Nilpotence of η in étale motivic spectra

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Abstract

We show that every object of the stable étale motivic homotopy category over any scheme is η-complete. In some cases we show that in fact the fourth power of η is null, whereas the third power of η is always nonvanishing, similar to the situation in topology.

Contents

Introduction

In topology, the Hopf map provides the first example of a nonzero element of a homotopy group of the form $\pi_n(S^{n-1})$, it is a fibration $S^3 \to S^2$ whose fibers are all isomorphic to S^1. A simple definition is as follows:

$$\eta_{\mathrm{top}}\colon S^3 \simeq \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1 \simeq S^2$$

is the projection, up to homotopy. The (desuspended) image of η_{top}, pointed at any point of S^3, in the category of spectra Sp is a map

$$\eta_{\mathrm{top}}\colon \Sigma\mathbb{S} \to \mathbb{S}$$

from the suspension of the sphere spectrum $\mathbb{S}$ to the sphere spectrum. It provides a generator for the first stable homotopy group of the sphere:

$$0 \neq \eta_{\mathrm{top}} \in \pi_1(\mathbb{S}) \simeq \mathbb{Z}/2\mathbb{Z}.$$

One may compute the powers of η_{top}, and show that $\eta_{\mathrm{top}}^4 = 0$. This is easy, as $\pi_4(\mathbb{S}) = 0$, which can be read of from the E_2-page of the Adams spectral sequence. It is true, but harder to see, that $\eta_{\mathrm{top}}^3 \neq 0 \in \pi_3(\mathbb{S}) = \mathbb{Z}/24\mathbb{Z}$. Indeed, this can be done by computations with Toda brackets, see [Tod62, Chapter V, Equation 5.5 and Proposition 5.6].

Motivic homotopy theory aims to imitate the methods of algebraic topology in the world of algebraic geometry. For that matter, many classical results of homotopy theory have now a version in algebraic geometry, see e.g. [Hoy15, AHW17, ABH23]. Let S be a scheme. Following Morel and Voevodsky, one considers the stable ∞-category $\mathcal{SH}(S)$ of $\mathbb{A}^1$-invariant motivic spectra. As in topology, we have an algebraic Hopf map given by the canonical projection

$$\eta\colon \mathbb{A}^2_S \setminus \{0\} \to \mathbb{P}^1_S$$

whose (desuspended) image in $\mathcal{SH}(S)$ is a map

$$\eta\colon \mathbb{G}_m \to \mathbb{S}$$

from the motivic sphere $\mathbb{G}_m$ to the motivic sphere spectrum $\mathbb{S}$. As a corollary to a theorem of Morel [Mor04, Corollary 6.4.5], one may compute over a perfect field k the endomorphisms of the η-inverted sphere $\mathbb{S}[\eta^{-1}] \coloneqq \operatorname{colim}(\mathbb{S} \xrightarrow{\eta} \mathbb{G}_m^{\otimes -1} \xrightarrow{\eta} \mathbb{G}_m^{\otimes -2} \to \cdots)$: we have

$$\operatorname{End}_{\mathcal{SH}(k)}(\mathbb{S}[\eta^{-1}]) \simeq W(k),$$

where $W(k)$ is the Witt ring of symmetric bilinear forms of k. In particular, by pulling back to fields, we see that the map η is *never nilpotent* in $\mathcal{SH}(S)$, for any (nonempty) scheme S. This implies that in $\mathcal{SH}(S)$, there exists many η-periodic objects, that is, objects M such that the map $\eta\colon \mathbb{G}_m \otimes M \to M$ is an equivalence.

In this short note, we observe that this discrepancy between motivic homotopy theory and classical homotopy theory disappears if ones works in the étale local stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}_{\text{ét}}(S)$ (see e.g. [Bac21, §5] for a definition, in the étale setting we will always work with hypersheaves). Our main result is the following:

Theorem A (Theorem 3.13)**.** *Let S be a scheme. Then for $X \in \mathcal{SH}_{\text{ét}}(S)$, the object $X[\eta^{-1}]$ is zero. In particular, every object of $\mathcal{SH}_{\text{ét}}(S)$ is η-complete, and η acts nilpotently on any compact object of $\mathcal{SH}_{\text{ét}}(S)$.*

A corollary of this result is the following:

Corollary B (Corollary 3.14)**.** *Let S be a scheme. The étale sheafification functor $L_{\text{ét}}\colon \mathcal{SH}(S) \to \mathcal{SH}_{\text{ét}}(S)$ factors canonically over $\mathcal{SH}(S)^{\wedge}_{\eta}$. In particular, any object of $\mathcal{SH}(S)$ that satisfies étale descent is already η-complete.*

In good cases, we can compute the index of nilpotence of η in $\mathcal{SH}_{\text{ét}}(S)$. For example, if k is an algebraically closed field, we show that in $\mathcal{SH}_{\text{ét}}(k)$ we have $\eta^4 = 0$. More generally:

Theorem C (Corollaries 3.7 and 3.11)**.** *Let S be any scheme. Then there exists a finite faithfully flat map $S' \to S$ such that η^4 is null in $\mathcal{SH}_{\text{ét}}(S')$.*

If there exists a map $f\colon S \to \operatorname{Spec}(k)$ where k is a field with $\operatorname{cd}_2(k) \leqslant 1$ and $\sup_{p\in\mathbb{P}} \operatorname{cd}_p(k) < \infty$ (e.g., any scheme defined over a finite field or an algebraically closed field), then η^4 is already null in $\mathcal{SH}_{\text{ét}}(S)$.

In fact, we conjecture the following:

Conjecture (Conjecture 3.9)**.** *For any scheme S we have $\eta^4 \cong 0$ in $\mathcal{SH}_{\text{ét}}(S)$.*

Remark. Since $f^*\eta^4 \cong \eta^4$ where $f\colon S \to \operatorname{Spec}(\mathbb{Z})$ is the unique morphism, it is of course enough to show that $\eta^4 \cong 0$ in $\mathcal{SH}_{\text{ét}}(\mathbb{Z})$.

On the other hand, we know that on almost all schemes that η^3 is not null:

Theorem D (Theorem 4.2)**.** *Let S be a scheme which has a point of characteristic not 2. Then η^3 is not null in $\mathcal{SH}_{\text{ét}}(S)$.*

Remark. If S is a scheme where all points are of characteristic 2, then η (and in particular η^3) is null in $\mathcal{SH}_{\text{ét}}(S)$ by Remark 4.3.

