10

Time-Invariant Electric Current in Solid and Liquid Conductors

10.1 Introduction

The term *time-invariant electric current* implies a steady, time-constant motion of a very large number of small charged particles. The term *current* is used because this motion is somewhat similar to the motion of a fluid. A typical example is the steady motion of free electrons inside a metallic conductor, but there are other types of time-invariant currents as well. What causes organized motion of large numbers of electrons (or other charges)? The answer is an electric field, which unlike in the electrostatic case, *does* exist inside current-carrying conductors. Time-invariant currents are frequently also called *direct currents*, abbreviated *dc*. A domain in which currents exist is known as the *current field*.

Inside a metallic conductor with no electric field present, a free electron (or any other type of free charge) moves chaotically in all directions, like a gas molecule. If there is an electric field inside the conductor, the electrons (negative charges) are accelerated in the direction opposite to that of the local vector **E**. This accelerated motion lasts until the electron collides with an atom. We can imagine that the electron

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then stops, transfers the acquired kinetic energy to the atom, is again accelerated in the opposite*E, and so on. So the electrons acquire an average "drift" velocity under the influence of the field, and the result of this organized motion is an electric current.

There are three important consequences of this fact:

1. In solid and liquid conductors, where the average path between two collisions is very short, the drift velocity is in the direction of the force, i.e., *the charges follow the lines of vector* **E**.
2. Charges constantly lose the acquired kinetic energy to the atoms they collide with. This results in a more vigorous vibration of the atoms, i.e., a higher temperature of the conductor. This means that in the case of an electric current in conductors, the energy of the electric field is constantly converted into heat. This heat is known as *Joule's heat*. It is also frequently called *Joule's losses* because it represents a loss of electric energy.
3. In the steady time-invariant state, the motion of electric charges is time-invariant. The electric field driving the charges must in turn be time-invariant, and is therefore due to a time-constant distribution of charges. Such an electric field is *identical to the electrostatic field of charges distributed in the same manner*. This is a conclusion of extreme importance. All the concepts we derived for the electrostatic field (scalar potential, voltage, etc.) are valid for time-invariant currents.

Liquid conductors have pairs of positive and negative *ions*, which move in opposite directions under the influence of the electric field. The electric current in liquid conductors is therefore made of two streams of charged particles moving in opposite directions, but we have the same mechanism and the same effect of energy loss (Joule's heat) of current flow as in the case of a solid conductor. There is an additional effect, however, known as *electrolysis*—chemical changes in any liquid conductor that always accompany electric current.

In a class of materials called *semiconductors*, there are two types of charge carriers—negatively charged electrons and positively charged holes. In this case, the electric ~~field~~ is due to both types of charges and depends very much on their concentrations. CURRENT

In gases, electric current is also due to moving ions, but the average path between two collisions is much longer than for solid and liquid conductors. The mechanism of current flow is therefore quite different.

In solid and liquid conductors the number of charges taking part in an electric current is extremely large. To understand this, recall that a solid or liquid contains on the order of 10^{28} atoms per cubic meter. It is not easy to understand these huge numbers. Perhaps it would help if we consider a volume of about $(0.1\ \text{mm})^3$ (a cube 0.1 mm on each side), which is barely visible by the naked eye. This tiny volume contains about 10^{12} atoms, which is more than one hundred times the number of humans on our planet! It is evident from this example that the term "electric *current*" is indeed appropriate.

Questions and problems: Q10.1

10.2 Current Density and Current Intensity: Point Form of Ohm's and Joule's Laws

Electric current in conductors is described by two quantities. The *current density vector*, $\mathbf{J}$, describes the organized motion of charged particles at a point. The *current intensity* is a scalar that describes this motion in an integral manner, through a surface.

Let a conductor have N free charges per unit volume, each carrying a charge Q and having an average (drift) velocity $\mathbf{v}$ at a given point. The current density vector at this point is then defined as

$$\mathbf{J} = NQ\mathbf{v} \qquad \text{amperes per m}^2\ (\text{A/m}^2). \tag{10.1}$$

(Definition of current density for one kind of charge carriers)

Note that this definition implies that the current density vector of equal charges of opposite sign moving in opposite directions is the same. Of course, motion of different charges in opposite directions physically is different. However, experiments indicate that practically all effects (Joule's heat, chemical effects, magnetic effects) of an electric current depend on the product $Q\mathbf{v}$, so it is convenient to adopt this definition for the current density vector.

If there are several types of free charges inside a conductor, the current density is defined as a vector sum of the expression in Eq. (10.1). For example, let the current be due to the motion of free charge carriers of charges $Q_1, Q_2, \ldots, Q_n$, moving with drift velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Let there be $N_1, N_2, \ldots, N_n$ of these charge carriers per unit volume, respectively. The current density vector is then given by

$$\mathbf{J} = \sum_{k=1}^{n} N_k Q_k \mathbf{v}_k \qquad (\text{A/m}^2). \tag{10.2}$$

(Definition of current density for several kinds of charge carriers)

The current intensity, I, through a surface is defined as the total amount of charge that flows through the surface during a small time interval, divided by this time interval. In counting this charge, opposite charges moving in opposite directions are added together. Thus

$$I = \frac{\mathrm{d}Q_{\text{through } S \text{ in } \mathrm{d}t}}{\mathrm{d}t} \qquad (\text{C/s} = \text{A}). \tag{10.3}$$

(Definition of current intensity through a surface)

This can also be expressed in terms of the current density vector, as follows.

Consider a surface element $\mathrm{d}S$ of the surface S in Fig. 10.1. Let the drift velocity of charges at $\mathrm{d}S$ be $\mathbf{v}$, their charge Q, and their number per unit volume N. During the

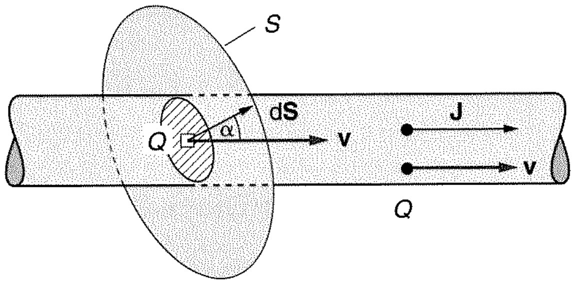


Figure 10.1 The current intensity I through S is equal to the flux of the current density vector $\mathbf{J}$ through S

time interval dt the charges move by a distance $v\,dt$ in the direction of $\mathbf{v}$. Therefore, the charge that crosses dS in dt is

$$dQ_{\text{through } dS \text{ during } dt} = dS\, v\, dt\, \cos\alpha\, NQ, \tag{10.4}$$

where α is the angle between the velocity vector and the normal to the surface element. The total charge through S during interval dt is obtained as a sum (integral) of these elemental charges over the entire surface, and the current intensity is obtained by dividing this sum by dt. Noting that $dS\, v\, \cos\alpha NQ = J\, dS\, \cos\alpha = \mathbf{J} \cdot d\mathbf{S}$, we obtain

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} \qquad \text{(A)}. \tag{10.5}$$

(Definition of current intensity through a surface in terms of the current density vector)

The unit for current is an *ampere* (A), equal to a coulomb per second (C/s). The unit for current density is A/m^2.

We now know that electric current in a conductor is produced by an electric field. We also know that in solid and liquid conductors the vectors $\mathbf{J}$ and $\mathbf{E}$ are in the same direction. For most conductors, vector $\mathbf{J}$ is a linear function of $\mathbf{E}$,

$$\mathbf{J} = \sigma \mathbf{E} \qquad [\sigma - \text{siemens per meter (S/m)}]. \tag{10.6}$$

[Point (local) form of Ohm's law]

Conductors for which (10.6) is valid are called *linear conductors*. The constant σ is known as the *conductivity* of the conductor. The unit for conductivity is *siemens per meter* (S/m).

The reciprocal value of σ is designated by ρ and is known as the *resistivity*. The unit for resistivity is *ohm · meter* ($\Omega \cdot$ m). Equation (10.6) can be written in the form

$$\mathbf{E} = \rho \mathbf{J} \qquad [\rho - \text{ohm} \cdot \text{meter } (\Omega \cdot \text{m})]. \tag{10.7}$$

[Point (local) form of Ohm's law]

Both Eqs. (10.6) and (10.7) are known as the *point form of Ohm's law* for linear conductors because they give a relationship between the two field quantities at every point inside a conductor.

