Multiple Integrals

Double Integral:

Def: Let $f(x,y)$ be a continuous and single valued function of two independent variables x, y defined on the region A of the xy plane. ie; $A = \iint_A f(x,y)\,dx\,dy$

Evaluation of Double integral:

(1) If x_1, x_2, y_1, y_2 are constants then, the order of integration is immaterial, provided the limits of integration are to be changed accordingly.

$$\iint_A f(x,y)\,dx\,dy = \left[\int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]dy\right] = \left[\int_{x_1}^{x_2}\left[\int_{y_1}^{y_2} f(x,y)\,dy\right]dx\right]$$

(2) If y_1, y_2 are functions of x say $y_1 = \phi_1(x), y_2 = \phi_2(x)$ and x_1, x_2 are constants then

$$\iint_A f(x,y)\,dx\,dy = \left[\int_{x_1}^{x_2}\left[\int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} f(x,y)\,dy\right]dx\right].$$

(3) If x_1, x_2 are functions of y say $x_1 = \phi_1(y), x_2 = \phi_2(y)$ and y_1, y_2 are constants, then

$$\iint_A f(x,y)\,dx\,dy = \left[\int_{y_1}^{y_2}\left[\int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\,dx\right]dy\right]$$

(4) If $f(x,y) = 1$ then the double integral $\iint_A dx\,dy$ gives the area of the region A.

Note: While integrating w.r.t x, treat y as constant and while integrating w.r.t y, treat x as constant.

Double integral in polar coordinates:

Let the region be defined by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the radii vectors $\theta = \alpha$ & $\theta = \beta$ then the double integral can be evaluated as

$$\int_{\theta=\alpha}^{\theta=\beta}\left[\int_{r=f_1(\theta)}^{r=f_2(\theta)} f(r,\theta)\,dr\right]d\theta.$$

① Evaluate $\int_0^3\int_1^2 xy(1+x+y)\,dy\,dx$

Sol: Given that

$$\int_0^3\int_1^2 xy(1+x+y)\,dy\,dx = \int_0^3\left[\int_1^2 (xy+x^2y+xy^2)\,dy\right]dx$$

$$= \int_0^3 \left(x\frac{y^2}{2}+x^2\frac{y^2}{2}+x\frac{y^3}{3}\right)_1^2 dx$$

$$= \int_0^3 \left[\left(\frac{4}{2}x+\frac{4}{2}x^2+\frac{8}{3}x\right)-\left(\frac{x}{2}+\frac{x^2}{2}+\frac{x}{3}\right)\right]dx$$

$$\frac{3x}{2}+\frac{7x}{3}$$

$$= \int_0^3 \left(2x+2x^2+\frac{8}{3}x-\frac{x}{2}-\frac{x^2}{2}-\frac{x}{3}\right)dx$$

$$= \int_0^3 \left(\frac{23}{6}x+\frac{3x^2}{2}\right)dx$$

$$= \left[\frac{23}{6}\cdot\frac{x^2}{2}+\frac{\cancel{3}}{2}\cdot\frac{x^3}{\cancel{3}}\right]_0^3$$

$$= \frac{23}{\cancel{6}\times 2}(\cancel{9}) + \frac{1}{2}(3)^3 - 0$$

$$= \frac{69}{4}+\frac{27}{2} = \frac{69+54}{4} = \frac{123}{4}$$

② Evaluate

(i) $$\int_0^2\int_0^x y\,dy\,dx = \int_0^2\left[\int_0^x y\,dy\right]dx$$

$$= \int_0^2 \left(\frac{y^2}{2}\right)_0^x dx$$

$$= \int_0^2 \left(\frac{x^2}{2}-0\right)dx$$

$$= \int_0^2 \frac{x^2}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right)_0^2$$

$$= \frac{1}{6}\left((8)-0\right) = \frac{1}{6}\times 8 = \frac{4}{3}$$

(ii) $\int_0^1\int_0^x y\,dy\,dx = \int_0^1\left[\int_0^x y\,dy\right]dx$

$= \int_0^1 \left(\frac{y^2}{2}\right)_0^x dx$

$= \frac{1}{2}\int_0^1 x^2\,dx$

$= \frac{1}{2}\left(\frac{x^3}{3}\right)_0^1 = \frac{1}{6}(1-0) = \frac{1}{6}$

(iii) $\int_0^3\int_0^2 (4-y)^2\,dy\,dx = \int_0^3 \left(\frac{(4-y)^3}{-3}\right)_0^2 dx$

