hw4_MK

April 9, 2022

0.0.1 1)

$$v'''(t) + v''(t) + 4v'(t) + 4v(t) = 4t^2 + 8t - 10,$$

$$v(0) = -3, \quad v'(0) = -2, \quad v''(0) = 2.$$

1.

$$v(t) = -\sin(2t) + t^2 - 3$$

B.c. verification:

$$v(0) = -\sin(2 \cdot 0) + 0^2 - 3 = -3$$

$$v'(t) = -2\cos(2t) + 2t$$

$$v'(0) = -2\cos(2 \cdot 0) + 2 \cdot 0 = -2$$

$$v''(t) = 4\sin(2t) + 2$$

$$v''(0) = 4\sin(2 \cdot 0) + 2 = 2$$

$$v'''(t) = 8\cos(2t)$$

The main equation verification:

$$v'''(t) + v''(t) + 4v'(t) + 4v(t) =$$

$$8\cos(2t) + 4\sin(2t) + 2 + 4(-2\cos(2t) + 2t) + 4(-\sin(2t) + t^2 - 3) =$$
$$8\cos(2t) + 4\sin(2t) + 2 - 8\cos(2t) + 8t - 4\sin(2t) + 4t^2 - 12 = 4t^2 + 8t - 10$$

Since everything in the equation is Lipschitz continuous over R, then there is a unique solution. Since $v(t)$ above is a solution, it's also unique.

2. Introduce new variables:

$$u_1(t) = v(t), u_2(t) = v'(t), u_3(t) = v''(t).$$

Then the equations take the form:

$$u_1'(t) = u_2(t), u_2'(t) = u_3(t), u_3'(t) = 4t^2 + 8t - 10 - u_3(t) - 4u_2(t) - 4u_1(t)$$

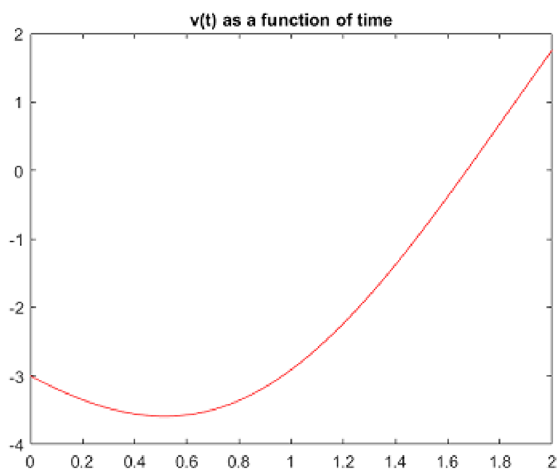
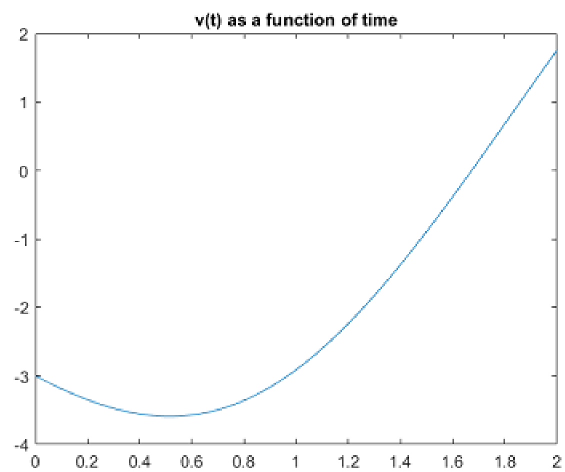
I.c.:

$$u_1(0) = v(0) = -3, u_2(0) = v'(0) = -2, u_3(0) = v''(0) = 2.$$

Hence:

$$\eta = [-3 \ -2 \ 2]^T$$

3. ode113 results (see odesample.m for the code):

True	Computed
	

4. odesampletest with ode113:

tol	max error	f evaluations
1.000e-01	6.271e-04	27
1.000e-02	4.875e-04	29
1.000e-03	6.338e-04	33
1.000e-04	1.196e-04	41
1.000e-05	1.996e-05	47
1.000e-06	7.727e-07	63
1.000e-07	2.087e-07	73
1.000e-08	1.283e-08	87
1.000e-09	4.231e-10	115
1.000e-10	6.668e-11	131
1.000e-11	6.143e-12	147
1.000e-12	1.576e-12	157

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 1.000e-13      5.462e-14        177
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5. odesampletest with ode45:

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   tol           max error    f evaluations

 1.000e-01      9.882e-06         67
 1.000e-02      1.024e-05         67
 1.000e-03      1.044e-05         67
 1.000e-04      9.925e-06         67
 1.000e-05      5.394e-06         85
 1.000e-06      5.069e-07        127
 1.000e-07      4.763e-08        199
 1.000e-08      4.573e-09        313
 1.000e-09      4.398e-10        493
 1.000e-10      4.359e-11        781
 1.000e-11      4.383e-12       1237
 1.000e-12      4.334e-13       1951
 1.000e-13      4.441e-14       3091
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0.0.2 2)

1.

