hw6_MK

May 12, 2022

0.0.1 1.

$$U = [U_0, U_1, \ldots, U_m]^T$$

(a)

$$D_- = \frac{1}{h}\begin{pmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}$$

$$D_-^2 = \frac{1}{h^2}\begin{pmatrix} 1\cdot 1 & & & (-1)\cdot(-1) & (-1)\cdot 1 + 1\cdot(-1) \\ (-1)\cdot 1 + 1\cdot(-1) & 1\cdot 1 & & & (-1)\cdot(-1) \\ (-1)\cdot(-1) & (-1)\cdot 1 + 1\cdot(-1) & 1\cdot 1 & & \\ & (-1)\cdot(-1) & (-1)\cdot 1 + 1\cdot(-1) & 1\cdot 1 & \\ & & (-1)\cdot(-1) & (-1)\cdot 1 + 1\cdot(-1) & 1\cdot 1 \end{pmatrix}$$

$$D_-^2 = \frac{1}{h^2}\begin{pmatrix} 1 & & & 1 & -2 \\ -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \end{pmatrix}$$

Hence the Taylor series method gives:

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n) + \frac{(ak)^2}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$$

where the index j runs from 0 to m with addition of indices performed mod $m+1$ to incorporate the periodic boundary conditions.

(b) Beam-Warming method:

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{(ak)^2}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$$

Subtracting the method from part (a) we get:

$$\tau^n = -\frac{ak}{2h}U_j^n + \frac{ak}{h}U_{j-1}^n - \frac{ak}{2h}U_{j-2}^n$$

$$\tau^n = -\frac{ak}{2h}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$$

Hence it is **first order accurate** compared to the Beam-Warming method.

(c)

$$D_+ = \frac{1}{h}\begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{pmatrix}$$

$$D_+ + D_- = \frac{1}{h}\begin{pmatrix} & 1 & & & -1 \\ -1 & & 1 & & \\ & -1 & & 1 & \\ & & -1 & & 1 \\ 1 & & & -1 & \end{pmatrix}$$

$$D_+D_- = \frac{1}{h^2}\begin{pmatrix} (-1)\cdot 1 + 1\cdot(-1) & 1\cdot 1 & & & (-1)\cdot(-1) \\ (-1)\cdot(-1) & (-1)\cdot 1 + 1\cdot(-1) & 1\cdot 1 & & \\ & (-1)\cdot(-1) & (-1)\cdot 1 + 1\cdot(-1) & 1\cdot 1 & \\ & & (-1)\cdot(-1) & (-1)\cdot 1 + 1\cdot(-1) & 1\cdot 1 \\ 1\cdot 1 & & & (-1)\cdot(-1) & (-1)\cdot 1 + 1\cdot(-1) \end{pmatrix}$$

$$D_+D_- = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix}$$

Hence the method gives:

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_{j+1}^n - U_{j-1}^n) + \frac{(ak)^2}{2h^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

which is exactly the standard **Lax-Wendroff method**.

0.0.2 2.

(a)

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1})$$

The trapezoidal method form:

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{1}{2}((-a)\frac{U_j^n - U_{j-1}^n}{h} + (-a)\frac{U_j^{n+1} - U_{j-1}^{n+1}}{h})$$

$$\frac{U^{n+1} - U^n}{k} = \frac{1}{2}(AU^n + AU^{n+1})$$

Hence in $U'(t) = AU(t)$:

$$A = -\frac{a}{h}\begin{pmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}$$

Since the index j runs from 1 to $m+1$ with addition of indices performed mod $m+1$ to incorporate the periodic boundary conditions A considers boundary conditions too. (This is illustrated for a grid with $m+1=5$ unknowns and $h=1/5$.)

