03/06/20

Graph Theory

→ End vertices: The two vertices associated with an edge ~~is~~ are called end vertices.

Loop (or) Self loop: If both the end vertices are same, the edge is called a loop or a self loop.

Incident point: It is a point where an edge meets the vertex.

Degree / Valency ($d(v_i)$): no of edges incident on a vertex is called degree of the vertex.

* pendent vertex: It is a vertex with degree '1'.
* Isolated vertex: It is a vertex with degree '0'.
* Every loop contributes 2 to the degree of the vertex.

Parallel edges:

If there are 2 or more edges associated with same end vertices, then those edges called parallel edges.

Null graph:

It is a graph in which degree of all vertices is 0.

i.e., set of isolated vertices.

Trivial graph: It is a null graph with only one vertex.

Adjacent vertices: Two vertices are said to be adjacent iff there is a direct path b/w the vertices.

Adjacent vertices: Two edges are said to be adjacent if they have one end vertex in common.

→ Based on parallel edges & self loops graphs, are classified into 3 types.

	Parallel edges	Self loops
Simple graph	×	×
Multi graph	✓	×
Psuedograph	✓	✓

Note:

→ Sum of degrees of all vertices is twice the no of edges

$$\boxed{\sum_{i=1}^{n} d(v_i) = 2|E|}$$

→ No of odd vertices in a graph is even.

→ Maximum degree of a vertex in a simple graph with n vertices is n-1.

→ Maximum no of edges in a simple graph with n vertices is

$$\boxed{{}^{n}C_2 = \frac{n(n-1)}{2}}$$

→ No of different graphs possible with n distinct vertices = $2^{\frac{n(n-1)}{2}}$.

→ No of different graphs possibles with n distinct vertices and 'e' edges is $\left[\frac{n(n-1)}{2}\right] C_e$

Degree Sequence:

If $v_1, v_2 \dots v_n$ are the vertices of a graph G, then then sequence of degrees of vertices $\{d_1, d_2 \dots d_n\}$ is called degree sequence of G. or a graphical sequence.

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Havel - Hakimi Result

Consider two sequences where (i) is in descending order

(i) $s, t_1, t_2 \cdots t_s, d_1, d_2 \cdots d_n$

(ii) $t_1-1, t_2-1 \cdots t_s-1, d_1, d_2 \cdots d_n$

sequence (i) is graphic $\Leftrightarrow$ sequence (ii) is graphic

* To find if a given sequence is graphic or not, we write sequence in form (i) and reduce it to (ii). Now we write (ii) in form (i) and reduce it again and continue this process until we reach a termination condition.

Note:

→ In a simple connected graph with n vertices, atleast two vertices will have same degree.

→ In a graph G with 'n' vertices and $|E|$ edges

$$\boxed{k(G) \le \lambda(G) \le \delta(G) \le \frac{2|E|}{n} \le \Delta(G) \le n-1}$$

where
- $k(G)$: vertex connectivity
- $\lambda(G)$: Edge connectivity
- $\delta(G)$: Minimum degree
- $\Delta(G)$: Maximum degree

Types of Graphs:

1) **Complete Graph:** (K_n) $(n \ge 1)$

A graph in which there is an edge b/w every pair of vertices.

* No of edges: $^nC_2 = \frac{n(n-1)}{2}$
* degree sequence: $n-1, n-1, \cdots n-1$ (n times)

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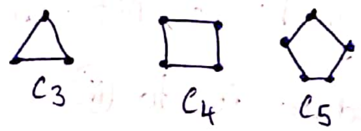
2) Regular Graph:

A graph in which degree of all vertices is same.

* If degree of every vertex is k, then it is called a k regular graph

3) Cycle graph (C_n) ($n \geq 3$)



* If given graph is cycle graph, then degree of every vertex is 2. But reverse need not to be true.
* No of edges : n

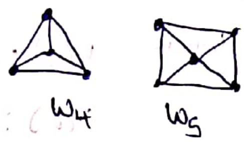
Cyclic graph : A graph with atleast one cycle (as subgraph)

Acyclic graph : A graph without cycles

4) Wheel graph (W_n) ($n \geq 4$):

A wheel graph W_n is obtained by adding a vertex (hub) to C_{n-1} such that the vertex is adjacent to all the vertices in C_{n-1}.