The trapezoidal method:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = \frac{1}{2}(f(u(t_n)) + f(u(t_{n+1})))$$

The local truncation error (LTE):

$$\tau^n = \frac{u(t_{n+1}) - u(t_n)}{k} - \frac{f(u(t_n)) + f(u(t_{n+1}))}{2} = \frac{u(t_{n+1}) - u(t_n)}{k} - \frac{u'(t_n) + u'(t_{n+1})}{2}$$

Taylor expansions:
Since:

$$u(t_{n+1}) = u(t_n) + ku'(t_n) + \frac{1}{2}k^2u''(t_n) + \frac{1}{6}k^3u'''(t_n) + O(k^4)$$

Then:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = u'(t_n) + \frac{1}{2}ku''(t_n) + \frac{1}{6}k^2u'''(t_n) + O(k^3)$$

Also, since:

$$u'(t_{n+1}) = u'(t_n) + ku''(t_n) + \frac{1}{2}k^2u'''(t_n) + O(k^3)$$

Then:

$$\frac{u'(t_n) + u'(t_{n+1})}{2} = u'(t_n) + \frac{1}{2}ku''(t_n) + \frac{1}{4}k^2u'''(t_n) + O(k^3)$$

From above LTE:

$$\tau^n = -\frac{1}{12}k^2u'''(t_n) + O(k^3)$$

2.

The 2-step BDF method:

$$\frac{3u(t_{n+1}) - 4u(t_n) + u(t_{n-1})}{2k} = f(u(t_{n+1}))$$

The local truncation error (LTE):

$$\tau^n = \frac{3u(t_{n+1}) - 4u(t_n) + u(t_{n-1})}{2k} - f(u(t_{n+1})) = \frac{3u(t_{n+1}) - 4u(t_n) + u(t_{n-1})}{2k} - u'(t_{n+1})$$

Taylor expansions:

Since:

$$u(t_{n+1}) = u(t_n) + ku'(t_n) + \frac{1}{2}k^2u''(t_n) + \frac{1}{6}k^3u'''(t_n) + O(k^4)$$

Then:

$$\frac{3u(t_{n+1}) - 3u(t_n)}{2k} = \frac{3}{2}u'(t_n) + \frac{3}{4}ku''(t_n) + \frac{1}{4}k^2u'''(t_n) + O(k^3)$$

Also, since:

$$u(t_{n-1}) = u(t_n) - ku'(t_n) + \frac{1}{2}k^2u''(t_n) - \frac{1}{6}k^3u'''(t_n) + O(k^4)$$

Then:

$$\frac{u(t_{n-1}) - u(t_n)}{2k} = -\frac{1}{2}u'(t_n) + \frac{1}{4}ku''(t_n) - \frac{1}{12}k^2u'''(t_n) + O(k^3)$$

Also:

$$u'(t_{n+1}) = u'(t_n) + ku''(t_n) + \frac{1}{2}k^2u'''(t_n) + O(k^3)$$

From above LTE:

$$\tau^n = -\frac{1}{3}k^2u'''(t_n) + O(k^3)$$

3.

The Runge-Kutta method:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = f(u(t_n) + \frac{1}{2}kf(u(t_n)))$$

The local truncation error (LTE):

$$\tau^n = \frac{u(t_{n+1}) - u(t_n)}{k} - f(u(t_n) + \frac{1}{2}kf(u(t_n))) = \frac{u(t_{n+1}) - u(t_n)}{k} - f(u(t_n) + \frac{1}{2}ku'(t_n))$$

Taylor expansions:

From previous parts:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = u'(t_n) + \frac{1}{2}ku''(t_n) + \frac{1}{6}k^2u'''(t_n) + O(k^3)$$