(b) Notice that $A = A_- + A_+$ where:

$$A_- = -\frac{a}{2h}\begin{pmatrix} & 1 & & & -1 \\ -1 & & 1 & & \\ & -1 & & 1 & \\ & & -1 & & 1 \\ 1 & & & -1 & \end{pmatrix}$$

$$A_+ = \frac{a}{2h}\begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix}$$

It is of the form shown in (10.15) of the textbook with $\epsilon = (ah)/2$. Hence according to the textbook, eigenvalues of A will be of the form:

$$\mu_p = -\frac{ia}{h}\sin(2\pi ph) - \frac{2\epsilon}{h^2}(1-\cos(2\pi ph))$$

$$\mu_p = -\frac{ia}{h}\sin(2\pi ph) - \frac{ah}{h^2}(1-\cos(2\pi ph))$$

$$\mu_p = -\frac{ia}{h}\sin(2\pi ph) - \frac{a}{h}(1-\cos(2\pi ph))$$

For the trapezoidal rule:

$$Re(z) \leq 0$$

where $z = k\mu_p$ and hence:

$$\frac{ak}{h}(\cos(2\pi ph) - 1) \leq 0$$

Since $\cos(2\pi ph) \leq 1$ for all p and h, we have $\cos(2\pi ph) - 1 \leq 0$. So the method is stable only when:

$$\frac{ak}{h} \geq 0$$

So since $k, h > 0$ when $a > 0$, the method **is stable** for all Courant numbers, and when $a < 0$, it **is not stable** for all Courant numbers.

(c) Replacing U_j^n and ak/h by $g(\xi)^n e^{i\xi jh}$ and ν in (6):

$$g(\xi)^{n+1}e^{i\xi jh} = g(\xi)^n e^{i\xi jh} - \frac{\nu}{2}(g(\xi)^n e^{i\xi jh} - g(\xi)^n e^{i\xi(j-1)h} + g(\xi)^{n+1}e^{i\xi jh} - g(\xi)^{n+1}e^{i\xi(j-1)h})$$

$$g(\xi) = 1 - \frac{\nu}{2}(1 - e^{-i\xi h} + g(\xi) - g(\xi)e^{-i\xi h})$$

$$g(\xi) = \frac{1 + \frac{\nu}{2}(e^{-i\xi h} - 1)}{1 - \frac{\nu}{2}(e^{-i\xi h} - 1)}$$

$|g(\xi)| \leq 1$ only when $Re(\nu(e^{-i\xi h} - 1)) \leq 0$. Hence:

$$\nu(\cos(\xi h) - 1) \leq 0$$

Since $\cos(\xi h) - 1 \leq 0$ for all ξ and h, the method is stable only if:

$$\nu \geq 0$$

(d) An implicit method such as (6) satisfies the CFL condition for any time step k. In this case the numerical domain of dependence is the positive half of the entire real line because the lower two-diagonal system couples together half of all points in such a manner that the solution at each point depends on the data at half of all points (i.e., half of the inverse of a lower two-diagonal is dense). Hence:

$$\frac{ak}{h} \geq 0$$

(e) We have a system of the form $U'(t) = AU(t) + g(t)$, where:

$$A = -\frac{a}{h}\begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}$$

$$g(t) = \begin{pmatrix} g_0(t)a/h \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

According to the textbook, A is a defective Jordan block with all its eigenvalues at the point $-a/h$. A is highly nonnormal. It is essentially a Jordan block of the sort discussed in Section D.5.1 of the textbook, and on a fine grid its ϵ-pseudospectra roughly fill up the circle of radius a/h about $-a/h$, even for very small ϵ. This is a case where we need to apply a more stringent requirement than simply requiring that $k\lambda$ be inside the stability region for all eigenvalues; we also need to require that

$$dist(k\lambda_\epsilon, S) \leq C\epsilon$$

where S is the stability region for Trapezoidal method:

$$S = \{z : Re(z) \leq 0\}$$

Hence:

$$\frac{ak}{h} \geq 0$$

For a CFL condition using the same reasoning as in the previous part on A_ϵ, since half of the ϵ-pseudoinverse of a lower two-diagonal is dense we have:

$$\frac{ak}{h} \geq 0$$

0.0.3 3.

Beam-Warming method:

$$U_j^{n+1} = U_j^n - \frac{ak}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{(ak)^2}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$$

Truncation error:

$$\tau^n = \frac{U_j^{n+1} - U_j^n}{k} + \frac{a}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) - \frac{a^2k}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$$

