Change of order of integration

→ In double integral, if the limits are constants, the order of integration is immaterial provided the limits are to be changed accordingly.

→ If the limits of integration are variables, the change of order of integration requires the change of limits also. This can be done by drawing a rough sketch of the region and changing limits accordingly.

① Evaluate the integral by changing the order of integration $\int_{0}^{\infty}\int_{x}^{\infty}\frac{e^{-y}}{y}\,dy\,dx$.

Sol: Given that $\int_{x=0}^{\infty}\int_{y=x}^{\infty}\frac{e^{-y}}{y}\,dy\,dx$

old	New
$x \to 0$ to ∞	$x \to 0$ to y.
$y \to x$ to ∞.	$y \to 0$ to ∞.

$y = x = 0, \infty$

changing the order of integration, the intergral can be written as

$$\int_{y=0}^{\infty}\int_{x=0}^{y}\frac{e^{-y}}{y}\,dx\,dy.$$

$$\therefore \int_{y=0}^{\infty}\int_{x=0}^{y}\frac{e^{-y}}{y}\,dx\,dy = \int_{y=0}^{\infty}\left[\int_{x=0}^{y} dx\right]\frac{e^{-y}}{y}\,dy.$$

$$= \int_{y=0}^{\infty}(x)_0^y\,\frac{e^{-y}}{y}\,dy.$$

$$= \int_{0}^{\infty}\cancel{y}\cdot\frac{e^{-y}}{\cancel{y}}\,dy$$

$$= \int_0^{\infty} e^{-y}\, dy$$

$$= \left(-e^{-y}\right)_0^{\infty}$$

$$= -e^{-\infty} + e^{-0}$$

$$= 0+1$$

$$= 1.$$

② Change the order of integration and evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$.

Sol: Given that $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$.

The limits are

$y = \frac{x^2}{4a}$, $y = 2\sqrt{ax}$ and $x = 0$ and $4a$.

$x^2 = 4ay$, $y^2 = 4ax$. | $x = 0$ and $4a$.

$\frac{y^2}{4a} = x$.

Old	New
x: 0 to $4a$	x: $\frac{y^2}{4a}$ to $2\sqrt{ay}$
y: $\frac{x^2}{4a}$ to $2\sqrt{ax}$	y: 0 to $4a$

$x^2 = 4ay$ | $x = \frac{y^2}{4a}$.

$x = 2\sqrt{ay}$

Changing the order of integration, the given integral can be written as $\int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\, dy$.

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$$\therefore \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_0^{4a}\left[\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_0^{4a} (x)_{\frac{y^2}{4a}}^{2\sqrt{ay}}\, dy.$$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= 2\sqrt{a}\left(\frac{y^{3/2}}{3/2}\right)_0^{4a} - \frac{1}{4a}\left(\frac{y^3}{3}\right)_0^{4a}.$$

$$= \frac{4\sqrt{a}}{3}\left[(4a)^{3/2}\right] - \frac{1}{12a}\left(4a\right)^3$$

$$= \cancel{\frac{16}{3}a^{1/2}a^{3/2}}$$

$$= \frac{4\sqrt{a}}{3}\times(4)^{3/2}a^{3/2} - \frac{\cancel{64}^{16}}{\cancel{12a}_{3}}a^{\cancel{3}2}.$$

$$= \frac{4\times(2^2)^{3/2}}{3}\cdot a^{1/2}\cdot a^{3/2} - \frac{16}{3}a^2.$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

③ change of order of integration and evaluate $\int_0^a\int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy$.

