{n!} e^{-(s+b)} . \quad (5)$$

Suppose for a certain $x_{\rm cut}$ one has $b = 0.5$ and we find there $n_{\rm obs} = 3$ events. The $p$-value of the background-only hypothesis is the probability, assuming $s = 0$, to find $n \geq n_{\rm obs}$.

$$p = P(n \geq n_{\rm obs}|s=0,b) = \sum_{n=n_{\rm obs}}^{\infty} \frac{b^n}{n!}e^{-b} = 1 - \sum_{n=0}^{n_{\rm obs}-1} \frac{b^n}{n!}e^{-b} \; . \tag{6}$$

Find the $p$-value and from this find the *significance* with which one can reject the $s = 0$ hypothesis, defined as

$$Z = \Phi^{-1}(1-p) \; , \tag{7}$$

where $\Phi$ is the standard cumulative Gaussian distribution and $\Phi^{-1}$ is its inverse (the standard Gaussian quantile). For more information see Sec. 10 of `arXiv:1307.2487`. You will need the cumulative chi-square distribution and the quantile of the Gaussian distribution, which from ROOT are available as `1 - TMath::Prob` and `TMath::NormQuantile`.

**1(f)** The expected (median) significance assuming the $s+b$ hypothesis of the test of the $s = 0$ hypothesis is a measure of sensitivity and this is what one tries to maximize when designing an experiment. It can be approximated with a number of different formulas. For $s \ll b$ one can use $\mathrm{med}[Z_b|s+b] = s/\sqrt{b}$. If $s \ll b$ does not hold, a better approximation is

$$\mathrm{med}[Z_b|s+b] = \sqrt{2\left((s+b)\ln\left(1+\frac{s}{b}\right) - s\right)} \; . \tag{8}$$

Using Eq. (8), find me median significance for $x_{\rm cut} = 0.1$. If you have time, try to write a program to find the value of $x_{\rm cut}$ that maximizes the median significance.

**1(g)** Now suppose that for each event we do not simply count the events having $x$ in a certain region but we design a test that exploits each measured value in the entire range $0 \leq x \leq 1$. The data thus consist of the number $n$ of events, which follows a Poisson distribution with mean of $s + b$, and the $n$ values $x_1, \ldots, x_n$.

We can define a test statistic to test the background-only hypothesis that is a monotonic function of the likelihood ratio $L_{s+b}/L_b$,

$$q = -2\ln\frac{L_{s+b}}{L_b} = -2\sum_{i=1}^{n} \ln\left[1 + \frac{s_{\rm tot}}{b_{\rm tot}}\frac{f(x_i|s)}{f(x_i|b)}\right] + C \; , \tag{9}$$

where $C$ is a constant that can be dropped. The motivation for this statistic is described further in Sec. 5.1 of `arXiv:1307.2487`.

From `http://www.pp.rhul.ac.uk/~cowan/stat/invisibles/mc/` download the program `invisibleMC.cc` and the makefile. Build and run the program. This will produce histograms of $q$ under the $s+b$ hypothesis, and also a histogram of $q$ (called `h_q_sb`) and it will find the median $q$, $\mathrm{med}[q|s+b]$.

You should add code in analogy with this that generates data according to the background-only ($s = 0$) hypothesis. Generate $10^7$ experiments and count how many have $q < \mathrm{med}[q|s+b]$. The fraction with $q < \mathrm{med}[q|s+b]$ is the median $p$-value of the background-only hypothesis. Find this and from it find the median significance $Z$ (the sensitivity). Compare to the values you found from Eq. (8).