From a second order accurate Taylor series expansion in time:

$$u(x, t+k) = u(x,t) + ku_t(x,t) + \frac{1}{2}k^2u_{tt}(x,t)$$

Using the advection equation replace u_t by $-au_x$ and u_{tt} by a^2u_{xx}:

$$u(x, t+k) = u(x,t) - kau_x(x,t) + \frac{1}{2}k^2a^2u_{xx}(x,t)$$

From a second order accurate Taylor series expansion in space:

$$u(x-h, t) = u(x,t) - hu_x(x,t) + \frac{1}{2}h^2u_{xx}(x,t)$$

$$u(x-2h, t) = u(x,t) - 2hu_x(x,t) + 2h^2u_{xx}(x,t)$$

Hence:

$$3u(x,t) - 4u(x-h,t) + u(x-2h,t) = 3u(x,t) - 4u(x,t) + 4hu_x(x,t) - \\ -2h^2u_{xx}(x,t) + u(x,t) - 2hu_x(x,t) + 2h^2u_{xx}(x,t)$$

$$\frac{1}{2h}(3u(x,t) - 4u(x-h,t) + u(x-2h,t)) = u_x(x,t)$$

Therefore from Taylor series we will get:

$$u(x, t+k) = u(x,t) - \frac{ak}{2h}(3u(x,t) - 4u(x-h,t) + u(x-2h,t)) + \frac{1}{2}k^2a^2u_{xx}(x,t)$$

Relabling and rearranging terms and using second order centered difference for u_{xx} we will get:

$$\frac{U_j^{n+1}-U_j^n}{k} = -\frac{a}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{a^2k}{2h^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

Replacing this in the truncation error we get:

$$\tau^n = -\frac{a}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{a^2k}{2h^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n) +$$

$$+\frac{a}{2h}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) - \frac{a^2k}{2h^2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$$

$$\tau^n = \frac{a^2k}{2h^2}(U_{j+1}^n - 3U_j^n + 3U_{j-1}^n + U_{j-2}^n)$$

Since the last term is just an average of two second order centered difference approximations in space and since initially Taylor series was a second order in time, we can conclude that the method is of a **second order**.

Inserting the formula $v(x,t)$ into the difference equation:

$$v(x,t+k) = v(x,t) - \frac{ak}{2h}(3v(x,t) - 4v(x-h,t) + v(x-2h,t)) + \frac{a^2k^2}{2h^2}(v(x,t) - 2v(x-h,t) + v(x-2h,t))$$

Form above:

$$\frac{1}{2h}(3v(x,t) - 4v(x-h,t) + v(x-2h,t)) = v_x - \frac{1}{3}h^2 v_{xxx}$$

Expanding these terms in Taylor series about (x,t) and simplifying gives:

$$v_t + \frac{1}{2}kv_{tt} + \frac{1}{6}k^2 v_{ttt} + a(v_x - \frac{1}{3}h^2 v_{xxx}) - \frac{1}{2}a^2k(v_{xx} - hv_{xxx}) = 0$$

$$v_t + av_x = -\frac{1}{2}kv_{tt} - \frac{1}{6}k^2 v_{ttt} + \frac{1}{3}ah^2 v_{xxx} + \frac{1}{2}a^2kv_{xx} - \frac{1}{2}a^2khv_{xxx}$$

If we keep the $O(k)$ terms:

$$v_t + av_x = -\frac{1}{2}kv_{tt} + \frac{1}{2}a^2kv_{xx}$$

Then taking derivative with respect to t:

$$v_{tt} = -av_{xt} - \frac{1}{2}kv_{ttt} + \frac{1}{2}a^2kv_{xxt}$$

With respect to t:

$$v_{ttt} = -av_{xtt} + O(k)$$

With respect to x:

$$v_{ttx} = -av_{xtx} + O(k)$$

With respect to x the initial equation:

$$v_{tx} = -av_{xx} - \frac{1}{2}kv_{ttx} + \frac{1}{2}a^2kv_{xxx}$$

With respect to x one more time:

$$v_{txx} = -av_{xxx} + O(k)$$

Combining these gives

$$v_{tt} = a^2v_{xx} + \frac{1}{2}akv_{ttx} - \frac{1}{2}a^3kv_{xxx} - \frac{1}{2}kv_{ttt} + \frac{1}{2}a^2kv_{xxt}$$

