5 Adjunctions via units and counits

Exercise 5.1. Let $A \underset{g}{\overset{f}{\rightleftarrows}} B$ be order-preserving maps between ordered sets. Prove directly that the flollowing conditions are equivalent:

a) for all $a \in A$ and $b \in B$,

$$f(a) \leq b \iff a \leq g(b)$$

b) $a \leq g(f(a))$ for all $a \in A$ and $f(g(b)) \leq b$ for all $b \in B$.

Proof. a) $\implies$ b). To see $a \leq g(f(a))$ by a) it suffices to prove $f(a) \leq f(a)$. That is clearly true and in the same manner one varifies that $f(g(b)) \leq b$ for all $b \in B$.

b) $\implies$ a). Assume $f(a) \leq b$. Then we can apply g (which is order-preserving) and get $g(f(a)) \leq g(b)$. By b) we know that $a \leq g(f(a))$ and are done (as $\leq$ is transitive). The same argument works in the other direction.

□

Exercise 5.2. a) Let
$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "G"] \end{tikzcd}$$
be an adjunction with unit η and counit ε. Write $\mathbf{Fix}(GF)$ for the full subcategory of $\mathscr{A}$ whose objects are those A such that η_A is an isomorphism and dually $\mathbf{Fix}(FG) \subset \mathscr{B}$. Prove that the adjunction restricts to an equivalence between $\mathbf{Fix}(GF)$ and $\mathbf{Fix}(FG)$

b) Give some examples of what happens in part a)

Proof. a) It suffices to prove that the restriction is well defined. So we have to prove that if η_A is an isomorphism than so is $\varepsilon_F(A)$ and if ε_B is an iso then so is $\eta_G(B)$. So let A be given such that η_A is an isomorphism. That means that $\eta_A : A \to G \circ F(A)$ is an isomorphism. We want to show that $\varepsilon_F(A) : F(A) \to (F \circ G)(F(A))$ is an isomorphism. But looking at the triangle identity

$$\begin{tikzcd} A \arrow[r, "F(\eta_A)"] \arrow[dr, "\mathrm{id}_{F(A)}"'] & FGF(A) \arrow[d, "\varepsilon_{F(A)}"] \\ & F(A) \end{tikzcd}$$

As functors respect isomorphisms we see that $F(\eta_A)$ and $\mathrm{id}_{F(A)}$ are isomorphisms. But than $\varepsilon_{F(A)}$ has to be an iso as well.

The argument works exactly the same for the other direction. (one probably could say something about duality here...)

b) Let us give one example. Recall that given the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ we have a left adjoint given by $I : \mathbf{Set} \to \mathbf{Top}$ that equipts a set with the discrete topologie.

That is $\begin{tikzcd} \mathbf{Top} \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend left, "I"] \end{tikzcd}$

In this case the map $U \circ I$ is the identity so $\mathbf{Fix}(UI) = \mathbf{Sets}$ is just everything. However the map $I \circ U$ has a problem if the topological space we start with is not discrete. It is easy to see that $\mathbf{Fix}(IU)$ is exactly the full subcategory given by the topological spaces with discrete topology. It is easy to see that both categories are equivalent.

$\square$

Exercise 5.3. Let $f\colon K \to L$ be a map of sets, and denote by $f^*\colon \mathcal{P}(L) \to \mathcal{P}(K)$ the map sending a subset S of L to its inverse image $f^{-1}S \subseteq K$. Then f^* is order-preserving with respect to the inclusion orderings on $\mathcal{P}(K)$ and $\mathcal{P}(L)$, and so can be seen as a functor. Find left and right adjoints to f^*.

Solution. Denote by $f_!\colon \mathcal{P}(K) \to \mathcal{P}(L)$ the map sending a subset T of K to its image $fT \subseteq L$. $f_!$ is order-preserving, hence can be seen as a functor. In the literature, this functor is often called "the exceptional image functor".

Denote by $f_*\colon \mathcal{P}(K) \to \mathcal{P}(L)$ the map sending a subset T of K to $L \setminus f(K \setminus T)$, i.e. to the complement of the image of the complement. In the literature, this functor is often called "the direct image functor"

We will show that $f_! \vdash f^* \vdash f_*$.

We start with $f_! \vdash f^*$. It suffices to show that for all $T \subseteq K$ we have $T \subseteq f^*(f_!(T))$ and for all $S \subseteq L$ we have $f_!(f^*(S)) \subseteq S$.

For the first inclusion, let $x \in T$. Then $f(x) = f(x) \in f(T) = f_!(T)$, hence $x \in f^{-1}(f_!(T)) = f^*(f_!(T))$.

For the second inclusion, let $y \in f_!(f^*(S))$. Than there exists $x \in f^*(S)$ such that $y = f(x)$. Since $x \in f^*(S) = f^{-1}(S)$, we get that $y \in S$.

This proves $f_! \vdash f^*$.

We now prove $f^* \vdash f_*$. It suffices to show that for all $S \subseteq L$ we have $S \subseteq f_*(f^*(S))$ and for all $T \subseteq K$ we have $f^*(f_*(T)) \subseteq T$.

For the first inclusion, let $x \in S$. We have $f_*(f^*(S)) = L \setminus f(K \setminus f^*(S))$. So it suffices to show that $x \notin f(K \setminus f^*(S))$. But it is clear that $S \cap f(K \setminus f^*(S)) = \emptyset$.

For the second inclusion, let $y \in f^*(f_*(T))$. This means that $f(y) \in f_*(T) = L \setminus f(K \setminus T)$. This implies, that $y \notin K \setminus T$, hence $y \in T$. $\square$