3 Natural transformations

Exercise 3.1. Give three more examples of natural transformations not mentioned above.

Solution.

- Consider the category $\mathscr{A}$ of n dimensional vector spaces over $\mathbb{R}$ with basis. So every object is a vectorspace together with the choice of a basis. As arrows we consider the linear maps $f : (V, (v_1, \dots, v_n)) \to (W, (w_1, \dots, w_n))$ such that $f(v_i) \subset < w_1, \dots, w_i >$. This is exactly defined in a way such that restriction works. So consider the category $\mathscr{B}$ of finite dimensional vectorspaces with basis. Here the arrows are just linear maps. Then we can consider $F, G : \mathscr{A} \to \mathscr{B}$ where F restricts to the span of the first $n-1$ basis elements and G restricts to the first $n-2$ to basis elements. For each object $(V, (v_1, \dots, v_n))$ in $\mathscr{A}$ we have a natural map $F(V) \to G(V)$. It is just restriction. We have to check commutativity of the following diagram for all $f : V \to W$.

$$\begin{tikzcd} < v_1, \dots, v_{n-1} > \arrow[r, "f|_{<v_1,\dots,v_{n-1}>}"] \arrow[d] & < w_1, \dots, w_{n-1} > \arrow[d] \\ < v_1, \dots, v_{n-2} > \arrow[r, "f|_{<v_1,\dots,v_{n-2}>}"] & < w_1, \dots, w_{n-2} > \end{tikzcd}$$

  Here the vertical arrows are just the natural projections. Note that this is well defined as f is an arrow in $\mathscr{A}$, i.e. it respects restriction. But if this diagram is well-defined it is clearly commutative.

- Let $F, G : \mathscr{A} \to \mathscr{B}$ be two constant functors. That means there are objects B_1, B_2 in $\mathscr{B}$ such that F maps evry object to B_1 and every map to the identity on B_1 and G does the same for B_2. Then a natural transformation is just the same as a morphism $B_1 \to B_2$ and it is a nautral isomorphism exactly if the morphism is an isomorphism.

- Let **Rings** be the category of Rings with ringmorphisms between them. Let F, G be the morphisms to the category of sets with basepoints that choose the 0 (respectivly 1) as basepoint. Then for each ring R we can consider the map $F(R) \to G(R)$ that swaps 1 and 0. It is easy to check that this is a natural transformation.

$\square$

Exercise 3.2. Prove lemma 1.3.11. Let $\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd}$ be a natural transformation. Then α is a natural isomorphism if and only if $\alpha_A \colon F(A) \to G(A)$ is an isomorphism for alle $A \in \mathscr{A}$.

Proof. Suppose α is a natural isomorphism. Then there exists a natural transformation $\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\beta\Uparrow"] & \mathscr{B} \end{tikzcd}$, such that $\alpha \circ \beta = \mathrm{id}_G$ and $\beta \circ \alpha = \mathrm{id}_F$. Let $A \in \mathscr{A}$ be an object. Then

$$\alpha_A \circ \beta_A = (\alpha \circ \beta)_A = (\mathrm{id}_G)_A = \mathrm{id}_{G(A)},$$

and similarly,

$$\beta_A \circ \alpha_A = (\beta \circ \alpha)_A = (\mathrm{id}_F)_A = \mathrm{id}_{F(A)}\,.$$

This just means that α_A is an isomorphism in $\mathscr{B}$.

Now the other direction: Suppose $\alpha_A \colon F(A) \to G(A)$ is an isomorphism for every $A \in \mathscr{A}$. Then we can define for every $A \in \mathscr{A}$ a morphism $\beta_A \coloneqq \alpha_A^{-1} \colon G(A) \to F(A)$.

We need to show, that β is a natural transformation $G \to F$, it then follows immediately from the definitions that β is an inverse to α, i.e. α is a natural isomorphism.

