State feedback control simulation of quadcopter model

Ts. Tengis*
Department of electronics
School of information and communication technology
MUST
Ulaanbaatar, Mongolia
*tengis@must.edu.mn

A. Batmunkh+
Department of electronics
School of information and communication technology
MUST
Ulaanbaatar, Mongolia
+abatmunkh@must.edu.mn

***Abstract* - There are many articles related to four rotor flying vehicle. Basic approaches of these articles are mathematical expressions describing dynamic model of the vehicle and proportional, integral and derivative control for manipulating the object in 3 dimensional space. This paper considers simulation of quadcopter control based on full state feedback technique with linearization in MATLAB environment and shows the results of the simulations.**

Keywords – quadcopter, rotor speed, full state feedback control, gain matrix

I. Introduction

During last decade with development of small size and high performance microcontrollers were developed many types of unmanned flying vehicles with multiple rotors. Among them the quadcopter takes important role due to its optimality, wide range of application.

Quadcopter is an air vehicle with four rotors uniformly located along a circle as shown in Figure 1 [2]. The machine mass is concentrated in the center of the circle. Each motor produces some force acting on the vehicle as lifting thrust. From Figure 1 seen that rotational direction of two rotors located diagonally is clockwise and other two's direction is counterclockwise. Rotating direction of the rotors not changed in time but their rotating velocity changed according to the movements in 3D space [1- 5]. Figure 2 shows direction of movement of the quadcopter controlled by speeds of four rotors which are the control inputs for the model. Bold circled arrow points the higher speed and the thin circled arrow notes lower speed.

Below shown brief description of several papers describing mathematical models of quadcopter based on state space approach and must be noted that in each of the articles have been used PID feedback control.

We will focus only on final state equations of the systems because all equations related to dynamics of quadcopter were derived in below considered papers.

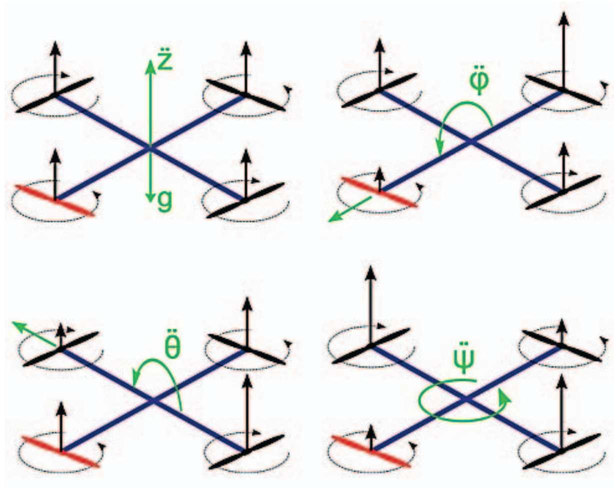

Fig 1. Forces acting on quadcopter and propeller rotational directions

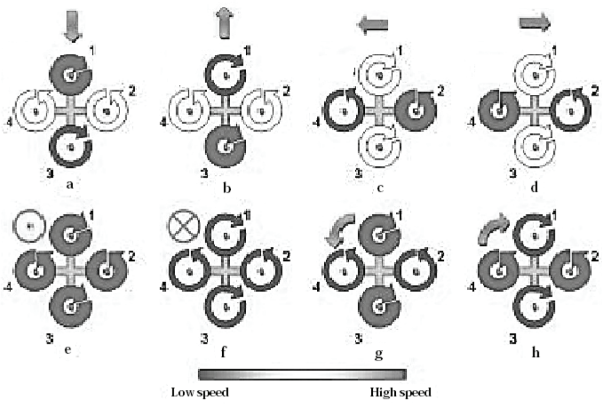

Fig 2. Rotor speed and various movement direction of the vehicle

A. Example one

In [1] using kinematics were derived state equations (1). Here, x_1, x_2 are 3D positions and linear velocities, x_3, x_4 are 3D angles and angular velocities in inertial frame, R, T_B, F_D are, accordingly, rotation matrix, thrust and force due to friction, $\tau_\varphi, \tau_\theta, \tau_\psi$ are torque and I_x, I_y, I_z are inertia along each axis and ω_x , ω_y , ω_z are the angular velocities of the quadcopter .

