2 Functors

Exercise 2.1. Give three more functors.

Solution. We go back to our categories from **??**.

- Consider the functor from the category of groups to the category $\mathscr{A}$ of sets with base point, where we map a group to the underlying set and choose the neutral element as basepoint. We map the group morphism to the corresponding map between sets and note that it respects the basepoint by definition.
- Recall our category $\mathscr{B}$ whose objects where natural numbers and where the arrows from n to m are given by the set $\{m-n, \dots, m\}$. From this category we define a contravariant functor to the category of (finite) sets in the following way. We map a natural number n to the set $\{1, \dots, n\}$. So 0 maps to the empty set. We map a morphism $k : n \to m$ to the map of sets $\{1, \dots, m\} \to \{1, \dots, n\}$ given by $i \mapsto \max\{i-k, 1\}$.
- Recall our category $\mathscr{C}$ of matrices over k we defined and the category whose objects are just the elements of k and whose arrows $x \to y$ are all the $r \in k$ such that $rx = y$. Then applying the determinant (both on objects and on arrows) gives a functor.

We only sketch a proof for the second example. The identity id_n in $\mathscr{B}$ is given by 0. Applying our functor yields the map $i \mapsto \max\{i-0, 1\}$ but that is just the identity on $\{1, \dots, n\}$. So let $n \leq m \leq k$ and $i : n \to m, j : m \to k$ we get the maps

$$\begin{aligned} i' : \{1, \dots, m\} &\to \{1, \dots, n\} \\ x &\mapsto \max\{x-i, 1\} \\ j' : \{1, \dots, k\} &\to \{1, \dots, m\} \\ x &\mapsto \max\{x-j, 1\}. \end{aligned}$$

As the compostion we optain the map

$$\begin{aligned} (j' \circ i')(x) &= \max\{\max\{x-i, 1\} - j, 1\} \\ &= \max\{x-i-j, 1\}. \end{aligned}$$

But thats exactly what we would get if we take the compostion first - so adding i and j - and than appling the functor.

Simply said the arrow/number i tells us how badly i' fails to be injective. And as we have defined our maps nicely the "uninjectivity" of $j' \circ i'$ is just given by the sum $i + j$. $\square$

Exercise 2.2. Show that functors preserve isomorphism.

Proof. Let $A \simeq A'$ be two isomorphic objects in a category $\mathscr{A}$. Let $f : A \to A'$ be the isomorphism with inverse g. Then for a functor F we have

$$F(f) \circ F(g) = F(f \circ g) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)} .$$

The same argument works in the other way showing that $F(f)$ and $F(g)$ are inverses proving that $F(A)$ and $F(A')$ are isormophic. $\square$

Exercise 2.3. Consider orderd sets A and B with correspoding categories $\mathscr{A}$ and $\mathscr{B}$. Then a functor $\mathscr{A} \to \mathscr{B}$ is the same as order preserving map $A \to B$.

Proof. Let F be a functor $\mathscr{A} \to \mathscr{B}$. By definition F defines a map $A \to B$. If $x \leq y$ in A we have a arrow $x \to y$ in $\mathscr{A}$. Applying the functor we get a arrow from $F(x)$ to $F(y)$ proving that $F(x) \leq F(y)$ so our map is order preserving.

Let f be a order preserving map $A \to B$. Then we can define our functor F on objects using f. As the map we started with was order preserving if there is an arrow $x \to y$ in $\mathscr{A}$ we find an arrow $f(y) \to f(y)$ in $\mathscr{B}$. As there is at most on arrow this gives us a natural definition for our functor on morphism. $\square$

Exercise 2.4. Given a topological space X let $C(X)$ be the ring of continuous real valued functions on X. For a continuous map between topologial spaces $f : X \to Y$ we define $f^* : C(Y) \to C(X)$. For a map $q : Y \to \mathbb{R}$ we define $f^*(q) : X \to \mathbb{R}$ as the compostion $q \circ f$. This defines a contravarient functor.

