Discretisation of Laplace's equation to numerically solve for fluid flow past a rotating cylinder

Krish Nigam, *Imperial College London* kn7811.com/archives/Fluid%20Flow

Motivation

From *temperature flow* to *voltage distributions* to *fluid flow*, **Laplace's equation** is the fundamental law which underpins all the mathematics involved in these scenarios.

ϕ satisfies *Laplace's* equation iff $\nabla^2\phi = 0$

Here, we concern ourselves specifically with the numerical approach to solving this problem by **discretising** our continuous region and we focus particularly on the case of **fluid flow**, in order to be able to model the fluid flow past a *rotating cylinder*.

Analytical Solution

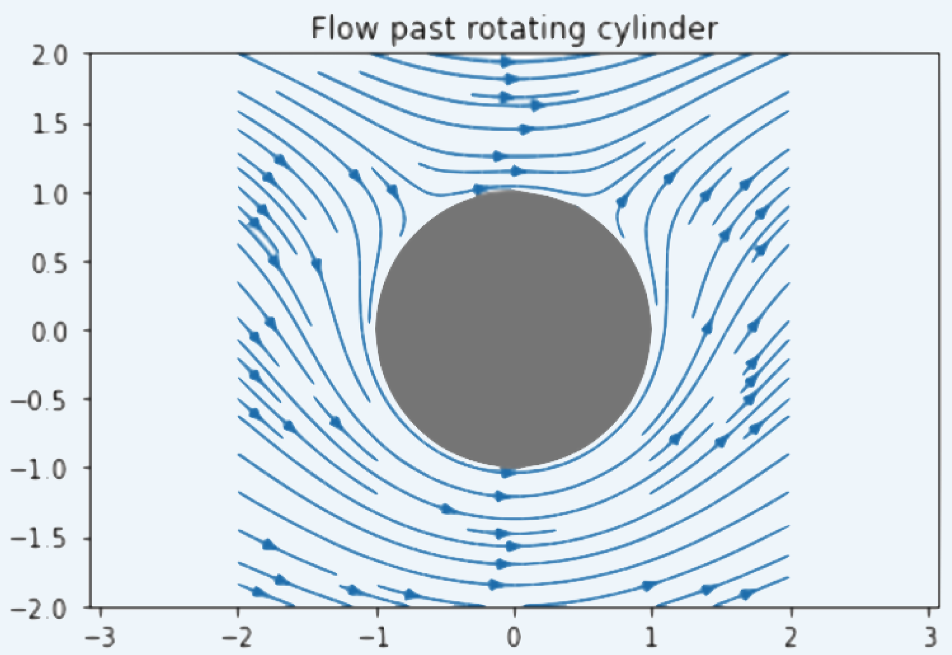
To compute an analytical solution, we assume throughout that the fluid flow is **inviscid**, **incompressible** and **irrotational**.
We introduce the complex potential

$$w(z) = \phi + i\psi$$

where ϕ, ψ satisfy Laplace's equation.
For a (possibly rotating) cylinder of radius a, centre $(0,0)$, its complex potential is given by

$$w(z) = V_\infty\left(z + \frac{a^2}{z}\right) + \frac{\Gamma}{2\pi}\log z$$

where $\Gamma \in \mathbb{R}$ is the constant modelling the rotation (*i.e.* $\Gamma = 0$ models the cylinder with no rotation).
We want to plot the velocity field, $\mathbf{u}$, in plotting streamlines where $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv$.