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \frac{1}{m} R T_B - \frac{1}{m} F_D$$
$$\dot{x}_3 = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\varphi & C_\theta S_\varphi \\ 0 & S_\varphi & C_\theta C_\varphi \end{bmatrix}^{-1} x_4 \quad (1)$$
$$\dot{x}_4 = \begin{bmatrix} \tau_\varphi / I_x \\ \tau_\theta / I_y \\ \tau_\psi / I_z \end{bmatrix} - \begin{bmatrix} (I_y - I_z)\omega_y \omega_z / I_x \\ (I_z - I_x)\omega_x \omega_z / I_y \\ (I_x - I_y)\omega_x \omega_y / I_z \end{bmatrix}$$

B. Example two

Dynamic behavior of the system expressed in [2] using Newton-Euler and Euler-Lagrange equations is

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{F}{m} - \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad (2)$$
$$\ddot{\eta} = \frac{d}{dt}\left(W_\eta^{-1}\right)\nu + W_\eta^{-1}\dot{\nu}$$

where η is the matrix of angular accelerations along three axis, A_1, A_2, A_3 are drag coefficients, F, m lifting force and mass respectively, ν is angular velocities in 3D space and W is transformation matrix from inertial to body frame.

C. Example three

Further, in [6] the equations of motion which represent dynamics of quadcopter in the inertial frame using Coriolis equation has been derived as (3).

$$[\ddot{p}] = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} + \begin{bmatrix} -C_\varphi S_\theta S_\psi - S\varphi S\psi \\ -C\varphi S_\theta S\psi + S\varphi C\psi \\ -C_\varphi C_\theta \end{bmatrix} F/m$$
$$\ddot{\varphi} = \tau_\varphi / I_x \quad (3)$$
$$\ddot{\theta} = \tau_\theta / I_y$$
$$\ddot{\psi} = \tau_\psi / I_z$$

where F, m are lifting force and mass, $[\ddot{p}] = [\ddot{x} \ \ , \ddot{y} \ \ , \ddot{z} \ \ ]^T$ is accordingly 3D linear acceleration of the vehicle, $\tau_\varphi, \tau_\theta, \tau_\psi$ are torques and I_x, I_y, I_z are inertia along each axis.

Each of the above articles uses PID control method due to its simplicity for controlling and tracking of a quadcopter.

In contrast with the above considered articles and basing on the above information and results of the experiments we set a goal to derive simplified linear dynamic equations for a quadcopter in the state space form with state feedback. It was assumed that the reference input parameters are various heights in meters, pitch and roll angle equal to zero in units of degree for the below considered problem. Under these conditions the vehicle will launch up and will be suspended at heights dictated by the reference and will track according to the various reference heights in form of step and sine function.

II. State space representation of a quadcopter

Nowadays, it is well known that one of main advantages of the state space method is modelling of multiple-input and multiple-output control system. When the equations of a system under control is highly nonlinear it is necessary to applicate linearization. But linearization could be done either slightly or highly depending on limits of state variables. As seen from (1) - (3) the state matrix contains sine, cosine functions which means the mathematical model is nonlinear.

It is well known that at small angles of about ± 25 °, the function $Sin\,(\theta)\ \approx \theta$ and $Cos\,(\theta)\ \approx 1$. Consequently can be achieved that the rotation matrix R can be replaced by the identity matrix by the aforementioned conditions with $\theta = 0$ and $\varphi\ =\ 0$ and the second component of the right-hand side of the second equation of (2) is zero because of its small values [4]. It can be seen in (3), each of the last three equations contains only one component of neglecting small inertia values.