Since:

$$f(u(t_n) + \frac{1}{2}ku'(t_n)) = f(u(t_n)) + \frac{1}{2}ku'(t_n)f'(u(t_n)) + \frac{1}{8}k^2(u'(t_n))^2f''(u(t_n)) + O(k^3)$$

And:

$$f(u(t_n)) = u'(t_n)$$

$$f'(u(t_n))u'(t_n) = u''(t_n)$$

$$f''(u(t_n))(u'(t_n))^2 + f'(u(t_n))u''(t_n) = u'''(t_n)$$

It follows that:

$$f(u(t_n) + \frac{1}{2}ku'(t_n)) = u'(t_n) + \frac{1}{2}ku''(t_n) + \frac{1}{8}k^2(u'''(t_n) - \frac{(u''(t_n))^2}{u'(t_n))}) + O(k^3)$$

From above LTE:

$$\tau^n = \frac{1}{24}k^2u'''(t_n) + \frac{1}{8}k^2\frac{(u''(t_n))^2}{u'(t_n))} + O(k^3)$$

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1.

2-step Adams-Moulton method:

$$\frac{u(t_{n+2}) - u(t_{n+1})}{k} = \frac{1}{12}(-f(u(t_n)) + 8f(u(t_{n+1})) + 5f(u(t_{n+2})))$$

The local truncation error (LTE):

$$\tau(t_{n+2}) = \frac{u(t_{n+2}) - u(t_{n+1})}{k} - \frac{-f(u(t_n)) + 8f(u(t_{n+1})) + 5f(u(t_{n+2}))}{12}$$

$$\tau(t_{n+2}) = \frac{u(t_{n+2}) - u(t_{n+1})}{k} - \frac{-u'(t_n) + 8u'(t_{n+1}) + 5u'(t_{n+2})}{12}$$

Matching corresponding coefficients:

$$\beta_0 = -\frac{1}{12}, \quad \beta_1 = \frac{2}{3}, \quad \beta_2 = \frac{5}{12}$$

2.

$$p(t) = [f_n] + [f_n, f_{n+1}](t - t_n) + [f_n, f_{n+1}, f_{n+2}](t - t_n)(t - t_{n+1})$$

where

$$t_n = -k, \quad t_{n+1} = 0, \quad t_{n+2} = k$$

and

$$f_{n+j} = f(u(t_{n+j}))$$

$$p(t) = A + B(t + k) + C(t + k)t$$

$$p(t) = (A + Bk) + (B + Ck)t + Ct^2$$

Then:

$$[f_n] = u'(t_n)$$

$$[f_n, f_{n+1}] = \frac{[f_{n+1}] - [f_n]}{t_{n+1} - t_n} = \frac{u'(t_{n+1}) - u'(t_n)}{k}$$

$$[f_{n+2}, f_{n+1}] = \frac{[f_{n+2}] - [f_{n+1}]}{t_{n+2} - t_{n+1}} = \frac{u'(t_{n+2}) - u'(t_{n+1})}{k}$$

$$[f_n, f_{n+1}, f_{n+2}] = \frac{[f_{n+2}, f_{n+1}] - [f_n, f_{n+1}]}{t_{n+2} - t_n} = \frac{u'(t_{n+2}) - 2u'(t_{n+1}) + u'(t_n)}{2k^2}$$

It follows that:

$$\int_{t_{n+1}}^{t_{n+2}} f(u(s))\, ds =$$

$$\int_0^k p(t)\, dt =$$

$$(A + Bk)k + \frac{B + Ck}{2}k^2 + \frac{C}{3}k^3 =$$

$$Ak + \frac{3}{2}Bk^2 + \frac{5}{6}Ck^3 =$$

$$u'(t_n)k + \frac{3}{2}\frac{u'(t_{n+1}) - u'(t_n)}{k}k^2 + \frac{5}{6}\frac{u'(t_{n+2}) - 2u'(t_{n+1}) + u'(t_n)}{2k^2}k^3 =$$

$$-\frac{1}{12}u'(t_n) + \frac{2}{3}u'(t_{n+1}) + \frac{5}{12}u'(t_{n+2})$$

Therefore:

$$\beta_0 = -\frac{1}{12}, \quad \beta_1 = \frac{2}{3}, \quad \beta_2 = \frac{5}{12}$$

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1.