For metallic conductors, conductivities range from about 10 MS/m (iron) to about 60 MS/m (silver). The conductivity of seawater is about 4 S/m, that of ground (soil) is between 10^{-2} and 10^{-4} S/m, and conductivities of good insulators are less than about 10^{-12} S/m.

We have already explained from a physical standpoint that there is a permanent transformation of electric energy into heat in every current field. Let us now derive the expression, known as *Joule's law in point form*, for the volume density of power in this energy transformation.

Let there be N charge carriers Q in the conductor, and let their local drift velocity be $\mathbf{v}$. The electric force on each charge is $Q\mathbf{E}$. The work done by the force when moving the charge during a time interval $\mathrm{d}t$ is equal to $Q\mathbf{E} \cdot (\mathbf{v}\,\mathrm{d}t)$. The work done in moving all the $N\,\mathrm{d}v$ charges inside a small volume $\mathrm{d}v$ is therefore

$$\mathrm{d}A_{\text{el.forces}} = Q\mathbf{E} \cdot (\mathbf{v}\,\mathrm{d}t)N\mathrm{d}v = \mathbf{J} \cdot \mathbf{E}\,\mathrm{d}v\,\mathrm{d}t \qquad (\mathrm{J}). \tag{10.8}$$

If we divide this expression by $\mathrm{d}v\,\mathrm{d}t$, we get the desired power per unit volume (volume power density)—the electric power that is lost to heat:

$$p_J = \frac{\mathrm{d}P_J}{\mathrm{d}v} = \mathbf{J} \cdot \mathbf{E} = \frac{J^2}{\sigma} = \sigma E^2 \quad \text{watts/m}^3 \ (\mathrm{W/m^3}). \tag{10.9}$$

(Joule's law in point form)

If we wish to determine the power of Joule's losses in a domain of space, we just have to integrate the expression in Eq. (10.9) over that domain:

$$P_J = \int_v \mathbf{J} \cdot \mathbf{E}\,\mathrm{d}v \quad \text{watts (W)}. \tag{10.10}$$

(Joule's losses in a domain of space)

Example 10.1—Fuses. Electrical devices are frequently protected from excessive currents by fuses, one type of which is sketched in Fig. 10.2. The fuse conductor is made to be much thinner than the circuit conductors elsewhere. For example, let the radius of the circuit conductor be n times that of the fuse. If a current of intensity I exists in the circuit, the volume density of Joule's losses in the thin conductor section is n^4 larger than those in the other section. In the case of excessive current, therefore, the thin conductor section melts long before the normal section is heated up. When the fuse melts, it becomes an open circuit and does not allow any

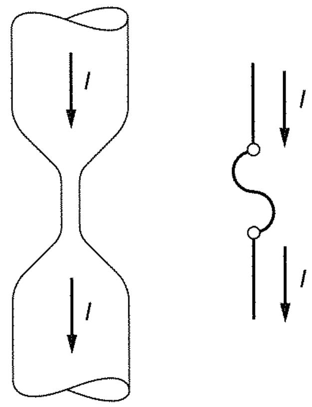


Figure 10.2 A simple model of a fuse

further current to flow and possibly damage the device protected by the fuse. Usually the thin conductor is a metal with a conductivity smaller than that of the thick wire.

Questions and problems: Q10.2 and Q10.3, P10.1 to P10.7

10.3 Current-Continuity Equation and Kirchhoff's Current Law

Experiments tell us that electric charge cannot be created or destroyed. This is known as the *law of conservation of electric charge*. The continuity equation is the mathematical expression of this law. Its general form is valid for time-varying currents, but it can easily be specialized for time-invariant currents.

Consider a closed surface S in a current field. Let $\mathbf{J}$ be the current density (a function of coordinates and, in the general case considered here, of time). The definition of current intensity applies to any surface, so it applies to a closed surface as well. The current intensity, $i(t)$, through S, with respect to the outward normal, is given by Eq. (10.3):

$$i(t) = \frac{\mathrm{d}q(t)_{\text{out of } S \text{ in } \mathrm{d}t}}{\mathrm{d}t}. \tag{10.11}$$

According to the law of conservation of electric charge, if some amount of charge leaves a closed surface, the charge of opposite sign inside the surface must increase by the same amount. So we can write Eq. (10.11) as

$$i(t) = -\frac{\mathrm{d}q(t)_{\text{inside } S \text{ in } \mathrm{d}t}}{\mathrm{d}t}. \tag{10.12}$$

The current intensity can also be written in the form of Eq. (10.5), and

$$\frac{\mathrm{d}q(t)_{\text{inside } S \text{ in } \mathrm{d}t}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_v \rho(t)\mathrm{d}v, \tag{10.13}$$

where v is the volume enclosed by S. Recall that by convention we always adopt the outward unit vector normal to a closed surface. Thus Eq. (10.12) can be rewritten in the form

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_v \rho(t)\mathrm{d}v. \tag{10.14}$$

(General form of the current continuity equation, where surface S may vary in time)

This is the *current continuity equation*. Note that we can imagine the surface S to change in time, in which case the form of the continuity equation in Eq. (10.14) must be used. This, however, is needed only in rare instances. If the surface S does not change in time, the time derivative acts on ρ only. Because ρ is a function of both time and space coordinates, the ordinary derivative needs to be replaced by a partial derivative, and we obtain a much more important form of the current continuity equation:

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\int_v \frac{\partial \rho(t)}{\partial t}\mathrm{d}v. \tag{10.15}$$

(Current continuity equation for a time-invariant surface)

Although the current continuity equation is not a field equation, it is of fundamental importance in the analysis of electromagnetic fields because only sources (charges and currents) satisfying this equation can be real sources of the field.

Now let the current field be constant in time, in which case the charge density is also constant in time. The partial derivative of ρ on the right side of Eq. (10.15) is then zero, and both forms of the current continuity equation reduce to

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = 0. \tag{10.16}$$

(Generalized Kirchhoff's current law)

This equation tells us that in time-constant current fields the amount of charge that flows into a closed surface is *exactly* the same as that which flows out of it. Equation (10.16) represents, in fact, the generalized form of the familiar *Kirchhoff's current law* from circuit theory. Indeed, if a surface S encloses a node of a circuit, there are currents only through the circuit branches, and Eq. (10.16) becomes

$$\sum_{k=1}^{n} I_k = 0. \tag{10.17}$$

We know that Kirchhoff's current law in this form is applied also to circuits with time-varying currents. Considering the preceding discussion, it should be clear that in such cases it is only approximate. (Can you explain why?)

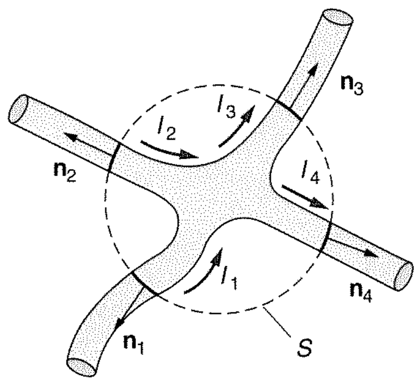


Figure 10.3 Application of generalized Kirchhoff's current law to a node with four wires

Example 10.2—Continuity equation applied to a circuit node. Let the surface S enclose a node with four wires (Fig. 10.3), with dc currents I_1, I_2, I_3, and I_4. The vector $\mathbf{J}$ is nonzero only over small areas of S where the wires go through the surface. There, the flux of $\mathbf{J}$ is simply the current intensity in that wire, so that Eq. (10.16) yields $-I_1 - I_2 + I_3 + I_4 = 0$, which is what we would get if we simply applied Kirchhoff's current law to the node. How are the signs of the currents determined and what do they correspond to in Eq. (10.16)?