Taking in account the drag force coefficients A_1, A_2, A_3 of (2) and considering above conditions and summarizing all information explained in the previous section the dynamic equations for the quadcopter can be written in the following main state space form (4).

$$\begin{bmatrix} \ddot{\varphi} \\ \ddot{\theta} \\ \ddot{\psi} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_x & 0 & 0 \\ 0 & 0 & 0 & 0 & A_y & 0 \\ 0 & 0 & 0 & 0 & 0 & A_z \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \frac{F}{m} \quad (4)$$

The full state space equations, (5), would be expressed for the open loop system by decomposing (4). Here, the state variables are $\varphi = x_1$, $\theta = x_3$, $\psi = x_5$ angles in body frame and $x = x_7$, $y = x_9$, $z = x_{11}$ positions in inertial frame. The force, F, consists of some constant and squared angular velocities, $[\omega_k^2]$, of rotors as described below.

$$
\begin{gathered}
\dot{x}_1 = x_2 \\
\dot{x}_2 = lk(\omega_4^2 - \omega_2^2)/I_{xx} \\
\dot{x}_3 = x_4 \\
\dot{x}_4 = lk(\omega_3^2 - \omega_1^2)/I_{yy} \\
\dot{x}_5 = x_6 \\
\dot{x}_6 = b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2)/I_{zz} \\
\dot{x}_7 = x_8 \\
\dot{x}_8 = A_x x_1 \\
\dot{x}_9 = x_{10} \\
\dot{x}_{10} = A_y x_3 \\
\dot{x}_{11} = x_{12} \\
\dot{x}_{12} = A_z x_5 - g + \frac{1}{m}(\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2)
\end{gathered} \tag{5}
$$

where ω_1^2, ω_2^2, ω_3^2, ω_4^2 are squares of angular velocities of rotors which are derived from propeller formula and produce the corresponding lifting forces, l, k, b, m, g are radius of quadcopter circle, lift and drag coefficients, mass of the vehicle and gravity constant respectively and I_{xx}, I_{yy}, I_{zz} are inertia for each axis. Consequently equation (5) can be rewritten in the standard matrix form of (6) and (7) for the open system.

$$
[\dot{x}] = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
A_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & A_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & A_z & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} [x] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -g \end{bmatrix} +
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -lk/I_{xx} & 0 & lk/I_{xx} \\
0 & 0 & 0 & 0 \\
-lk/I_{yy} & 0 & lk/I_{yy} & 0 \\
0 & 0 & 0 & 0 \\
b/I_{zz} & -b/I_{zz} & b/I_{zz} & -b/I_{zz} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
k & k & k & k
\end{bmatrix} [\omega] \tag{6}
$$

$$
[y] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} [x] \tag{7}
$$

where $[x] = [x_1, x_2, \ldots, x_{12}]^T$ -the internal states such as positions in inertial frame, angles around each axis and their first derivatives, $[\omega] = [\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2]^T$-squared angular velocities of rotors and $[y] = [y_1, y_2 \ldots y_6]^T$ -the outputs.

III. STATE FEEDBACK CONTROL

As described in introduction, control and tracking of quadcopter carried out using PID control method, but in this article we tried to consider usage of full state feedback control and its simulation in MATLAB. As known, the reference input of the closed loop system is the control input for tracking and control for the quadrotor system. Through the reference input will be dictated 3D motion of sixth degree of freedom in space. Figure 3 illustrates block scheme of the state feedback. The control input, u, consists of 4 angular velocities, $[\omega]$, of rotors and it is formed by the sum of reference input multiplied by the gain matrix *G* and internal states multiplied by the feedback matrix *K*.