The predictor-corrector method:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = \frac{f(u(t_n)) + f(u(t_n) + kf(u(t_n)))}{2}$$

The local truncation error (LTE):

$$\tau^n = \frac{u(t_{n+1}) - u(t_n)}{k} - \frac{f(u(t_n)) + f(u(t_n) + kf(u(t_n)))}{2} = \frac{u(t_{n+1}) - u(t_n)}{k} - \frac{u'(t_n) + f(u(t_n) + ku'(t_n))}{2}$$

Taylor expansions:

From previous parts:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = u'(t_n) + \frac{1}{2}ku''(t_n) + \frac{1}{6}k^2u'''(t_n) + O(k^3)$$

Since:

$$f(u(t_n)+ku'(t_n)) = f(u(t_n)) + ku'(t_n)f'(u(t_n)) + \frac{1}{2}k^2(u'(t_n))^2 f''(u(t_n)) + O(k^3)$$

And:

$$f(u(t_n)) = u'(t_n)$$

$$f'(u(t_n))u'(t_n) = u''(t_n)$$

$$f''(u(t_n))(u'(t_n))^2 + f'(u(t_n))u''(t_n) = u'''(t_n)$$

It follows that:

$$f(u(t_n)+ku'(t_n)) = u'(t_n) + ku''(t_n) + \frac{1}{2}k^2(u'''(t_n) - \frac{(u''(t_n))^2}{u'(t_n)}) + O(k^3)$$

From above LTE:

$$\tau^n = -\frac{1}{12}k^2u'''(t_n) + \frac{1}{4}k^2\frac{(u''(t_n))^2}{u'(t_n))} + O(k^3)$$

It follows that the predictor-corrector method is second order accurate.

2.

The 2nd predictor-corrector method:

$$\frac{u(t_{n+2})-u(t_{n+1})}{k} = \frac{-f(u(t_n)) + 8f(u(t_{n+1})) + 5f(u(t_{n+1}) + \frac{k}{2}(-f(u(t_n)) + 3f(u(t_{n+1}))))}{12}$$

The local truncation error (LTE):

$$\tau^{n+1} = \frac{u(t_{n+2})-u(t_{n+1})}{k} - \frac{-f(u(t_n)) + 8f(u(t_{n+1})) + 5f(u(t_{n+1}) + \frac{k}{2}(-f(u(t_n)) + 3f(u(t_{n+1}))))}{12}$$

$$= \frac{u(t_{n+2})-u(t_{n+1})}{k} - \frac{-u'(t_n) + 8u'(t_{n+1}) + 5f(u(t_{n+1}) + \frac{k}{2}(-u'(t_n) + 3u'(t_{n+1})))}{12}$$

Taylor expansions:

From previous parts:

$$\frac{u(t_{n+2})-u(t_{n+1})}{k} = u'(t_{n+1}) + \frac{1}{2}ku''(t_{n+1}) + \frac{1}{6}k^2u'''(t_{n+1}) + O(k^3)$$

Since:

$$f(u(t_{n+1}) + \frac{k}{2}(-u'(t_n) + 3u'(t_{n+1}))) = f(u(t_{n+1})) + \frac{k}{2}(-u'(t_n) +$$

$$3u'(t_{n+1}))f'(u(t_{n+1})) + \frac{k^2}{8}(-u'(t_n) + 3u'(t_{n+1}))^2 f''(u(t_{n+1})) + O(k^3)$$

And:

$$f(u(t_{n+1})) = u'(t_{n+1})$$

$$f'(u(t_{n+1}))u'(t_{n+1}) = u''(t_{n+1})$$

$$f''(u(t_{n+1}))(u'(t_{n+1}))^2 + f'(u(t_{n+1}))u''(t_{n+1}) = u'''(t_{n+1})$$

Also, since:

$$u'(t_n) = u'(t_{n+1}) - ku''(t_{n+1}) + \frac{1}{2}k^2u'''(t_{n+1}) + O(k^3)$$

Then:

$$u'(t_{n+1}) - u'(t_n) = ku''(t_{n+1}) - \frac{1}{2}k^2u'''(t_{n+1}) + O(k^3)$$

From above:

$$\frac{k}{2}(-u'(t_n) + 3u'(t_{n+1}))f'(u(t_{n+1})) =$$

$$\frac{k}{2}(ku''(t_{n+1}) + O(k^2) + 2u'(t_{n+1}))\frac{u''(t_{n+1})}{u'(t_{n+1})} =$$

$$ku''(t_{n+1}) + \frac{k^2}{2}\frac{(u''(t_{n+1}))^2}{u'(t_{n+1})} + O(k^3)$$

And:

