MAT1002 Lab Experiment No. 9

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Aim: Stability Analysis of Difference Equation Systems using z−transform

z−transform:

• The z−transform of a sequence x_n is given by : $Zx_n = X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$

• The inverse Z−transform is : $x_n = Z^{-1}\{X(z)\}$

• The z−transform maps a discrete sequence xn from a sample domain [n] into a complex plane z.

Transfer Function:

• Consider the generalised input-output difference equation

$$(a_0E^N + a_1E^{N-1} + \ldots + a_{N-1}E_1 + a_N)x_n = \\ (b_0E^M + b_1E^{M-1} + \ldots + n_{M-1}E_1 + a_M)u_n$$

• Taking z−transform on both sides of the above difference equation, the given difference equations transforms into a transfer function, also known as the Pulse Transfer Function H(z), which is the ratio of the z−tranform of the state-response $\{x_n\}$ to the z−transform of the input $\{u_n\}$ when all initial conditions are zero. The transfer function can be expressed mathematically as :

$$H(z) = \frac{b_0 + b_1z^{-1} + b_2z^{-2} + \ldots + b_Mz^{-M}}{a_0 + a_1z^{-1} + a_2z^{-2} + \ldots + a_Nz^{-N}} \\ = z^{N-M}\frac{b_0z^M + b_1z^{M-1} + b_2z^{M-2} + \ldots + b_M}{a_0z^N + a_1z^{N-1} + a_2z^{N-2} + \ldots + a_N}$$

Poles, Zeros, and Gain:

• If the numerator and denominator of the proper form of a transfer function are factored, – the roots of the numerator polynomial are called zeros – the roots of the denominator polynomial are called poles – the leading coefficient in the numerator is called gain.
• A system is said to be stable if all of its poles lie inside of a unit circle on the z−plane.

Finding Poles, Zeros and Gain using MATLAB:

• The MATLAB provides a number of functions to assist int the zero-pole-gain analysis of the system.

• The command [z p k] = tf2zpk(b,a) finds the zeros z, poles p, and gain k from the coefficients b and a of the transfer function H(z).

• The command zplane(b,a) plots the zeros and poles of the system along with a reference unit circle. A o is used for zeros and and x is used for poles. If all the x lies inside the unit circle, then the given system is stable.

Example:

Consider the difference equation given by $x_n + 02.x_{n-2} = u_n + 2u_{n-1}$ with all zero initial conditions.

Applying the z−transform on both sides, we obtain $X(z) + 0.2z^{-2}X(z) = U(z) + 2z^{-1}U(z).$

Thus we have, $H(z) = \dfrac{X(z)}{U(z)} = \dfrac{1+2z^{-1}}{1+0.2z^{-2}} = \dfrac{z^2+2z}{z^2+0.2}$

MATLAB Implementation :

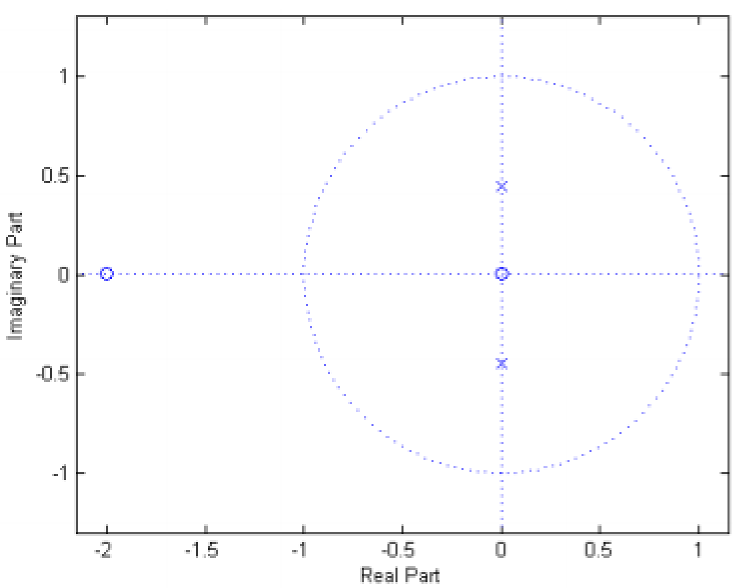
The MATLAB representation of the above transfer function is

```
num = [1 2 0];
den = [1 0 0.2];
```