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1 Recollections on stable étale motivic homotopy theory

Definition 1.1. Let S be a scheme. We say that S is *étale bounded* if

$$\mathrm{sup}_{x\in X, p\in\mathbb{P}}\mathrm{cd}_p(\kappa(x)) < \infty,$$

where $\mathrm{cd}_p(k)$ is the mod-p-Galois cohomological dimension of a field k, and $\mathbb{P}$ is the set of all prime numbers. Similarly, we say that S is *étale locally étale bounded* if there exists an étale cover $S' \to S$ such that S' is étale bounded.

Example 1.2. *By [Bac21, Example 2.14] the scheme* $\mathrm{Spec}(\mathbb{Z})$ *is étale locally étale bounded, and the scheme* $\mathrm{Spec}(\mathbb{Z}[i]) = \mathrm{Spec}(\mathbb{Z}[x]/(x^2+1))$ *is étale bounded.*

Lemma 1.3. *Let $X \to S$ be a morphism of finite type with S quasi-compact. If S is (étale locally) étale bounded, the same is true for X.*

Proof. Suppose that S is étale locally étale bounded. Choose an étale cover $S' \to S$ so that S' is étale bounded. In particular, we get an étale cover $S' \times_S X \to X$, such that $S' \times_S X \to S'$ is of finite type.

Hence, we may assume that S is étale bounded, and we have to show that the same is true for X. For this, see [Mat25, Lemma 2.24] (the extra assumptions given in the reference that S is of finite Krull dimension and that $X \to S$ is smooth are not necessary). □

Lemma 1.4. *Let S be a scheme that is étale locally étale bounded. Then $\mathcal{SH}_{\text{ét}}(S)$ is compactly generated. If S is moreover étale bounded, then $\Sigma^\infty_+ X \in \mathcal{SH}_{\text{ét}}(S)$ is compact for every qcqs $X \in \mathrm{Sm}_S$.*

Proof. This is [AGV22, Proposition 2.4.22] (see also Remark 2.4.23 of *ibid*). □

Theorem 1.5 (Rigidity)**.** *Let S be a scheme and ℓ a prime. Then there are canonical equivalences*

$$\mathcal{SH}_{\text{ét}}(S)^\wedge_\ell \cong \mathcal{SH}_{\text{ét}}(S[1/\ell])^\wedge_\ell \cong \mathrm{Shv}_{\text{ét}}(S[1/\ell], \mathrm{Sp})^\wedge_\ell,$$

where we write $S[1/\ell] \coloneqq S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[1/\ell])$.

Proof. Consider the open immersion $S[1/\ell] \to S$, with closed reduced complement Z of characteristic ℓ. By [Ayo07, Corollaire 4.5.4] there is a recollement $\mathcal{SH}_{\text{ét}}(S[1/\ell])^\wedge_\ell \to \mathcal{SH}_{\text{ét}}(S)^\wedge_\ell \to \mathcal{SH}_{\text{ét}}(Z)^\wedge_\ell$ (note that Ayoub implicitly fixes a topology, which is allowed to be the étale topology, see the beginning of [Ayo07, Section 4.5]; the fact that Ayoub's result implies that we have an ∞-categorical recollement is classical, see [Rob14, Proposition 9.4.20]). Moreover, $\mathcal{SH}_{\text{ét}}(Z)^\wedge_\ell = 0$ by [BH21, Theorem A.1]. This implies the first equivalence. The second equivalence in this generality is [BH21, Theorem 3.1]. □

2 Recollections on completions and periodizations

In this section, let $\mathcal{E}$ be a stable presentably symmetric monoidal category with unit $\mathbb{S}$, and $\nu\colon T \to \mathbb{S}$ be a map with T tensor-invertible. Consider the functor $(-)/\!\!/\nu$ that is given as the cofiber of $T \otimes - \xrightarrow{\nu} -$.

Definition 2.1. We say that a map $f\colon X \to Y$ is a ν-equivalence, if $f/\!\!/\nu$ is an equivalence. Write $(-)^\wedge_\nu\colon \mathcal{E} \to \mathcal{E}$ for the associated Bousfield localization at ν-equivalences, called *ν-completion*. We write $\mathcal{E}^\wedge_\nu$ for the essential image of $(-)^\wedge_\nu$.

Definition 2.2. We say that an object $X \in \mathcal{E}$ is *ν-periodic* if $X/\!\!/\nu = 0$ (equivalently, $T \otimes X \xrightarrow{\nu} X$ is an equivalence). The ν-periodization functor is the Bousfield localization $(-)[\nu^{-1}]\colon \mathcal{E} \to \mathcal{E}$ with essential image the subcategory of ν-periodic objects. Since ν-periodic objects are evidently closed under limits and colimits (for limits use that $T \otimes -$ preserves them since T is invertible), this localization exists by the adjoint functor theorem.

Lemma 2.3. *The localization functor $(-)[\nu^{-1}]$ is smashing, i.e. for all $X \in \mathcal{E}$ we have $X[\nu^{-1}] \cong \mathbb{S}[\nu^{-1}] \otimes X$.*

Proof. By [AI22, Lemma A.5.2] it is enough to show that for all $X, Y \in \mathcal{E}$ such that X is ν-periodic, then so is $X \otimes Y$ and $\underline{\mathrm{map}}(Y, X)$. Here, $\underline{\mathrm{map}}(-,-)$ denotes the internal hom object in $\mathcal{E}$ (which exists by the adjoint functor theorem). But now both $\nu \otimes X \otimes Y$ and $\nu \otimes \underline{\mathrm{map}}(Y, X) \cong \underline{\mathrm{map}}(Y, \nu \otimes X)$ are equivalences, since already $\nu \otimes X$ is one. $\square$

Lemma 2.4. *Let $f\colon X \to Y$ be a map in $\mathcal{E}$. Then f is a ν-equivalence if and only if* $\mathrm{fib}(f)$ *is ν-periodic.*

Proof. The map f is a ν-equivalence if and only if $f /\!\!/ \nu$ is an equivalence, i.e., if and only if $0 = \mathrm{fib}(f /\!\!/ \nu) \cong \mathrm{fib}(f) /\!\!/ \nu$. But the latter is zero if and only if $\mathrm{fib}(f)$ is ν-periodic. $\square$

We now try to describe the ν-periodization functor explicitly. For this, recall the following definition:

Definition 2.5. Let $X \in \mathcal{E}$. We define the *mapping telescope* $M_\nu(X)$ as the filtered colimit

$$\mathrm{colim}\, X \xrightarrow{\nu} T^{\otimes -1} \otimes X \xrightarrow{\nu} T^{\otimes -2} \otimes X \to \cdots$$

Since the tensor product is compatible with colimits, we see that $M_\nu(X) \cong M_\nu(\mathbb{S}) \otimes X$.