$$\frac{k^2}{8}(-u'(t_n) + 3u'(t_{n+1}))^2 f''(u(t_{n+1})) = \frac{k^2}{8}(O(k) + 2u'(t_{n+1}))^2 f''(u(t_{n+1})) =$$

$$\frac{k^2}{8}(O(k^2) + 4O(k)u'(t_{n+1}) + 4(u'(t_{n+1}))^2)f''(u(t_{n+1})) =$$

$$\frac{k^2}{2}(u'(t_{n+1}))^2 f''(u(t_{n+1})) + O(k^3) =$$

$$\frac{k^2}{2}(u'''(t_{n+1}) - \frac{(u''(t_{n+1}))^2}{u'(t_{n+1})}) + O(k^3)$$

It follows that:

$$f(u(t_{n+1}) + \frac{k}{2}(-u'(t_n) + 3u'(t_{n+1}))) = u'(t_{n+1}) + ku''(t_{n+1}) + \frac{k^2}{2}u'''(t_{n+1}) + O(k^3)$$

Hence:

$$-u'(t_n) + 8u'(t_{n+1}) + 5f(u(t_{n+1}) + \frac{k}{2}(-u'(t_n) + 3u'(t_{n+1}))) =$$

$$7u'(t_{n+1}) + ku''(t_{n+1}) - \frac{1}{2}k^2u'''(t_{n+1}) + O(k^3) + 5u'(t_{n+1}) +$$

$$5ku''(t_{n+1}) + \frac{5k^2}{2}u'''(t_{n+1}) + O(k^3) = 12u'(t_{n+1}) + 6ku''(t_{n+1} + 2k^2u'''(t_{n+1}) + O(k^3)$$

and:

$$\frac{-u'(t_n) + 8u'(t_{n+1}) + 5f(u(t_{n+1}) + \frac{k}{2}(-u'(t_n) + 3u'(t_{n+1})))}{12} =$$

$$u'(t_{n+1}) + \frac{1}{2}ku''(t_{n+1} + \frac{1}{6}k^2u'''(t_{n+1}) + O(k^3)$$

From above LTE:

$$\tau^{n+1} = O(k^3)$$

It follows that this predictor-corrector method is third order accurate.

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1.

The 3-step Adams-Bashforth method:

$$\frac{u(t_{n+3}) - u(t_{n+2})}{k} = \frac{5f(u(t_n)) - 16f(u(t_{n+1})) + 23f(u(t_{n+2}))}{12}$$

$$\rho(\zeta) = \zeta^3 - \zeta^2$$

Hence:

$$\alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = -1, \quad \alpha_3 = 1$$

and:

$$\sum_{j=0}^{3} \alpha_j = 0 + 0 - 1 + 1 = 0$$

$$\sum_{j=0}^{3} j\alpha_j = 0 \cdot 0 + 1 \cdot 0 - 2 \cdot 1 + 3 \cdot 1 = 1$$

Also:

$$\sigma(\zeta) = \frac{1}{12}(5 - 16\zeta + 23\zeta^2)$$

Hence:

$$\beta_0 = \frac{5}{12}, \quad \beta_1 = -\frac{16}{12}, \quad \beta_2 = \frac{23}{12}, \quad \beta_3 = 0$$

and:

$$\sum_{j=0}^{3} \beta_j = \frac{5}{12} - \frac{16}{12} + \frac{23}{12} + 0 = 1 = \sum_{j=0}^{3} j\alpha_j$$

2.

The 3-step Adams-Moulton method:

$$\frac{u(t_{n+3}) - u(t_{n+2})}{k} = \frac{f(u(t_n)) - 5f(u(t_{n+1})) + 19f(u(t_{n+2})) + 9f(u(t_{n+3}))}{24}$$

$$\rho(\zeta) = \zeta^3 - \zeta^2$$

Hence:

$$\alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = -1, \quad \alpha_3 = 1$$

and:

$$\sum_{j=0}^{3} \alpha_j = 0 + 0 - 1 + 1 = 0$$

$$\sum_{j=0}^{3} j\alpha_j = 0 \cdot 0 + 1 \cdot 0 - 2 \cdot 1 + 3 \cdot 1 = 1$$

Also:

$$\sigma(\zeta) = \frac{1}{24}(1 - 5\zeta + 19\zeta^2 + 9\zeta^3)$$

Hence:

$$\beta_0 = \frac{1}{24}, \quad \beta_1 = -\frac{5}{24}, \quad \beta_2 = \frac{19}{24}, \quad \beta_3 = \frac{9}{24}$$

and:

$$\sum_{j=0}^{3} \beta_j = \frac{1}{24} - \frac{5}{24} + \frac{19}{24} + \frac{9}{24} = 1 = \sum_{j=0}^{3} j\alpha_j$$

3.