* no of edges : $2(n-1)$



* Degree sequence : $\underset{\text{hub}}{n-1}, \underbrace{3, 3, 3 \ldots 3}_{n-1 \text{ times}}$

5) Line graph ($L(G)$):

For a graph G, Line graph $L(G)$ is constructed as follows:

(i) Create a vertex in $L(G)$ corresponding to each edge in G.

(ii) Two vertices of $L(G)$ are made adjacent, if their corresponding edges are adjacent in graph G.

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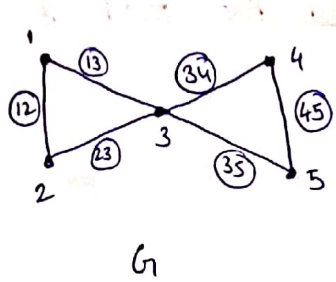
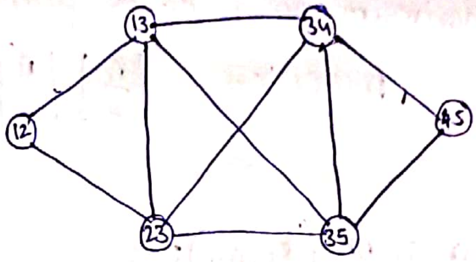
Eg:





Properties of line graph:

* If a graph is euler graph, then its line graph is both euler graph and hamiltonian graph.
* In a graph, a vertex of degree k contributes $\frac{k(k-1)}{2}$ no of edges to line graph.
* No of edges in a line graph is given by

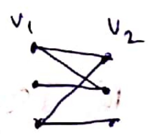
$$= \boxed{\frac{\Sigma d_i^2 - \Sigma d_i}{2} = \frac{\Sigma d_i^2}{2} - |E|}$$

where d_i is degree of i^{th} vertex in the graph
$|E|$ is no of edges in the graph.

* Line graph of a star graph ~~is a complete~~ of n vertices is a complete graph of (n-1) vertices.

6) Biparlite Graph:

A graph in which vertices can be divided into two sets, such that no vertices of same set are adjacent.

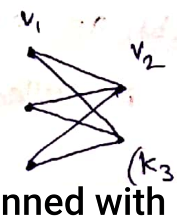


* A graph is biparlite ⇔ 2-colorable ⇔ no odd length cycle

7) Complete Biparlite Graph ($K_{m,n}$):

A biparlite graph in which each vertex in set V_1 is adjacent to each vertex in set V_2.

In $K_{m,n}$: no of edges: $|V_1||V_2| = mn$
no of vertices: $|V_1| + |V_2| = m+n$



($K_{3,2}$)

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* Maximum no of edges possible in bipartite graph of n vertices = $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$

8) <u>Star graph</u> ($K_{1,n-1}$):

It is complete bipartite graph with one vertex in one set and rest of the vertices in the other.

$K_{1,5}$

* A star graph ($K_{1,n-1}$) with n vertices is a complete bipartite graph with n vertices and minimum no of edges.

9) <u>Complement of a graph</u> ($\bar{G}$):

Complement of a graph G, denoted by $\bar{G}$, is a graph which contains only edges that are not present in G and doesn't contain edges that are present in G.

* $G + \bar{G} = K_n$ (complete graph)
* $e(G) + e(\bar{G}) = \frac{n(n-1)}{2}$

→ If G is a graph ~~with~~ with 'n' vertices and degree sequence $d_1, d_2 \dots d_n$, then

then the degree sequence of $\bar{G}$ is given by

$n-1-d_1, n-1-d_2, \dots\dots n-1-d_n$

10) <u>Isomorphic graphs</u>:

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic, if there exists a bijective function 'f' from V_1 to V_2 such that 'a' and 'b' are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all 'a' and 'b' in V_1. The function 'f' is called an isomorphism.

* ~~Is~~ two simple graph are isomorphic, if there is one to one correspondence b/w vertices of the two graphs that preserves adjacency relationship.

* Isomorphism is an equivalence relation.