The zero-pole-gain analysis and plotting of zeros-poles of the system can be done by using the commands

```
[z p k] = tf2zpk(num,den)
zplane(z,p)
```

MATLAB Output :



Inverse z−transform:

• Inverse z−transform can be found using three techniques:

– Partial Fraction Method

– Power Series/Long Division Method

• We can use the MATLAB built-in functions to find the inverse z−transform by using the both of the above methods.

Partial Fraction Method **:**

Consider the transfer function $H(z) = \dfrac{18z^3}{18z^3 + 3z^2 - 4z - 1}$.

To find the partial fraction expansion of H(z) we use MATLAB's bulit-in function `residuez(num, den)`

```
num = [18 0 0 0];
den = [18 3 -4 -1];
[r,p,k] = residuez(num,den);
disp('Residues'); disp(r');
disp('Poles'); disp(p');
disp('Constants'); disp(k)
```

OUTPUT:

```
Residues
0.3600   0.2400   0.4000

Poles
0.5000  -0.3333  -0.3333

Constants
0
```

The result of the program can be read as:

$$H(z) = \frac{0.36}{1 - 0.5z^{-1}} + \frac{0.24}{1 + 0.3333z^{-1}} + \frac{0.4}{(1 + 0.3333z^{-1})^2}$$

Power Series/Long Division Method: To find the z−transform by power series/long division method, one may use the built-in MATLAB function filter . To illustrate this, consider the transfer function :

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}.$$

MATLAB Code :

```
L = 11;
num = [1 2];
den = [1 0.4 -0.12];
u = [1 zeros(1,L-1)];
x = filter(num,den,u);
disp('Coefficients of the power series expansion: ');
disp(x);
```

MATLAB Output:

```
Coefficients of the power series expansion:

        Columns 1 through 10

1.0000  1.6000  -0.5200  0.4000  -0.2224  0.1370  -0.0815
0.0490 -0.0294 0.0176

Column 11

        -0.0106
```

<u>Impulse and Step Responses</u> :

• In electronic engineering and control theory, step response is the time behaviour of the outputs of a general system when its inputs change from zero to one in a very short time.

• In signal processing, the impulse response, of a dynamic system is its output when presented with a brief input signal, called an impulse.

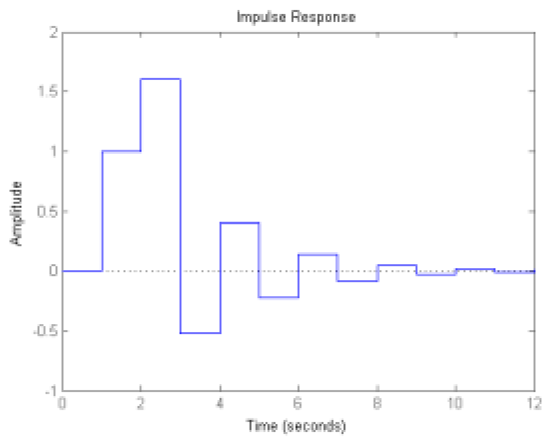
• If the transfer function of a system is given by H(z), then the impulse response of a system is given by h(n) where h(n) is the inverse z−transform of H(z).