$= -\frac{1}{3}\int_0^3 \left[(4-2)^3 - (4-0)^3\right]dx$

$= -\frac{1}{3}\int_0^3 (8-64)\,dx$

$= -\frac{1}{3} \cdot \cancel{-56}\int_0^3 1\cdot dx$

$= \frac{56}{3}(x)_0^3$

$= \frac{56}{3}\times(3-0) = \frac{56}{3}\times 3 = 56$

(iv) $\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$

G.T. $\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx = \int_0^a\left[\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\right]dx$

$= \int_0^a \left(x^2y + \frac{y^3}{3}\right)_0^{\sqrt{a^2-y^2}} dx$

$= \int_0^a \left(x^2\sqrt{a^2-y^2} + \frac{(a^2-y^2)\sqrt{a^2-y^2}}{3}\right)dx$

$= \sqrt{a^2-y^2}\int_0^a \left[x^2 + \frac{(a^2-y^2)}{3}\right]dx$

$= \sqrt{a^2-y^2}\left[\frac{x^3}{3} + \frac{a^2-y^2}{3}x\right]_0^a$

$= \sqrt{a^2-y^2}\left[\frac{a^3}{3} + \frac{(a^2-y^2)a}{3}\right]$

$= \sqrt{a^2-y^2}\left[\frac{a^3}{3} + \frac{a^3}{3} - \frac{ay^2}{3}\right]$

$= \sqrt{a^2-y^2}\left[\frac{2a^3}{3} - \frac{ay^2}{3}\right]$

$= \sqrt{a^2-y^2}\left[\frac{a}{3}(2a^2-y^2)\right]$

$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy = \int_{x=0}^{1} \left[\int_{y=x}^{\sqrt{x}} (x^2+y^2)\,dy \right] dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left\{ \left[x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3} \right] - \left[x^3 + \frac{x^3}{3} \right] \right\} dx$$

$$= \int_0^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3} \right] dx$$

$$= \left[\frac{x^{7/2}}{\frac{7}{2}} + \frac{1}{3}\frac{x^{5/2}}{\frac{5}{2}} - \frac{4}{3}\cdot\frac{x^4}{4} \right]_0^1$$

$$= \left(\frac{2}{7}(1) + \frac{2}{15}(1) - \frac{1}{3} \right) - 0$$

$$= \frac{30+14-35}{105} = \frac{\cancel{9}\,3}{\cancel{105}\,35} = \frac{3}{35}$$

$$\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \left[\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx \right] \frac{dy}{\sqrt{1-y^2}}$$

$$= \int_0^1 (\sin^{-1} x)_0^1\, dy$$

$$= \int_0^1 \left[\sin^{-1}(1) - \sin^{-1}(0) \right] \frac{dy}{\sqrt{1-y^2}}$$

$$= \int_0^1 \left(\frac{\pi}{2} - 0 \right) \frac{dy}{\sqrt{1-y^2}}$$

$$= \frac{\pi}{2} \cancel{(y)_0^1} = \cancel{\frac{\pi}{2}(1-0)} =$$

$$= \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy$$

$$= \frac{\pi}{2} (\sin^{-1} y)_0^1$$

$$= \frac{\pi}{2} \left[\sin^{-1}(1) - \sin^{-1}(0) \right]$$

$$= \frac{\pi}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$

③

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2} = \int$$

Let $p = \sqrt{1+x^2} \Rightarrow 1+x^2 = p^2$

$$\therefore \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2}\right] dx = \int_0^1 \left[\int_0^{p} \frac{dy}{p^2+y^2}\right] dx.$$

$$= \int_0^1 \left[\frac{1}{p}\tan^{-1}\left(\frac{y}{p}\right)\right]_0^p dx$$

$$= \cancel{\frac{1}{p}} \int_0^1 \frac{1}{p}\left[\tan^{-1}\left(\frac{p}{p}\right) - \tan^{-1}(0)\right] dx.$$

$$= \cancel{\frac{1}{p}} \int_0^1 \frac{1}{p}\left[\tan^{-1}(1) - 0\right] dx.$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{p}\, dx.$$

$$= \cancel{\boxed{\frac{\pi}{4p}\left(\frac{x}{\sqrt{1+x^2}}\right)_0^1 = \frac{\pi}{4p}}}$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx.$$

