

Seminar program:

Algebraic K -theory and chromatic homotopy theory

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TALK 0 (MAXIME): INTRODUCTORY TALK

TALK 1 (TIMO): INTRODUCTION TO ALGEBRAIC K -THEORY 1/2

References: [Heb21, Sections III and IV], [Lur14, Lectures 16-20], [Hil].

- Coordinate with the speaker of Talk 2 to split the material between the two of you.
- Define K -theory of stable categories ([Lur14, Lecture 16]).
- Define K -theory of schemes as $K(\text{Perf}(-))$ and explain how this induces transfer maps (i.e., the maps induced by pushforward on perfect complexes).
- Define K -theory of terms of group completion ([Lur14, Lecture 18]), state the group completion theorem ([Lur14, Lecture 18, Proposition 4], [Nik17]).
- For K -theory in the group completion sense, show that $K(-)$ is lax symmetric monoidal and preserves filtered colimits. Mention that this is also true for K -theory of stable infinity categories.
- Explain the additivity and Verdier localization properties [Lur14, Lecture 17], [Heb21, IV.29] or [Heb, 5.8.1].
- State the resolution theorem [Lur14, Lecture 18-20].

TALK 2 (ANDI): INTRODUCTION TO ALGEBRAIC K -THEORY 2/2

- Coordinate with the speaker of Talk 1 to split the material between the two of you.

TALK 3 (ANTONIO): THE NISNEVICH AND ÉTALE TOPOLOGIES, DESCENT

References: [Lur18], [Lur09].

- Define the Nisnevich and étale topologies.
- Define the notion of descent in general [Lur09, §6.2.2] and characterize Nisnevich descent via distinguished squares (see e.g. [AHW17, Theorem 3.2.5] combined with [BH21, Proposition A.2(d)], but proving this probably takes too long).
- Prove that having Nisnevich and Galois descent is equivalent to having étale descent [Lur18, Theorem B.6.4.1].
- Show that K -theory has Nisnevich descent by reducing to Nisnevich descent for $\text{QCoh}(-)$ (the ∞ -category of complexes with quasi-coherent cohomology). See e.g. [Kha17, Lectures 2-5, especially Lecture 5, Theorem 7.3]

TALK 4 (ANTON): EQUIVARIANT HOMOTOPY THEORY AND GALOIS DESCENT FOR RATIONAL K -THEORY

References: [CMNN22, Section 3 and Appendix A], [Ram, Sections 1-2, 6-7].

- Define genuine G -spectra in terms of Mackey functors [CMNN22, Appendix A].
- Give a construction of equivariant K -theory following [CMNN22, 2.12, 2.13].
- Formulation of descent in terms of Mackey functors, and [CMNN22, Criterion 3.4].
- Explain 3.1, 3.2, 3.4 and 3.6. Avoid anything that mentions (F, ϵ) -nilpotence or telescopic localization.

TALK 5 (MANUEL): TOPOLOGICAL K -THEORY KU AND COMPLEX ORIENTATIONS

References: [Cno24, Chapter 5], [Ada74, II.2], [Lur10, Lecture 4].

- Define KU , ku , explain their relation to BU , and state Bott periodicity ([Cno24, Chapter 5]). Define $K(1)$ as KU/p (without mentioning other heights). Mention the existence of a homotopy ring map $KU \rightarrow K(1)$.
- Explain what complex orientations are (definition with \mathbb{CP}^∞), and prove that KU and $K(1)$ are complex-oriented.
- Recall the notion of formal group laws, their relationship to complex oriented ring spectra, and compute those of KU , ku , $K(1)$ (see [Ada74, II.2]).

TALK 6 (DOMINIK): THE $K(1)$ -HOMOLOGY OF BC_p AND AMBIDEXTENSITY

References: [nLa26], [HKR00], [GS96], [LH13], [CSY20], [RW80].

- Using the Gysin sequence (see [nLa26]), compute the E -homology of BC_p for a complex oriented ring spectrum E , in terms of the formal group law of E [HKR00, Section 5]. Apply this to $E = K(1)$.
- Introduce the norm map for BC_p .
- Prove the following Lemma: If $K(1)^*BC_p$ is finite dimensional over $K(1)^*$, then the norm map is an equivalence in $\mathbf{Sp}_{K(1)}$ (a reference is [GS96], but see the end of the program for an easier, more modern argument).
- Combining the previous points, conclude that the norm map for BC_p is a $K(1)$ -local equivalence.