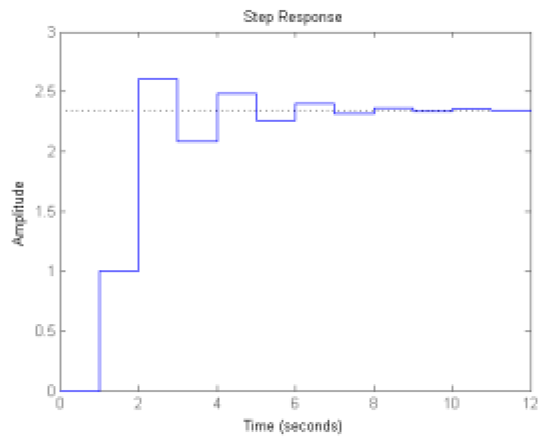
Impulse and Step Responses using MATLAB:

The impulse response response and unit step response of difference equation represented by the transfer function H(z), can be plotted using MATLAB with the help of the command `dimpulse(num,den,nooftimpesteps)` (for impulse response) and `dstep(num,den,nooftimesteps)` (for unit step response). For the previous transfer function the impulse response and step response for 13 time steps can be plotted using

```
figure; dimpulse(num,den,13);
figure; dstep(num,den,13)
```

MATLAB Output :





Aim : Stability Analysis of Motor Driven Gear Train

Theory :

• A gear train is a mechanical system formed by mounting gears on a frame so that the teeth of the gears engage.

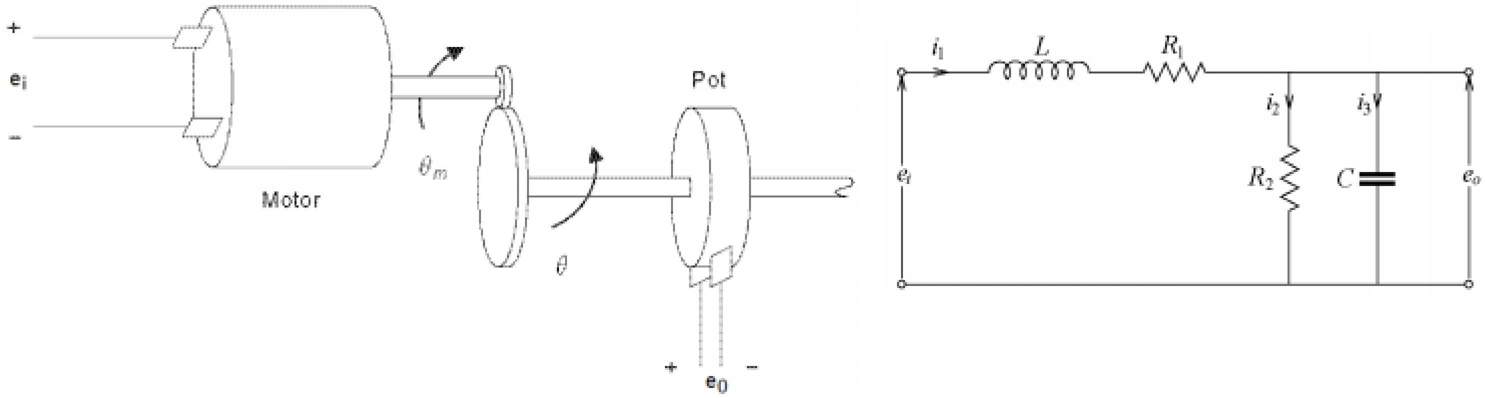
• Gear train system occurs in many of the automobile drivetrains, which generally have two or more major areas where gearing is used.

• Gearing is employed in the transmission, which contains a number of different sets of gears that can be changed to allow a wide range of vehicle speeds.

• Gearing is also used in the differential, which facilitates the splitting torque equally between two wheels while permitting them to have different speeds when travelling in a curved path.

Mathematical Formulation :

Consider a motor driving a rotary load through a gear train and the simple RLC circuit representing the motor connection, as shown in below figure



Applying the basic circuit laws for voltage and currents, we obtain

$$e_i(t) = L\frac{di_1(t)}{dt} + R_1 i_1(t) + e_0(t) \;\ldots\ldots\ldots\ldots\ldots\ldots..\; (1)$$

and

$$e_0(t) = R_2 i_2(t) = \frac{1}{C}\int_0^t i_3(\tau)d\tau + v_c(0) \implies i_3(t) = C\frac{de_0(t)}{dt} \;\ldots\ldots\ldots(2)$$

But, we have

$$i_1(t) = i_2(t) + i_3(t) = \frac{1}{R_2}e_0(t) + C\frac{de_0(t)}{dt} \qquad \ldots\ldots\ldots\ldots\; (3)$$