Questions and problems: Q10.4 and Q10.5

10.4 Resistors: Ohm's and Joule's Laws

A resistor is a resistive body with two equipotential contacts. A resistor of general shape is shown in Fig. 10.4. Assume that the material of the resistor is linear. We know that the resistivity, ρ, for linear materials does not depend on the current density. Then the current density, $\mathbf{J}$, is proportional at all points of the resistor to the current intensity I through its terminals. Therefore, the electric field vector in the

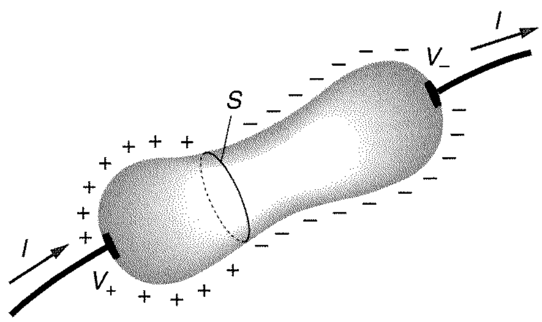


Figure 10.4 A resistor consists of a resistive body with two metallic (equipotential) contacts (the resistor terminals)

resistor material, $\mathbf{E} = \rho\mathbf{J}$, and the potential difference between its terminals are also proportional to the current intensity,

$$V_+ - V_- = RI \qquad [R - \text{ohms } (\Omega)], \tag{10.18}$$

where R is a constant. This equation is known as *Ohm's law*. Resistors for which this equation holds are called *linear resistors*. The constant R is called the *resistance* of the resistor. In some instances it can be computed starting from the defining formula in Eq. (10.18), but it can always be measured. The unit for resistance is the *ohm* (Ω).

The reciprocal of resistance is called the *conductance*, G. Its unit is called the *siemens* (S). In the United States, sometimes the mho (ohm backwards) is used instead, but this is not a legal SI unit and we will not use it.

Example 10.3—Resistance of a straight wire segment. As an example of calculating resistance, consider a straight wire of resistivity ρ, length l, and cross-sectional area S. Let the current intensity in the wire be I. The current density vector is parallel to the wire axis, and its magnitude is $J = I/S$. The electric field vector is therefore also parallel to the wire axis, and its magnitude is $E = \rho J = \rho I/S$. The potential difference between the ends of the wire segment is $V_1 - V_2 = El = \rho Il/S$. So the resistance of the wire segment is

$$R = \rho\frac{l}{S} \qquad (\Omega). \tag{10.19}$$

Consider now a resistor of resistance R. Let the current intensity in the resistor be I, and the voltage between its terminals V. During a time interval t, a charge equal to $Q = It$ flows through the resistor. This charge is transported by electric forces from one end of the resistor to the other end. From the definition of voltage, the work done by electric forces is

$$A_{\text{el.forces}} = QV = VIt \qquad (\text{J}). \tag{10.20}$$

Because of energy conservation, an energy equal in magnitude to this work is transformed into heat inside the resistor:

$$W = VIt = RI^2t = \frac{V^2}{R}t \qquad (\text{J}). \tag{10.21}$$

Since the process of transformation of electric energy into heat is constant in time, the power of this transformation of energy is W/t, that is,

$$P = VI = RI^2 = \frac{V^2}{R} \qquad (\text{W}). \tag{10.22}$$

(Joule's law)

This is the familiar *Joule's law* from circuit theory. It is named after the British physicist James Prescott Joule (1818–1889), who established this law experimentally.

Questions and problems: Q10.6 to Q10.9, P10.8 to P10.14

10.5 Electric Generators

We know that actual sources of the electric field are electric charges. We also know that we must remove some charges from a body, or put them on a body, in order to obtain excess electric charges on it. This obviously cannot be done by the electric forces themselves. Devices that do this must use some nonelectric energy to separate one type of charge from another. Such devices are known as *electric generators.*

An electric generator is sketched in Fig. 10.5. In a region of the generator there are nonelectric forces, known as *impressed forces,* that separate charges of different signs on the two generator terminals, or electrodes, denoted in the figure by "−" (negative) and "+" (positive). These forces can be diverse in nature. For example, in chemical batteries these are chemical forces; in thermocouples these are forces due to different mobilities of charge carriers in the two conductors that make the connection; in large rotating generators these are magnetic forces acting on charges inside conductors.

Impressed forces, by definition, act only on charges. Therefore they can always be represented as a product of the charge on which they act and a vector quantity that must have the dimension of the electric field strength:

$$\mathbf{F}_{\text{impressed}} = Q\mathbf{E}_{\text{i}}. \tag{10.23}$$

The vector $\mathbf{E}_{\text{i}}$ is not necessarily an electric field strength (although it does have the same dimension and unit). It is known as the *impressed electric field strength.* The region of space it exists in is known as the *impressed electric field.* The impressed electric field is a concept used often in electromagnetic theory. For example, in the analysis of radar wave scattering from a radar target, the radar wave is considered to be a field impressed on the target.

The *electromotive force,* or *emf,* of a generator is defined as the work done by impressed forces in taking a unit charge through the generator, from its negative to

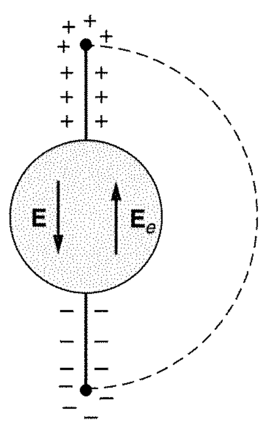


Figure 10.5 Sketch of an electric generator

its positive terminal:

$$\mathcal{E} = \left\{ \int_{-}^{+} \mathbf{E}_{\mathrm{i}} \cdot \mathrm{d}\mathbf{l} \right\}_{\text{through the generator}} \qquad \text{(definition of emf).} \tag{10.24}$$

By simple reasoning, the emf of a generator can be expressed in terms of the voltage between open-circuited generator terminals, as follows. Assume that the generator is open-circuited (no current is flowing through the generator). Then at all its points, the electric field strength due to separated charges and the impressed electric field strength must have equal magnitudes and opposite directions, i.e., $\mathbf{E} = -\mathbf{E}_i$ (otherwise free charges in the conducting material of the generator would move). If we substitute this into Eq. (10.24) and exchange the integration limits, we get

$$\mathcal{E} = \left\{ \int_{+}^{-} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \right\}_{\text{any path}} = V_{+} - V_{-}. \tag{10.25}$$

The electric field strength **E** is due to time-constant charges, i.e., it is an electrostatic field. The line integral can therefore be taken along *any* path, including one outside the generator (dashed line in Fig. 10.5). This means that we can measure the emf of a generator by a voltmeter with the voltmeter leads connected in any way to the terminals of an open-circuited generator. We will see that this is not the case for time-varying voltages.

Because the generator is always made of a material with nonzero resistivity, some energy is transformed into heat in the generator itself. Therefore, we describe a generator by its internal resistance also. This is simply the resistance of the generator in the absence of the impressed field.

Questions and problems: Q10.10 to Q10.12, P10.15 and P10.16

10.6 Boundary Conditions for Time-Invariant Currents

Current fields often exist in media of different conductivities separated by boundary surfaces. We now formulate boundary conditions for this case.

