6 Representables

Exercise 6.1. Find one more represetable functor not mentioned above.

Solution. Let $\mathscr{G}$ be a group that we consider as a Category with one element we call G. Consider a functor $F : \mathscr{G} \to \mathbf{Set}$. This functor corresponds to a G-set, i.e. the set $M = F(G)$ is a $\mathscr{G}$-set. This means that given an element g in the group i.e. a morhphism $g\colon \mathscr{G} \to \mathscr{G}$ and an element $m \in M$ we can define $g \cdot m = F(g)(m)$. As F is a functor we get $\mathrm{id}_G \cdot m = m$ and $(g \circ h) \cdot m = g \cdot (h \cdot m)$.

A natural transformation is precicely a morphism of G-sets, and a natural ismomorphism is an isomorphism of G-sets.

We say that a $\mathscr{G}$-set M is *free*, if whenever there exists $m \in M$ such that $g \cdot m = m$, we already have $g = 1_G$.

We say that M is *transitive* or *1-transitive*, if for each pair $m, m' \in M$ there exists a $g \in G$ with $g \cdot m = m'$.

We call M a *G-torsor*, if M is free and transitive.

We will show that the representable functors $G \to \mathbf{Set}$ are precicely the G-torsors.

Note that since there is only one object in G, there is only one representable functor, and hence all G-torsors are isomorphic. One can see G as a G-set via left multiplication, this is also a G-torsor.

We will first show that a representable functor is a G-torsor. So let $F \coloneqq H_G$, and $M \coloneqq F(G) = \mathscr{G}(G, G)$. In other words, M is the set of elements of G. The G-set structure is given by $g \cdot h \coloneqq F(g)(h) = g \circ h$, i.e. by left multiplication.

This is a G-torsor: Indeed, if $g \cdot h = h$, then $g = g \cdot h \cdot h^{-1} = h \cdot h^{-1} = 1_G$. (This shows that M is free.) If $g, h \in M$, then $(hg^{-1}) \cdot g = h$. (This shows transitivity.)

Now let M' be any other G-torsor. We will show that $M \cong M'$, this will immediately give the desired isomorphism of functors. Choose any element $m_0 \in M'$. Then we define a morphism $f\colon M \to M', g \mapsto g \circ m_0$.

This is a morphism of G-sets:

$$f(h \circ g) = h \circ g \circ m_0 = h \circ f(g).$$

It is injective: Let $g, h \in M$ such that $g \circ m_0 = f(g) = f(h) = h \circ m_0$. Then $h^{-1} \circ g \circ m_0 = m_0$, since M' is free, we get $h^{-1} g = 1_G$, in other words $g = h$.

It is surjective: Let $m \in M'$ arbitrary. Since M' is transitive, there exists a $g \in G$ with $g \cdot m_0 = m$. Hence $f(g) = g \cdot m_0 = m$, i.e. f is surjective.

This concludes $M \cong M'$.

It is obvious, that if N is a G-set that is not a torsor, it cannot be ismomorphic to a torsor, this concludes the claim. $\square$

Exercise 6.2. Let $\mathscr{A}$ be a (locally small) category, and let $A, A' \in \mathscr{A}$ with $H_A \cong H_{A'}$. Prove directly that $A \cong A'$.

Proof. Let $\eta\colon H_A \to H_{A'}$ an isomorphism of functors. Consider the isomorphism

$$\eta_A\colon \mathscr{A}(A,A) = H_A(A) \to H_{A'}(A) = \mathscr{A}(A,A').$$

Let $f := \eta_A(id_A)\colon A \to A'$, and define $g := \eta_{A'}^{-1}(id_{A'})\colon A' \to A$.