From which, we obtain the following second order differential equation, relating input and output of the system, representing mathematical model of the circuit given in the figure

$$\frac{d^2e_0(t)}{dt^2} + \left(\frac{L + R_1R_2C}{R_2LC}\right)\frac{de_0(t)}{dt} + \left(\frac{R_1 + R_2}{R_2LC}\right)e_0(t) = \frac{1}{LC}e_i(t) \;\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

Denoting $e_0(t)$ by x(t) and $e_i(t)$ by u(t), and using the approximations dx/dt ≈ $x_t - x_{t-1}$ and d^(2)x/dt^(2) ≈ $x_t - 2x_{t-1} + x_{t-2}$, the above differential equation (4) reduces to the following second order difference equation,

$$x_t = (a+1)x_{t-1} - ax_{t-2} + bu_{t-1} \;\ldots\ldots\ldots\ldots\ldots\ldots..\; (5)$$

where $a = \dfrac{R_2LC}{(R_1 + L)(R_2C + 1) + R_2}$ and $b = \dfrac{R_2}{(R_1 + L)(R_2C + 1) + R_2}$

The transfer function for the above difference equation (5) is given by

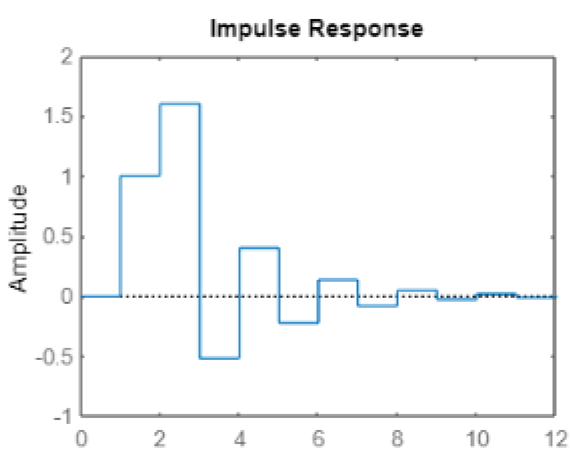
$$H(z) = \frac{bz}{z^2 - (1+a)z + a} \quad \text{....................} \ (6)$$

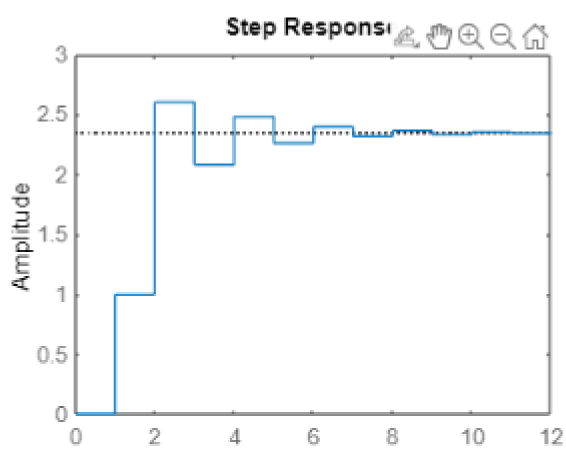
Exercise :

Determine the poles, zeros and stability of the above transfer function for $R_1 = 1\ \Omega$, $R_2 = 1\ \Omega$, L = 2 Henry and C = 2 Farad. Also plot the impulse and unit step responses.

MATLAB Output :

```
z =

      0

p =

      1.0000

      0.4000

k =

       0.1000

coefficients of the power series expansion:

     Columns 1 through 10

          0   0.1000   0.1400   0.1560   0.1624   0.1650   0.1660

          0.1664   0.1666   0.1666


     Column 11

          0.1666
```





Exercise (MAT1002_ Lab Question_22 April 2021):

Q.] 4*Y(n)-4*Y(n-1)+Y(n+2) = X(n) – X(n-1)
i) Find zeros and poles of the function and visualize it
ii) Check stability of the system