Replacing v_{xtt}:

$$v_{tt} = a^2v_{xx} - \frac{1}{2}a^2kv_{xxt} - \frac{1}{2}a^3kv_{xxx} - \frac{1}{2}kv_{ttt} + \frac{1}{2}a^2kv_{xxt} + O(k^2)$$

$$v_{tt} = a^2v_{xx} - \frac{1}{2}a^3kv_{xxx} - \frac{1}{2}kv_{ttt} + O(k^2)$$

Insertingthis in the intial equation:

$$v_t + av_x = -\frac{1}{2}k(a^2v_{xx} - \frac{1}{2}a^3kv_{xxx} - \frac{1}{2}kv_{ttt}) - \frac{1}{6}k^2v_{ttt} + \frac{1}{3}ah^2v_{xxx} + \frac{1}{2}a^2kv_{xx} - \frac{1}{2}a^2khv_{xxx} + O(k^3)$$

$$v_t + av_x = \frac{1}{4}a^3k^2v_{xxx} + \frac{1}{12}k^2v_{ttt} + \frac{1}{3}ah^2v_{xxx} - \frac{1}{2}a^2khv_{xxx} + O(k^3)$$

From previous results:

$$v_{ttt} = -av_{xtt} + O(k) = a^2v_{xtx} + O(k) = -a^3v_{xxx} + O(k)$$

Finally inserting it in the previous equation:

$$v_t + av_x = \frac{1}{6}a^3k^2v_{xxx} + \frac{1}{3}ah^2v_{xxx} - \frac{1}{2}a^2khv_{xxx} + O(k^3)$$

$$v_t + av_x = \frac{1}{6}ah^2(2 - \frac{3ak}{h} + \frac{a^2k^2}{h^2})v_{xxx} + O(k^3)$$

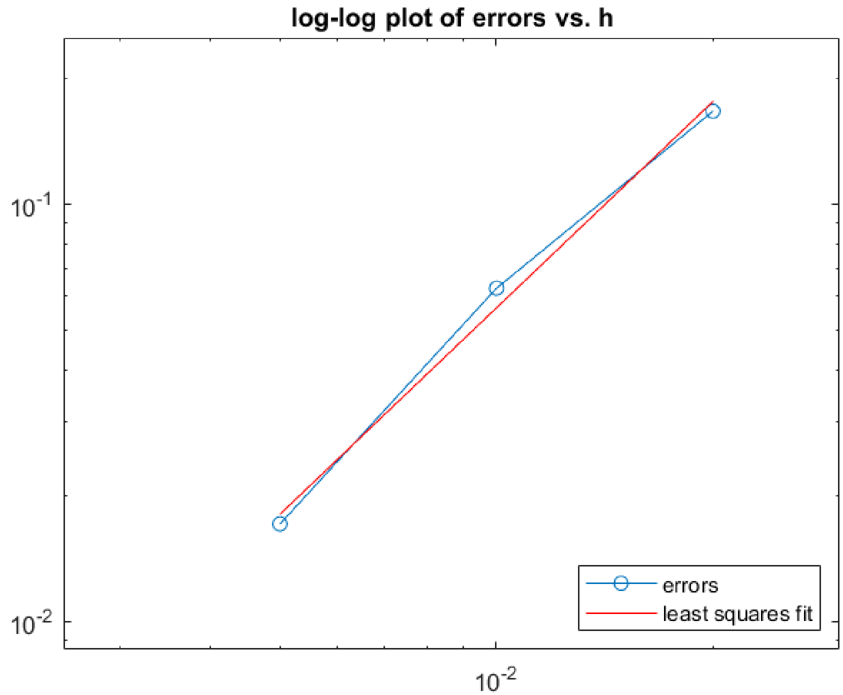
which is exactly (10.47) and of third order accuracy.

0.0.4 4.

(a) The results of error_table(hvals, E):

```
h          error        ratio        observed order
0.02000   1.66622e-01        NaN                NaN
0.01000   6.27543e-02   2.65514            1.40879
0.00500   1.70902e-02   3.67195            1.87654
```

The results of error_loglog(hvals, E):
Least squares fit gives E(h) = 108.653 * h^1.64267