2-step Simpson's method:

$$\frac{u(t_{n+2}) - u(t_n)}{k} = \frac{f(u(t_n)) + 4f(u(t_{n+1})) + f(u(t_{n+2}))}{3}$$

$$\rho(\zeta) = \zeta^2 - 1$$

Hence:

$$\alpha_0 = -1, \quad \alpha_1 = 0, \quad \alpha_2 = 1$$

and:

$$\sum_{j=0}^{2} \alpha_j = -1 + 0 + 1 = 0$$

$$\sum_{j=0}^{2} j\alpha_j = -0 \cdot 1 + 1 \cdot 0 + 2 \cdot 1 = 2$$

Also:

$$\sigma(\zeta) = \frac{1}{3}(1 + 4\zeta + \zeta^2)$$

Hence:

$$\beta_0 = \frac{1}{3}, \quad \beta_1 = \frac{4}{3}, \quad \beta_2 = \frac{1}{3}$$

and:

$$\sum_{j=0}^{2} \beta_j = \frac{1}{3} + \frac{4}{3} + \frac{1}{3} = 2 = \sum_{j=0}^{r} j\alpha_j$$

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1.

Runge's 3rd order method:

$$c_1 = 0, a_{11} = 0, a_{12} = 0, a_{13} = 0$$

$$\sum_{j=1}^{3} a_{1j} = 0 + 0 + 0 = 0 = c_1$$

$$c_2 = 1/2, a_{21} = 1/2, a_{22} = 0, a_{23} = 0$$

$$\sum_{j=1}^{3} a_{2j} = 1/2 + 0 + 0 = 1/2 = c_2$$

$$c_3 = 1, a_{31} = 0, a_{32} = 1, a_{33} = 0$$

$$\sum_{j=1}^{3} a_{3j} = 0 + 1 + 0 = 1 = c_3$$

$$c_4 = 1, a_{41} = 0, a_{42} = 0, a_{43} = 1$$

$$\sum_{j=1}^{3} a_{4j} = 0 + 0 + 1 = 1 = c_4$$

$$b_1 = 1/6, b_2 = 2/3, b_3 = 0, b_4 = 1/6$$

$$\sum_{j=1}^{4} b_j = 1/6 + 2/3 + 0 + 1/6 = 1$$

$$\sum_{j=1}^{4} b_j c_j = 1/6 \cdot 0 + 2/3 \cdot 1/2 + 0 \cdot 1 + 1/6 \cdot 1 = 1/2$$

$$\sum_{j=1}^{4} b_j c_j^2 = 1/6 \cdot 0^2 + 2/3 \cdot (1/2)^2 + 0 \cdot 1^2 + 1/6 \cdot 1^2 = 1/3$$

$$\sum_{i=1}^{4} \sum_{j=1}^{4} b_i a_{ij} c_j = b_2 a_{21} c_1 + b_3 a_{32} c_2 + b_4 a_{43} c_3 = 1/6 \cdot 1 \cdot 1 = 1/6$$

From above (5.35), (5.38), and (5.39) are satisfied and hence the method is third order. Applying one step of the method:

$$Y_1 = U^n$$

$$Y_2 = U^n + k\frac{1}{2}f(Y_1, t_n + \frac{k}{2}) = U^n + k\frac{1}{2}\lambda Y_1 = U^n + k\frac{1}{2}\lambda U^n,$$

$$Y_3 = U^n + kf(Y_2, t_n + k) = U^n + k\lambda Y_2 = U^n + k\lambda U^n + k^2\frac{1}{2}\lambda^2 U^n,$$

$$Y_4 = U^n + kf(Y_3, t_n + k) = U^n + k\lambda Y_3 = U^n + k\lambda U^n + k^2\lambda^2 U^n + k^3\frac{1}{2}\lambda^3 U^n,$$

$$U^{n+1} = U^n + \frac{k}{6}f(Y_1, t_n) + \frac{2k}{3}f(Y_2, t_n + \frac{k}{2}) + \frac{k}{6}f(Y_4, t_n + k)$$

$$U^{n+1} = U^n + \frac{k\lambda}{6}U^n + \frac{2k\lambda}{3}(U^n + k\frac{1}{2}\lambda U^n) +$$
$$\frac{k\lambda}{6}(U^n + k\lambda U^n + k^2\lambda^2 U^n + k^3\frac{1}{2}\lambda^3 U^n) =$$
$$U^n + k\lambda U^n + \frac{k^2\lambda^2}{2}U^n + \frac{k^3\lambda^3}{6}U^n + O(k^4)$$

