8 Consequences of the Yoneda Lemma

Exercise 8.1 (Lemma 4.3.8)**.** Let $J\colon A \to B$ be a full and faithful functor and $A, A' \in A$. Then:

a) a map f in $\mathscr{A}$ is an isomorphism if and only if the map $J(f)$ in $\mathscr{B}$ is an isomorphism;

b) for any isomorphism $g\colon J(A) \to J(A')$ in $\mathscr{B}$, there is a unique isomorphism $f\colon A \to A'$ in $\mathscr{A}$ such that $J(f) = g$;

c) the objects A and A' of $\mathscr{A}$ are isomorphic if and only if the objects $J(A)$ and $J(A')$ of $\mathscr{B}$ are isomorphic.

Proof. to a). We have already seen that functors respect isomorphisms so it only remains to prove the other direction. So let $f : A \to A'$ be a map in $\mathscr{A}$ such that $J(f)$ is an isomorphism. This means there is an inverse map $J(f)^{-1}\colon J(A') \to J(A)$. As we are full we find $g\colon A' \to A$ such that $J(g) = J(f)^{-1}$. We claim that $f \circ g = id_{A'}$ and $g \circ f = id_A$. Note that $J(f \circ g) = J(f) \circ J(g) = J(f) \circ J(f)^{-1} = \mathrm{id}_J(A')$. As $J(\mathrm{id}_{A'}) = \mathrm{id}_{J(A')}$ and J is faithful we get $f \circ g = \mathrm{id}_{A'}$ as we wanted. ($g \circ f = \mathrm{id}_A$ follows in the same way)

to b). As J is full and faithful we find a unique map $f\colon A \to A'$ such that $J(f) = g$. As g is an isomorphism, f has to be an isomorphism by a).

to c) If A and A' are isomorphic than for any functor $F(A)$ and $F(A')$ are isomorphic. The other direction follows immediately from b). □

Exercise 8.2. $\mathscr{B}$ be a category and $J\colon \mathscr{C} \to \mathscr{D}$ a functor. There is an induced functor $J \circ -\colon [\mathscr{B}, \mathscr{C}] \to [\mathscr{B}, \mathscr{D}]$ defined by composition with J.

- Show that if J is full and faithful then so is $J \circ -$.
- Deduce that if J is full and faithful and $G, G'\colon \mathscr{B} \to \mathscr{C}$ with $J \circ G \cong J \circ G'$, then $G \cong G'$.
- Now deduce that right adjoints are unique: If $F\colon A \to B$ and $G, G'\colon \mathscr{B} \to \mathscr{A}$ with $F \vdash G$ and $F \vdash G'$ then $G \cong G'$.

Proof. To (a): Assume J is full und faithful. Fix two functors $F, G\colon \mathscr{B} \to \mathscr{C}$. From a natural transformation $\eta\colon F \to G$ we construct the natural transformation $J \circ \eta : J \circ F \to J \circ G$ that is defined by $(J \circ \eta)_B = J(\eta_B)$ for all $B \in \mathscr{B}$.

We start by showing that $J \circ -$ is faithful. So consider two naturaltransformations η, γ from $F \to G$ such that for all $B \in \mathscr{B}$ we have $(J \circ \eta)_B = (J \circ \gamma)_B$. By definition this is just $J(\eta_B) = J(\gamma_B)$. As γ_B and η_B are both morphisms from $F(B) \to G(B)$ in $\mathscr{C}$ and J was faithful we see $\eta_B = \gamma_B$ for all B and hence $\eta = \gamma$.

To see that $J \circ -$ is full let $\eta : J \circ F \to J \circ G$ be a natural transformation. That means that for all B we have a morphism $\eta_B \colon J \circ F(B) \to J \circ G(B)$. There is only one thing we can do namly define a natural transformation for each $\gamma_B \colon F(B) \to G(B)$ as the unique morphism $g_B \colon F(B) \to G(B)$ such that $J(g_B) = \eta_B$. By definition $J(\gamma_B) = \eta_B$ (or in other words $J \circ \gamma = \eta$) so it only remains to see that γ is in fact a natural transformation. For this let $f : B \to B'$ be a morphism and consider the diagram

$$\begin{tikzcd} F(B) \arrow[r, "F(f)"] \arrow[d, "\gamma_B"] & F(B') \arrow[d, "\gamma_{B'}"] \\ G(B) \arrow[r, "G(f)"] & G(B'). \end{tikzcd}$$

We need to see that this commutes, i.e. $\gamma_{B'} \circ F(f) = G(f) \circ \gamma_B$. Here we will use that J is faithful. To conclude the proof it then surffices to see that $J(\gamma_{B'} \circ F(f)) = J(G(f) \circ \gamma_B)$. Using that J is a functor and that $J(\gamma_B) = \eta_B$ this can be written into $\eta_{B'} \circ J(F(f)) = J(G(f)) \circ \eta_B$. This now follows directly from the commutativity of the following diagram, i.e. by the fact that η is a natural transformation.

$$\begin{tikzcd} J \circ F(B) \arrow[r, "J \circ F(f)"] \arrow[d, "\eta_B"] & J \circ F(B') \arrow[d, "\eta_{B'}"] \\ J \circ G(B) \arrow[r, "J \circ G(f)"] & J \circ G(B'). \end{tikzcd}$$

Remark: Note that we needed both full and faithful in order to prove that $J \circ -$ is full. We needed faithful to see that the γ we constructed was in fact a natural transformation. This again shows that the word natural makes sense, since faithfulness gave us a natural (in fact a unique) choice for γ. Without faithfulness we would have needed to make a (unnatural) choice for γ.

To (b): Suppose $J \circ G \cong J \circ G'$. Then there is a natural isomorphism $\eta \colon J \circ G \to J \circ G'$. From (a), we know that $J \circ -$ is fully faithful, so there is exactly one $a \colon G \to G'$ such that $J \circ a = \eta$, and exactly one $b \colon G' \to G$ such that $J \circ b = \eta^{-1}$. But then $J \circ (a \circ b) = (J \circ a) \circ (J \circ b) = \eta \circ \eta^{-1} = \mathrm{id}_{J \circ G'} = J \circ \mathrm{id}_{G'}$. Again, by fully faithfulness of $J \circ -$, we obtain $a \circ b = \mathrm{id}_{G'}$, and by a similar argument, $b \circ a = \mathrm{id}_G$. Hence, a is a natural isomorphism with inverse b, and we get $G \cong G'$.

To (c): Consider the functor $J = H_\bullet \colon \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Sets}], A \mapsto \mathrm{Hom}_{\mathscr{A}}(-, A)$, the Yoneda embedding. We have seen that J is fully faithful. Consider $J \circ - \colon [\mathscr{B}, \mathscr{A}] \to [\mathscr{B}, [\mathscr{A}^{\mathrm{op}}, \mathbf{Sets}]]$.

Then we get $J \circ G = \mathrm{Hom}_{\mathscr{A}}(-, G(-)) \cong \mathrm{Hom}_{\mathscr{B}}(F(-), -) \cong \mathrm{Hom}_{\mathscr{A}}(-, G'(-)) = J \circ G'$, where the isomorphisms are given by adjunction. (Note that we first plug in a $B \in \mathscr{B}$ in $G(-)$, and the an $A \in \mathscr{A}^{\mathrm{op}}$ in the other place. Sadly, our notation fails at recognizing something like that, but this is not a problem since inserting into functors does not depend on the order, and the place where to insert each variable is usually clear from the context.)

Using (b), we get that $G \cong G'$ $\square$