Seminar program:
Algebraic K-theory and chromatic homotopy theory

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Talk 0 (Maxime): Introductory Talk

Talk 1 (Timo): Introduction to algebraic K-theory 1/2

References: [Heb21, Sections III and IV], [Lur14, Lectures 16-20], [Hil].

- Coordinate with the speaker of Talk 2 to split the material between the two of you.
- Define K-theory of stable categories ([Lur14, Lecture 16]).
- Define K-theory of schemes as $K(Perf(-))$ and explain how this induces transfer maps (i.e., the maps induced by pushforward on perfect complexes).
- Define K-theory of terms of group completion ([Lur14, Lecture 18]), state the group completion theorem ([Lur14, Lecture 18, Proposition 4], [Nik17]).
- For K-theory in the group completion sense, show that $K(-)$ is lax symmetric monoidal and preserves filtered colimits. Mention that this is also true for K-theory of stable infinity categories.
- Explain the additivity and Verdier localization properties [Lur14, Lecture 17], [Heb21, IV.29] or [Heb, 5.8.1].
- State the resolution theorem [Lur14, Lecture 18-20].

Talk 2 (Andi): Introduction to algebraic K-theory 2/2

- Coordinate with the speaker of Talk 1 to split the material between the two of you.

Talk 3 (Antonio): The Nisnevich and étale topologies, descent

References: [Lur18], [Lur09].

- Define the Nisnevich and étale topologies.
- Define the notion of descent in general [Lur09, §6.2.2] and characterize Nisnevich descent via distinguished squares (see e.g. [AHW17, Theorem 3.2.5] combined with [BH21, Proposition A.2(d)], but proving this probably takes too long).
- Prove that having Nisnevich and Galois descent is equivalent to having étale descent [Lur18, Theorem B.6.4.1].
- Show that K-theory has Nisnevich descent by reducing to Nisnevich descent for QCoh$(-)$ (the ∞-category of complexes with quasi-coherent cohomology). See e.g. [Kha17, Lectures 2-5, especially Lecture 5, Theorem 7.3]

Talk 4 (Anton): Equivariant homotopy theory and Galois descent for rational K-theory

<u>References:</u> [CMNN22, Section 3 and Appendix A], [Ram, Sections 1-2, 6-7].

- Define genuine G-spectra in terms of Mackey functors [CMNN22, Appendix A].
- Give a construction of equivariant K-theory following [CMNN22, 2.12, 2.13].
- Formulation of descent in terms of Mackey functors, and [CMNN22, Criterion 3.4].
- Explain 3.1, 3.2, 3.4 and 3.6. Avoid anything that mentions (F, ϵ)-nilpotence or telescopic localization.

Talk 5 (Manuel): Topological K-theory KU and complex orientations

<u>References:</u> [Cno24, Chapter 5], [Ada74, II.2], [Lur10, Lecture 4].

- Define KU, ku, explain their relation to BU, and state Bott periodicity ([Cno24, Chapter 5]). Define $K(1)$ as KU/p (without mentioning other heights). Mention the existence of a homotopy ring map $\mathsf{KU} \to K(1)$.
- Explain what complex orientations are (definition with $\mathbb{CP}^\infty$), and prove that KU and $K(1)$ are complex-oriented.
- Recall the notion of formal group laws, their relationship to complex oriented ring spectra, and compute those of KU, ku, $K(1)$ (see [Ada74, II.2]).

Talk 6 (Dominik): The $K(1)$-homology of BC_p and ambidexterity

<u>References:</u> [nLa26], [HKR00], [GS96], [LH13], [CSY20], [RW80].

- Using the Gysin sequence (see [nLa26]), compute the E-homology of BC_p for a complex oriented ring spectrum E, in terms of the formal group law of E [HKR00, Section 5]. Apply this to $E = K(1)$.
- Introduce the norm map for BC_p.
- Prove the following Lemma: If $K(1)^* BC_p$ is finite dimensional over $K(1)^*$, then the norm map is an equivalence in $\mathsf{Sp}_{K(1)}$ (a reference is [GS96], but see the end of the program for an easier, more modern argument).
- Combining the previous points, conclude that the norm map for BC_p is a $K(1)$-local equivalence.