Sol: The limits are $y = \frac{x}{a}$, $y = \sqrt{x/a}$

$x = ay$, $\frac{ay^2}{\cancel{a}} = x$

and $x = 0, a$

old	new
x: 0 to a	x: ay^2 to ay.
y: $\frac{x}{a}$ to $\sqrt{\frac{x}{a}}$	y: 0 to 1

changing the order of integration, the given integral can be written as

$$\int_{y=0}^{1}\int_{x=ay^2}^{ay}(x^2+y^2)\,dx\,dy$$

$$\therefore \int_{y=0}^{1}\int_{ay^2}^{ay}(x^2+y^2)\,dx\,dy = \int_0^1\left[\int_{x=ay^2}^{ay}(x^2+y^2)\,dx\right]dy$$

$$= \int_0^1 \left(\frac{x^3}{3}+xy^2\right)_{ay^2}^{ay} dy$$

$$= \int_0^1 \left[\frac{1}{3}\left[(ay)^3-(ay^2)^3\right]+(ay-ay^2)y^2\right]dy$$

$$= \int_0^1 \left[\frac{1}{3}a^3y^3(1-y^3)+a(1-y)y^3\right]dy$$

$$= \frac{1}{3}\left[\int_0^1 (a^3y^3-a^3y^6)\,dy\right]+\int_0^1 (ay^3-ay^4)\,dy$$

$$= \frac{1}{3}\left[a^3\left(\frac{y^4}{4}\right)_0^1 - a^3\left(\frac{y^7}{7}\right)_0^1\right] + a\left(\frac{y^4}{4}\right)_0^1 - a\left(\frac{y^5}{5}\right)_0^1$$

$$= \frac{1}{12}a^3(1-0) - \frac{a^3}{21}(1-0) + \frac{a}{4} - \frac{a}{5}$$

$$= \frac{7a^3-4a^3}{84} + \frac{5a-4a}{20}$$

$$= \frac{3a^3}{84} + \frac{a}{20}$$

$$= \frac{a^3}{28} + \frac{a}{20}$$

[illegible]

(4) $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ (14)

Sol: The limits are $x=1$, $x=\sqrt{4-y} \Rightarrow y = 4-x^2$.

$y=0, 3$.

Old	New
x: 1 to $\sqrt{4-y}$	x: 1 to 2
y: 0 to 3	y: 0 to $4-x^2$

($\because$ when $y=0 \rightarrow x=\sqrt{4-0}=2$ big
$y=3 \rightarrow x=\sqrt{4-3}=1$. small).

$y=0,\ 4-x^2$.

By changing the order of integration, the given intergral can be written as

$$\int_{x=1}^{2}\int_{y=0}^{4-x^2} (x+y)\,dy\,dx$$

$$\therefore \int_{x=1}^{2}\int_{y=0}^{4-x^2} (x+y)\,dy\,dx = \int_1^2 \left[\left(xy+\frac{y^2}{2}\right)_0^{4-x^2} \right] dx.$$

$$= \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2} \right] dx.$$

$$= \int_1^2 \left[4x - x^3 + \frac{16+x^4-8x^2}{2} \right] dx.$$

$$= \left(\frac{4x^2}{2} - \frac{x^4}{4} + 8x + \frac{x^5}{10} - 4\frac{x^3}{3} \right)_1^2.$$

$$= \left(2(2)^2 - \frac{2^4}{4} + 8(2) + \frac{2^5}{10} - \frac{4}{3}(2)^3\right) - \left(2 - \frac{1}{4} + 8 + \frac{1}{10} - \frac{4}{3}\right)$$

$$= \left(8 - 4 + 16 + \frac{16}{5} - \frac{32}{3}\right) - \left(10 - \frac{1}{4} + \frac{1}{10} - \frac{4}{3}\right)$$

$$= \left(20 + \frac{16}{5} - \frac{32}{3}\right) - \left(\frac{600-15+6-80}{60}\right).$$

$$= \frac{300+48-160}{15} - \frac{511}{60}.$$

$$= \frac{188 \times 4 - 511}{60} = \frac{241}{60}.$$

1212 - 2 | 4, 10, 3
2, 5, 3
190
1022
248
160
188
× 4
752
511
241.

680 606
95
519

⑤ $\int_0^b \int_0^{\frac{a\sqrt{b^2-y^2}}{b}} xy\,dx\,dy$

Sol: The limits are $x=0$, $x=\frac{a\sqrt{b^2-y^2}}{b}$

and $y=0, y=b$.