$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1}\right)\right]_0^1 \text{ (or) } \frac{\pi}{4}\left(\sinh^{-1}x\right)_0^1$$

$$= \frac{\pi}{4}\log\left(1+\sqrt{2}\right) \text{ (or) } \frac{\pi}{4}\sinh^{-1}1.$$

$$\int_0^2 \int_0^x (x+y)\,dy\,dx = \int_0^2 \left[\int_{y=0}^{x} (x+y)\,dy\right] dx.$$

$$= \int_0^2 \left(xy + \frac{y^2}{2}\right)_0^x dx.$$

$$= \int_0^2 \left(x^2 + \frac{x^2}{2}\right) dx$$

$$= \int_0^2 \frac{3x^2}{2}\, dx$$

$$= \frac{3}{2} \times \left(\frac{x^3}{3}\right)_0^2$$

$$= \frac{8}{2} = 4.$$

→ $\int_0^2 \int_0^x e^{x+y}\, dy\, dx = \int_0^2 \left[\int_0^x e^x \cdot e^y\, dy \right] dx.$

$= \int_0^2 (e^y)_0^x\, e^x\, dx$

$= \int_0^2 (e^x - 1)\, e^x\, dx$

$= \int_0^2 (e^{2x} - e^x)\, dx$

$= \left(\frac{e^{2x}}{2} - e^x \right)_0^2 = \left(\frac{e^4}{2} - e^2 \right) - \left(\frac{1}{2} - 1 \right)$

$= \frac{e^4}{2} - e^2 + \frac{1}{2}$

$= \frac{1}{2}\left[e^4 - 2e^2 + 1 \right]$

$= \frac{(e^2-1)^2}{2}$ //.

→ Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

Sol: ~~put $x = r\cos\theta,\ y = r\sin\theta \Rightarrow x^2+y^2 = r^2,\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$~~
~~$x = 0 \rightarrow r =$~~
~~$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$~~

$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \int_0^\infty \left(e^{-x^2} \cdot e^{-y^2}\, dx \right) dy$

$= \int_0^\infty e^{-y^2} \left(\int_0^\infty e^{-x^2}\, dx \right) dy$

$\because \boxed{\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}}$

$= \int_0^\infty e^{-y^2} \left(\frac{\sqrt{\pi}}{2} \right) dy$

$= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-y^2}\, dy$

$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$ //

(5)

(4)

Q Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2=4x$ and $x^2=4y$.

Sol: Given parabolas are $y^2=4x \rightarrow (1)$

$x^2=4y \rightarrow (2)$

From (1) & (2)

$$\left(\frac{y^2}{4}\right)^2=4y$$

$$\Rightarrow \frac{y^4}{16}-4y=0$$

$$\Rightarrow y(y^3-64)=0$$

$$\Rightarrow y=0 \text{ or } y^3=64 \Rightarrow y=4$$

$\therefore$ $y=0 \rightarrow x=0 \rightarrow (0,0)$

$y=4 \rightarrow x^2=16 \Rightarrow x=4 \rightarrow (4,4)$

(i) $$\iint_R y\,dx\,dy = \int_{x=0}^{4}\left[\int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\right]dx$$

$$=\int_0^4 \left(\frac{y^2}{2}\right)_{\frac{x^2}{4}}^{2\sqrt{x}} dx$$

$$=\frac{1}{2}\int_0^4 \left[(2\sqrt{x})^2-\left(\frac{x^2}{4}\right)^2\right]dx$$

$$=\frac{1}{2}\left[\int_0^4\left(4x-\frac{x^4}{16}\right)dx\right]$$

$$=\frac{1}{2}\left[\cancel{4}^{2}\cdot\frac{x^2}{\cancel{2}}-\frac{x^5}{80}\right]_0^4$$

$$=\frac{1}{2}\left[2(16)-\frac{4^5}{80}\right]-0$$

$$=\frac{1}{2}\left[32-\frac{64}{5}\right]$$

$$=\frac{1}{2}\left[\frac{160-64}{5}\right]$$

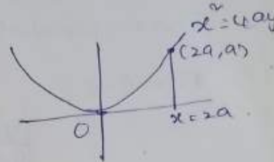
$$=\frac{\cancel{96}^{48}}{\cancel{2}\times 5}$$

$$=\frac{48}{5}$$

→ Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis, ordinate $x=2a$ & $x^2=4ay$.