A. Pole placement

Feedback gain matrix *K* would be defined after pole placement procedure of the characteristic equation of the system. The procedure of pole placement set such that the rise time, overshoot and settling time are set in the same manner as in [7, 8, 9]. We present, intuitively, the settling time for real model of quadcopter about 2 seconds and overshoot about no more 4%. Then using the pole dominant criterion we can choose the poles which are one of many variants as -2+1.96j; -2-1.96j; two -2.0; two -4.0; two -5.0; two -8.0; and two -3.0;

B. Reference input gain

For the system the state feedback control references are pitch, roll angles equal to zero and the height of the copter is h. Using state feedback law the relationship between plant input, *u*, and reference, *ref*, is written as (8).

$$u = G \cdot ref - Kx \tag{8}$$

Assuming that all state variables are readable (measureable) and using state feedback law, (8), and allowing the reference input, ref, we get the state feedback control system with reference input as shown in figure 2. The system on the figure demonstrates quadcopter structure with input $u = [\omega_1^2\ \omega_2^2\ \omega_3^2\ \omega_4^2]^T$ and reference $ref = [\psi_r\ \theta_r\ \varphi_r\ x_r\ y_r\ z_r]^T$. To get full correspondence between feedback states and reference input there should be used relation, *G*, that is expressed as follows (9).

$$G = -B^{-1}(A - BK)C^{-1} \tag{9}$$

Inserting (8) into standard state space equations would be created closed loop feedback equations of the control system that is follows as (10).

$$
\begin{gathered}
\dot{x} = (A - BK)x + BG \cdot ref \\
y = Cx
\end{gathered} \tag{10}
$$

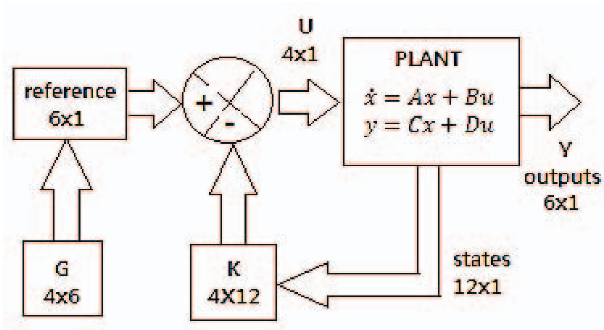

Fig 3. State feedback control with reference

IV. Discrete time state space model

In the last section we derived the linearized and simplified model of the quadcopter dynamics in continuous time domain. Now it is necessary to discretize the model as if it was a real model based on digital microcontroller system. It is known, that discretized version of continuous time system is derived through the Laplace transform and Z transform and using the corresponding MATLAB functions equations (8), (9), (10) are converted into discrete time form of (11), (12).

$$\begin{gathered} P_d = \exp(T_s[p]) \\ K_d = c2d(K) \\ G_d = -B_d^{-1}(A_d - B_d K_d)C_d^{-1} \end{gathered} \tag{11}$$

$$\begin{gathered} x[n+1] = (A_d - B_d K_d)x[n] + B_d G_d u[n] \\ y[n] = C_d x[n] \end{gathered} \tag{12}$$

where P_d is poles on Z plan, A_d, B_d, C_d, K_d, G_d are discrete time matrices and $x[n], y[n], u[n]$ are values of the states, outputs and control inputs at $n-th$ time step correspondingly.

V. Simulation

The simulation has been done in two steps. First is open loop control and second is closed loop control with reference. In the simulation some constant values as drag coefficients $A_x = 0.25$, $A_y = 0.25$, $A_z = 0.25$ and inertia $I_M = 3.357E-5, b = 1.14E-7, k = 2.98E-6$ are taken from [2]. Values of mass of the copter, *m,* distance from center of the copter to any of the rotor center, *l*, are could be chosen randomly. The sample time step $t_s = 0.01\,sec$ and duration of simulation time is within 10 seconds.

A. Open loop control

This simulation was done to show effects of rotor speed control for controlling movement of quadcopter in 3D space. The numbers shown in TABLE 1 are only relative speed of each rotors and Figure 4 shows various positioning of the copter in accordance with the values of the speed of rotors.