Consider the boundary surface sketched in Fig. 10.6. If we apply the continuity equation for time-invariant currents to the small, coinlike closed surface, we immediately see that the normal components of the current density vector in the two media must be the same:

$$J_{1\mathrm{n}} = J_{2\mathrm{n}}, \qquad \text{or} \qquad \sigma_1 E_{1\mathrm{n}} = \sigma_2 E_{2\mathrm{n}}. \tag{10.26}$$

We know that the electric field in a current field has the same properties as the electrostatic field. Therefore the same boundary condition for the tangential component of the vector **E** on a boundary applies:

$$E_{1\mathrm{t}} = E_{2\mathrm{t}}, \qquad \text{or} \qquad \frac{J_{1\mathrm{t}}}{\sigma_1} = \frac{J_{2\mathrm{t}}}{\sigma_2}. \tag{10.27}$$

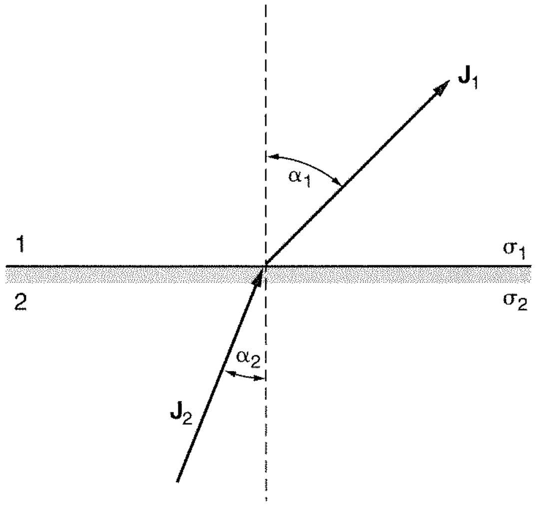


Figure 10.6 Boundary surface between two conducting media

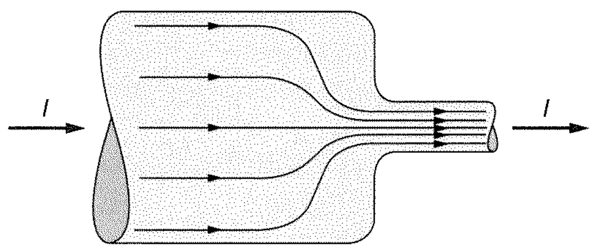


Figure 10.7 The normal component of vector **J** in a conductor adjacent to an insulator is zero.

Example 10.4—Boundary conditions between a conductor and an insulator. If one of the two media in Fig. 10.6, say, medium 1, is an insulator (e.g., air), what are the expressions for the boundary conditions?

We know that there can be no current in the insulator. Hence from Eq. (10.26) we conclude that the normal component of the current density vector in a conductor adjacent to the insulator must be zero. Tangential components of the vector **E** are the same in the two media, but of course J_{1t} does not exist. Lines of vector **J** in such a case are sketched in Fig. 10.7.

Questions and problems: Q10.13

10.7 Grounding Electrodes and an Image Method for Currents

Consider an electrode (a conducting body) of arbitrary shape buried in a poorly conducting medium (for example, soil) near the flat boundary surface between the poor conductor and some insulator (for example, air). Suppose that a current of intensity I is flowing from the electrode into the conducting medium, and is supplied from a distant current source through a thin insulated wire, as shown in Fig. 10.8. According to the boundary conditions, the current flow lines are tangential to the surface.

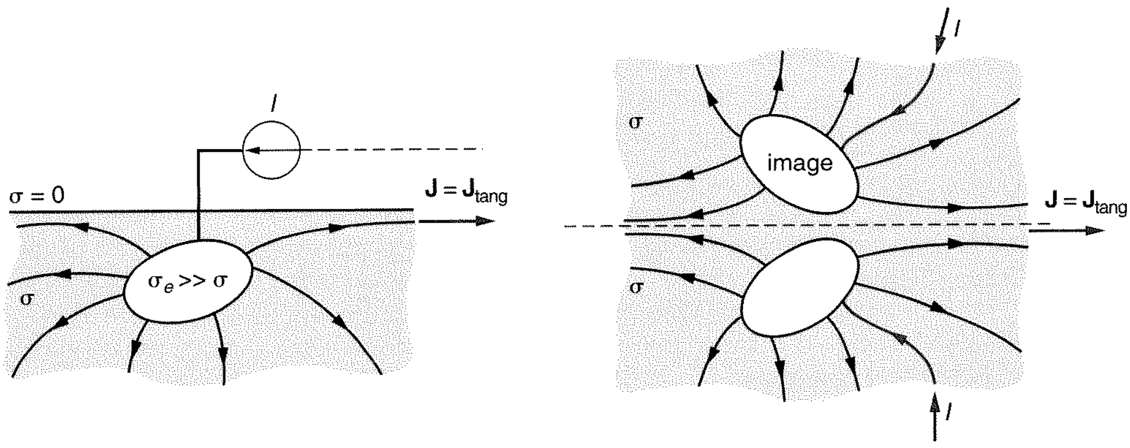


Figure 10.8 Image method for dc currents flowing out of an electrode into a poor conductor near the interface with an insulator

Imagine that the entire space is filled with the poor conductor. If an image electrode *with the current of the same intensity flowing out of it* is placed in the upper half of the space, as shown in the figure, the resulting current flow in both half spaces will be tangential to the plane of symmetry (the former boundary surface). Therefore, the current distribution in the lower half space is the same as in the actual case. Consequently, the influence of the boundary surface may be replaced by the image electrode, provided the current through the image electrode has the same intensity *and direction* as that through the real electrode. This method is very useful for analyzing current flow from electrodes buried under the surface of the earth. Such electrodes are often used for grounding purposes.

Example 10.5—Hemispherical grounding electrode. Suppose that a hemispherical electrode of very high conductivity is buried in poorly conducting soil of conductivity σ, as shown in Fig. 10.9. We are interested in

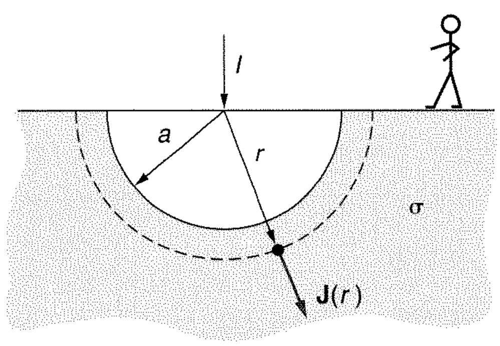


Figure 10.9 A hemispherical grounding electrode

- The resistance of such a grounding system
- The intensity of the electric field at all points on the earth's surface if a current I flows from the electrode into the earth
- What happens if a person in noninsulating shoes approaches this grounding electrode

We use the image method. Let the current I flow out of the hemispherical electrode. If all space is filled with earth, the image is another hemispherical electrode, so we get a spherical electrode with current $2I$ in a homogeneous conducting medium. Due to symmetry, the current from the spherical electrodes is radial. Consequently, the current from the original hemispherical electrode is also radial. The current density at a distance r from the center of the hemisphere is therefore

$$J = \frac{2I}{4\pi r^2} = \frac{I}{2\pi r^2}.$$

The magnitude of the electric field is

$$E = \frac{J}{\sigma} = \frac{I}{2\pi\sigma r^2},$$

so that the potential of the electrode with respect to a reference point at infinity is obtained as

$$V_a = \int_a^\infty E\,\mathrm{d}r = \int_a^\infty \frac{I}{2\pi\sigma r^2}\mathrm{d}r = \frac{I}{2\pi\sigma a}.$$

Is it possible to define the resistance of this grounding system? We need two terminals for a resistor, and the hemispherical electrode has (seemingly) just one. However, the current must be collected somewhere at a distant point by a generator and returned to the electrode through a wire. The distant point is the other "resistor" contact. Usually this other contact is a large grounding system of a power plant, with a large contact area with the ground, so that the principal contribution to the resistance of this resistor comes from the hemispherical electrode. We can therefore define the resistance of the hemispherical grounding electrode as

$$R = \frac{V_a}{I} = \frac{1}{2\pi\sigma a}.$$

The potential at any point on the surface of the earth due to the current flow is

$$V_r = \int_r^\infty E(r)\mathrm{d}r = \frac{I}{2\pi\sigma r},$$

and the potential difference between two points at a distance $\Delta r = d$ apart is given by

$$\Delta V = \frac{I}{2\pi\sigma r} - \frac{I}{2\pi\sigma(r+d)}.$$

Suppose a person approaches the grounding electrode when a large current of $I = 1000$ A is flowing through it. The potential difference between his feet can be very large and even fatal. For example, if $\sigma = 10^{-2}$ S/m, $r = 1$ m, and the person's step is $d = 0.75$ m long, the potential difference between the two feet will be 6820 volts.

Real grounding electrodes are usually in the shape of a plate, a rod, or a thin metal mesh and are buried in the ground. The variation in the conductivity of the soil has a large effect on the behavior of the grounding electrode. For that reason the conductivity around it is sometimes purposely increased by, for example, adding salt to the soil. The example of a HEMI ~~semi~~spherical electrode is useful because it gives us an idea of the order of magnitude, but it is certainly not precise.