First tree terms at the end are the first three terms of the Taylor expansion of e^{ky}, hence the method is thrid order accurate.

2.

Heun's 3rd order method:

$$c_1 = 0, a_{11} = 0, a_{12} = 0$$

$$\sum_{j=1}^{2} a_{1j} = 0 + 0 = 0 = c_1$$

$$c_2 = 1/3, a_{21} = 1/3, a_{22} = 0$$

$$\sum_{j=1}^{2} a_{2j} = 1/3 + 0 = 1/3 = c_2$$

$$c_3 = 2/3, a_{31} = 0, a_{32} = 2/3$$

$$\sum_{j=1}^{2} a_{3j} = 0 + 2/3 = 2/3 = c_3$$

$$b_1 = 1/4, b_2 = 0, b_3 = 3/4$$

$$\sum_{j=1}^{3} b_j = 1/4 + 0 + 3/4 = 1$$

$$\sum_{j=1}^{3} b_j c_j = 1/4 \cdot 0 + 0 \cdot 1/3 + 3/4 \cdot 2/3 = 1/2$$

$$\sum_{j=1}^{3} b_j c_j^2 = 1/4 \cdot 0^2 + 0 \cdot (1/3)^2 + 3/4 \cdot (2/3)^2 = 1/3$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} b_i a_{ij} c_j = b_2 a_{21} c_1 + b_3 a_{32} c_2 = 3/4 \cdot 2/3 \cdot 1/3 = 1/6$$

From above (5.35), (5.38), and (5.39) are satisfied and hence the method is third order. Applying one step of the method:

$$Y_1 = U^n$$

$$Y_2 = U^n + k\frac{1}{3}f(Y_1, t_n + \frac{k}{3}) = U^n + k\frac{1}{3}\lambda Y_1 = U^n + k\frac{1}{3}\lambda U^n,$$

$$Y_3 = U^n + k\frac{2}{3}f(Y_2, t_n + \frac{2k}{3}) = U^n + k\frac{2}{3}\lambda Y_2 =$$

$$U^n + k\frac{2}{3}\lambda U^n + k^2\frac{2}{9}\lambda^2 U^n,$$

$$U^{n+1} = U^n + \frac{k}{4}f(Y_1, t_n) + \frac{3k}{4}f(Y_3, t_n + \frac{2k}{3})$$

$$U^{n+1} = U^n + \frac{k\lambda}{4}U^n + \frac{3k\lambda}{4}(U^n + k\frac{2}{3}\lambda U^n +$$

$$k^2\frac{2}{9}\lambda^2 U^n) = U^n + k\lambda U^n + \frac{k^2\lambda^2}{2}U^n + \frac{k^3\lambda^3}{6}U^n$$

First tree terms at the end are the first three terms of the Taylor expansion of e^{ky}, hence the method is thrid order accurate.

0.0.7 7)

1.

The trapezoidal method:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = \frac{1}{2}(f(u(t_n)) + f(u(t_{n+1})))$$

With $f(u) = \lambda u$:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = \frac{1}{2}(\lambda u(t_n) + \lambda u(t_{n+1}))$$

$$(1 - \lambda k/2)u(t_{n+1}) = (1 + \lambda k/2)u(t_n)$$

$$U^{n+1} = \frac{1 + z/2}{1 - z/2}U^n$$

2. Neumann series expansion:

$$\frac{1}{1 - z/2} = 1 + \frac{z}{2} + \frac{z^2}{4} + O(z^3)$$

Multiply by $(1 + z/2)$:

$$\frac{1 + z/2}{1 - z/2} = 1 + \frac{z}{2} + \frac{z^2}{4} + O(z^3) + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + O(z^4) = 1 + z + \frac{z^2}{2} + O(z^3)$$

Taylor series expansion of e^z:

$$e^z = 1 + z + \frac{z^2}{2} + O(z^3)$$

Hence:

$$R(z) = e^z + O(z^3)$$

0.0.8 8)