So let $f \colon A \to A'$ be a morphism in $\mathscr{A}$. Then

$$\begin{aligned}
\beta_{A'} \circ G(f) = & \\
& \downarrow \text{Definition of } \beta_{A'} \\
& = \alpha_{A'}^{-1} \circ G(f) \\
& = \alpha_{A'}^{-1} \circ G(f) \circ \mathrm{id}_{G(A)} \\
& \downarrow \alpha_A \text{ is an isomorphism} \\
& = \alpha_{A'}^{-1} \circ G(f) \circ \alpha_A \circ \alpha_A^{-1} \\
& \downarrow \alpha \text{ is a natural transformation} \\
& = \alpha_{A'}^{-1} \circ \alpha_{A'} \circ F(f) \circ \alpha_A^{-1} \\
& \downarrow \alpha_{A'} \text{ is an isomorphism} \\
& = \mathrm{id}_{F(A')} \circ F(f) \circ \alpha_A^{-1} \\
& = F(f) \circ \alpha_A^{-1}.
\end{aligned}$$

This is just the naturality condition, hence β is a natural transformation and we are done.

(Note: This proof is very similiar to the proof that the inverse of a bijective linear map is again bijective, or that the inverse of a bijective group homomorphism is again a group homomorphism) □

Exercise 3.3. Let A and B be sets, and denote by B^A the set of functions from A to B. Write down:

1. a canonical function $A \times B^A \to B$;
2. a canonical function $A \to B^(B^A)$.

Solution. 1. we have a canonical function $(a, f) \mapsto f(a)$

2. we have a canonical function $a \mapsto (f \mapsto f(a))$

Note that this is very similar to the construction of the embedding $V \to V^{**}$ of an k-vectorspace into its bi-dual space.

Also note that both solutions are "essentially the same map". The procedure of switching between both solutions is called *currying*. $\square$

Exercise 3.4. In this exercise, you will prove Proposition 1.3.18. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a functor.

- Suppose that F is an equivalence. Prove that F is full, faithful and essentially surjective on objects.

- Now suppose instead that F is full, faithful and essentially surjective on objects. For each $B \in \mathscr{B}$, choose an object $G(B)$ of A and an isomorphism $\epsilon_B\colon F(G(B)) \to B$. Prove that G extends to a functor in such a way that $(\epsilon_B)_{B\in\mathscr{B}}$ is a natural isomorphism $FG \to 1_{\mathscr{B}}$. Then construct a natural isomorphism $1_{\mathscr{A}} \to GF$, thus proving that F is an equivalence.

Proof.

- Let F be an equivalence. Then there is a functor $G\colon \mathscr{B} \to \mathscr{A}$, and natural isomorphisms $\eta\colon 1_{\mathscr{A}} \to G \circ F$ and $\epsilon\colon F \circ G \to 1_{\mathscr{B}}$. We first show that F is faithful and full. For this, let $A, B \in \mathscr{A}$. We need to show that $\mathscr{A}(A,B) \to \mathscr{B}(F(A), F(B))$ is injective and surjective. We consider the composition

$$\mathscr{A}(A,B) \xrightarrow{F} \mathscr{B}(F(A), F(B)) \xrightarrow{G} \mathscr{A}(G(F(A)), G(F(B))) \to \mathscr{A}(A,B),$$

  where the last arrow is given by $f \mapsto \eta_B^{-1} \circ f \circ \eta_A$. By definition of η, this composition is just the identity on $\mathscr{A}(A,B)$. Since η_A, η_B are isomorphisms, we see that $\mathscr{A}(A,B) \xrightarrow{G\circ F} \mathscr{A}(G(F(A)), G(F(B)))$ is bijective. This implies that $\mathscr{A}(A,B) \to \mathscr{B}(F(A), F(B))$ is injective. Surjectivity follows similar.

  It remains to show that F is essentially surjective. Let $X \in \mathscr{B}$. Let $A := G(X)$. Then $F(A) = F(G(X)) \cong X$, where the last isomorphism is given by ϵ_X. Hence F is essentially surjective.