TALK 7 (JONAS): ÉTALE DESCENT FOR $K(1)$ -LOCAL K -THEORY

References: [CSY20], [Hop14].

- The goal of this talk is to prove that $K(1)$ -local K -theory has étale descent, following [CSY20].
- Show that π_0 of a $K(1)$ -local \mathbb{E}_∞ -ring spectrum admits a δ -ring structure. See [CSY20, Section 5.2 and Theorem 5.2.2] or [Hop14].
- Recalling that δ -rings are torsion-free, deduce the height 1 May nilpotence conjecture: an \mathbb{E}_∞ -ring spectrum that is p -local and rationally zero, is also $K(1)$ -locally zero [CSY20, Theorem 5.2.6].
- Deduce that $K(1)$ -local K -theory has étale descent by applying the methods of Talks 3 and 4.

TALK 8 (EMANUELE): HYPERDESCENT

References: [CM21].

- Define hyperdescent, hypersheaves and hypercompleteness.
- State and prove the criteria [CM21, 1.6, 1.8] (reduction to fields, and a concrete criterion for hyperdescent over fields).

TALK 9 (LUCA): THE NORM-RESIDUE ISOMORPHISM THEOREM AND HYPERDESCENT FOR $K(1)$ -LOCAL K -THEORY 1/2

References: [CM21], [HW19].

- Coordinate with the speaker of Talk 10 to split the material between the two of you.
- State the norm-residue isomorphism theorem (see [HW19]).
- Use it to prove that $K(1)$ -local K -theory has étale hyperdescent, following [CM21, Section 6.2 and Theorem 6.13].

TALK 10 (LORENZO): THE NORM-RESIDUE ISOMORPHISM THEOREM AND HYPERDESCENT FOR $K(1)$ -LOCAL K -THEORY 2/2

References: [CM21].

- Coordinate with the speaker of Talk 9 to split the material between the two of you.

TALK 11 (MAXIME): QUILLEN-SUSLIN RIGIDITY

References: [Sus83].

- We proved in Talks 9 and 10 that $K(1)$ -local K -theory was an étale hypersheaf. The goal of this talk is to compute its stalks.
- State (but do not prove) Gabber rigidity.
- Reduce to algebraically closed fields [Sus83].
- Compute it in this case, following Quillen (case $\overline{\mathbb{F}}_p$).

TALK 12 (TOM): MITCHELL'S THEOREM

References: [ENY20], [Mit90].

- Briefly introduce $K(2)$ (note that it will be more thoroughly introduced in Talk 13).
- State and prove Mitchell's theorem: the $K(2)$ -local K -theory of a discrete ring vanishes. This can be done using étale hyperdescent, following [ENY20].

TALK 13 (JULIE): REDSHIFT AND PURITY 1/2

References: [Lur10], [LMMT24], [CMNN22].

- Coordinate with the speaker of Talk 14 to split the material between the two of you.
- Introduce the Morava E - and K - theory spectra [Lur10, Lecture 22], either using Landweber exactness or blackboxing their existence, and only mention the description as quotients of MU .
- Explain the notion of redshift, explaining how this generalizes Mitchell’s theorem from Talk 12.
- Prove purity (see [LMMT24, CMNN22]), which gives the upper bound for redshift.

TALK 14 (KLAUS): REDSHIFT AND PURITY 2/2

References: [Lur10], [LMMT24], [CMNN22].

- Coordinate with the speaker of Talk 13 to split the material between the two of you.

Argument for Talk 6

- By monoidality of the norm map, it suffices to prove that $\mathbb{S}_{K(1)}^{tC_p} \simeq 0$.
- Reduce to proving that KU^{tC_p} is p -completely 0, i.e. prove that $K(1)^{tC_p} \simeq 0$.
- Note that the assumptions guarantee that $K(1)^{tC_p}$ is a compact $K(1)$ -module (as a cofiber of compact $K(1)$ -modules) and hence since it is $K(1)^{BC_p}[e^{-1}]$ where e is the Euler class, we deduce that the canonical map $K(1)^{BC_p} \rightarrow K(1)^{tC_p}$ splits, which proves that the norm map is also split injective, and hence an iso for dimension reasons.

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