Now, in a variety of situations, the mapping telescope agrees with the ν-periodization.

Lemma 2.6. *Suppose that there exists a compactly generated presentably symmetric monoidal stable category $\mathcal{D}$ with unit $\widetilde{\mathbb{S}}$, and a symmetric monoidal left adjoint $L\colon \mathcal{D} \to \mathcal{E}$. Suppose moreover that there exists a map $\widetilde{\nu}\colon \widetilde{T} \to \widetilde{\mathbb{S}}$ in $\mathcal{D}$ with $\widetilde{T}$ tensor invertible, such that $L(\widetilde{\nu}) \simeq \nu$.*

Then for all $X \in \mathcal{E}$ we have $M_\nu(X) \cong X[\nu^{-1}]$.

Proof. We have $M_\nu(X) \cong M_\nu(\mathbb{S}) \otimes X$ and $X[\nu^{-1}] \cong \mathbb{S}[\nu^{-1}] \otimes X$ (the latter holds since the localization is smashing). Hence, it suffices to prove the result for $X = \mathbb{S}$. It is clear that $\mathbb{S} \to M_\nu(\mathbb{S})$ is sent to an equivalence by the functor $(-)[\nu^{-1}]$. Thus it suffices to prove that $M_\nu(\mathbb{S})$ is ν-periodic. For this, as $L(M_{\widetilde{\nu}}(\widetilde{\mathbb{S}})) \cong M_\nu(L\widetilde{\mathbb{S}}) \cong M_\nu(\mathbb{S})$, it suffices to prove the statement in $\mathcal{D}$, which is compactly generated. Then the result is [Bac18, Lemma 17]. $\square$

Remark 2.7. The last lemma holds for example if $\mathcal{E}$ is compactly generated.

Lemma 2.8. *Let $X \in \mathcal{E}$ be a compact object such that $M_\nu(X) = 0$. Then there exists $n \gg 0$ such that $\nu^n\colon X \to T^{\otimes -n} \otimes X$ is null.*

Proof. Since X is compact, we have a filtered colimit of abelian groups

$$0 = \pi_0 \mathrm{Map}_{\mathcal{E}}(X, M_\nu(X)) \cong \operatorname*{colim}_n \pi_0 \mathrm{Map}_{\mathcal{E}}(X, T^{\otimes -n} \otimes X).$$

In particular, the vanishing of the canonical map $X \to M_\nu(X)$ is witnessed on some finite stage, whence we obtain $\nu^n \simeq 0$ for some $n \gg 0$. $\square$

3 Nilpotence of η

Definition 3.1. Let S be a scheme. The algebraic Hopf map over S is the map

$$\eta\colon \mathbb{G}_m \to \mathbb{S}$$

in $\mathcal{SH}(S)$ obtained as the $\mathbb{P}^1$-desuspension of the quotient map $\mathbb{A}^2_S \setminus \{0\} \to \mathbb{P}^1_S$.

Remark 3.2. By smooth base change, if $f\colon T \to S$ is any map of schemes, then $f^*\eta \simeq \eta$ in $\mathcal{SH}(T)$.

We immediately get the following equivalence between the η-periodization and the mapping telescope.

Proposition 3.3. *Let S be a scheme and $X \in \mathcal{SH}_{\text{ét}}(S)$. Then $X[\eta^{-1}] \cong M_\eta(X)$.*

Proof. By the last remark, for the canonical left adjoint $f^*\colon \mathcal{SH}_{\text{ét}}(\mathrm{Spec}(\mathbb{Z})) \to \mathcal{SH}_{\text{ét}}(S)$, we have $f^*\eta \simeq \eta$. Hence, the result follows from Lemma 2.6, since $\mathcal{SH}_{\text{ét}}(\mathrm{Spec}(\mathbb{Z}))$ is compactly generated by Lemma 1.4. $\square$

Lemma 3.4. *Assume that -1 is a sum of squares on a scheme S. Then $\eta = 0$ in $\mathcal{SH}(S)[\frac{1}{2}]$.*

Proof. If -1 is a sum of n squares in S, then there is a map of rings

$$\mathbb{Z}[x_1, \dots, x_n]/(x_1^2 + \dots x_n^2 + 1) \to \mathcal{O}_S(S).$$

Thus, we may assume that S is of finite type over $\mathrm{Spec}(\mathbb{Z})$. By [CD19, Lemma 16.2.3] (in *loc. cit.* they invert all primes, but the proof works *verbatim* with only 2 inverted, see their [CD19, Remark 16.2.12]), there is some idempotent element $\varepsilon \in \mathrm{End}_{\mathcal{SH}(S)[\frac{1}{2}]}(\mathbb{S}[\frac{1}{2}])$ such that $\eta = \varepsilon\eta$. In particular, $\mathcal{SH}(S)[\frac{1}{2}] \simeq \mathcal{SH}(S)[\frac{1}{2}]^+ \times \mathcal{SH}(S)[\frac{1}{2}]^-$ where the + part (*resp.* the $-$ part) consists of modules over $\mathrm{Im}\frac{1-\varepsilon}{2}$ (*resp.* modules over $\mathrm{Im}\frac{1+\varepsilon}{2}$). The image of η in $\mathcal{SH}(S)[\frac{1}{2}]^+$ is zero, since $\frac{1-\epsilon}{2}\eta = \frac{\eta-\eta}{2} = 0$. Thus, it suffices to show that $\mathcal{SH}(S)[\frac{1}{2}]^- = 0$. By [CD19, Proposition 4.3.17] (that we may apply thanks to [AGV22, Proposition 2.5.11]) we may assume that $S = \mathrm{Spec}(k)$ is the spectrum of a field.

Over a field, the splitting of $\mathcal{SH}(k)[\frac{1}{2}]$ is induced by a splitting of the endomorphisms of the unit $GW(k)[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}] \times W(k)[\frac{1}{2}]$, where $GW(k)$ and $W(k)$ are the Grothendieck-Witt ring and the Witt ring of k (see e.g. [BH20, §2.7.3]). Hence, it suffices to show that under our assumptions $W(k)$ has 2-power torsion. This is for example proven in [Sch12, Chapter 2, Theorem 7.1]. $\square$

Proposition 3.5. *Let $S = \mathrm{Spec}(\overline{\mathbb{Z}})$ be the spectrum of the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$. The map $\eta^4 \colon \mathbb{G}_m^{\otimes 4} \to \mathbb{S}$ is null in $\mathcal{SH}_{\text{ét}}(S)$.*

Proof. We begin with the observation that S is étale bounded: its residual fields are $\overline{\mathbb{Q}}$ and copies of $\overline{\mathbb{F}}_p$ for all prime numbers p, which, as they are algebraically closed, have étale cohomological dimension 0. Consider the arithmetic fracture square