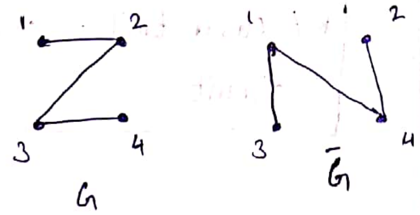
* To determine if the given two graph are isomorphic we check if ~~same~~ both the graphs have
  - same no of vertices
  - same no of edges
  - same degree sequences
  - same no of cycles with length of each cycle matching

Even if all the above conditions are satisfied we cannot conclude that given graphs are isomorphic.

11) <u>Self-Complement Graph</u>:

A graph is said to be self-complement graph if it is ~~isom~~ isomorphic to its complement graph.

Eg:



no of edges in a self-complement graph, $\boxed{e = \frac{n(n-1)}{4}}$

Thus for a self complement graph with 'n' vertices

$$\boxed{n \equiv 0 \text{ (or) } 1 \pmod 4}$$

12) <u>Hypercube (Q_n)</u>:

A hypercube graph Q_n is a graph with 2^n vertices where each vertex corresponding a n-length binary string. Also vertices are adjacent iff their corresponding binary strings differ exactly in one bit position.

Degree of each vertex : n

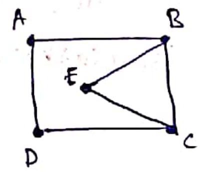
no of edges : $\frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$

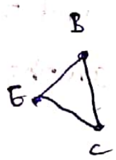
* Every cycle in hypercube is of even length.
  So every hypercube is a bipartite graph.

13) <u>Clique</u>:

If a subgraph of a graph is complete, then it is called clique.

Eg :





is a clique

<u>Connectivity</u>:

	Repetition of vertices	Repetition of edges
walk	✓	✓
trail	✓	×
path (or) line	×	×

* A closed trail is called circuit

<u>Connected graph</u> : there exists path b/w any pair of vertices.

<u>Disconnected graph</u> : there exists pair of vertices such that there is no path b/w them.

* The connected subparts of a disconnected graph are known as components.

* If G is disconnected, $\bar{G}$ is neccessarily connected
  If G is connected, $\bar{G}$ may or may not be connected.

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* In a connected graph
  - no of connected components, $k = 1$
  - no of edges, e ranges

$$n-1 \le e \le \frac{n(n-1)}{2}$$

* In a disconnected graph,
  - no of components, $k > 1$
  - no of edges, e ranges b/w

$$\boxed{n-k \le e \le \frac{(n-k+1)(n-k)}{2}}$$

Tree :

* Tree is a simple connected graph without cycle.

  no of edges: $n-1$

* In a tree, there exists a unique path b/w any two pair of vertices.

Forest :

* Forest is collection of trees. (or) A disconnected acyclic graph.

  (or)

  It is disconnected graph in which each component is minimally connected (tree)

Note:

* In a simple graph of n vertices, if $\delta(G) \ge \left\lceil \frac{n}{2} \right\rceil$, then the graph is necessarily connected.
* A simple graph is necessarily connected if it has more than ${}^{(n-1)}C_2$ edges.
* A simple graph is necessarily disconnected if it has less than $(n-1)$ edges.

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Cut edge / Bridge :

The edge whose removal from a connected graph makes the graph disconnected is called cut edge or bridge.

Cut vertex (or) Cut point (or) Articulation point

The vertex whose removal from a connected graph, makes the graph disconnected is called a cut vertex.

* In a connected graph.

if a cut edge exists; then cut vertex also exists.

The end vertices of a ~~edge~~ cut edge are cut vertex.

However, existence of cut vertex can't guarentee the existence of cut edge.

Cut edge set : (or) Cut set :

It is set of edges whose removal from a connected graph, makes the graph disconnected and no proper subset of a cutset shall disconnect the graph.

Cut vertex set :

It is set of vertices whose removal from a connected graph, makes the graph disconnected or reduces graph into ~~a~~ trivial graph and no proper subset of a cut vertex set shall disconnect the graph.

Edge Connectivity ($\lambda(G)$):

It is minimum no of edges to be removed to make a graph disconnected.

Vertex Connectivity ($\kappa(G)$) :

It is minimum no of vertices to be removed to make a graph disconnected.