Sol: Given that $x^2 = 4ay \rightarrow (1)$

$x = 2a$.



From (1) & (2)

$(2a)^2 = 4ay$.

$\Rightarrow 4a^2 - 4ay = 0$

$\Rightarrow 4a(a-y) = 0$.

$\Rightarrow a = y;\ a = 0$.

$\Rightarrow \therefore y = 0 \rightarrow x = 0 \rightarrow (0,0)$

$x = 2a \Rightarrow y = a \rightarrow (2a, a)$

The limits are y ~~varies~~ varies from 0 to $\frac{x^2}{4a}$.

x varies from ~~2a to a~~ 0 to $2a$.

$$\therefore \int_{x=0}^{2a}\int_{y=0}^{x^2/4a} xy\,dy\,dx = \int_0^{2a}\left[\int_0^{\frac{x^2}{4a}} y\,dy\right] x\,dx.$$

$$= \int_0^{2a}\left[\left(\frac{y^2}{2}\right)_0^{x^2/4a}\right] x\,dx.$$

$$= \int_0^{2a}\frac{1}{2}\left[\left(\frac{x^2}{4a}\right)^2 - 0\right] x\,dx$$

$$= \frac{1}{2}\int_0^{2a}\frac{x^4 \times x}{16a^2}\,dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left(\frac{x^6}{6}\right)_0^{2a}$$

$$= \frac{1}{32 \times 6a^2}(2a)^6$$

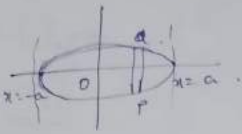
$$= \frac{1}{32 \times 6}\,\frac{64a^6}{a^2}$$

$$= \frac{a^4}{3}$$

→ Find $\iint (x+y)^2 \, dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol: Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\Rightarrow y = \pm \frac{b}{a}\sqrt{a^2-x^2}$.



We have $-a \le x \le a$, $-\frac{b}{a}\sqrt{a^2-x^2} \le y \le \frac{b}{a}\sqrt{a^2-x^2}$

$$\therefore I = \iint_R (x+y)^2 \, dx\,dy = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dx\,dy.$$

$$= \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy + \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} 2xy\,dx\,dy$$

$$= 2\int_{-a}^{a}\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx + 0.$$

$(\because x^2+y^2$ is even & xy is an odd$)$.

$$= 2\int_{-a}^{a}\left(x^2y+\frac{y^3}{3}\right)_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\cdot\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx.$$

$$= 4\int_{0}^{a}\left[\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

$x=0 \Rightarrow \theta=0$ & $x=a \Rightarrow \theta=\frac{\pi}{2}$.

$$I = 4\int_{0}^{\pi/2}\left[\frac{b}{a}a^2\sin^2\theta\left(\sqrt{a^2-a^2\sin^2\theta}+\frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{3/2}\right]a\cos\theta\,d\theta$$

$$I = 4\int_{0}^{\pi/2}\left[ab\sin^2\theta\, a\cos\theta+\frac{b^3}{3a^3}a^3\cos^3\theta\right]a\cos\theta\,d\theta$$

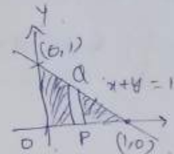
$$= 4\int_{0}^{\pi/2}\left[a^3b\sin^2\theta\cos^2\theta+\frac{b^3a}{3}\cos^4\theta\right]d\theta.$$

$$I = 4\left[a^3b\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{ab^3}{\cancel{3}}\cdot\frac{\cancel{3}}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \cancel{4}\times\frac{\pi}{\cancel{4}\times4}\left[a^3b+ab^3\right]$$

$$= \frac{\pi}{4}ab(a^2+b^2)$$

→ Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region in the +ve quadrant for which $x+y \le 1$.