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{S}_2^\wedge \arrow[d] \\ \mathbb{S}[1/2] \arrow[r] & \mathbb{S}_2^\wedge[1/2] \end{tikzcd}$$

in $\mathcal{SH}_{\text{ét}}(S)$ (cf. e.g. [Mat24, Corollary 7.3]). Mapping into this from $\mathbb{G}_m^{\otimes 4}$ gives the following cartesian square of mapping spectra:

$$\begin{tikzcd} \mathrm{map}(\mathbb{G}_m^{\otimes 4}, \mathbb{S}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{map}(\mathbb{G}_m^{\otimes 4}, \mathbb{S}_2^\wedge) \arrow[d] \\ \mathrm{map}(\mathbb{G}_m^{\otimes 4}, \mathbb{S}[1/2]) \arrow[r] & \mathrm{map}(\mathbb{G}_m^{\otimes 4}, \mathbb{S}_2^\wedge[1/2]). \end{tikzcd}$$

On homotopy groups we get the following (part of a) long exact sequence:

$$\pi_1(\mathrm{map}(\mathbb{G}_m^{\otimes 4}, \mathbb{S}_2^\wedge[1/2])) \longrightarrow \pi_0(\mathrm{map}(\mathbb{G}_m^{\otimes 4}, \mathbb{S})) \longrightarrow \pi_0(\mathrm{map}(\mathbb{G}_m^{\otimes 4}, \mathbb{S}_2^\wedge)) \oplus \pi_0(\mathrm{map}(\mathbb{G}_m^{\otimes 4}, \mathbb{S}[1/2])).$$

By Theorem 1.5, we have $\mathcal{SH}_{\text{ét}}(S)_2^\wedge \simeq \mathcal{SH}_{\text{ét}}(S[1/2])_2^\wedge \simeq \mathrm{Shv}_{\text{ét}}(S[1/2], \mathrm{Sp})_2^\wedge$, and the equivalence sends $(\mathbb{G}_m)_2^\wedge$ to the object $\Sigma\mathbb{S}_2^\wedge$: indeed by [Bac21, Theorem 6.5] $(\mathbb{G}_m)_2^\wedge$ is equivalent to the twisting spectrum $\hat{\mathbb{1}}_2(1)[1]$, and by [Bac21, Theorem 3.6] this twisting spectrum is equivalent to $\mathbb{S}_2^\wedge[1]$ when S has all 2-power roots of unity. Recall also that $\mathbb{G}_m^{\otimes 4}$ is compact in $\mathcal{SH}_{\text{ét}}(S)$ (cf. Lemma 1.4). This allows us to rewrite the above exact sequence as:

$$\pi_5(\mathrm{R\Gamma}(S[1/2]_{\text{ét}}, \mathbb{S}_2^\wedge))[1/2] \to \pi_0(\mathrm{map}(\mathbb{G}_m^{\otimes 4}, \mathbb{S})) \to \pi_4(\mathrm{R\Gamma}(S[1/2]_{\text{ét}}, \mathbb{S}_2^\wedge)) \oplus \pi_0(\mathrm{map}(\mathbb{G}_m^{\otimes 4}[1/2], \mathbb{S}[1/2])).$$

Write (f, g) for the image of η^4 under the right map.

It suffices to show that $f = 0 = g$, and moreover that $\pi_5(\mathrm{R\Gamma}(S[1/2]_{\text{ét}}, \mathbb{S}_2^\wedge)) \cong 0$. Note first that because S has all roots of unity, the map g vanishes by Lemma 3.4. We compute $\pi_i(\mathrm{R\Gamma}(S[1/2]_{\text{ét}}, \mathbb{S}_2^\wedge))$ for $i > 0$. Because the étale cohomological dimension of $S[1/2]$ is zero, the descent spectral sequence ([CM21, Proposition 2.13], that we may apply because étale hypersheaves of spectra on S are indeed Postnikov complete by [Bac21, Lemma 2.16], using that S is étale bounded)

$$E_2^{p,q} = \mathrm{H}^p(S[1/2]_{\text{ét}}, \pi_{-q}((\mathbb{S}_{\text{top}})_2^\wedge)) \Rightarrow \pi_{-p-q}(\mathrm{R\Gamma}(S[1/2]_{\text{ét}}, \mathbb{S}_2^\wedge))$$

ensures that

$$\pi_i(\mathrm{R}\Gamma(S[1/2]_{\text{ét}}, \mathbb{S}_2^\wedge)) \simeq \mathrm{H}^0(S[1/2]_{\text{ét}}, \pi_i((\mathbb{S}_{\text{top}})_2^\wedge)),$$

with $\mathbb{S}_{\text{top}} \in \mathrm{Sp}$ the topological sphere spectrum. Now, for both $i = 4$ and $i = 5$ we have that $\pi_i((\mathbb{S}_{\text{top}})_2^\wedge) \cong 0$ (see e.g. the table after [Rav03, Definition 1.1.6]), which implies $\pi_i(\mathrm{R}\Gamma(S[1/2]_{\text{ét}}, \mathbb{S}_2^\wedge)) \cong 0$. This finishes the proof. □

Using a similar technique, we also have the following:

Proposition 3.6. *Let k be a field with $\mathrm{cd}_2(k) \leqslant 1$ and $\sup_{p\in\mathbb{P}} \mathrm{cd}_p(k) < \infty$ (e.g. a finite field, cf. [Ser94, Chapter II, §3.3 (a)], or a separably closed field). Then in $\mathcal{SH}_{\text{ét}}(k)$, we have $\eta^4 = 0$.*

Proof. First note that because the 2-cohomological dimension of k is finite, -1 is a sum of squares in k: Indeed, suppose not. Then k is orderable, and the absolute Galois group of k contains an element of order 2. Thus, the 2-cohomological dimension is infinite, cf. [Ser94, Chapitre II, §4.1, Proposition 10']. We begin as in Proposition 3.5 (note that $\mathrm{Spec}(k)$ is étale bounded by assumption): there is a short exact sequence

$$\pi_5(\mathrm{R}\Gamma(k_{\text{ét}}, \hat{\mathbb{1}}_2(-4)))[1/2] \to \pi_0(\mathrm{map}_{\mathcal{SH}_{\text{ét}}(k)}(\mathbb{G}_m^{\otimes 4}, \mathbb{S})) \to \pi_4(\mathrm{R}\Gamma(k_{\text{ét}}, \hat{\mathbb{1}}_2(-4))) \oplus \pi_0(\mathrm{map}(\mathbb{G}_m^{\otimes 4}[1/2], \mathbb{S}[1/2])),$$