Proof. Everything is continuous and so this is well defines. If id_X is the identity on X applying our functor yields the map $C(X) \to C(X)$ that precomposites with the identity. But thats doing nothing i.e. it is the identity map on $C(X)$. Consider

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[d, "q"] \\ & & \mathbb{R}. \end{tikzcd}$$

Then pulling back with g^* gives us $g^*(q)$ which we can pullback with f^* to get $f^*(g^*(q))$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f^*(g^*(q))", blue] & Y \arrow[r, "g"] \arrow[d, "g^*(q)", red] & Z \arrow[d, "q"] \\ \mathbb{R} \arrow[r, "\mathrm{id}"] & \mathbb{R} \arrow[r, "\mathrm{id}"] & \mathbb{R}. \end{tikzcd}$$

Here the red map is defined such that the right diagram commutes and the blue one is defined such that the left diagram commutes. But then the blue one makes the whole diagram commute. But that is exactly the defining the property of $(g \circ f)^*$ and so we may conclude that $f^*(g^*(q)) = (g \circ f)^*$. This diagram is a bit "forced" and a simple calculation would do the job aswell. $\square$

Exercise 2.5. Which functors are full/faithfull. Instead of checking the ones from the book we consider our examples.

Solution.

- The first example from groups to sets with basepoint is the forgetful functor. It is faithful, because equality of maps of groups is *defined* as equality of maps of sets. But it is not full: Consider for example the map of pointed sets $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$, sending an equivalence class to its smallest non-negative representative. This map is clearly not a group homomorphism.
- The second example is cleary not full as we only reach very specific maps with our definition. However it is faithful as the maps $x \mapsto \max\{x - i, 1\}$ is uniquely determined by i.
- The last example is neither full nor faithful (unless n=1, then both categories are isomorphic via this functor). To see that it is not faithful consider the case where $A = B = 0$ then we find (as $n \neq 1$) two matrices $M_1 \neq M_2$ with the same determinant. This means that both map to the same arrow $0 \to 0$ and our functor is not faithful.

  To see that it is not full consider the case where $A = 0$ and $B \neq 0$ is a matrix with determinant 0 (we can find such a matrix as $n \neq 1$). Then there is no matrix M such that $M \cdot A = B$. However every element in k induces an arrow from $0 \to 0$. Clearly this has no chance of beeing full.

□

Exercise 2.6.

- What are the subcategories of an ordered set? Which are full?
- What are the subcategories of a group? Which are full?

Solution.

- Let A be an ordered set. As a category, the objects are just the set A, and there exists a morphism between two objects x and y if and only if $x \leq y$. Specifying a subcategory amounts to choosing a subset $B \subseteq A$, and a relation R on B, such that for all $x, y, z \in B$:
  1. xRy implies $x \leq y$ (because B is a subcategory)
  2. xRy and yRz implies xRz (because composition must be transitive)
  3. xRx (because the composition must have identity arrows)

  If we have xRy and yRx, this implies $x \leq y$ and $y \leq x$, hence $x = y$, since $\leq$ is anti-symmetric. Hence R is a partial order, and in fact, it is a suborder of $\leq|_B$, the order of A restricted to B. It is clear that the subcategory is full if and only if $R = \leq|_B$.
- Let G be a group. As a category G has one object, and some arrows (one for each group element) which are all isomorphisms. So specifying a non-empty subcategory amounts to chosing arrows. However we have to be careful as our subcategory still needs the identity on our objects and has to be closed under composition. In the terminology of groups this means exactly we need to keep the neutral element (the

identity) and if we have chosen two elements we also have to add their product. However we are not required to choose the inverse map i.e. the inverse of our element in the group. This means we don't neccessarily get a subgroup. However we at least get that our subcategory corresponds to a submonoid. It is full if and only if it is the whole category again.

□