Sol: $$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{y=1-x}(x^2+y^2)\,dx\,dy$$

$$= \int_0^1\left[\int_0^{1-x}(x^2+y^2)\,dy\right]dx$$

$$= \int_0^1\left(x^2y+\frac{y^3}{3}\right)_0^{1-x}dx$$

$$= \int_0^1\left[x^2(1-x)+\frac{(1-x)^3}{3}\right]dx$$

$$= \int_0^1\left[x^2-x^3+\frac{(1-x)^3}{3}\right]dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{(1-x)^4}{12}\right]_0^1$$

$$= \left(\frac{1}{3}-\frac{1}{4}+0\right)-\left(0-0-\frac{1}{12}\right)$$

$$= \frac{1}{12}+\frac{1}{12}$$

$$= \frac{2}{12}$$

$$= \frac{1}{6}$$

HW ***

Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Double integrals in polar coordinates

→ Evaluate $\int_0^{\pi} \int_0^{a\sin\theta} r\,dr\,d\theta$

Sol. G.T $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$

$$= \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a\sin\theta} d\theta$$

$$= \int_0^{\pi}\frac{1}{2}\left(a^2\sin^2\theta - 0\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[(\pi - \cancel{0}) - 0\right] \quad (\because \sin 2\pi = 0)$$

$$= \frac{a^2}{8}\cancel{(2\pi - 1)}$$

$$= \frac{a^2}{4}(\pi)$$

$$= \frac{a^2\pi}{4}$$

(2) Evaluate $\int_0^\infty \int_0^{\pi/2} e^{-r^2} r \, d\theta \, dr$

Sol:

$$\int_0^\infty \int_0^{\pi/2} e^{-r^2} r \, d\theta \, dr = \int_0^\infty e^{-r^2} r \, (\theta)_0^{\pi/2} \, dr$$

$$= \int_0^\infty e^{-r^2} r \, \frac{\pi}{2} \, dr$$

$$= \frac{\pi}{2} \int_0^\infty e^{-r^2} r \, dr$$

Let $-r^2 = t$

$-2r\,dr = dt$

$r\,dr = -\frac{dt}{2}$

$$= \frac{\pi}{2}\left[\int_0^\infty e^{t} \, -\frac{dt}{2}\right]$$

$$= -\frac{\pi}{4} \int_0^\infty e^{t} \, dt$$

$$= -\frac{\pi}{4} \left(e^{-r^2}\right)_0^\infty$$

$$= -\frac{\pi}{4}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{\pi}{4}(0-1)$$

$$= \frac{\pi}{4}$$

(3) $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta \, dr \, d\theta$

Sol: GT $\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta \, dr \, d\theta = \int_0^{\pi} \sin\theta \left(\int_0^{a\cos\theta} r \, dr\right) d\theta$

$$= \int_0^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \, a^2\cos^2\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta\cos^2\theta \, d\theta$$

$\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$\theta = 0 \rightarrow t = 1$

$\theta = \pi \rightarrow t = -1$

$$= -\frac{a^2}{2}\int_1^{-1} t^2 \, dt$$

$$= +\frac{a^2}{2}\cdot 2\int_0^1 t^2 \, dt = -a^2\left(\frac{t^3}{3}\right)_0^1 = +\frac{a^2}{3}(1-0)$$

$$= a^2/3$$

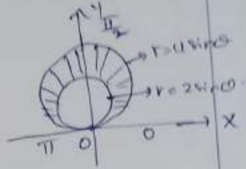
$$\cancel{= -\frac{a^2}{2}\left(\frac{t^3}{3}\right)_0^{\pi}}$$

$$\cancel{= -\frac{a^2}{6}\left(\cos^3\theta\right)_0^{\pi}}$$

$=$

→ Evaluate $\iint r^3 \, dr \, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Sol: Given that r varies from $r = 2\sin\theta$ to $4\sin\theta$ and cover the whole region θ varies from 0 to π.



$$\therefore \iint r^3 \, dr \, d\theta = \int_0^{\pi} \left[\int_{2\sin\theta}^{4\sin\theta} r^3 \, dr \right] d\theta$$

$$= \int_0^{\pi} \left(\frac{r^4}{4} \right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4} \int_0^{\pi} (256\sin^4\theta - 16\sin^4\theta) \, d\theta$$

$$= \frac{1}{4} \times 240 \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta \qquad \left(\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x) = f(x) \right)$$

$$= 120 \cdot \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2}$$

$$= \cancel{120}^{15} \times \frac{3}{\cancel{4}} \times \frac{1}{\cancel{2}} \times \frac{\pi}{\cancel{2}}$$

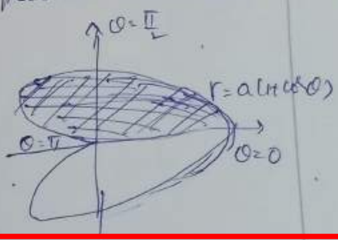
$$= \frac{45\pi}{2}$$

→ Evaluate $\iint r\sin\theta \, dr \, d\theta$ over the cardioid $r = a(1+\cos\theta)$ above the initial line.