Solution (By MATLAB): Y(z) = [(z^2)*(2*z-1)] / [(z-1)*(4*z^2-4*z+1)]

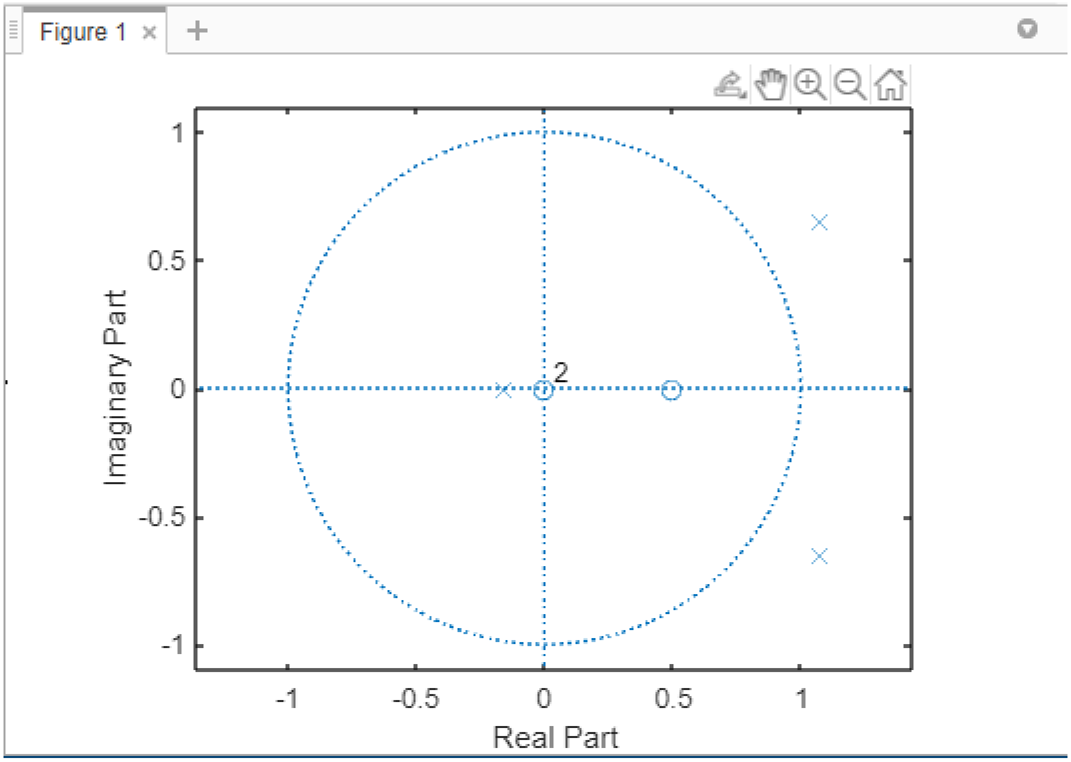
ZerosAndPoles.m

```
clc;
clear all;
close all;
num=input('Input the numerator polynomial:');
den=input('Input the denominator polynomial:');
tranf=tf(num,den);
[zeros, poles, gain]=tf2zp(num,den);
disp('Poles of transfer function are:');
disp(poles);
disp('Zeros of transfer function are:');
disp(zeros);
disp('Gain of transfer function are:');
disp(gain);
zplane(num,den);
```

MATLAB Output :

Command Window

```
Input the numerator polynomial:
[2 -1]
Input the denominator polynomial:
[4 -8 5 1]
Poles of transfer function are:
   1.0786 + 0.6526i
   1.0786 - 0.6526i
  -0.1573 + 0.0000i

Zeros of transfer function are:
    0.5000

Gain of transfer function are:
    0.5000
```



To check stability (Using MATLAB command window):

Command Window

```
>> sys = tf([2 -1],[4 -8 5 1],0.1)

sys =

          2 z - 1
  -----------------------
  4 z^3 - 8 z^2 + 5 z + 1

Sample time: 0.1 seconds
Discrete-time transfer function.

>> B = isstable(sys)

B =

  logical

   0
```

Observation : Since B = 0, the system is unstable.