- Suppose now that F is a fully faithful, essentially surjective functor. For every $B \in \mathscr{B}$, we choose an object $G(B) \in \mathscr{A}$ and an isomorphism $\epsilon_B\colon F(G(B)) \to B$. This is possible since F is essentially surjective. Thus we have defined our G on objects. We need to say what it does to morphisms. So let $f\colon B \to B'$ a morphism in $\mathscr{B}$. Let $f' := \epsilon_{B'}^{-1} \circ f \circ \epsilon_B\colon F(G(B)) \to F(G(B'))$. Since F is fully faithful, there exists a unique morphism $G(f)$, such that $F(G(f)) = f'$. Thus we have defined G on morphisms. It is easy to see that G preserves identites and compositions.

  The next thing we need to show is, that $\epsilon\colon F \circ G \to \mathrm{id}_{\mathscr{B}}$ is a natural isomorphism. Since ϵ_B is an ismorphism for every B by construction, the only thing we need to check is the naturality. So let $f\colon B \to B'$ a morphism in $\mathscr{B}$. Then

$$\begin{aligned}\epsilon_{B'} \circ F(G(f)) = \\ &\downarrow\text{Definition of } G \\ &= \epsilon_{B'} \circ \epsilon_{B'}^{-1} \circ f \circ \epsilon_B \\ &= f \circ \epsilon_B \\ &= 1_{\mathscr{B}}(f) \circ \epsilon_B.\end{aligned}$$

This is just the naturality condition.

To conclude, we must construct a natural isomorphism $\eta\colon 1_{\mathscr{A}} \to G \circ F$. Let $A \in \mathscr{A}$. Then we have an ismorphism $\epsilon_{F(A)}\colon F(G(F(A))) \to F(A)$. As F is fully faithful, this gives us a unique ismorphism $G(F(A)) \to A$, we denote by $\eta_A\colon A \varnothing G(F(A))$ its inverse. From this construction it is clear that η is a natural ismorphism $1_{\mathscr{A}} \to G \circ F$ if we can prove naturality. The prove of naturality is similar to the prove of naturality of ϵ, and we omit it here.

□

Exercise 3.5. For a fixed field k we consider the category of **Mat** whose objects are the natural numbers and where $\mathbf{Mat}(m, n) = \{n \times m \,\text{matrices over}\, k\}$. We show that **Mat** is equivalent to **FDVect**.

Proof. Composition of matrices is of course just matrix multiplication. Consider the functor F : **FDVect** $\to$ **Mat** that maps are vectorspace to its dimension. For each vectorspace we chose a basis (for k^n we chose the standard basis) and then given a linear map $V \to W$ we have representation of that map as a matrix. For the functor G : **Mat** $\to$ **FDVect** we map n to the vector space k^n (with the standard basis). Given a $n \times m$ matrix we think of it as a representation of a linear map $k^m \to k^n$ with respect to the standard basis. It is easy to see that $F \circ G$ is the identityfunctor on **Mat**. The other direction is a bit harder as $G \circ F$ is clearly not the identityfunctor. Any vectorspace of dimension n gets mapped to k^n. But that is okay as we have a natural map from V to k^n as soon as we have chosen a basis on both of them. But that is something we have done in the very beginning. Or a little bit mor precise for each V we consider the map $V \to k^n$ induced by the basis. Then this induces a natural equivalence between the identityfunctor on **FDVect** and our functor $G \circ F$.

In this proof we have seen that there is a canonical way to go from **Mat** to **FDVect**. But to get the map in the other direction we have to chose a basis for each vector space. So it is not canonical. Another way to say what is going on is that (similar to the example of the sets {1,. . . ,n} as canonical elements in **Sets**) we have that the subcategory of **FDVect** given by the vectorspaces k^n with maps between them is equivalent to **FDVect**.

□