and we denote by (f, g) the image of η^4 by the right map. By Lemma 3.4 we know that $g = 0$. We will now show that $f = 0$ be showing that the whole group $\pi_4(\mathrm{R}\Gamma(k_{\text{ét}}, \hat{\mathbb{1}}_2(-4)))$ vanishes. First, we need the following fact: $\pi_k(\hat{\mathbb{1}}_2(-4))$ is 2-power torsion for all $k > 0$ and vanishes if $k = 4$ or $k = 5$. For this, consider the t-exact conservative stalk functor $\rho^*\colon \mathrm{Shv}_{\text{ét}}(k_{\text{ét}}, \mathrm{Sp}) \to \mathrm{Shv}_{\text{ét}}((k^{\text{sep}})_{\text{ét}}, \mathrm{Sp}) = \mathrm{Sp}$. Note that $\rho^*(\hat{\mathbb{1}}_2(-4)) \cong \hat{\mathbb{1}}_2(-4) \cong \hat{\mathbb{1}}_2$, the (2-completed) sphere spectrum, where we used Theorem 1.5 and [Bac21, Theorem 3.6 (2) and (3)] (note that we do not need to re-2-complete after pulling back along ρ because the identification can be made in the ∞-category of proétale sheaves, where ρ^* commutes with limits as it is a slice). But now $\pi_k(\mathbb{S})$ is torsion for every $k > 0$ by Serre's finiteness theorem [Rav03, Theorem 1.1.8], and $\pi_4(\mathbb{S}) = \pi_5(\mathbb{S}) = 0$.

Consider the descent spectral sequence ([CM21, Proposition 2.13], again our sheaves are Postnikov complete by [Bac21, Lemma 2.16])

$$E_2^{p,q} = \pi_{-p}\mathrm{R}\Gamma(k_{\text{ét}}, \pi_{-q}(\hat{\mathbb{1}}_2(-4))) \Rightarrow \pi_{-p-q}\mathrm{R}\Gamma(k_{\text{ét}}, \hat{\mathbb{1}}_2(-4)).$$

Hence, to see that $\pi_4\mathrm{R}\Gamma(k_{\text{ét}}, \hat{\mathbb{1}}_2(-4)) = 0$, it suffices to show (using that $k_{\text{ét}}$ is of 2-cohomological dimension ≤ 1 and that $\pi_n(\hat{\mathbb{1}}_2(-4))$ is 2-power torsion for all $n > 0$) that $\pi_i(\hat{\mathbb{1}}_2(-4)) = 0$ for $i = 4$ and $i = 5$. This we have seen above.

Hence, we see that η^4 comes from an element in $\pi_5(\mathrm{R}\Gamma(k_{\text{ét}}, \hat{\mathbb{1}}_2(-4)))[1/2]$. Since η (and hence η^4) is 2-torsion by Lemma 3.4, we conclude that $\eta^4 = 0$. □

Corollary 3.7. *Let S be a scheme with a map to a field k with $\mathrm{cd}_2(k) \leqslant 1$ and $\sup_{p\in\mathbb{P}} \mathrm{cd}_p(k) < \infty$ (e.g. any scheme of equicharacteristic $p > 0$, or any scheme defined over an algebraically closed field). The map $\eta^4\colon \mathbb{G}_m^{\otimes 4} \to \mathbb{S}$ is null in $\mathcal{SH}_{\text{ét}}(S)$.*

Proof. Since η^4 pulls back to η^4 along the map $S \to \mathrm{Spec}(k)$, we may assume that $S = \mathrm{Spec}(k)$, in which case the result is Proposition 3.6. □

Remark 3.8. One would hope that a similar proof shows that $\eta^4 = 0$ over $\mathbb{Z}[i]$. This does not quite work, as in the spectral sequence we get an additional nonzero term given by the étale cohomology group $H^2_{\text{ét}}(\mathrm{Spec}(\mathbb{Z}[i]), \pi_6(\hat{\mathbb{1}}_2(-4))) \neq 0$.

Nonetheless, we conjecture the following:

Conjecture 3.9. *For any scheme S we have $\eta^4 \cong 0$ in $\mathcal{SH}_{\text{ét}}(S)$.*

Remark 3.10. 1. If the conjecture holds for S, and $f\colon S' \to S$ is a morphism, then the conjecture also holds for S': Indeed, $\eta^4 \simeq f^*\eta^4 \simeq 0$.

2. As noted in Remark 3.8, the obstruction for the argument in Proposition 3.6 to work for $\mathbb{Q}(i)$ lies in the apparition of a $\mathrm{H}^2_{\text{ét}}(\mathbb{Q}(i), \pi_6(\hat{\mathbb{1}}_2(-4)))$. In fact, as the stalk of $\pi_6(\hat{\mathbb{1}}_2(-4))$ is $\mathbb{Z}/2$ (thus $\pi_6(\hat{\mathbb{1}}_2(-4))$ is the constant sheaf $\mathbb{Z}/2$ as $\mathrm{Aut}(\mathbb{Z}/2) = \{1\}$), and as $\pi_5(\mathrm{R}\Gamma(k_{\text{ét}}, \hat{\mathbb{1}}_2(-4)))[1/2]$ is always zero because it is the localization at 2 of a 2-torsion abelian group (this can again be read in the spectral sequence), we obtain an isomorphism
$$\pi_0(\mathrm{map}_{\mathcal{SH}_{\text{ét}}(\mathbb{Q}(i))}(\mathbb{G}_m^{\otimes 4}, \mathbb{S})) \simeq \mathrm{Br}(\mathbb{Q}(i))[2]$$
between the group in which lives η^4 and the 2-torsion in the Brauer group of $\mathbb{Q}(i)$. By the Albert–Brauer–Hasse–Noether short exact sequence [Hür92, Theorem 4], one may identify this right-hand side with
$$\mathrm{Br}(\mathbb{Q}(i))[2] \simeq \ker(\bigoplus_p \mathrm{Br}(\mathbb{Q}_p(i))[2] \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z})$$
and even better as each $\mathrm{Br}(\mathbb{Q}_p(i))[2]$ is isomorphic to $\mathbb{Z}/2$ ([CF67, Chapter VI, §1.1, Theorem 1 and Corollary]), we see that we have an equivalence
$$\pi_0(\mathrm{map}_{\mathcal{SH}_{\text{ét}}(\mathbb{Q}(i))}(\mathbb{G}_m^{\otimes 4}, \mathbb{S})) \simeq \ker(\bigoplus_p \mathbb{Z}/2 \xrightarrow{\text{sum}} \mathbb{Z}/2).$$
The maps $\mathrm{Br}(\mathbb{Q}(i))[2] \to \mathrm{Br}(\mathbb{Q}_p(i))[2]$ are given by restriction, thus by the functoriality of the spectral sequence we used, we see that *there is only finitely many prime numbers p such that η^4 is nonzero in $\mathcal{SH}_{\text{ét}}(\mathbb{Q}_p(i))$*. Moreover, that number of primes is even. This also implies that to prove Conjecture 3.9 for $\mathrm{Spec}(\mathbb{Q}(i))$ it suffices to do it for $\mathrm{Spec}(\mathbb{Q}_p(i))$ for all prime numbers. See Remark 4.6 for an additional result in this direction.