TABLE I. Rotor speed

	Relative speed of rotors				
	back	*forward*	*left*	*right*	*up*
ω_1	20	20	10	40	30
ω_2	10	40	20	20	30
ω_3	20	20	40	10	30
ω_4	40	10	20	20	30

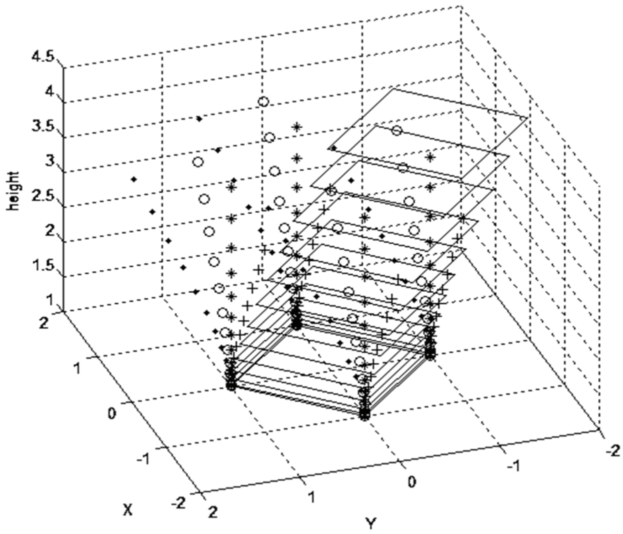

Fig 4. Results of 3D representation of open loop control in correspondence with the values in TABLE 1. Shown five directions of movement denoted by '.', '+', 'o' and square.

B. Closed loop control and reference input

Below shown part of MATLAB code for the system illustrated in Figure 3 and the "FOR" statement demonsrates the program loop of state feedback control.

```
%------------- OS, settling time-------------
Tsetl=2;
OS=4;
zt=-log(OS/100)/sqrt(pi^2+log(OS/100)^2);
wn=Tsetl/zt;
fn=wn/2/pi;
bt=sqrt(1-zt^2);
wd=wn*bt;
p=[-8 -8 -4 -4 -2 -2 -3 -3 -5 -5-zt*wn+bt*wn*i  -zt*wn-bt*wn*i];
dt=0.01;
t=0:dt:10;
hold on
for i=1:length(t)
    Rt=rotate(x(1,1),x(3,1),x(5,1));
    [a,b,c]=square(l,x(7,1),x(9,1),x(11,1));
    e=[a;b;c];
    ee=Rt*e;
%      plot3(x(7,1),x(9,1),x(11,1),'r.')
       plot3(e(1,:),e(2,:),e(3,:),'k')
%        plot(i*dt,x(7,1),i*dt,x(9,1),'-',i*dt,x(11,1))
%   plot(i*dt,u(1),'+',i*dt,u(2),'.',i*dt,u(3),'--',i*dt,u(4),'*')
     u=(Gd*ref(:,i)-Kd*x);
     x=Ad*x+Bd*u;
end
```

Here, the *square()* subroutine draws quadcopter body square frame, *rotate()* subroutine translates from body to inertial frame. The reference signal could be chosen of any form. As an example, the copter starts at position (0, 0, 0) and launches up to 80 meters and moves one circle of motion with radius of 8 meters and remains at position (5.76, -5.77, 78). Below shown the reference input matrix.

$$ref = \begin{bmatrix} zeros(size(t)) \\ zeros(1, length(t)) \\ zero(1, length(t)) \\ 8 * sin(2 * pi * t/T) \\ 8 * cos(2 * pi * t/T) \\ 80 * ones(1, length(t)) \end{bmatrix}$$

C. *Simulation results in graphics*

Figures 5, 6 correspondingly show timing diagrams and Figure 7 shows 3D imagination of trajectory of the plant.