Sol: Given that $r = a(1+\cos\theta) \rightarrow (1)$ is symmetrical about the initial line and it passes through the pole O when $\theta = \pi$

θ varies from 0 to π.

r varies from 0 to $a(1+\cos\theta)$



$$\therefore \iint r\sin\theta \, dr \, d\theta = \int_0^{\pi} \left[\int_0^{a(1+\cos\theta)} r \, dr \right] \sin\theta \, d\theta$$

$$= \int_0^{\pi} \left(\frac{r^2}{2} \right)_0^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi} [a(1+\cos\theta)]^2 \sin\theta \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} (1+\cos\theta)^2 \sin\theta \, d\theta.$$

put $1+\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$\sin\theta \, d\theta = -dt$

$$= \frac{a^2}{2} \int_0^{\pi} t^2 (-dt)$$

$$= -\frac{a^2}{2} \left(\frac{t^3}{3} \right)_0^{\pi}$$

$$= -\frac{a^2}{6} \left((1+\cos\theta)^3 \right)_0^{\pi}$$

$$= -\frac{a^2}{6} \left[(1+\cos\pi)^3 - (1+\cos 0)^3 \right]$$

$$= -\frac{a^2}{6} [0 - 8].$$

$$= \frac{\cancel{8}^{4} a^2}{\cancel{6}_{3}}$$

$$= \frac{4a^2}{3}$$

Change of Variables in Double integral

→ To change cartesian coordinates to polar coordinates:

$x = r\cos\theta, \quad y = r\sin\theta.$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r,\theta)\, r\,dr\,d\theta.$$

(1) Transform the integral into polar coordinates and hence evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx.$

Sol: The region of integration is $x=0$ & $x=a$.

$y=0$ and $y=\sqrt{a^2-x^2}$.

$\Rightarrow y^2 = a^2-x^2$

$\Rightarrow x^2+y^2 = a^2.$

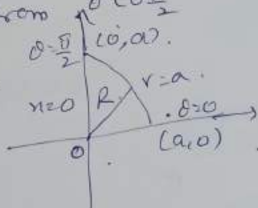
ie; the given region is a quardrant of the circle $x^2+y^2=a^2$.

put $x = r\cos\theta$, $y = r\sin\theta$.

we have $x^2+y^2=r^2$ and $dx\,dy = r\,dr\,d\theta$.

The limits are : r varies from 0 to a.

and θ varies from 0 to $\frac{\pi}{2}$



$$\therefore \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r.r\,dr\,d\theta.$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_0^a d\theta.$$

$$= \frac{1}{3}\int_0^{\pi/2} a^3\,d\theta.$$

$$= \frac{a^3}{3}(\theta)_0^{\pi/2} = \frac{a^3}{3} \times \frac{\pi}{2} = \frac{\pi a^3}{6}$$

② Evaluate the double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dy\, dx$ by changing into polar coordinates.

Sol: ~~The~~ ~~Leq~~ Given that the region of integration is

~~y~~ x = 0 & y = a and x = 0 & $x = \sqrt{a^2-y^2}$

$\rightarrow x^2+y^2=a^2$.

put $x = r\cos\theta$, $y = r\sin\theta$.

Then $dx\,dy = r\,dr\,d\theta$

The limits are $r: 0 \rightarrow a$.

$\theta: 0 \rightarrow \frac{\pi}{2}$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\, dy\, dx = \cancel{\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} x \right.}$$

$$= \int_0^{\pi/2} \int_0^a r^2\, r\, dr\, d\theta.$$

$$= \int_0^{\pi/2} \left[\int_0^a r^3\, dr \right] d\theta.$$

$$= \int_0^{\pi/2} \left(\frac{r^4}{4} \right)_0^a d\theta.$$

$$= \frac{a^4}{4} (\theta)_0^{\pi/2}.$$

$$= \frac{a^4}{4} \left(\frac{\pi}{2} \right)$$

$$= \frac{\pi a^4}{8}$$

(3)

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$

Sol: The region of integration is $y=0$, $y=\sqrt{2x-x^2}$, $x=0$ & $x=2$

i.e; $y=0$, $y^2=2x-x^2 \Rightarrow x^2+y^2=2x$ represents the circle with centre $(1,0)$ and radius 1.