Using the result about $\mathrm{Spec}(\overline{\mathbb{Z}})$, we get the following weak version of the conjecture:

Corollary 3.11. *Let S be a scheme. Then there exists a finite faithfully flat map $X \to S$ such that $\eta^4 = 0$ in $\mathcal{SH}_{\text{ét}}(X)$.*

Proof. We claim that it is enough to show this for $\mathrm{Spec}(\mathbb{Z})$. Indeed, suppose we have found a finite faithfully flat map $X \to \mathrm{Spec}(\mathbb{Z})$ such that $\eta^4 = 0$ in $\mathcal{SH}_{\text{ét}}(X)$. Then consider the pulled back finite faithfully flat map $X \times S \to S$. Since η^4 in $\mathcal{SH}_{\text{ét}}(X \times S)$ is pulled back from $\mathcal{SH}_{\text{ét}}(X)$, it vanishes.

By Proposition 3.5, the map $\widetilde{X} := \mathrm{Spec}(\overline{\mathbb{Z}}) \to \mathrm{Spec}(\mathbb{Z})$ is such that $\eta^4 = 0$ in $\mathcal{SH}_{\text{ét}}(\widetilde{X})$. Note that $\widetilde{X} \to \mathrm{Spec}(\mathbb{Z})$ factors through $\mathrm{Spec}(\mathbb{Z}[i])$. In particular, we may write $\overline{\mathbb{Z}}$ as a filtered colimit of finite algebras A_α over $\mathbb{Z}[i]$, which are thus all étale bounded by Lemma 1.3 and Example 1.2.

Hence, if we denote by $X_\alpha = \mathrm{Spec}(A_\alpha)$, the categories $\mathcal{SH}_{\text{ét}}(X_\alpha)$ and $\mathcal{SH}_{\text{ét}}(\widetilde{X})$ are compactly generated, with $\mathbb{G}_m$ and $\mathbb{S}$ are compact in them, cf. Lemma 1.4. Moreover, by continuity ([AGV22, Proposition 2.5.11]), the map
$$\mathrm{colim}_\alpha\, \pi_0(\mathrm{map}_{\mathcal{SH}_{\text{ét}}(X_\alpha)}(\mathbb{G}_m^{\otimes 4}, \mathbb{S})) \to \pi_0(\mathrm{map}_{\mathcal{SH}_{\text{ét}}(\widetilde{X})}(\mathbb{G}_m^{\otimes 4}, \mathbb{S}))$$
is an isomorphism of abelian groups (see [DFJK21, A.10]). As η^4 vanishes in the colimit, there exists a finite level, say X_{α_0}, such that $\eta^4 = 0$ in $\mathcal{SH}_{\text{ét}}(X_{\alpha_0})$. This finishes the proof. □

Corollary 3.12. *In $\mathcal{SH}_{\text{ét}}(\mathrm{Spec}(\mathbb{Z}))$, the η-periodisation $\mathbb{S}[\eta^{-1}]$ vanishes.*

Proof. Since $\mathrm{Spec}(\mathbb{Z})$ is étale locally étale bounded, cf. Example 1.2, pulling back along the geometric points of $\mathrm{Spec}(\mathbb{Z})$ is conservative, cf. [Bac21, Corollary 5.12]. As for any map of schemes $f\colon S \to T$ we have $f^*\mathbb{S}[\eta^{-1}] \simeq \mathbb{S}[\eta^{-1}]$ (since $f^*\eta \cong \eta$, f^* commutes with colimits and the η-periodization is given by the mapping telescope, cf. Proposition 3.3), the claim follows from the version for algebraically closed fields, cf. Corollary 3.7. □

With the above we can prove:

Theorem 3.13. *Let X be a scheme. Then for any $M \in \mathcal{SH}_{\text{ét}}(X)$, we have $M[\eta^{-1}] \cong 0$, and the map $M \to M^\wedge_\eta$ to its η-completion is an equivalence. In particular, if M is a compact object in $\mathcal{SH}_{\text{ét}}(X)$, there exists an integer n such that $\eta^n\colon M \otimes \mathbb{G}_m^{\otimes n} \to M$ vanishes.*

Proof. Recall that the η-periodization is given by the mapping telescope, cf. Proposition 3.3. For the first part, using that the fiber of the map $M \to M^\wedge_\eta$ is η-periodic (cf. Lemma 2.4), we see that it suffices to check that any η-periodic object vanishes. If $N \in \mathcal{SH}_{\text{ét}}(X)$ is an η-periodic object, we have $N = N \otimes \mathbb{S}[\eta^{-1}]$, and denoting by $f \colon X \to \operatorname{Spec}(\mathbb{Z})$ the structural morphism, we have $\mathbb{S}[\eta^{-1}] \simeq f^*\mathbb{S}[\eta^{-1}] = 0$, thus $N = 0$. The second part now follows from Lemma 2.8. $\square$

Corollary 3.14. *Let S be a scheme. The étale sheafification functor $L_{\text{ét}} \colon \mathcal{SH}(S) \to \mathcal{SH}_{\text{ét}}(S)$ factors canonically over $\mathcal{SH}(S)^\wedge_\eta$. In particular, any object of $\mathcal{SH}(S)$ that satisfies étale descent is already η-complete.*

Proof. We have to see that $L_{\text{ét}}$ inverts every morphism $f \colon E \to F$ such that $f /\!\!/ \eta$ is an equivalence. We know from Lemma 2.4 that $\operatorname{fib}(f)$ is η-periodic, which implies that $L_{\text{ét}}\operatorname{fib}(f) \cong 0$, hence $L_{\text{ét}}f$ is an equivalence. $\square$

4 Nonvanishing of η^3

We proved in Theorem 3.13 that for every scheme X and any object $M \in \mathcal{SH}_{\text{ét}}(X)$, the map $M \to M^\wedge_\eta$ is an equivalence. Moreover, by Corollary 3.7, we know that for any scheme S defined over a field k of small étale cohomological dimension, we have $\eta^4 = 0$ in $\mathcal{SH}_{\text{ét}}(S)$. Even better, Corollary 3.11, we know that if X is any scheme, there exists a finite faithfully flat map $Y \to X$ such that $\eta^4 = 0$ in $\mathcal{SH}_{\text{ét}}(Y)$. It is surprisingly hard to descent the homotopy witnessing that $\eta^4 = 0$ over Y to X. On the other hand, in topology it is true that η^3 is not null. In this section we show that this still holds in $\mathcal{SH}_{\text{ét}}(S)$ for any scheme S not of equicharacteristic 2.