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* Relation b/w vertex connectivity & edge connectivity

$$\boxed{k(G) \le \lambda(G) \le \delta(G) \le \frac{2e}{n} \le \Delta(G) \le n-1}$$

* Cut edge never belongs to a cycle of graph.
* If degree of every vertex in a connected graph is even, then every edge will be a part of cycle and hence cut edge doesn't exist.
* If a graph has cut vertex, then it is not a hamiltonian graph.
* If a graph has cut edge, then it is not a hamiltonian graph and not a euler graph
* A connected graph can have atmost $(n-1)$ cut edges and atmost $(n-2)$ cut vertices.

04/08/20

Coloring:

Coloring is assining colors to vertices of a graph such that no two adjacent vertices has same color.

Chromatic Number ($\chi(G)$):

It is minimum no of colors required to paint all the vertices of a graph such that no two adjacent vertices have same color.

A graph with chromatic number k is called k-colorable graph.

* Chromatic number of a cycle graph, $C_n = \begin{cases} 2, & \text{if } n \text{ is even.} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$

* Chromatic number of a wheel graph, $W_n = \begin{cases} 4, & \text{if } n \text{ is } \cancel{\text{odd}} \text{ even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$

* Chromatic number of a complete graph, $K_n = n$

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* 2-Colorable graph → Tree → Bipartite graph
  2-Colorable graph → Even length Cycles only → Bipartite graph

Welsh Powell's Algorithm:

* This algorithm is used to find chromatic ~~no~~ number of a graph.

Step 1: List out degree sequence of the graph in descending order.

Step 2: Color the first vertex with some unused color and go down the list and color all the vertices with the same color, if the vertex is not ~~add~~ adjacent to any of the vertices with that color.

Step 3: Now remove all the colored vertices and repeat the above step② with a different color.

Independent Set:

* It is set of non-adjacent vertices.

maximal independent set:

* It is an independent set to which we can't add any new element.

Independence Number ($\beta(G)$):

* It is number of vertices present in largest maximal independent set.

* If the graph is colored, the vertices of independent set need not to be of same colored vertices.

Dominating Set:

* Dominating set $D \subseteq V$ such that any vertex in V is either belongs to D or adjacent to some vertex which belongs to D.

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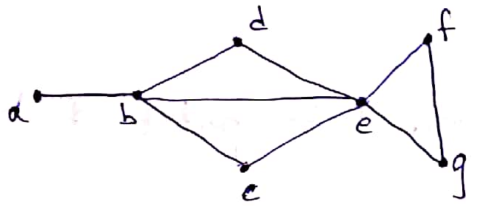
Minimal dominating set: The dominating set from which we cannot remove any element such that the set is still a dominating set.

Domination Number: ($\alpha(G)$): It is number of vertices present in smallest ~~of~~ minimal dominating set.

Eg: Consider the below graph



* { }, {a}, {a,c}, {a,c,d}, {a,c,d,f} ...etc are independent sets.
* {b,f}, {a,e}, ~~{...}~~ {a,c,d,g} ... etc are maximal, independent sets.
* Independence number, $\beta(G) = 4$
* {a,b,c,d,e,f,g}, {b,c,d,e,f,g}, {a,c,d,g}, {b,e} ...etc are dominating sets.
* $\underbrace{\{a,c,d,f\}}_{\text{MIS \& MDS}}$, $\underbrace{\{b,e\}}_{\text{only MDS}}$, $\underbrace{\{a,e\}}_{\text{MIS \& MDS}}$ .... etc. are minimal dominating sets.
* Domination number, $\alpha(G) = 2$.

Note:

* Every maximal independent set is a minimal dominating set.
  But Reverse need not to be true.
  $\therefore$ $\boxed{\alpha(G) \le \beta(G)}$

Matching:

* Matching is a set of non-adjacent edges. It is also known as Independent edge set.
* The vertices involved in matching are said to be matched.

Maximal Matching Set

It is a matching set such that we cannot add a new edge into the set such that it is still a matching.

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Matching Number (M(G)):

It is no of edges present in largest maximal matching set.

No of edges in a matching, for an n-vertex graph $\leq \lfloor \frac{n}{2} \rfloor$

Induced degree:

The degree of a vertex in a matching is called induced degree.

Perfect matching:

A matching is said to be perfect if every vertex is ~~mapped~~ matched.

