4 Definitions and Examples of Adjoints

Exercise 4.1. Find three examples of initial objects not mentioned above.

Solution.

- Let A be a partially ordered set. Consider the associated category $\mathscr{A}$. Then an initial object $I \in \mathscr{A}$ is an object with the property $I \leq J$ for alle $J \in \mathscr{A}$, i.e. a minimal element with respecto to the partial order. Note that such a minimum need not exists (not even if A is totally ordered). A concrete example would be the ordered set $\mathbb{N}$, this has as inital object the number 1. On the other hand, the ordered set $\mathbb{Z}$ has no initial object at all.

- While the inital object in the category **Set** is the empty set the inital objects in the category of sets with basepoint are given by any set with one element. In this case the terminal and inital objects are exactly the same.

- Fix a topological space X. Consider the category $\mathscr{A} := [\mathcal{O}(X)^{\mathrm{op}}, \mathrm{Set}]$ of presheaves on X. Then an initial presheaf is the presheaf given by the rule $U \mapsto \emptyset$. (The restriction maps are just the identity on $\emptyset$.) This is because $\emptyset$ is inital in the catagory of sets.

  More generally, consider a category $\mathscr{C}$ with an inital object I, and a category $\mathscr{A}$. Then the category $[\mathscr{A}, \mathscr{C}]$ has an initial object, it is the functor given by $A \mapsto I$, $f \mapsto \mathrm{id}_I$.

$\square$

Exercise 4.2. Show that the naturality equations can be replaced by the single equation

$$\overline{A' \xrightarrow{p} A \xrightarrow{f} G(B) \xrightarrow{G(q)} G(B')} = (F(A') \xrightarrow{f(p)} F(A) \xrightarrow{\bar{f}} B \xrightarrow{q} B').$$

Proof. We assume

$$\overline{A' \xrightarrow{p} A \xrightarrow{f} G(B)} = (F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\bar{f}} B)$$

and

$$\overline{(F(A) \xrightarrow{g} B \xrightarrow{q} B')} = (A \xrightarrow{\bar{g}} G(B) \xrightarrow{G(q)} G(B')).$$

We then see that

$$\begin{aligned}
\overline{A' \xrightarrow{p} A \xrightarrow{f} G(B) \xrightarrow{G(q)} G(B')} &= \overline{A' \xrightarrow{f \circ p} G(B) \xrightarrow{G(q)} G(B')} \\
&= (F(A) \xrightarrow{\overline{f \circ p}} B \xrightarrow{q} B') \\
&= (F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\bar{f}} B \xrightarrow{q} B').
\end{aligned}$$

Where we used the second naturality equation and then the first. Note you have to play around with the $\overline{(\,\cdot\,)}$ a bit to apply these rules. To go in the other direction assume that we have the long naturality equation. The setting $A = A'$ and $p = \mathrm{id}_A$ gives the second equation and setting $B = B'$ and $q = \mathrm{id}_B$ the first. $\square$

Exercise 4.3. Show that left adjoints preserve initial objects.

Proof. Consider the following pair of adjoint functors: $\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "G"] \end{tikzcd}$

Let I be an initial object of $\mathscr{A}$. This means that $\mathscr{A}(I, X) \cong \{*\}$ is a singleton set for every object $X \in \mathscr{A}$. We get $\mathscr{B}(F(I), Y) \cong \mathscr{A}(I, G(Y)) \cong \{*\}$ is a singleton set for every object $Y \in \mathscr{B}$. Here the first isomorphism is given by the adjunction, and the second because I is initial in $\mathscr{A}$. This proves that $F(I)$ is initial in $\mathscr{B}$. $\square$

Exercise 4.4. Fix a topological space X, and write $\mathcal{O}(X)$ for the poset of open subsets of X, ordered by inclusion. Let $\Delta : \mathrm{Set} \to [\mathcal{O}(X)^{\mathrm{op}}, \mathrm{Set}]$ be the functor assigning to a set A the presheaf ΔA with constant value A. Exhibit a chain of adjoint functors $\Lambda \vdash \Pi \vdash \Delta \vdash \Gamma \vdash \nabla$.

Solution. Let $\Gamma : [\mathcal{O}(X)^{\mathrm{op}}, \mathrm{Set}] \to \mathrm{Set}$ the functor sending a presheaf $\mathcal{F}$ to $\Gamma(\mathcal{F}) := \mathcal{F}(X)$. This is called the global sections functor. Then, if A is a set, and $\mathcal{F}$ is a presheaf, let $f\colon \Delta A \to \mathcal{F}$ be a morphism of presheafs (i.e. a natural transformation of functors) Then the natural transformation at the object X gives us a morphism $f_X\colon A = (\Delta A)(X) \to (\mathcal{F})(X) = \Gamma(\mathcal{F})$.

On the other hand, if $g\colon A \to \Gamma(\mathcal{F})$ is a map of sets, we define a morphism of presheafs η in the following way: if $U \subseteq X$ is an open set of X, then $\eta_U\colon A = (\Delta A)(U) \to (\mathcal{F})(U)$ via $a \mapsto \mathrm{res}^X_U(g(a))$. (Note that this clearly commutes with the restriction maps, hence is a morphism of presheafs.) We omit the proof that those mappings are mutually inverse, and natural. This gives us a natural isomorphism $\mathrm{Hom}(\Delta A, \mathcal{F}) \to \mathrm{Hom}(A, \Gamma(\mathcal{F}))$, hence Δ is left adjoint to Γ.

Let $\nabla\colon \mathrm{Set} \to [\mathcal{O}(X)^{\mathrm{op}}, \mathrm{Set}]$ be the functor given on objects by $A \mapsto \nabla A$, where ∇A is the presheaf on X, given by $U \mapsto (\nabla A)(U) = \{*\}$ if $U \neq X$, and $(\nabla A)(X) = A$. (with the obvious restriction maps).

Then giving a map $\Gamma(\mathcal{F}) \to A$ is the same as giving a map $\mathcal{F} \to \nabla A$, because on any level $U \neq X$, there is only one map $\mathcal{F}(U) \to (\nabla A)(U) = \{*\}$, and on the level X, we just use our given map $\mathcal{F}(X) = \Gamma(\mathcal{F}) \to A = (\nabla A)(X)$. (We omit a formal proof that this indeed gives the desired adjunction).

Π will be the functor $\mathcal{F} \mapsto \mathcal{F}(\emptyset)$, because we can build a map $\mathcal{F} \to \Delta A$ on level U via $f_U = g \circ res^U_\emptyset : \mathcal{F}(U) \to A = \Delta A(U)$. We omit further verivications that this approach gives the desired result.

Λ will then be the functor $A \mapsto \Lambda A$, where ΛA is the presheaf given by $(\Lambda A)(U) = \emptyset$ if $U \neq \emptyset$, And $(\Lambda A)(\emptyset) = A$. The proof is similar to the proof that ∇ is a right adjoint of Γ. $\square$