The idea is now to use the naturality condition to show that f and g are mutually inverse. Consider the commutative diagram.

$$\begin{tikzcd}
H_A(A) \arrow[r, "H_A(g)"] \arrow[d, "\eta_A"] & H_A(A') \arrow[d, "\eta_{A'}"] \\
H_{A'}(A) \arrow[r, "H_{A'}(g)"] & H_{A'}(A').
\end{tikzcd}$$

Using this diagram for id_A we get going down then right

$$(H_{A'}(g) \circ \eta_A)(\mathrm{id}_A) = H_{A'}(g)(\eta_A(\mathrm{id}_A)) = H_{A'}(g)(f) = f \circ g.$$

Going right then down we get

$$(\eta_{A'} \circ H_A(g))(\mathrm{id}_A) = \eta_{A'}(\mathrm{id}_A \circ g) = \eta_{A'}(g) = \mathrm{id}_{A'}\,.$$

As the diagram commutes (as all diagrams should...) we get $f \circ g = \mathrm{id}_{A'}$. Arguing in the same way (i.e. plugging in $\mathrm{id}_{A'}$) with the commutative diagram

$$\begin{tikzcd}
H_{A'}(A') \arrow[r, "H_{A'}(f)"] \arrow[d, "\eta_{A'}^{-1}"] & H_{A'}(A) \arrow[d, "\eta_A^{-1}"] \\
H_A(A') \arrow[r, "H_A(f)"] & H_A(A).
\end{tikzcd}$$

we obtain $g \circ f = \mathrm{id}_A$ This concludes the proof. $\square$

Exercise 6.3. Prove that the forgetful functor $U\colon \mathbf{CRing} \to \mathbf{Set}$ is isomorphic to $F := \mathbf{CRing}(\mathbb{Z}[x], -) = H_{\mathbb{Z}[X]}$.

Proof. We define a natural isomorphism $\eta\colon U \to F$. For this, let $R \in \mathbf{CRing}$ a commutative ring. Then define $\eta_R\colon U(R) \to F(R)$ via $r \mapsto f_r$, where we denote by f_r the map

$$\begin{aligned} f_r\colon \mathbb{Z}[X] &\to R \\ g(X) &\mapsto g(r) \end{aligned}$$

η_R is clearly an ismorphism as it posseses an inverse $f \mapsto f(X)$. (Here we use the fact that ring morphisms map 1 to 1 and are additive and multiplicative)

So we only need to show that η is natural. So let $\psi\colon R \to S$ be a morhism of rings. Then we get for each $r \in R$

$$\begin{aligned}\eta_S(U(\psi)(r)) &= \eta_S(\psi(r)) \\ &= f_{\psi(r)} \\ &= \psi \circ f_r \\ &= \psi \circ \eta_R(r) \\ &= F(\psi)(\eta_R(r)).\end{aligned}$$

This shows naturality. (Here we used that for a polynomial $g \in \mathbb{Z}[X]$ we have $g(\psi(r)) = \psi(g(r)))$ ☐

Exercise 6.4. The Sierpinski space is the two-point topological space S in which one of the singleton subsets is open but the other is not. Prove that for any topological space X, there is a canonical bijection between the open subsets of X and the continuous maps $X \to S$. Use this to show that the functor $O : \mathbf{Top}^{\mathrm{op}} \to \mathit{Set}$ of Example 4.1.19 is represented by S.

Proof. Let $S = \{x_0, \eta\}$ be the Sirpinski space, such that $\{\eta\}$ open.

We will show that ther is a natural isomorphism $\phi\colon \mathcal{O} \to H_S$.

For a topological space X, we define

$$\begin{aligned}\phi_X\colon \mathcal{O}(X) &\to H_S(X) \\ U &\mapsto [x \mapsto \begin{cases}\eta & \text{if } x \in U \\ x_0 & \text{if } x \notin U\end{cases}]\end{aligned}$$

This is well defined: Cleary $\phi_X(U)$ is a map $X \to S$. It is continuous, because $\phi_X(U)^{-1}(\{\eta\}) = U$ is open, and $\{\eta\}$ is the only non-trivial open subset of S.

It is an ismomorphism, since an inverse is given by $g \mapsto g^{-1}(\{\eta\})$.

So the only thing we need to check is naturality. We will show naturality for ϕ^{-1} because this is easier.

Let $f\colon X \to Y$ a continuous map of topological spaces. Then for $g \in H_S(X)$, i.e. $g\colon X \to S$ continuous, we have

$$\begin{aligned}\mathcal{O}(f)(\phi_X^{-1}(g)) &= f^{-1}(g^{-1}(\{\eta\})) \\ &= (g \circ f)^{-1}(\{\eta\}) \\ &= \phi_{X'}^{-1}(g \circ f) \\ &= \phi_{X'}^{-1}(H_f(g)).\end{aligned}$$

This is the naturality condition for ϕ^{-1}, hence we are done. ☐