(or)

Induced degree of all vertices = 1.

* Every perfect matching is maximal, but not vice-versa.
* If perfect matching exists, then no of vertices = even, but reverse need not to be true.
* Total no of perfect matchings ~~of co~~ for a complete graph with 2n vertices is

$$(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1 = \frac{(2n)!}{2^n \cdot n!}$$

while solving problems do not forget to take no of vertices = 2n but not n

Complete Matching:

In a bipartite graph G having a vertex partition $\{V_1, V_2\}$ a complete matching from V_1 to V_2 is matching in which ~~there~~ ~~is~~ ~~an~~ every vertex in V_1 is matched.

* If a complete matching exists from V_1 to V_2, then $|V_1| \leq |V_2|$.

Hall's Theorem:

A complete matching of V_1 into V_2 exists $\Leftrightarrow$ every subset of r vertices in V_1 is collectively adjacent to r (or) more vertices in V_2 for all values of r.

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* In a bipartite graph with vertex partition $\{V_1, V_2\}$

If $\delta(V_1) \geq \Delta(V_2)$ $\forall v_1 \in V_1$ & $\forall v_2 \in V_2$ then

a complete matching from V_1 to V_2 exists.

However, the above condition is sufficient but not necessary.

Covering set (or) Edge Covering set:

It is set of edges such that all vertices should incident on atleast on edge.

Minimal covering set:

It is a covering set from which we can't remove any more elements (edges) such that it is still a covering set.

Covering Number (C(G)):

It is no of edges present in the smallest covering set.

Note:

* Every perfect matching is a minimal covering set.
* Every edges with one of end vertices as pendant vertex must be inculded in the covering set.

Vertex Covering Set:

* It is set of vertices such the every edge of the graph must incedent on atleast one vertex in the set.

Minimal vertex covering set:

It is a vertex Covering set from which we can't remove any vertices such that the property holds.

Note:

Sum of size of minimum vertex cover and size of maximum independent set = no of vertices in the graph.

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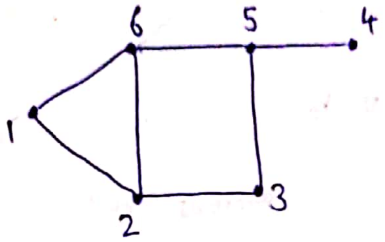
Eg: Consider the below graph, G



* {16, 12, 65, 23, 26, 35, 45}, {16, 12, 53, 54} ... etc are edge covering sets
* {16, 12, 53, 54}, {16, 23, 54} .. etc are minimal edge covering sets.
  perfect matching &
* Covering Number, $C(G) = 3$
* {1, 2, 3, 4, 5, 6}, {1, 6, 5, 3} ..... etc are vertex covering sets
* {1, 6, 5, 3}, {~~4, 2, 5, 3~~} {2, 5, 6} ... etc are minimal vertex covering sets.
* Size of minimum vertex covering set = 3
* In a star graph, $K_{1,n-1}$

  Size of minimum edge covering set = $n-1$ (all edges)

  Size of minimum vertex covering set = 1 (centre vertex)
* Every vertex covering set is a dominating set.

Euler Path or Euler line:

→ A euler path in a multi-graph is a ~~trail~~ path which includes each edge of the multigraph exactly once and intersects each vertex atleast once.

Euler Circuit (or) Euler Cycle:

→ A closed euler path (start vertex = end vertex) is called euler circuit.

* A graph euler circuit is called eulerian graph (or) euler graph.
* A graph containing euler path but not euler circuit is called

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Semi-eulerian graph.

* A graph has euler circuit if degree of all the vertices is even.
* A graph has euler path if degree of all the vertices ~~can~~ is even or there must be exactly 2 vertices of odd degree.
* If a graph as exactly two odd vertices, then the euler path starts at one of the odd vertices and ends at the other.
* A graph containing euler path is said to be traversable.
* A null graph is considered eulerian graph.

Hamiltonian Path:

It is an open path that covers all the vertices exactly once.

Hamiltonian Circuit (or) Hamiltonian cycle:

It is a closed path that covers all the vertices exactly once.

* The graph containing hamiltonian circuit is called hamiltonian graph.
* Every hamiltonian graph contains hamiltonian path also.
* ~~Ever~~ If G is a g euler graph, then L(G) [line graph] is both euler & hamiltonian graph.