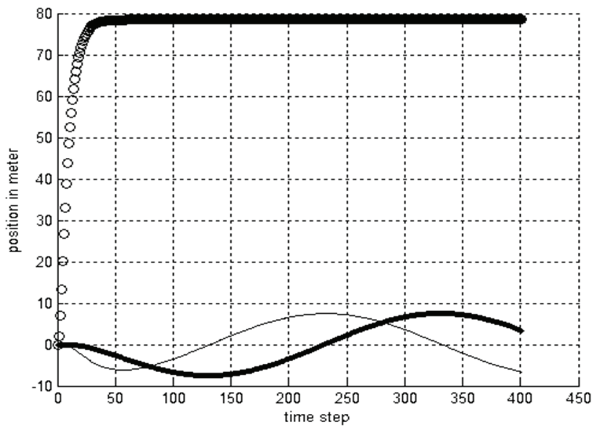

Fig 5. Result of Plot(i*dt,x(7,1),i*dt,x(9,1),'-',i*dt,x(11,1)) line. Small circle denotes the height or z-position, thin line x-position end thick line shows y-position.

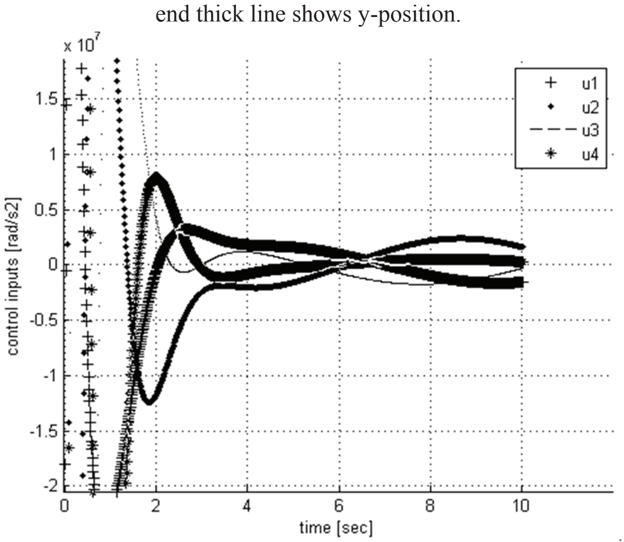

Fig 6. Vertically zoomed in timing diagrams of 4 control inputs. Result of plot(i*dt,u(1),'+',i*dt,u(2),'.',i*dt,u(3),'--',i*dt,u(4),'*') line

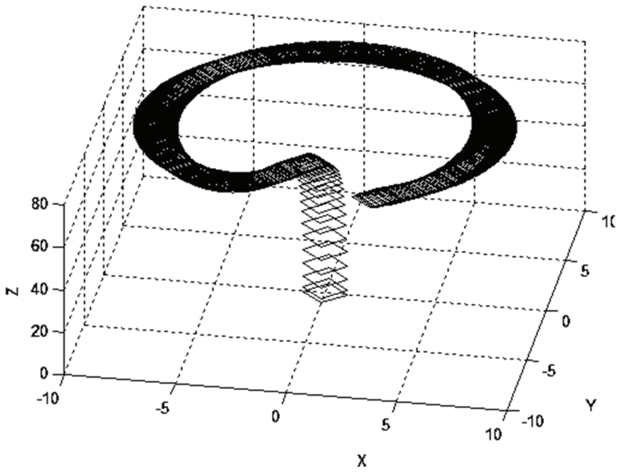

Fig 7. Trajectory in 3D space. Result of plot3(e(1,:),e(2,:),e(3,:),'k') line.

CONCLUTION

We tried to show by the simulation quadcopter model based on state feedback control. The method uses simplified mathematical model considering several simplification as linearization, zeroing of some nonlinear functions. Although the model was highly simplified but the simulation result shows relatively realistic dynamics. The linearized model is simulated in open loop and closed loop modes. During the simulation there was chosen reference input control as vertical launch, circular motion. The results are simulated and shown graphically in MATLAB environments.

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