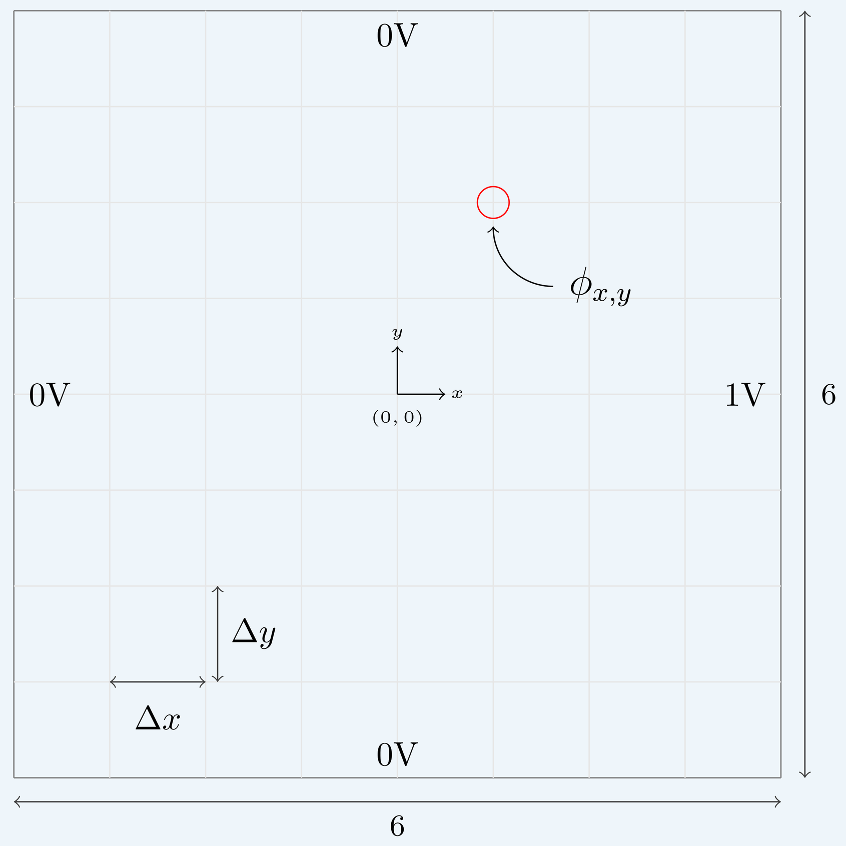
The details for the mathematical rigour behind these results can be found in [1].

Finite-Difference Method

We use a method called '*finite differences*' to solve the numerical version of Laplace's equation.

$$\nabla^2\phi = 0 \iff \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Consider a grid of **discrete** points with $\phi_{x,y}$ being the 'potential' value at (x, y).



We can find the gradient and replace Laplace's equation with the discrete version:

$$\frac{\phi_{x+1,y} - 2\phi_{x,y} + \phi_{x-1,y}}{(\Delta x)^2} + \frac{\phi_{x,y+1} - 2\phi_{x,y} + \phi_{x,y-1}}{(\Delta y)^2} = 0$$

but since we make our region a square $\Delta x = \Delta y$ reducing to

$$\boxed{\phi_{x,y} = \frac{1}{4}\left(\phi_{x+1,y} + \phi_{x-1,y} + \phi_{x,y+1} + \phi_{x,y-1}\right)}$$

This tells us that the *potential* at each point is the **average** of its neighbouring points!

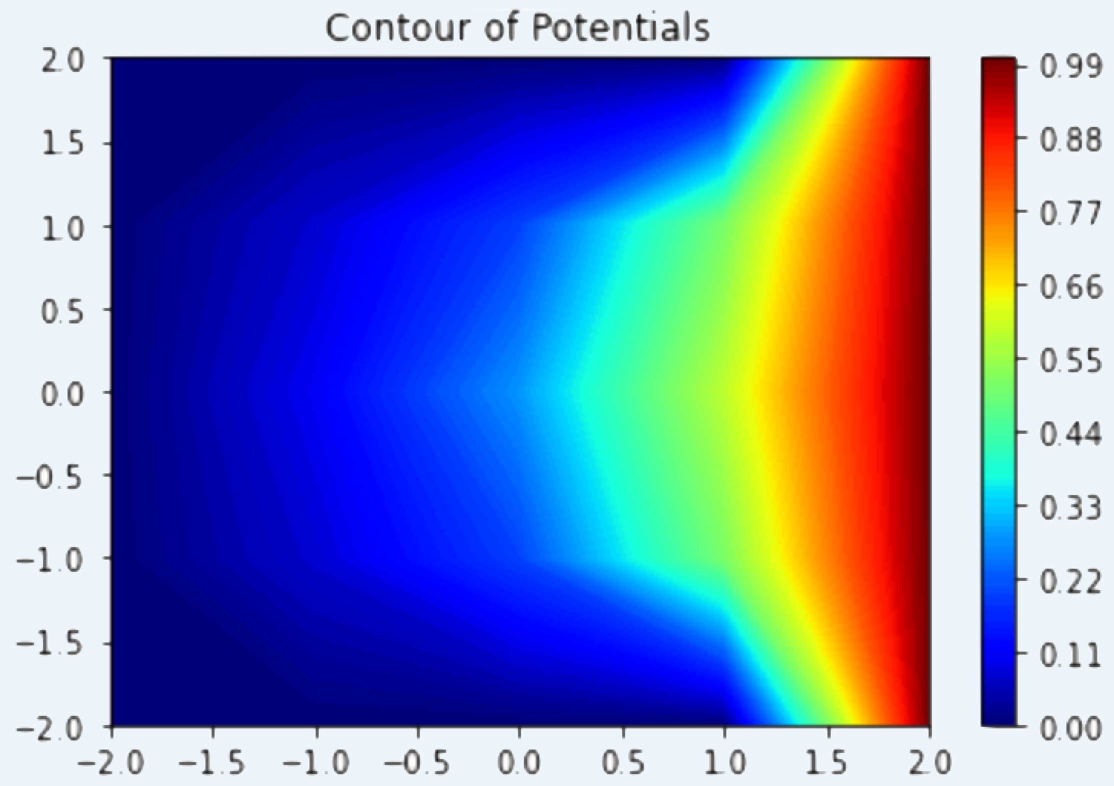
Hence, instead of only the 4 neighbouring points, we may also consider all 8 neighbouring points instead for faster *convergence*.