Determining the existence of Hamiltonian cycle:

We don't have any necessary conditions for confirming the existence of hamiltonian cycle.

* However we have a few ~~confirmin~~ sufficient conditions to confirm or deny the existence of hamiltonian cycle.

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Dirac's theorem:

* A simple graph with n vertices ($n \geq 3$) and $\delta(G) \geq \frac{n}{2}$ has hamiltonian cycle.

Ore's theorem:

A graph G has hamiltonian cycle if for any two vertices u and v which are not adjacent

$$\deg(u) + \deg(v) \geq n$$

* Minimum number of edges than can be draw such that an n-vertex graph is ~~non~~ not hamiltonian is $\frac{(n-1)(n-2)}{2} + 1$

Thus if no of edges $\geq \frac{(n-1)(n-2)}{2} + 2$, then the graph necessarily contains hamiltonian cycle.

* For a complete graph K_n, $\forall n \geq 3$

no of different hamiltonian cycles $= \frac{(n-1)!}{2}$

* For a complete graph K_n such that n is odd and $n \geq 3$

no of edge disjoint hamiltonian cycles $= \frac{n-1}{2}$

* For a complete bipartite graph $K_{n,n}$, $\forall n \geq 2$

no of ~~bo~~ different hamiltonian cycles $= \frac{n!(n-1)!}{2}$

* For a complete bipartite graph $K_{m,n}$

hamiltonian cycle exists $\Leftrightarrow m = n$

* If a graph ~~G~~ has a pendent vertex, then it is not a hamiltonian graph.

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Directed Graphs:

* $G=(V,E)$ is a directed graph where the elements of edge set E are ordered pairs.

indegree & outdegree:

* Indegree of a vertex is no of edges incidented on the vertex. It is denoted as $deg^+(v)$.
* Outdegree of a vertex is no of edges going away from the vertex. It is denoted as $deg^-(v)$
* A loop in digraph contribute '1' to indegree and 1 to outdegree.
* $\sum_{i=1}^{n} deg^+(v_i) = \sum_{i=1}^{n} deg^-(v) = |E|$

Note:

* A pair of vertices of a digraph are weakly connected if there is a non-directed path between them.
* A pair of vertices in a digraph are unilaterally connected if there a directed path from first vertex to 2nd vertex or from 2nd vertex to 1st vertex.
* A pair of vertices in a digraph are strongly connected if there is a directed path from 1st vertex to 2nd vertex and 2nd vertex to first vertex.
* Two vertices u and v of a directed graph are said to be quasi strongly connected if there is a vertex w from which there is a directed path to u and directed path to v.

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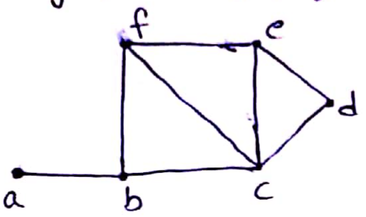
A few more useful terminologies:

Consider below graph $G=(V,E)$



Distance: Distance b/w two vertices u and v is denoted by $d(u,v)$

$d(u,v)$ = no of edges in a shortest path b/w u & v

Eg: $d(a,h)=3$ $d(b,a)=1$ $d(b,d)=2$

Eccentricity: Eccentricity of a vertex v is -

$$e(v) = \max(d(u,v)) \quad \forall u \in V$$

i.e., ~~dis~~ distance b/w v and farthest vertex from v.

$e(a)=3$ $e(b)=2$ $e(c)=2$

Radius: Radius of a connected graph

$$r(G) = \text{minimum}(e(v)) \quad \forall v \in V$$

$r(G)=2$

Diameter: $d(G) = \text{maximum}(e(v)) \quad \forall v \in V$

$d(G)=3$

central point: A vertex v is called center if its ~~ec~~ eccentricity = radius of G

$$e(v) = r(G)$$

b, c, f are central points.

Centre: Set of all central points of G.

$$C(G) = \{b, c, f\}$$

Circumference of graph: It is no of edges in longest cycle

In above graph, circumference = 5

Girth: It is no of edges in shortest cycle.

In above graph, girth $g(G)=5$

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