Old	New
$x: 0$ to $\frac{a\sqrt{b^2-y^2}}{b}$	$x: 0$ to a
$y: 0$ to b	$y: 0$ to $\frac{b}{a}\sqrt{a^2-x^2}$

we have $x=\frac{a\sqrt{b^2-y^2}}{b} \Rightarrow \frac{bx}{a}=\sqrt{b^2-y^2}$

$$\Rightarrow \frac{b^2x^2}{a^2}=b^2-y^2$$

$$\Rightarrow y^2=b^2-\frac{b^2x^2}{a^2}$$

$$\Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}$$

By changing the order of integration, the given integral can be written as

$$\int_0^a \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dy\,dx = \int_0^a \left[\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy\right] x\,dx$$

$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x\left[\left(\left(\frac{b}{a}\left(\sqrt{a^2-x^2}\right)\right)^2-0\right)\right]dx$$

$$= \frac{1}{2}\int_0^a x\frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$= \frac{1}{2}\left(b^2\frac{x^2}{2}-\frac{b^2}{a^2}\frac{x^4}{4}\right)_0^a$$

$$= \frac{1}{2}\left[\frac{a^2b^2}{2}-\frac{b^2}{a^2}\cdot\frac{a^4}{4}\right]$$

$$= \frac{1}{2}\frac{a^2b^2}{2}\left(1-\frac{1}{2}\right) = \underline{\frac{a^2b^2}{8}}$$

6 By changing the order of integration, evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx.$$

Sol: The limits are $x=0,1$

$y=0, \sqrt{1-x^2}$.

old	new
x: 0 to 1	x: 0 to $\sqrt{1-y^2}$.
y: 0 to $\sqrt{1-x^2}$	y: 0 to 1.

$y=\sqrt{1-x^2} \Rightarrow y^2=1-x^2 \Rightarrow x=\sqrt{1-y^2}$.

By changing the order of integration, the given integral can be written as $\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} y^2\,dx\,dy$.

$$\therefore \int_0^1 \int_{x=0}^{\sqrt{1-y^2}} y^2\,dx\,dy = \int_0^1 y^2\left[\int_0^{\sqrt{1-y^2}} 1.\,dx\right]dy.$$

$$= \int_0^1 y^2\,(x)_0^{\sqrt{1-y^2}}\,dy.$$

$$= \int_0^1 y^2\left(\sqrt{1-y^2}-0\right)dy.$$

$$= \int_0^1 y^2\sqrt{1-y^2}\,dy.$$

put $y=\sin\theta \Rightarrow dy=\cos\theta\,d\theta$.

$y=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=0$.

$y=1 \Rightarrow \sin\theta=1 \Rightarrow \theta=\frac{\pi}{2}$.

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$$

$\sin\theta=t$

$\cos\theta\,d\theta=dt$

$\int t^2\,dt$.

$$= \cancel{\left(\frac{t^3}{3}\right)_0^{\frac{\pi}{2}}} = \cancel{\frac{1}{3}}($$

$$= \frac{1}{4}\int_0^{\pi/2} \sin^2 2\theta \, d\theta.$$

$$= \frac{1}{4}\left[\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \frac{1}{8}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/2}$$

$$= \frac{1}{8}\left[\frac{\pi}{2} - \frac{1}{2}(0-0)\right]$$

$$= \frac{1}{8}\times\frac{\pi}{2}$$

$$= \frac{\pi}{16}$$

(7) $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx.$$

Sol: The limits are $y = 0$ to $\sqrt{a^2-x^2}$
$y=0$ & $y = \sqrt{a^2-x^2}$
$x^2+y^2=a^2$

and $x=0$ ~~and~~ & $x=a$.