1.

$$R(z) = \frac{1 + \frac{1}{3}z}{1 - \frac{2}{3}z + \frac{1}{6}z^2} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots + Cz^{p+1} + O(z^{p+2})$$

$$1 + \frac{1}{3}z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} - \frac{2z}{3} - \frac{2z^2}{3} - \frac{z^3}{3} - \frac{z^4}{9} + \frac{z^2}{6} + \frac{z^3}{6} + \frac{z^4}{12} + \frac{z^5}{36} + \frac{z^4}{24} + \cdots + Cz^{p+1} + O(z^{p+2}) =$$

$$1 + \frac{1}{3}z + \frac{1}{24}z^4 + \cdots + Cz^{p+1} + O(z^{p+2})$$

and so

$$Cz^{p+1} = -\frac{1}{24}z^4 + \ldots$$

from which we conclude that $p = 3$.

2.

The backward Euler method:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = f(u(t_{n+1}))$$

With $f(u) = \lambda u$:

$$\frac{u(t_{n+1}) - u(t_n)}{k} = \lambda u(t_{n+1})$$

$$(1 - \lambda k) u(t_{n+1}) = u(t_n)$$

$$U^{n+1} = \frac{1}{1-z} U^n$$

$$R(z) = \frac{1}{1-z} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + Cz^{p+1} + O(z^{p+2})$$

$$1 = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} - z - z^2 - \frac{z^3}{2} - \frac{z^4}{6} + \cdots + Cz^{p+1} + O(z^{p+2}) = 1 - \frac{1}{2} z^2 + \cdots + Cz^{p+1} + O(z^{p+2})$$

and so

$$Cz^{p+1} = \frac{1}{2} z^2 + \ldots$$

from which we conclude that $p = 1$.

3.

The TR-BDF2 method:

$$Y_1 = U^n,$$

$$Y_2 = U^n + \frac{k}{4}(\lambda Y_1 + \lambda Y_2) = U^n + \frac{k}{4}(\lambda U_n + \lambda Y_2),$$

$$Y_2 = \frac{1 + \frac{\lambda k}{4}}{1 - \frac{\lambda k}{4}} U^n$$

$$Y_3 = U^n + \frac{k}{3}(\lambda Y_1 + \lambda Y_2 + \lambda Y_3) = U^n + \frac{k}{3}(\lambda U^n + \lambda \frac{1 + \frac{\lambda k}{4}}{1 - \frac{\lambda k}{4}} U^n + \lambda Y_3),$$

$$Y_3 = \frac{1 + \frac{\lambda k}{3}}{1 - \frac{\lambda k}{3}} U^n + \frac{\lambda k}{3} \frac{1 + \frac{\lambda k}{4}}{(1 - \frac{\lambda k}{3})(1 - \frac{\lambda k}{4})} U^n$$

$$U^{n+1} = Y_3 = (\frac{1 + \frac{z}{3}}{1 - \frac{z}{3}} + \frac{\frac{z}{3} + \frac{z^2}{12}}{(1 - \frac{z}{3})(1 - \frac{z}{4})}) U^n = \frac{1 + \frac{z}{3} - \frac{z}{4} - \frac{z^2}{12} + \frac{z}{3} + \frac{z^2}{12}}{1 - \frac{7z}{12} + \frac{z^2}{12}} U^n = \frac{1 + \frac{5z}{12}}{1 - \frac{7z}{12} + \frac{z^2}{12}} U^n$$

$$R(z) = \frac{1 + \frac{5z}{12}}{1 - \frac{7z}{12} + \frac{z^2}{12}} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + Cz^{p+1} + O(z^{p+2})$$

$$1 + \frac{5z}{12} = 1 - \frac{7z}{12} + \frac{z^2}{12} + z - \frac{7z^2}{12} + \frac{z^3}{12} + \frac{z^2}{2} - \frac{7z^3}{24} + \frac{z^4}{24} + \frac{z^3}{6} - \frac{7z^4}{72} + \frac{z^5}{72} + \cdots + Cz^{p+1} + O(z^{p+2}) =$$

$$1 + \frac{5z}{12} - \frac{1}{24}z^3 + \cdots + Cz^{p+1} + O(z^{p+2})$$

and so

$$Cz^{p+1} = \frac{1}{24}z^3 + \ldots$$

from which we conclude that $p = 2$.