Questions and problems: Q10.14 to Q10.17, P10.17 to P10.19

10.8 Chapter Summary

1. The basic quantities that describe electric current are the current density vector, $\mathbf{J}$, describing current flow *at any point*, and the current intensity, I, *describing current flow through a surface.*
2. The current density vector $\mathbf{J}$ is most frequently a linear function of the local electric field strength, $\mathbf{J} = \sigma\mathbf{E}$ (point form of Ohm's law), also expressed as $\mathbf{E} = \rho\mathbf{J}$, where σ is the conductor conductivity, and ρ its resistivity.
3. The volume power density of transformation of electric energy into heat in conductors is described by the point form of Joule's law, $p_J = \mathbf{J} \cdot \mathbf{E}$.
4. The current continuity equation is a mathematical expression of the experimental law of conservation of electric charges. Its form for time-invariant currents is just the Kirchhoff's current law of circuit theory.
5. A resistor is an element, made of a resistive material, with two terminals, each equipotential. Ohm's and Joule's laws for resistors known from circuit theory are derived from field theory.
6. It is not possible to maintain a current field without devices known as electric generators, which convert some other form of energy into electric energy, i.e., energy of separated electric charges. Electric generators are characterized by their electromotive force and internal resistance.
7. Based on boundary conditions for current fields, an image method can be formulated for these fields, similar to that in electrostatics. However, in this case the boundary conditions are satisfied by a "positive" image. The image method can be used, for example, to determine the field and resistance of grounding electrodes.

QUESTIONS

Q10.1. What do you think is the main difference between the motion of a fluid and the motion of charges constituting an electric current in conductors?

Q10.2. Describe in your own words the mechanism of transformation of electric energy into heat in current-carrying conductors.

Q10.3. Is Eq. (10.5) valid also for a closed surface, or must the surface be open? Explain.

Q10.4. A closed surface S is situated in the field of time-invariant currents. What is the charge that passes through S during a time interval dt?

Q10.5. Is a current intensity on the order of 1 A frequent in engineering applications? Are current densities on the order of $1\,\mathrm{mA/m^2}$ or $1\,\mathrm{kA/m^2}$ frequent in engineering applications? Explain.

Q10.6. What is the difference between linear and nonlinear resistors? Can you think of an example of a nonlinear resistor?

Q10.7. A wire of length l, cross-sectional area S, and resistivity ρ is made to meander very densely. The lengths of the successive parts of the meander are on the order of the wire radius. Is it possible to evaluate the resistance of such a wire accurately using Eq. (10.19)? Explain.

Q10.8. Explain in your own words the statement in Eq. (10.20).

Q10.9. Assume that you made a resistor in the form of an uninsulated metal container (one resistor contact) with a conducting liquid (e.g., tap water with a small amount of salt), and a thin wire dipped into the liquid (the other resistor contact). If you change the level of water, but keep the length of the wire in the liquid constant, will this produce a substantial variation of the resistor resistance? If you change the length of the wire in the liquid, will this produce a substantial variation of the resistor resistance? Explain.

Q10.10. What is meant by "nonelectric forces" acting on electric charges inside electric generators?

Q10.11. List a few types of electric generators, and explain the nonelectric (impressed) forces acting in them.

Q10.12. Does the impressed electric field strength describe an *electric* field? What is the unit of the impressed electric field strength?

Q10.13. Prove that on a boundary surface in a time-invariant current field, $\mathbf{J}_{1\mathrm{norm}} = \mathbf{J}_{2\mathrm{norm}}$.

Q10.14. Where do you think the charges producing the electric field in the ground in Fig. 10.9 are located? (These charges cause the current flow in the ground.)

Q10.15. Explain in your own words what is meant by the grounding resistance, and what this resistor is physically.

Q10.16. Is it possible to define the grounding resistance if the generator is not grounded? Explain.

Q10.17. Assume that a large current is flowing through the grounding electrode. Propose at least three different ways to approach the electrode with a minimum danger of electric shock.

PROBLEMS

P10.1. Prove that the current in any homogeneous cylindrical conductor is distributed uniformly over the conductor cross section.

P10.2. Uniformly distributed charged particles are placed in a liquid dielectric. The number of particles per unit volume is $N = 10^9\,\mathrm{m^{-3}}$, and each is charged with $Q = 10^{-16}$ C. Calculate the current density and the current magnitude obtained when such a liquid moves with a velocity of $v = 1.2\,\mathrm{m/s}$ through a pipe of cross section area $S = 1\,\mathrm{cm^2}$. Is this current produced by an electric field?

P10.3. A conductive wire has the shape of a hollow cylinder with inner radius a and outer radius b. A current I flows through the wire. Plot the current density as a function of radius, $J(r)$. If the conductivity of the wire is σ, what is the resistance of the wire per unit length?

P10.4. A conductor of radius a is connected to one with radius b. If a current I is flowing through the conductor, find the ratio of the current densities and of the densities of Joule's losses in both parts of the conductor if the conductivity for both parts is σ.

P10.5. The homogeneous dielectric inside a coaxial cable is not perfect. Therefore, there is some current, $I = 50\,\mu\text{A}$, flowing through the dielectric from the inner toward the outer conductor. Plot the current density inside the cable dielectric, if the inner conductor radius is $a = 1\,\text{mm}$, the outer radius $b = 7\,\text{mm}$, and the cable length $l = 10\,\text{m}$.

P10.6. Find the expression for the current through the rectangular surface S in Fig. P10.6 as a function of the surface width x.

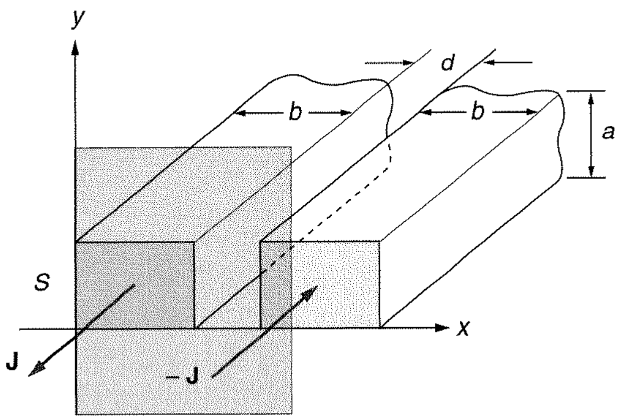


Figure P10.6 Calculating current intensity

P10.7. Find the expression for the current intensity through a circular surface S shown in Fig. P10.7 for $0 < r < \infty$.

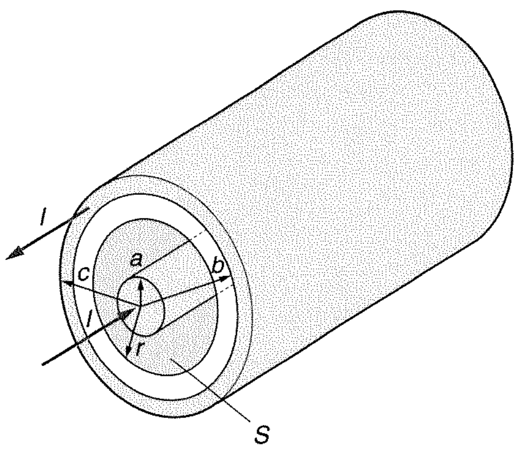


Figure P10.7 A coaxial cable

P10.8. The resistivity of a wire segment of length l and cross-sectional area S varies along its length as $\rho(x) = \rho_0(1 + x/l)$. Determine the wire segment resistance.

P10.9. Find the resistance between points 2 and 2′ of the resistor shown in Fig. P10.9.

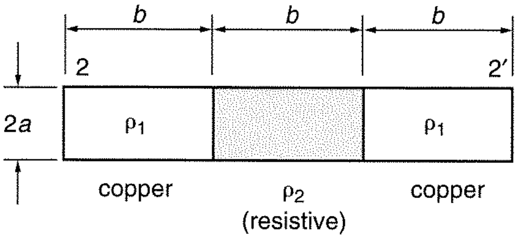


Figure P10.9 An idealized resistor CROSSECTION OF RADIUS a.