Old	New
x: 0 to a	x: 0 to $\sqrt{a^2-y^2}$
y: 0 to $\sqrt{a^2-x^2}$	y: 0 to a

By changing the order of integration, the given integral can be written as

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy = \int_0^a \left(\int_0^p \sqrt{p^2-x^2}\, dx\right) dy \quad \text{where } p^2 = a^2-y^2.$$

Put $x = p\sin\theta \Rightarrow dx = p\cos\theta\, d\theta$.
If $x=0 \Rightarrow \theta=0$, $x=p \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$= \int_0^a \left[\int_0^{\pi/2} p\cos\theta \cdot p\cos\theta\, d\theta\right) dy$$

$$= \int_0^a p^2 \left(\int_0^{\pi/2} \cos^2\theta \, d\theta \right) dy$$

$$= \int_0^a p^2 \left(\frac{2-1}{2} \cdot \frac{\pi}{2} \right) dy$$

$$= \frac{\pi}{4} \int_0^a (a^2 - y^2)\, dy$$

$$= \frac{\pi}{4} \left(a^2 y - \frac{y^3}{3} \right)_0^a$$

$$= \frac{\pi}{4} \left[\left(a^3 - \frac{a^3}{3} \right) - 0 \right]$$

$$= \frac{\pi}{\cancel{4}_2} \times \frac{\cancel{2}a^3}{3}$$

$$= \frac{\pi}{6} a^3 //.$$

HW

$$\int_0^1 \int_1^{2-x} xy \, dx \, dy.$$

$y = 1$ to $2-x$.

$x = 0$ to 1.

old	New
$x: 0$ to 1	$x: 0$ to $2-y$
$y: 1$ to $2-x$	$y: 1$ to 2

$y = 2 - x \Rightarrow x = 2 - y$.

Triple integrals

① Evaluate $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$.

Sol: $\int_1^e \int_1^{\log y} \left(\int_1^{e^x} \log z \, dz \right) dx \, dy$

$$= \int_1^e \int_1^{\log y} \left(z \log z - z \right)_1^{e^x} dx\,dy$$

$$= \int_1^e \int_1^{\log y} \left[(e^x x - e^x) - (0-1) \right] dx\,dy$$

$$= \int_1^e \int_1^{\log y} (e^x x - e^x + 1)\, dx\,dy$$

$$= \int_1^e \left[\int_1^{\log y} (x e^x - e^x + 1)\, dx \right] dy$$

$$= \int_1^e \left[(x e^x - e^x - e^x + x)_1^{\log y} \right] dy$$

$$= \int_1^e \left[(y \log y - 2y + \log y) - (e - 2e + 1) \right] dy$$

$$= \int_1^e \left[y \log y - 2y + \log y + e - 1 \right] dy$$

$$= \left[\frac{y^2}{2} \log y - \frac{y^2}{4} - \cancel{2} \frac{y^2}{\cancel{2}} + y \log y - y + (e-1) y \right]_1^e$$

$$= \left[\frac{e^2}{2}(1) - \frac{e^2}{4} - e^2 + \cancel{e} - \cancel{e} + (e-1)e \right] - \left[0 - \frac{1}{4} - 1 + 0 - 1 + (e-1) \right]$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \cancel{e^2} + \cancel{e^2} + \frac{1}{4} + 1 + 1 - e + 1$$

$$= \frac{e^2}{8} - 2e + \frac{13}{4} = \frac{1}{8}(e^2 - 8e + 13)$$

2) $\int_0^1 \int_1^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$

Sol: $\int_0^1 \int_1^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$

$$= \int_0^1 \int_1^{\sqrt{1-x^2}} \left[\int_0^{\sqrt{1-x^2-y^2}} z \, dz \right] xy \, dy \, dx.$$

$$= \int_0^1 \int_1^{\sqrt{1-x^2}} \left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} xy \, dy \, dx$$

$$= \int_0^1 \int_1^{\sqrt{1-x^2}} \frac{1}{2}(1-x^2-y^2) \, xy \, dy \, dx.$$

$$= \frac{1}{2} \int_0^1 \left[\int_1^{\sqrt{1-x^2}} (xy - x^3y - xy^3) \, dy \right] dx.$$

$$= \frac{1}{2} \int_0^1 \left[\frac{xy^2}{2} - x^3\frac{y^2}{2} - \frac{xy^4}{4} \right]_1^{\sqrt{1-x^2}} dx.$$

$$= \frac{1}{4} \int_0^1 \left[x(1-x^2) - x^3(1-x^2) - \frac{x}{2}(1-x^2)^2 \right] dx.$$