Put $x=r\cos\theta$, $y=r\sin\theta$.

The limits are ~~$r=0$, $r=2$~~.

$y=0 \Rightarrow r\sin\theta=0 \Rightarrow \theta=0$ ($\because r\neq 0$).

$x=0 \Rightarrow r\cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}$ ($r\neq 0$).

$y=\sqrt{2x-x^2} \Rightarrow x^2+y^2=2x$

$\Rightarrow r^2=2r\cos\theta \Rightarrow r=2\cos\theta$

$r=0 \to 2\cos\theta$

$\theta: 0 \to \frac{\pi}{2}$

$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta}{r}\cdot r\,dr\,d\theta \quad (\because dx\,dy = r\,dr\,d\theta)$$

$$= \int_0^{\pi/2}\left[\int_0^{2\cos\theta} r\,dr\right]\cos\theta\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_0^{2\cos\theta}\cos\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4\cos^2\theta\cdot\cos\theta\,d\theta$$

$$= \frac{4}{2}\int_0^{\pi/2}\cos^3\theta\,d\theta$$

$$= 2\cdot\frac{3-1}{3} = 2\times\frac{2}{3}$$

$$= \frac{4}{3}$$

→ (4) $\int_0^a \int_y^a \frac{x\,dy\,dx}{x^2+y^2}$

Sol: The region of integration is

$x=y,\ x=a,\ y=0,\ y=a \rightarrow (1)$.

put $x=r\cos\theta,\ y=r\sin\theta$.

From (1), $x=y \Rightarrow r\cos\theta=r\sin\theta \Rightarrow \cos\theta=\sin\theta \Rightarrow \theta=\frac{\pi}{4}$.

$x=a \Rightarrow r\cos\theta=a \Rightarrow r=a\sec\theta$.

$y=0 \Rightarrow r\sin\theta=0 \Rightarrow \theta=0$.

$y=a \Rightarrow r\sin\theta=a \Rightarrow r=$ $dxdy=r\,dr\,d\theta$.

~~y=a ⇒ r cosθ=~~

$\therefore \int_0^a \int_y^a \frac{x\,dy\,dx}{x^2+y^2} = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r\cos\theta}{r^2}\cdot r\,dr\,d\theta$

$= \int_0^{\pi/4} (r)_0^{a\sec\theta} \cos\theta\,d\theta.$

$= \int_0^{\pi/4} a\sec\theta\cdot\cos\theta\,d\theta$

$= a\int_0^{\pi/4} 1\cdot d\theta$

$= a\cdot(\theta)_0^{\pi/4}$

$= a\left(\frac{\pi}{4}-0\right)$

$= \frac{\pi}{4}a.$

(5) S.T $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = 8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$

OR Evaluate $\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$ by changing to polar coordinates.

Sol: The region of integration is $x=\frac{y^2}{4a},\ x=y,\ y=0,\ y=4a$.

put $x=r\cos\theta,\ y=r\sin\theta$ then $dxdy=r\,dr\,d\theta$.

$x=\frac{y^2}{4a} \Rightarrow y^2=4ax \Rightarrow r^2\sin^2\theta=4ar\cos\theta$.

$r=\frac{4a\cos\theta}{\sin^2\theta}.$

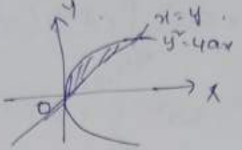
and $y = x$ ⇒

⇒ $r\cos\theta = r\sin\theta$

⇒ $\theta = \frac{\pi}{4}$.

$y = 0 \Rightarrow r\cos\theta = 0$

⇒ $\theta = \frac{\pi}{2}$.



Also $x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$ and $x^2 + y^2 = r^2$.

∴ The limits are $r : 0 \to \frac{4a\cos\theta}{\sin^2\theta}$

$\theta : \frac{\pi}{4} \to \frac{\pi}{2}$.