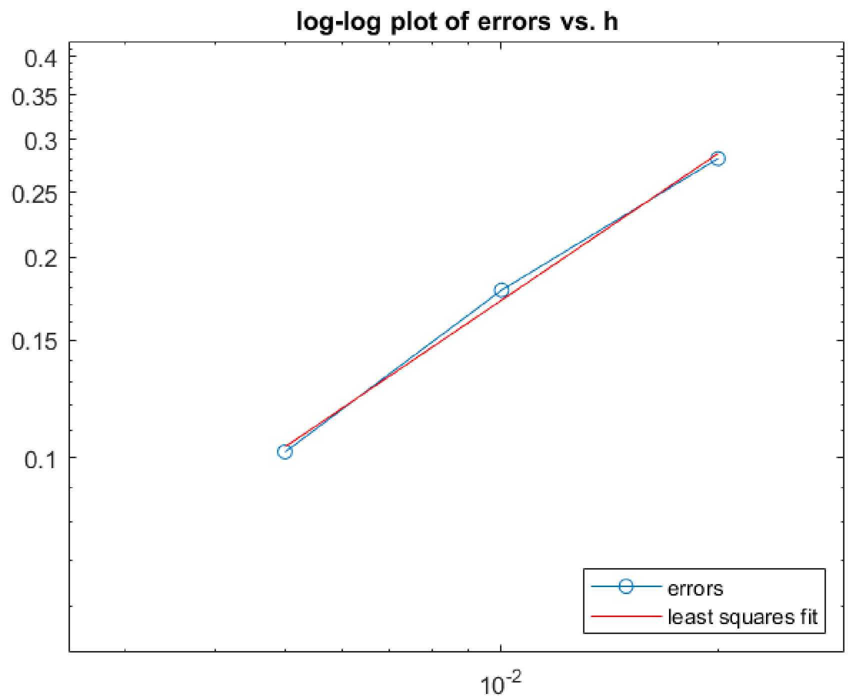


Corresponding plot:
Indeed since 1.64267 ~ 2, second-order accuracy is observed.

(b) The results of error_table(hvals, E):

h	error	ratio	observed order
0.02000	2.81062e-01	NaN	NaN
0.01000	1.78422e-01	1.57526	0.65559
0.00500	1.02013e-01	1.74902	0.80654

The results of error_loglog(hvals, E):
Least squares fit gives E(h) = 4.99386 * h^0.731067



Corresponding plot:
Indeed since 0.731067 ~ 1, first-order accuracy is observed.

(c) The results of error_table(hvals, E) as we decrease k:

h	error	ratio	observed order
0.02000	2.81062e-01	NaN	NaN
0.02000	4.95204e-01	0.56757	-Inf
0.02000	5.47426e-01	0.90460	-Inf

Indeed accuracy is decreased as $k \to 0$.
The modified equation for the upwinde method:

$$v_t + av_x = \frac{1}{2}ah(1 - \frac{ak}{h})v_{xx}$$

Since we are solving the advection equation $v_t + av_x = 0$ we can consider the RHS to be an error term:

$$E(k) = \frac{1}{2}ah(1 - \frac{ak}{h})v_{xx}$$

Let's say $k = ch$ then:

$$E(ch) = \frac{1}{2}ah(1 - \frac{ach}{h})v_{xx}$$

$$E(ch) = \frac{1}{2}ah(1 - ac)v_{xx}$$

Notice that for reasonable a, c values $(1 - ac) \in [0, 1]$ and as $c \to 0$ we have $(1 - ac) \to 1$. In other words as $k \to 0$ our error increases causing the mentioned paradox.

0.0.5 5.

$$e^{\frac{Ak}{2}}e^{Bk}e^{\frac{Ak}{2}} = (I + \frac{1}{2}kA + \frac{1}{8}k^2A^2 + \dots)(I + kB + \frac{1}{2}k^2B^2 + \dots)(I + \frac{1}{2}kA + \frac{1}{8}k^2A^2 + \dots)$$

$$= (I + \frac{1}{2}kA + \frac{1}{8}k^2A^2 + kB + \frac{1}{2}k^2AB + \frac{1}{2}k^2B^2 + \dots)(I + \frac{1}{2}kA + \frac{1}{8}k^2A^2 + \dots)$$

$$= I + \frac{1}{2}kA + \frac{1}{8}k^2A^2 + kB + \frac{1}{2}k^2AB + \frac{1}{2}k^2B^2 + \frac{1}{2}kA + \frac{1}{4}k^2A^2 + \frac{1}{2}k^2BA + \frac{1}{8}k^2A^2 + \dots$$

$$= I + k(A + B) + \frac{1}{2}k^2(A^2 + AB + BA + B^2) + \dots$$

Since Taylor expansions are exactly the same up to quadratic term, we can conclude that the Strang splitting is second oreder accurate on the problem (11.18).