$$= \frac{1}{4} \int_0^1 \left[x - x^3 - x^3 + x^5 - \frac{x}{2}(1+x^4-2x^2) \right] dx$$

$$= \frac{1}{4} \int_0^1 \left(x - 2x^3 + x^5 - \frac{x}{2} - \frac{x^5}{2} + \frac{\cancel{2}x^3}{\cancel{2}} \right) dx.$$

$$= \frac{1}{4} \int_0^1 \left(\frac{x}{2} - x^3 + \frac{x^5}{2} \right) dx.$$

$$= \frac{1}{4} \left[\frac{x^2}{4} - \frac{x^4}{4} + \frac{x^6}{12} \right]_0^1$$

$$= \frac{1}{4} \left[\left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{4}} + \frac{1}{12} \right) - 0 \right].$$

$$= \frac{1}{4} \times \frac{1}{12}$$

$$= \frac{1}{48}.$$

③ $\int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{2}} r\,dz\,dr\,d\theta$

Sol: $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{2}} r\,dz\,dr\,d\theta$

$$= \int_0^{\pi/2}\int_0^{a\sin\theta}\left[\int_0^{\frac{a^2-r^2}{2}} 1.\,dz\right] r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{a\sin\theta} (z)_0^{\frac{a^2-r^2}{2}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{a\sin\theta} \left(\frac{a^2-r^2}{2}\right) r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left[\int_0^{a\sin\theta} (a^2r-r^3)\,dr\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{a^2r^2}{2}-\frac{r^4}{4}\right)_0^{a\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(a^2r^2-\frac{r^4}{2}\right)_0^{a\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\left(a^2a^2\sin^2\theta-\frac{a^4\sin^4\theta}{2}\right) d\theta$$

$$= \frac{a^4}{4}\left[\int_0^{\pi/2}\sin^2\theta\,d\theta-\frac{1}{2}\int_0^{\pi/2}\sin^4\theta\,d\theta\right]$$

$$= \frac{a^4}{4}\left[\frac{2-1}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\cdot\frac{4-1}{4}\cdot\frac{4-3}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{a^4\pi}{16}\left[1-\frac{3}{4}\cdot\frac{1}{2}\right]$$

$$= \frac{5\pi a^4}{16\times 8} = \underline{\underline{\frac{5\pi a^4}{128}}}$$

3. Evaluate $\iiint (xy+yz+zx)\,dx\,dy\,dz$ where V is the region of space bounded by $x=0, x=1, y=0, y=2, z=0, z=3$.

Sol: $\iiint_V (xy+yz+zx)\,dx\,dy\,dz = \int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{1}(xy+yz+zx)\,dx\,dy\,dz$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left(\frac{x^2}{2}y + xyz + \frac{zx^2}{2}\right)_0^1 dy\,dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{2}\left(\frac{y}{2}+yz+\frac{z}{2}\right)dy\,dz$$

$$= \int_0^3\left(\frac{y^2}{4}+\frac{y^2}{2}z+\frac{z}{2}y\right)_0^2 dz$$

$$= \int_0^3\left[\left(\frac{4}{4}+\frac{4}{2}z+z\right)-0\right]dz$$

$$= \int_0^3 (1+3z)\,dz$$

$$= \left(z+3\frac{z^2}{2}\right)_0^3$$

$$= 3+\frac{27}{2}-0$$

$$= \frac{33}{2}$$

4. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\,dy\,dx$.

Sol: $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{1-x^2-y^2-z^2}}\,dz\,dy\,dx$

$$= \int_0^1\int_0^{\sqrt{1-x^2}}\left[\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{\sqrt{\left(\sqrt{1-x^2-y^2}\right)^2-z^2}}\,dz\right]dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}}\left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}}dy\,dx \quad \left(\because \int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)\right)$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx$$

$$= \frac{\pi}{2} \int_0^1 (y)_0^{\sqrt{1-x^2}}\, dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{\sin^{-1}x}{2} \right]_0^1$$

$$= \frac{\pi}{2} \left[\frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$= \frac{\pi^2}{8}$$

(4) Evaluate $\iiint xyz\, dx\, dy\, dz$ over the +ve octant of the sphere $x^2+y^2+z^2=a^2$

