1 Categories

Exercise 1.1. Some more categories

Solution.

- The category $\mathscr{A}$ of sets with basepoint. The objects are nonempty sets with a special element. The morphisms are maps of sets that map the special element to the special element.
- The category $\mathscr{B}$ whose objects are the the natural numbers (without zero). The arrows from n to m are given by the set $\{m-n,\dots,m\}$. So there are no morphisms if $m < n$. Given integers $n \leq m \leq k$ aswell as arrows $i : n \to m, j : m \to k$ we define $j \circ i : n \to k$ by $i + j$.
- Fix a field k and some positive integer n (If you want to be fancy you can replace k by a ring). Consider the category $\mathscr{C}$ whose objects are given as $n \times n$ matrices over k. For two matrices A, B the morphisms from A to B are all the matirces M such that $M \cdot A = B$. The compostion is just matrix multiplication.

We omit the proof that all of the above a really categories. □

Exercise 1.2. Show that a map in a category can have at most one inverse.

Proof. Let $f : A \to B$ be a morphism between two objects in some category. Assume that $g, g' : B \to A$ are both inverse to f. Then

$$g = g \circ \mathrm{id}_A = g \circ (f \circ g') = (g \circ f) \circ g' = \mathrm{id}_A \circ g' = g'.$$

□

Exercise 1.3. For categoires $\mathscr{A}, \mathscr{B}$ we consider the categorie $\mathscr{A} \times \mathscr{B}$. Give the definition of compostion and the identities.

Solution. Let (A, B) be an object in $\mathscr{A} \times \mathscr{B}$. Then $(\mathrm{id}_A, \mathrm{id}_B)$ is the identity on (A, B). For two morphism $(f, g) : (A, B) \to (C, D)$ and $(h, j) : (C, D) \to (E, F)$ we define the compostion by $(h, j) \circ (f, g) = (h \circ f, j \circ g)$. We omit the proof that this works. □