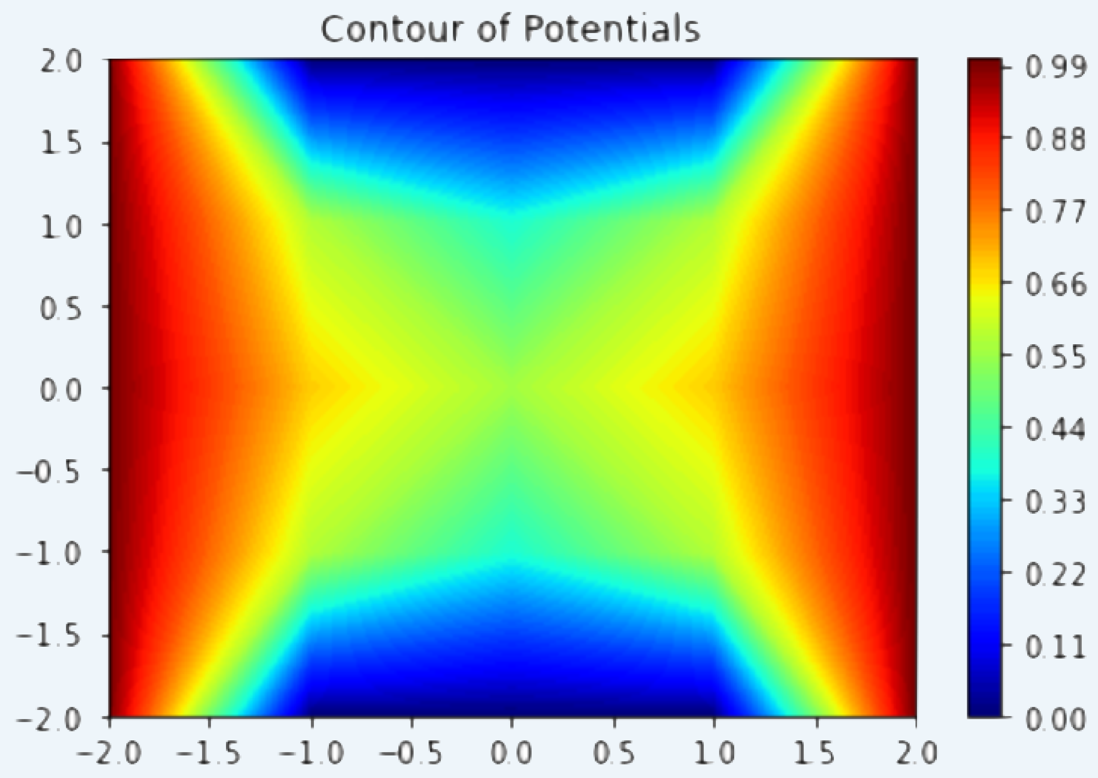
The Python code exploits this and is broadly based off the ideas in [2].

Results 1 - Potential distribution with no obstacle

We use the method of finite differences to plot the contours as shown. For instance, the first scenario may be interpreted as the right boundary being at one volt and all other boundaries grounded. Then the plot shows the voltage distribution.

Equally, we could say that the right boundary is at temperature 1°C and all other boundaries are at 0°C and the plot then shows the temperature distribution.