Talk 7 (Jonas): Étale descent for $K(1)$-local K-theory

<u>References:</u> [CSY20], [Hop14].

- The goal of this talk is to prove that $K(1)$-local K-theory has étale descent, following [CSY20].
- Show that π_0 of a $K(1)$-local $\mathbb{E}_\infty$-ring spectrum admits a δ-ring structure. See [CSY20, Section 5.2 and Theorem 5.2.2] or [Hop14].
- Recalling that δ-rings are torsion-free, deduce the height 1 May nilpotence conjecture: an $\mathbb{E}_\infty$-ring spectrum that is p-local and rationally zero, is also $K(1)$-locally zero [CSY20, Theorem 5.2.6].
- Deduce that $K(1)$-local K-theory has étale descent by applying the methods of Talks 3 and 4.

Talk 8 (TBD): Hyperdescent

References: [CM21].

- Define hyperdescent, hypersheaves and hypercompleteness.
- State and prove the criteria [CM21, 1.6, 1.8] (reduction to fields, and a concrete criterion for hyperdescent over fields).

Talk 9 (Luca): The norm-residue isomorphism theorem and hyperdescent for $K(1)$-local K-theory 1/2

References: [CM21], [HW19].

- Coordinate with the speaker of Talk 10 to split the material between the two of you.
- State the norm-residue isomorphism theorem (see [HW19]).
- Use it to prove that $K(1)$-local K-theory has étale hyperdescent, following [CM21, Section 6.2 and Theorem 6.13].

Talk 10 (Lorenzo): The norm-residue isomorphism theorem and hyperdescent for $K(1)$-local K-theory 2/2

References: [CM21].

- Coordinate with the speaker of Talk 9 to split the material between the two of you.

Talk 11 (Maxime): Quillen-Suslin rigidity

References: [Sus83].

- We proved in Talks 9 and 10 that $K(1)$-local K-theory was an étale hypersheaf. The goal of this talk is to compute its stalks.
- State (but do not prove) Gabber rigidity.
- Reduce to algebraically closed fields [Sus83].
- Compute it in this case, following Quillen (case $\overline{\mathbb{F}}_p$).

Talk 12 (TBD): Mitchell's theorem

References: [ENY20],[Mit90].

- Briefly introduce $K(2)$ (note that it will be more thoroughly introduced in Talk 13).
- State and prove Mitchell's theorem: the $K(2)$-local K-theory of a discrete ring vanishes. This can be done using étale hyperdescent, following [ENY20].

Talk 13 (TBD): Redshift and purity 1/2

References: [Lur10], [LMMT24], [CMNN22].

- Coordinate with the speaker of Talk 14 to split the material between the two of you.
- Introduce the Morava E- and K- theory spectra [Lur10, Lecture 22], either using Landweber exactness or blackboxing their existence, and only mention the description as quotients of MU.
- Explain the notion of redshift, explaining how this generalizes Mitchell's theorem from Talk 12.
- Prove purity (see [LMMT24, CMNN22]), which gives the upper bound for redshift.

Talk 14 (TBD): Redshift and purity 2/2

References: [Lur10], [LMMT24], [CMNN22].

- Coordinate with the speaker of Talk 13 to split the material between the two of you.

Argument for Talk 6

- By monoidality of the norm map, it suffices to prove that $\mathbb{S}_{K(1)}^{tC_p} \simeq 0$.
- Reduce to proving that KU^{tC_p} is p-completely 0, i.e. prove that $K(1)^{tC_p} \simeq 0$.
- Note that the assumptions guarantee that $K(1)^{tC_p}$ is a compact $K(1)$-module (as a cofiber of compact $K(1)$-modules) and hence since it is $K(1)^{BC_p}[e^{-1}]$ where e is the Euler class, we deduce that the canonical map $K(1)^{BC_p} \to K(1)^{tC_p}$ splits, which proves that the norm map is also split injective, and hence an iso for dimension reasons.

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