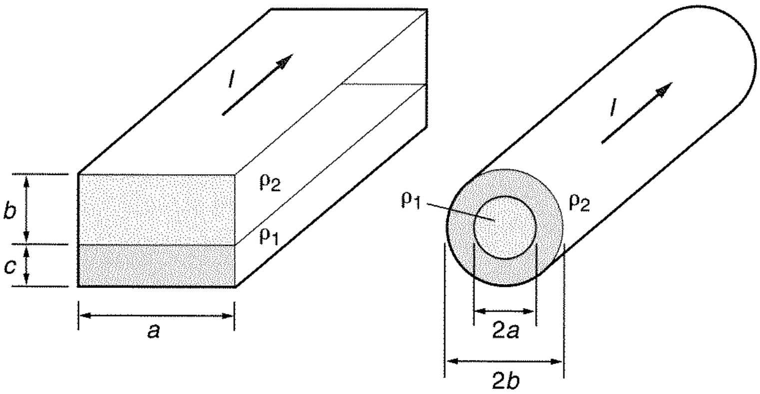


Figure P10.10 Two inhomogeneous conductors

P10.10. Show that the electric field is uniform in the case of both inhomogeneous conductors in Fig. P10.10. Find the resistance per unit length of these conductors, the ratio of currents in the two layers, and the ratio of Joule's losses in the two layers.

P10.11. The dielectric in a coaxial cable with inner radius a and outer radius b has a very large, but finite, resistivity ρ. Find the conductance per unit length between the cable conductors. Specifically, find the conductance between the conductors of a cable $L = 1\,\text{km}$ long with $a = 1\,\text{cm}$, $b = 3\,\text{cm}$, and $\rho = 10^{11}\,\Omega \cdot \text{m}$.

P10.12. A lead battery is shown schematically in Fig. P10.12. The total surface area of the lead plates is $S = 3.2\,\text{dm}^2$, and the distance between the plates is $d = 5\,\text{mm}$. Find the approximate internal resistance of the battery, if the resistivity of the electrolyte is $\rho = 0.016\,\Omega \cdot \text{m}$.

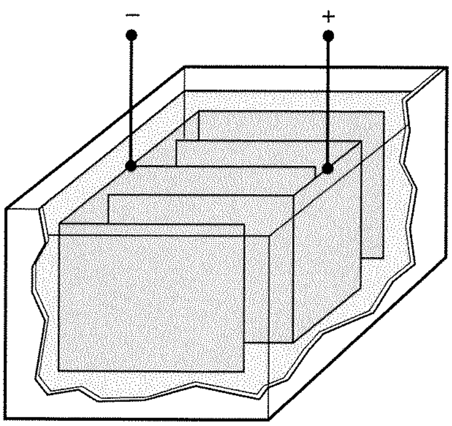


Figure P10.12 A lead battery

P10.13. Calculate approximately the resistance between cross sections 1 and 2 of the nonuniform strip conductor sketched in Fig. P10.13. The resistivity of the conductor is ρ. Why can the resistance be calculated only approximately?

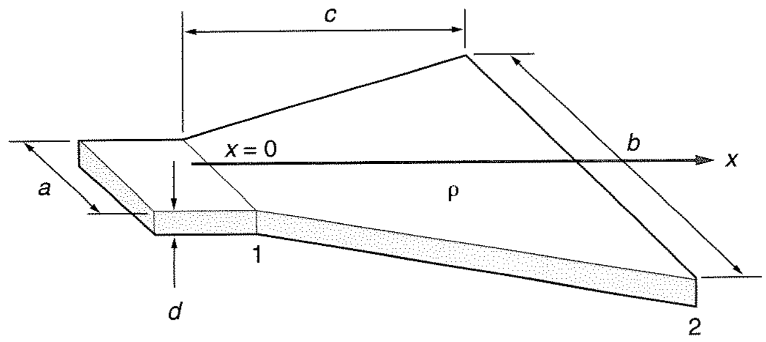


Figure P10.13 A nonuniform strip conductor

P10.14. Calculate approximately the resistance between cross sections 1 and 2 of the conical part of the conductor sketched in Fig. P10.14. The resistivity of the conductor is ρ.

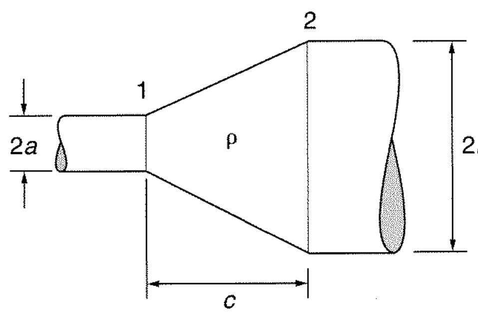


Figure P10.14 A conical conductor

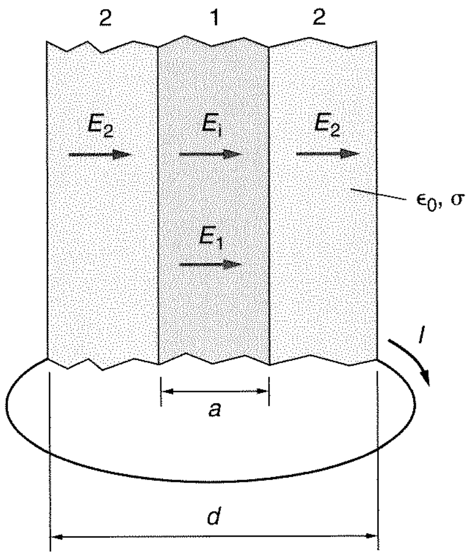


Figure P10.15 A resistor with an impressed field

P10.15. In the darker shaded region of the very large conducting slab of conductivity σ and permittivity ϵ_0 shown in Fig. P10.15, a uniform impressed field $\mathbf{E}_i$ acts as indicated. The end surfaces of the slab are coated with a conductor of conductivity much greater than σ, and connected by a wire of negligible resistance. Determine the current density, the electric field intensity, and the charge density at all points of the system. Ignore the fringing effect.

P10.16. Repeat problem P10.15 assuming that the permittivity of the slab is ϵ (different from ϵ_0). Note that in that case polarization charges are also present.

P10.17. Determine the resistance of a hemispherical grounding electrode of radius a if the ground is not homogeneous, but has a conductivity σ_1 for $a < r < b$, and σ_2 for $r > b$, where $b > a$, and r is the distance from the grounding electrode center.

P10.18. Determine the resistance between two hemispherical grounding electrodes of radii R_1 and R_2, which are a distance d ($d \gg R_1, R_2$) apart. The ground conductivity is σ.

P10.19. A grounding sphere of radius a is buried at a depth d ($d \gg a$) below the surface of the ground of conductivity σ (Fig. P10.19). Determine the points at the surface of the ground at which the electric field intensity is the largest. Determine the electric field intensity at these points if the intensity of the current through the grounding sphere is I.

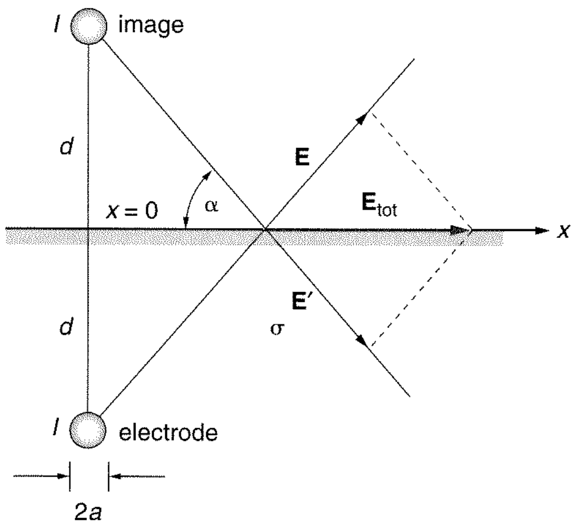


Figure P10.19 A deeply buried spherical electrode

11

Some Applications of Electrostatics

11.1 Introduction

At this point, we will briefly study some common instances and applications of electrostatic fields. The earth's strongest electric field is the atmospheric field between charged clouds, and between charged clouds and the ground. We will briefly describe fields, voltages, and currents for the extreme case of thunderstorms. Commercial applications of electrostatics include pollution-control filters, xerography, printing, electrostatic separation, coating, and some medical and biological applications. We will also look at some applications where understanding of dc current density is needed, such as probes for characterizing semiconductor materials. We will review only the basic principles of these processes and include some examples of engineering design parameters that directly apply the knowledge gained in the previous chapters. Even at that level, the examples in this chapter will show clearly how powerful the knowledge we have gained is for understanding existing devices and for discovering and designing new ones.