Proposition 4.1. *Let S be a nonempty scheme of characteristic zero. Then in $\mathcal{SH}_{\text{ét}}(S)$, the map η^3 is not null. In fact, its image in $\mathcal{SH}_{\text{ét}}(S)^\wedge_2$ is not null.*

Proof. Indeed, let $x \colon \operatorname{Spec}(k) \to S$ be a geometric point of S. It suffices to prove that η^3 is not null in $\operatorname{Spec}(k)$. If k admits an embedding into the field $\mathbb{C}$ of the complex numbers, because the Betti realisation of η is the topological Hopf map, we see that $\eta^3 \neq 0$ (cf. [Tod62, Chapter V, Equation 5.5 and Proposition 5.6]). Even better, we have that $\eta^3 \neq 0$ after 2-completion (as $\pi_3(\mathbb{S}_{\text{top}})$ is 2-torsion). Unfortunately, it might be possible that k is too big to be embedded in $\mathbb{C}$. In this case we work as follows: we have a map $f \colon \operatorname{Spec}(k) \to \operatorname{Spec}(\overline{\mathbb{Q}})$, which induces an equivalence on étale sheaves of spectra, thus in particular on 2-completed étale sheaves of spectra:

$$\mathrm{Sp}^\wedge_2 \simeq \mathrm{Shv}_{\text{ét}}(\overline{\mathbb{Q}}, \mathrm{Sp})^\wedge_2 \simeq \mathrm{Shv}_{\text{ét}}(k, \mathrm{Sp})^\wedge_2.$$

Moreover, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{SH}_{\text{ét}}(k) \arrow[d, "\rho_2"] & \mathcal{SH}_{\text{ét}}(\overline{\mathbb{Q}}) \arrow[l, "f^*"'] \arrow[r, "\rho_{\mathrm{B}}"] \arrow[d, "\rho_2"] & \mathrm{Sp} \arrow[d, "(-)^\wedge_2"] \\
\mathrm{Shv}_{\text{ét}}(k, \mathrm{Sp})^\wedge_2 & \mathrm{Shv}_{\text{ét}}(\overline{\mathbb{Q}}, \mathrm{Sp})^\wedge_2 \arrow[l, "\simeq", "f^*"'] \arrow[r, "\simeq"'] & \mathrm{Sp}^\wedge_2
\end{tikzcd}$$

where ρ_{B} is the Betti realisation and ρ_2 is the 2-adic étale realisation (see the proof of [Ayo25, Proposition 6.10]). In particular, we see that:

$$(f^*)^{-1}(\rho_2(\eta^3_{\operatorname{Spec}(k)})) \simeq \rho_2(\eta^3_{\operatorname{Spec}(\overline{\mathbb{Q}})}) \simeq \rho_{\mathrm{B}}(\eta^3_{\operatorname{Spec}(\overline{\mathbb{Q}})})^\wedge_2 \neq 0,$$

where we know that $\rho_{\mathrm{B}}(\eta^3_{\operatorname{Spec}(\overline{\mathbb{Q}})})^\wedge_2 \neq 0$ by the first part of the proof. This proves that η^3 is not null in $\mathcal{SH}_{\text{ét}}(k)$. $\square$

Theorem 4.2. *Let S be nonempty scheme which is not of equicharacteristic 2. Then in $\mathcal{SH}_{\text{ét}}(S)$, the map η^3 is not null.*

Proof. We will use rigid analytic étale motives. As in the proof of Proposition 4.1, it suffices to prove that $\eta^3 \neq 0$ in $\mathcal{SH}_{\text{ét}}(k)$ for k an algebraically closed field. If k is of characteristic zero, this is Proposition 4.1.

Thus, we assume that $\mathrm{char}(k) = p$ for some prime number $p \neq 2$. Let K' be the fraction field of the ring $W(k)$ of p-typical Witt vectors in k, and let K be the completion of it algebraic closure. As in the proof of [Ayo25, Proposition 6.7], there is a commutative diagram of symmetric monoidal functors (see Recollection 4.4)

$$\begin{tikzcd} \mathcal{SH}_{\text{ét}}(k) \arrow[r, "\xi^*"] \arrow[dr, "\rho_2"'] & \mathrm{Rig}\mathcal{SH}_{\text{ét}}(K) \arrow[d, "\rho_2"] & \mathcal{SH}_{\text{ét}}(K) \arrow[l, "\mathrm{Rig}^*"'] \arrow[dl, "\rho_2"] \\ & \mathrm{Sp}_2^{\wedge}. & \end{tikzcd}$$

We have that $\xi^*\eta_{\mathrm{Spec}(k)} \simeq \mathrm{Rig}^*\eta_{\mathrm{Spec}(K)}$. Indeed, this is shown in Lemma 4.5 below. In particular, we see that $\rho_2(\eta_{\mathrm{Spec}(k)}) \simeq \rho_2(\eta_{\mathrm{Spec}(K)})$. The third power of the latter is not null by Proposition 4.1. This finishes the proof. □

Remark 4.3. If S is of equicharacteristic 2, then there is a map of schemes $S \to \mathrm{Spec}(\mathbb{F}_2)$, and as $\mathcal{SH}_{\text{ét}}(\mathbb{F}_2) \simeq \mathcal{SH}_{\text{ét}}(\mathbb{F}_2)[\frac{1}{2}]$ by [BH21, Lemma A.1], we see that $\eta = 0$ in $\mathcal{SH}_{\text{ét}}(S)$ by Lemma 3.4.

Recollection 4.4. Let k an algebraically closed field of characteristic $p > 0$. Let K be the completion of the algebraic closure of the fraction field of the ring $W(k)$ of p-typical Witt vectors in k. This is an algebraically closed valued field (see the proof of [Sch17, Remark 1.4.1] that works in this generality) with ring of integers (elements of norm $\leqslant 1$) that we denote K°, which has residual field $K^\circ/\mathfrak{m} \simeq k$ (by [Neu99, Chapter II, Proposition 4.3]).