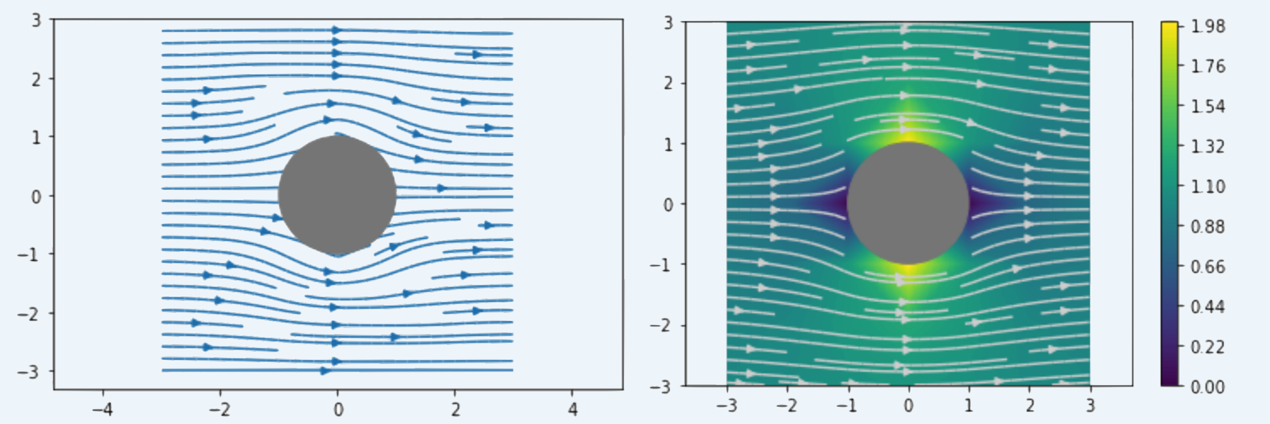




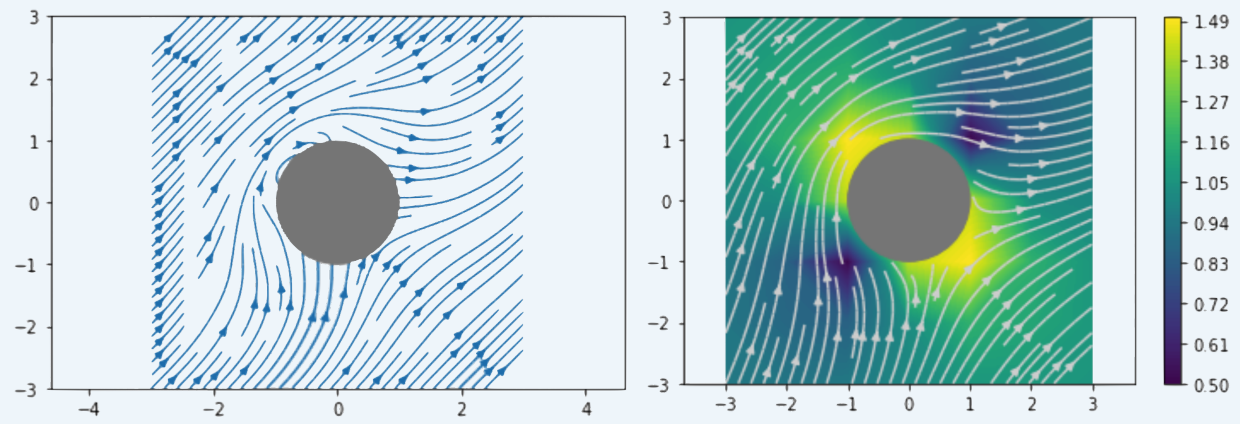
Results 2 - Fluid flow past a rotating cylinder

We now introduce an obstacle into our region, in this case a cross section of a cylinder (*i.e.* a circle). This obstacle in itself is also a boundary so we must fix our function's values there. Since we are discretising the problem, we fix the values at the grid points within the square made by $(-1,-1)$, $(-1,1)$, $(1,1)$, $(1,-1)$. We then iterate our algorithm considering left, right, bottom and top regions separately as we require the inner boundary conditions to be fixed.

For a cylinder with no rotation, we obtain the following plots (left - discretised; right - analytical):



Streamline plots for fluid flow past cylinder with no rotation.



Streamline plots for fluid flow past cylinder rotating clockwise.

Notice how that in both the cases the analytical plots have a contour background. These represent the magnitude of the velocity at each point in the fluid. Notice how while the approximations are not perfect, the numerical solutions plotted are both very similar to the analytical plots.

Further Applications

Our models for fluid flow can be extended past more funky geometries for which a complex potential function may not exist. Instead, boundary conditions must be established either through experimental means or intuition. We can further use this method beyond fluid flow to other physical systems as previously described, *e.g.* the flow past a cylinder is analgous to the electric field past a perfect insulator.

References

[1] Ruban, A. and Gajjar, J., 2014. *Fluid Dynamics Part 1: Classical Fluid Dynamics*. Oxford University Press, pp.129-232.

[2] *Using Python to Solve Computational Physics Problems*. Available at: <https://www.codeproject.com/Articles/1087025/Using-Python-to-Solve-Computational-Physics-Proble> [Accessed: 2022-06]

[3] *Potential flow around a circular cylinder*. Wikipedia. Available at: <https://en.wikipedia.org/wiki/Potential_flow_around_a_circular_cylinder> [Accessed: 2022-06]