Sol: Given that $x^2+y^2+z^2=a^2$

$$z = \sqrt{a^2-x^2-y^2}$$

The limits are z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

z " " 0 to a.

$$\therefore \iiint_V xyz\, dx\, dy\, dz = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} z\, xy\, dz\, dx\, dy$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} xy\, dx\, dy$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2-x^2-y^2)\, xy\, dx\, dy$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3)\, dy\, dx$$

$$= \frac{1}{2} \int_0^a \left[\left(\frac{a^2xy^2}{2} - \frac{x^3}{2}y^2 - \frac{x}{4}y^4 \right)_0^{\sqrt{a^2-x^2}} \right] dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^2x(a^2-x^2)}{2} - \frac{x^3}{2}(a^2-x^2) - \frac{x}{4}(a^2-x^2)^2\right]dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^4x - a^2x^3}{2} - \frac{x^3a^2}{2} + \frac{x^5}{2} - \frac{x}{4}(a^4+x^4-2a^2x^2)\right]dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^4x}{2} - \frac{a^2x^3}{2} - \frac{x^3a^2}{2} + \frac{x^5}{2} - \frac{a^4x}{4} - \frac{x^5}{4} + \frac{2a^2x^3}{4}\right)dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^4x}{4} + \frac{a^2x^3}{2} + \frac{x^5}{4}\right)dx$$

$$= \frac{1}{8}\int_0^a (a^4x - 2a^2x^3 + x^5)\,dx$$

$$= \frac{1}{8}\left[a^4\left(\frac{x^2}{2}\right) + 2a^2\frac{x^4}{4} + \frac{x^6}{6}\right]_0^a$$

$$= \frac{1}{8}\left[\cancel{a^4}\cdot\frac{a^2}{2} + \cancel{2a^2}\frac{a^4}{\cancel{4}} + \frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left[\cancel{a^6} + \frac{a^6}{6}\right]$$

$$= \frac{a^6}{48}$$

valuate $\displaystyle\int_0^1\int_0^2\int_{\sqrt{x^2+y^2}}^{y} xyz\,dz\,dy\,dx$

$$\int_0^1\int_0^2\int_{z=\sqrt{x^2+y^2}}^{y} xyz\,dz\,dy\,dx = \int_0^1\int_0^2\left[\int_{z=\sqrt{x^2+y^2}}^{y} z\,dz\right]xy\,dy\,dx$$

$$= \int_0^1\int_0^2\left(\frac{z^2}{2}\right)_{\sqrt{x^2+y^2}}^{y} xy\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_{y=0}^2 (y^2-(x^2+y^2))\,xy\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_{y=0}^2 (\cancel{y^3x} - x^3y - \cancel{y^3x})\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^2 x^3 y\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{y^2}{2}\right)_0^2 x^3\,dx$$

$$= -\frac{1}{4}\int_0^1 \left(y^2\right)_0^2 x^3\,dx$$

$$= -\frac{1}{4}\int_0^1 (4-0)x^3\,dx$$

$$= -\frac{1}{4}\times 4\int_0^1 x^3\,dx$$

$$= -1\left(\frac{x^4}{4}\right)_0^1$$

$$= -\left(\frac{1}{4}-0\right) = \underline{\underline{-\frac{1}{4}}}$$

$\rightarrow$ $$\int_0^1\int_0^1\int_0^y xyz\,dx\,dy\,z = \int_0^1\int_0^1\int_{z=0}^y z\,\cancel{dz}\;xy\,dy\,dx$$

$$= \int_0^1\int_0^1 \left(\frac{z^2}{2}\right)_0^y xy\,dy\,dx$$

$$= \int_0^1\int_0^1 \frac{1}{2}(y^2-0)xy\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^1 xy^3\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^4}{4}\right]_0^1 dx$$

$$= \frac{1}{8}\int_0^1 x(1-0)\,dx$$

$$= \frac{1}{8}\left(\frac{x^2}{2}\right)_0^1$$

$$= \frac{1}{16}(1-0) = \underline{\underline{\frac{1}{16}}}$$