11.2 Atmospheric Electricity and Storms

Thunderstorms are the most obvious manifestations of electrical phenomena on our planet. A typical storm cloud carries about 10 to 20 coulombs of each type of charge, at an average height of 5 km above the earth's surface. Where does the electric energy of a cloud come from? The main source of energy on our planet is the sun: huge water masses on earth are heated by the sun's energy. The evaporated water, which contains energy originated from the sun, forms clouds. A small portion of the energy of a cloud is turned into electric energy. So thunderstorms are huge (but quite inefficient) thermoelectric generators. Some of the suspected mechanisms of cloud electrification are the breakup of raindrops, freezing, frosting, and friction between drops or crystals. Movements inside a cloud eventually make the cloud into an electric dipole, with negative charges in the lower part and positive charges in the upper part. The lower part induces positive charges on the earth below the cloud (Fig. 11.1a). The induced charge density is greatest on tall, sharp objects.

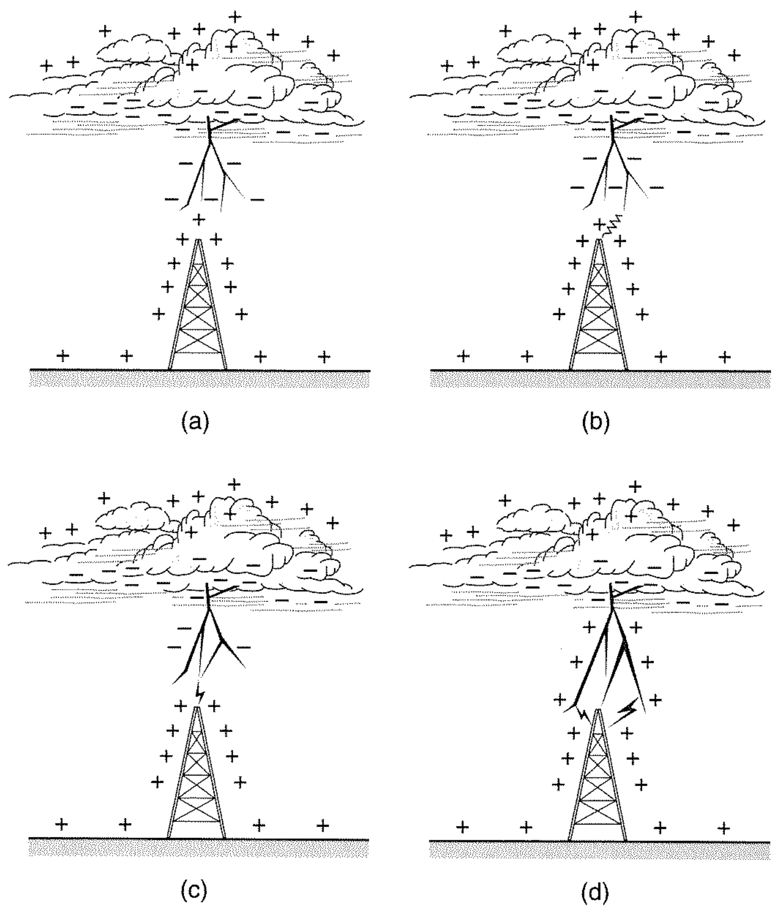


Figure 11.1 Formation of a lightning bolt. (a, b) The stepped-leader down-flowing stroke defines the path for the visible return stroke (c, d).

The beginning of a cloud-to-ground lightning bolt is an invisible discharge, called the *stepped leader*. The stepped-leader air breakdown is initiated at the bottom of the cloud. It moves in discrete steps, each about 50 m long and lasting for about 1 μs. Because of this discharge, electrons are released from the lower part of the cloud. These electrons are attracted by the induced positive charges on the ground, and they move downward (Fig. 11.1b). As the negatively charged stepped leader approaches the ground, it induces even more positive charges, especially on protruding objects. When the leader is about 100 m above the ground, a spark moves up from the ground to meet it. Once a conductive path is established, huge numbers of electrons flow from the cloud to the ground. To balance the charge flow, positive charges move up toward the cloud, trying to neutralize the huge negative charge at the bottom of the cloud. This is the discharge that we see and is called the *return stroke*. The return stroke lasts only about 100 μs, so it looks as if the entire channel lights up at the same time.

The currents in the return stroke are typically on the order of 10 kA but can be as large as 200 kA. The temperatures associated with the Joule's heat of such a current are very high—temperatures in the return stroke reach 30,000 K. The air does not have time to expand in volume, so its pressure rises to several million pascals, causing a sound shock—thunder.

At any time, there are about 1800 thunderstorms across the earth, and about 100 lightning flashes per second. The National Center for Health Statistics estimates that the death toll of lightning, about 150 people every year in the United States, is bigger than that of hurricanes and tornadoes. Lightning also causes considerable damage. Lightning rods (connected to grounding electrodes) protect exposed structures from damage by routing the strokes to the ground through the rod rather than through the structure. Benjamin Franklin suggested the use of lightning rods, and they were in place in the United States and France as early as the middle of the 18th century. It is estimated that for rural buildings containing straw and protected by a rod, the danger of fire caused by lightning is reduced by a factor of 50.

Questions and problems: Q11.1 to Q11.3, P11.1 and P11.2

11.3 Electric Current in a Vacuum and in Gases

In lightning, current flows from the clouds to the earth through atmosphere, which is a gas. There are two major differences between electric currents in solid and liquid conductors and those in a vacuum and in rarefied (low-pressure) gases. First, the electric charges in a vacuum or in gases are most often moved by the electric field of some stationary charges only; there is usually no impressed electric field. Second, in a vacuum and in gases there is no relation similar to the point form of Ohm's law. This is obvious in the case of a vacuum: charges do not collide with atoms but move under the influence of the electric forces and forces of inertia only. Consequently they do not follow the lines of vector **E** except if the field lines are straight.

In the case of gases, particularly if rarefied, an accelerated ion has a relatively long path between collisions, so it can acquire a considerable kinetic energy. As a consequence, in rarefied gases Ohm's law is not valid. Various other effects can occur, however, due to the possibility of a chain production of new pairs of ions by collisions of high-velocity ions with neutral molecules. Electrostatic discharge due to high voltages is one such effect that is of practical interest and will be described briefly in this chapter.

Example 11.1—Motion of electric charges in the electric field. Let a charge Q of mass m move in an electrostatic field in a vacuum. The equation of motion of the charge has the form

$$m\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = Q\mathbf{E}(\mathbf{r}), \tag{11.1}$$

where $\mathbf{r}(t)$ is the position vector (variable in time) of the charge, and $\mathbf{E}$ is the electric field strength, a function of coordinates (that is, of $\mathbf{r}$).

Let us now look specifically at the motion of a charged particle in a uniform electric field. Assume that a charge Q ($Q < 0$) leaves with a negligibly small velocity the negative plate of a charged parallel-plate capacitor in which the electric field strength is E. Let the plates be a distance d apart. We wish to determine the position and velocity of the charge as a function of time, if the charge left the plate at $t = 0$.