Denote by $\mathrm{SmFSch}_{\mathrm{Spf}(K^\circ)}$ the category of $\mathfrak{m}$-adic smooth formal schemes over $\mathrm{Spf}(K^\circ)$, and by SmRig_K the category of smooth rigid analytic varieties over K. From the two categories $\mathrm{SmFSch}_{\mathrm{Spf}(K^\circ)}$ and SmRig_K one can define, as in algebraic geometry, categories of formal motives $\mathrm{F}\mathcal{SH}_{\text{ét}}(\mathrm{Spf}(K^\circ))$ and rigid analytic motives $\mathrm{Rig}\mathcal{SH}_{\text{ét}}(K)$, say with the étale topology, suitably modified [AGV22, Definitions 3.1.3 and 2.1.15]. One can define three functors on these categories:

1. The *special fiber functor* $(-)_\sigma\colon \mathrm{SmFSch}_{\mathrm{Spf}(K^\circ)} \to \mathrm{Sm}_k$ that associates to a formal scheme $\mathcal{X} \in \mathrm{SmFSch}_{\mathrm{Spf}(K^\circ)}$ its restriction along $\mathrm{Spec}(k) \to \mathrm{Spf}(K^\circ)$, and its induced functor $\sigma^*\colon \mathrm{F}\mathcal{SH}_{\text{ét}}(\mathrm{Spf}(K^\circ)) \to \mathcal{SH}_{\text{ét}}(k)$ [AGV22, Notations 1.1.6 and 3.1.9].

2. The *generic fiber functor* $(-)^{\mathrm{rig}}\colon \mathrm{SmFSch}_{\mathrm{Spf}(K^\circ)} \to \mathrm{SmRig}_K$ that goes to the category of smooth rigid analytic varieties over K, and its induced functor $\xi^*\colon \mathrm{F}\mathcal{SH}_{\text{ét}}(\mathrm{Spf}(K^\circ)) \to \mathrm{Rig}\mathcal{SH}_{\text{ét}}(K)$ [AGV22, Notations 1.1.8 and 3.1.12].

3. The *analytification functor* $(-)^{\mathrm{an}}\colon \mathrm{Sm}_K \to \mathrm{SmRig}_K$ that sends a variety to its rigid-analytic version, and its induced functor $\mathrm{Rig}^*\colon \mathcal{SH}_{\text{ét}}(K) \to \mathrm{Rig}\mathcal{SH}_{\text{ét}}(K)$, see [AGV22, Construction 1.1.15 and Remark 2.2.6].

Moreover, witnessing the phenomenon that "$\mathbb{A}^1$-invariant motives do not see thickenings", the special fiber functor σ^* is an equivalence $\mathrm{F}\mathcal{SH}_{\text{ét}}(\mathrm{Spf}(K^\circ)) \simeq \mathcal{SH}_{\text{ét}}(k)$, see [AGV22, Theorem 3.1.10] for this. To summarize, there is a commutative diagram

$$\begin{tikzcd} \mathrm{Sm}_k \arrow[d] & \mathrm{SmFSch}_{\mathrm{Spf}(K^\circ)} \arrow[l, "(-)_\sigma"'] \arrow[r, "(-)^{\mathrm{rig}}"'] \arrow[d] & \mathrm{SmRig}_K \arrow[d] & \mathrm{Sm}_K \arrow[l, "(-)^{\mathrm{an}}"'] \arrow[d] \\ \mathcal{SH}_{\text{ét}}(k) & \mathrm{F}\mathcal{SH}_{\text{ét}}(\mathrm{Spf}(K^\circ)) \arrow[l, "\simeq"'] \arrow[r, "\xi^*"'] & \mathrm{Rig}\mathcal{SH}_{\text{ét}}(K) & \mathcal{SH}_{\text{ét}}(K). \arrow[l, "\mathrm{Rig}^*"'] \end{tikzcd}$$

Finally, note that K°-scheme X gives an $\mathfrak{m}$-adic formal scheme $\mathcal{X}$ through the $\mathfrak{m}$-adic formal completion functor that sends X to $\mathcal{X} \coloneqq \widehat{X} = X \times_{\mathrm{Spec}(K^\circ)} \mathrm{Spf}(K^\circ)$, the fiber product being taken in the category of formal schemes. In particular, if X is a smooth K°-scheme, then under the equivalence $\mathcal{SH}_{\text{ét}}(k) \simeq \mathrm{F}\mathcal{SH}_{\text{ét}}(\mathrm{Spf}(K^\circ))$ the motive $\Sigma^\infty_+ X_k$ of the k-scheme $X \times_{\mathrm{Spec}(K^\circ)} \mathrm{Spec}(k)$ corresponds to $\Sigma^\infty_+ \widehat{X}$.

Lemma 4.5. *In the setting of Recollection 4.4, we have* $\xi^*\eta_{\mathrm{Spec}(k)} \simeq \mathrm{Rig}^*\eta_{\mathrm{Spec}(K)}$.

Proof. Indeed, by [Ayo15, Corollaire 1.3.5], the map

$$(\widehat{\mathbb{A}^2_{K^\circ} \setminus \{0\}})^{\mathrm{rig}} \to (\mathbb{A}^2_K \setminus \{0\})^{\mathrm{an}}$$

is an equivalence in $\mathrm{Rig}\mathcal{SH}(K)$. As $(\widehat{\mathbb{P}^1_{K^\circ}})^{\mathrm{rig}} \to (\mathbb{P}^1_K)^{\mathrm{an}}$ is already an equivalence as rigid analytic varieties (see [AGV22, Definition 2.2.11]), the commutative square

$$\begin{array}{ccc} (\widehat{\mathbb{A}^2_{K^\circ} \setminus \{0\}})^{\mathrm{rig}} & \longrightarrow & (\mathbb{A}^2_K \setminus \{0\})^{\mathrm{an}} \\ \downarrow & & \downarrow \\ (\widehat{\mathbb{P}^1_{K^\circ}})^{\mathrm{rig}} & \longrightarrow & (\mathbb{P}^1_K)^{\mathrm{an}} \end{array}$$

induces an equivalence $\xi^*\eta_{\mathrm{Spec}(k)} \simeq \mathrm{Rig}^*\eta_{\mathrm{Spec}(K)}$ in $\mathrm{Rig}\mathcal{SH}_{\mathrm{\acute{e}t}}(K)$. ☐

Remark 4.6. The Lemma 4.5 does not require any of the fields k and K to be algebraically closed. In particular, we see that this applies to $K = \mathbb{Q}_p(i)$, which is interesting in view of Remark 3.10 (4). Using Proposition 3.6 and Lemma 4.5, we see that $\mathrm{Rig}^*(\eta^4_{\mathrm{Spec}\,\mathbb{Q}_p(i)})$ is zero.

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