7 The Yoneda Lemma

Exercise 7.1. State the dual of the Yoneda Lemma.

Proof. See https://math.stackexchange.com/questions/2573288/how-do-you-explicitly-form-the-dual-of-the-yoneda-lemma $\square$

Exercise 7.2. Let M be a monoid. The underlying set of M can be given a right M-action by multiplication. This M-set is called the right regular representation of M and is written as $\underline{M}$.

a) When M is regarded as a one-object category, functors $M^{op} \to \mathbf{Set}$ correspond to right M-sets (Example 1.2.14). Show that the M-set corresponding to the unique representable functor $M^{op} \to \mathbf{Set}$ is the right regular representation.

b) Now let X be any right M-set. Show that for each $x \in X$, there is a unique map $\alpha\colon \underline{M} \to X$ of right M-sets such that $\alpha(1) = x$. Deduce that there is a bijection between $\{$maps $\underline{M} \to X$ of right M-sets$\}$ and X.

c) Deduce the Yoneda lemma for one-object categories.

Proof. To a). The unique representable functor $M^{op} \to \mathbf{Set}$ is the functor corresponding to $M^{op}(\star, -) = M(-, \star)$, where $\star$ is the one element in M. For any contravarient functor F the corresponding M-set is just the image of the functor of the unique point so in our case it is $M(\star, \star)$ which is just M (in a natural way). The right action works like this. For each $g \in F(\star)$ and $m \in M$ we define $g \cdot m = F(m)(g)$. In our case this means the right action is given as $x \cdot m = H_\star(m)(x) = x \cdot m$.

To b). First let $x \in X$ be a right M-set with basepoint. We define a map $\alpha : M \to X$ by $\alpha(m) = x \cdot m$. It is easy to see that this gives us $\alpha(m) \cdot m' = \alpha(m \cdot m')$ i.e. it is a map of right G-sets. It is also unique with the property $\alpha(1) = x$ as the rest follows from the fact that it is a morphisms of G-sets. The bijection is just $x \mapsto (m \mapsto x \cdot m)$.

To c). Let X be a functor from $M^{op} \to \mathbf{Set}$ (every one object category can be considerd as a monoid). As there is only one object in M we only need to show that X is isomorphic to the set of transformations from $H_\star$ to X. Note that everything here is a set so isomorphic just means bijective. This now follows from b) after we understand that transformations of functors corespond to maps of G-sets under our identification (maybe we have already seen this?). So let $\alpha\colon M \to X'$ be a map of G-sets (X' is the G-set corresponding to X). We then can define a transformation $\eta : H_\star \to X$ as follows.

$$\begin{aligned} \eta : (H_\star(\star)) = M &\to X' = X(\star) \\ m &\mapsto \alpha(m) \end{aligned}$$

Naturality follows from the fact that α was a morphism G-sets and it is easy to see that this is bijective. $\square$