$$\therefore \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_0^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{\cancel{r^2}(\cos^2\theta - \sin^2\theta)}{\cancel{r^2}}\cdot r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left(\frac{r^2}{2}\right)_0^{\frac{4a\cos\theta}{\sin^2\theta}} \cdot (\cos^2\theta - \sin^2\theta)\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{1}{2}\left(\frac{4a\cos\theta}{\sin^2\theta}\right)^2 (\cos^2\theta - \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\times \overset{8}{\cancel{16}}a^2 \int_{\pi/4}^{\pi/2} \frac{\cos^2\theta}{\sin^4\theta}(\cos^2\theta - \sin^2\theta)\,d\theta.$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} (\cot^4\theta - \cot^2\theta)\,d\theta.$$

$$\cancel{= 8a^2\left[\int_{\pi/4}^{\pi/2} \cot^2\theta\cdot\cot^2\theta\,d\theta - \int_{\pi/4}^{\pi/2}(\operatorname{cosec}^2\theta - 1)\,d\theta\right]}$$

$$= 8a^2\left[\int_{\pi/4}^{\pi/2} \cot^2\theta\cdot(\operatorname{cosec}^2\theta - 1)\,d\theta - \left(-\cot\theta - \theta\right)_{\pi/4}^{\pi/2}\right]$$

$$= 8a^2\left[\int_0^1 t^2.\right.$$

$\cot\theta = t$

$-\operatorname{cosec}^2\theta\,d\theta = dt$

$\theta = \frac{\pi}{4} \Rightarrow t = 1$

$\theta = \frac{\pi}{2} \Rightarrow t = 0$.

$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta\,(\cot^2\theta - 1)\,d\theta$$

$$= 8a^2 \int_{\pi/4}^{\pi/2} \cot^2\theta\,(\csc^2\theta - 2)\,d\theta$$

$$= 8a^2 \left[\int_{\pi/4}^{\pi/2} \cot^2\theta \csc^2\theta\,d\theta - 2\int_{\pi/4}^{\pi/2} \cot^2\theta\,d\theta \right]$$

$$= 8a^2 \left[\int_0^1 t^2\,dt - 2\int_{\pi/4}^{\pi/2} (\csc^2\theta - 1)\,d\theta \right]$$

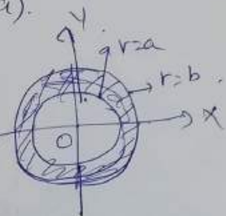
$$= 8a^2 \left[\left(\frac{t^3}{3}\right)_0^1 - 2\left(-\cot\theta\right)_{\pi/4}^{\pi/2} + 2(\theta)_{\pi/4}^{\pi/2} \right]$$

$$= 8a^2 \left[\left(\frac{1}{3}\right) + 2(0-1) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \right.$$

$$= 8a^2 \left[\frac{1}{3} - 2 + \frac{\cancel{2}\pi}{\cancel{4}_2} \right]$$

$$= 8a^2 \left[\frac{\pi}{2} - \frac{5}{3} \right]$$

→ By changing into polar coordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the annular region b/w the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ $(b>a)$.



Sol: G.T $x^2+y^2=a^2$ & $x^2+y^2=b^2$.

Put $x = r\cos\theta$, $y = r\sin\theta$.

Now $x^2+y^2=a^2 \Rightarrow r^2(1)=a^2 \Rightarrow r=a$.

$x^2+y^2=b^2 \Rightarrow r^2=b^2 \Rightarrow r=b$.

and $\theta : 0 \to 2\pi$

Hence $\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{\cancel{r^2}\cos^2\theta \cdot r^2\sin^2\theta}{\cancel{r^2}(1)} \cdot r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_a^b r^3 \sin^2\theta \cos^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \sin^2\theta \cos^2\theta \left(\frac{r^4}{4}\right)_a^b d\theta .$$

$$= \int_0^{2\pi} \frac{b^4-a^4}{4} \sin^2\theta \cos^2\theta \, . d\theta .$$

$$= \frac{b^4-a^4}{4\times 4} \int_0^{2\pi} (2\sin\theta\cos\theta)^2 d\theta .$$

$$= \frac{b^4-a^4}{16} \int_0^{2\pi} \sin^2 2\theta \, d\theta .$$

$$= \frac{b^4-a^4}{16} \int_0^{2\pi} \frac{1-\cos 4\theta}{2} d\theta .$$

$$= \frac{b^4-a^4}{32} \left[(\theta)_0^{2\pi} - \left(\frac{\sin 4\theta}{4}\right)_0^{2\pi} \right]$$

$$= \frac{b^4-a^4}{32} \left\{ (2\pi - 0) - \frac{1}{4}(0-0) \right]$$

$$= \frac{b^4-a^4}{\cancel{32}\,16} \times \cancel{2}\pi$$

$$= \frac{b^4-a^4}{16} \pi$$