Let the x axis be perpendicular to the plates, with $x = 0$ at the negative plate (Fig. 11.2). The charge will move parallel to the x axis. According to Eq. (11.1), the equation of motion is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = QE,$$

that is, the charge moves with constant acceleration QE/m. The charge velocity as a function of time is given by

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{QE}{m}t,$$

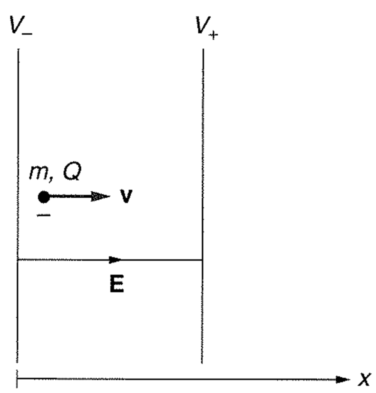


Figure 11.2 Motion of a charged particle in a uniform electric field between two parallel charged plates

since the initial velocity is zero. From this equation, the position of the particle is given by

$$x = \frac{QE}{2m}t^2.$$

Assume that the voltage between the plates is $V = 1000\,\text{V}$, that the charge is an electron ($e \simeq -1.6 \cdot 10^{-19}\,\text{C}$, $m \simeq 9.1 \cdot 10^{-31}\,\text{kg}$), and that the distance between the plates is $d = 10\,\text{cm}$. The electric field strength between the plates is then $E = V/d = 10{,}000\,\text{V/m}$.

The charge will reach the positive plate after a time obtained from the last equation, in which we set $x = d$. This yields $t = 1.07 \cdot 10^{-8}$ s. The velocity of the electron before impact with the positive electrode is now obtained if we insert this particular time into the expression for the velocity. The result is $v = 18.8 \cdot 10^6\text{m/s}$, a very large velocity (close to those velocities for which relativistic corrections in particle mass need to be made). In conclusion, electrons can be accelerated to remarkably high velocities with voltages that are not difficult to produce.

The preceding equations were derived for a uniform electric field. For arbitrary $\mathbf{E}(\mathbf{r})$ no analytical solution of Eq. (11.1) for the position vector $\mathbf{r}$ in time is known, but it can always be found approximately by numerical methods. Often the exact trajectory of the particle is not of interest. Instead only the magnitude of the velocity of the particle is required, and it can easily be calculated.

Let the charged particle (of charge Q and mass m) leave point 1, which is at potential V_1, with a velocity of magnitude v_1. We wish to determine the magnitude v_2 of its velocity when it reaches point 2, which is at a potential V_2. In moving the particle from point 1 to point 2, the electric forces perform work $Q(V_1 - V_2)$, and due to energy conservation, we know that the particle kinetic energy must have increased by precisely that amount. Thus

$$\frac{mv_2^2}{2} - \frac{mv_1^2}{2} = Q\int_1^2 \mathbf{E} \cdot d\mathbf{l} = Q(V_1 - V_2). \tag{11.2}$$

The magnitude of the velocity at point 2 is

$$v_2 = \sqrt{v_1^2 + \frac{2Q(V_1 - V_2)}{m}}. \tag{11.3}$$

In the particular, but common, case when $v_1 = 0$, this becomes

$$v_2 = \sqrt{\frac{2Q(V_1 - V_2)}{m}}. \tag{11.4}$$

(Velocity of a charge accelerated from zero velocity by potential difference $V_1 - V_2$)

Example 11.2—Velocity of an electron accelerated by 1 kV. As a numerical example, let us determine the velocity of an electron accelerated from zero velocity by a 1000-V voltage. Using Eq. (11.4) we get $v = 18.8 \cdot 10^6\,\text{m/s}$, as in Example 11.1 for the special case of a uniform field. Note that this result is valid for *any* electric field (not necessarily uniform), in which the electron covers a voltage of 1000 V.

Questions and problems: Q11.4 to Q11.9, P11.3 to P11.7

11.4 Corona and Spark Discharge

In gases, and particularly in air at normal atmospheric pressure, specific steady discharging currents may occur in certain circumstances. A necessary condition for this process is a region of electric field with intensity greater than the dielectric strength of air (about 30 kV/cm). In this region the air becomes ionized, i.e., conducting, which is equivalent to an enlargement of the electrode. If the electric field intensity outside this enlarged "electrode" is less than the dielectric strength of air, the process stops. The cloud of charges around the electrode stays permanently, and it forms a source of ions that are propelled toward the electrodes of the opposite sign. As a result, there are steady discharging currents between the electrodes. The ionized cloud is known as a *corona*.

In some instances, however, the electric field strength may not be decreased when a corona is formed around an electrode. The process then does not stop, but instead spans the whole region between the electrodes. Violent discharge of the electrodes occurs, known as *spark discharge*. The spark discharge is not, of course, a time-invariant phenomenon.

Normally corona is not desirable, because it results in losses of charge on charged conductors. In some instances, however, it is of great use. For example, discharging of an aircraft that is charged during flight by friction is performed by encouraging corona discharges at several positions on the aircraft. We will see in the next sections that corona discharge is done on purpose when small neutral metal or dielectric particles need to be charged, for example in electrostatic painting of cars.

Spark discharge is also usually undesirable. For example, in manufacturing plants or coal mines where explosive gases may exist, electrostatically charged bodies may discharge through sparks, which in turn may have enough energy to initiate a large-scale explosion. Spark discharge is a relatively frequent cause of explosions involving loss of human lives and property.

Questions and problems: Q11.10 to Q11.12

11.5 Electrostatic Pollution-Control Filters

Electrostatic filters are used in environmental control for removing fine particles from exhaust gases. In the filtering (or precipitation) process, the particles are charged, separated from the rest of the gas by a strong electric field, and finally attracted to a pollutant-collecting electrode. In the United States there are a few thousand large industrial electrostatic filters, and a large number of small units used for indoor air cleaning. Electric power generation plants in the United States generate about 2×10^7 tons of coal fly ash every year. This ash accounts for most of the use of electrostatic filters, although steel and cement production, paper processing, sulfuric acid manufacturing, petroleum refining, and phosphate and other chemical processing also use electrostatic filters.

A simplified diagram of a filter (this type is referred to as the "tubular" precipitator) is shown in Fig. 11.3. The polluted gas flow enters the bottom of the cylinder and the small ash particles are charged by ionized air around a high-potential

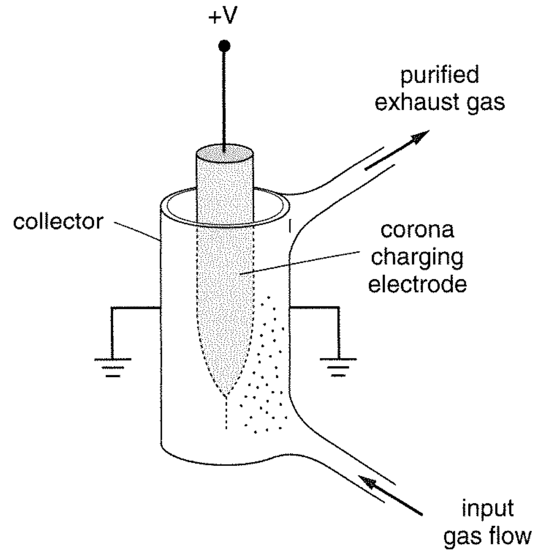


Figure 11.3 Simplified diagram of an electrostatic tubular filter

electrode (to learn more about this mechanism of gas ionization, the reader is encouraged to refer to, e.g., A.D. Moore, ed., *Electrostatics and its applications*, John Wiley, 1973). The electric field between the high-potential electrode and the coaxial grounded cylinder causes the charged particles to be attracted to the cylinder, which acts as the collecting electrode. The purified gas flows through the clean-gas exhaust at the top of the cylinder.

To understand the engineering design problems in electrostatic filters, let us consider first an uncharged idealized spherical conductive particle in a uniform electric field (Fig. 11.4a). Due to the presence of the external field, induced charges are created on the surface of the sphere. Using knowledge we gained in Chapter 6, it can be shown that the resulting electric field at points A and B on the sphere is three times that of the external field (see Example 11.3).

This means that the uniform external field has been changed by the presence of the uncharged particle and is now nonuniform, as shown in Fig. 11.4a. (The field inside the particle is not shown, because it is different for conductive or dielectric particles, as will be discussed shortly.) We said that in an electrostatic filter these particles encounter ionized air when they enter the electric field. This means that they are not strictly in a vacuum, since there are occasional charged ions in the space around them. These ions will be attracted to the particle if they are found in a certain region close to it, as shown in Fig. 11.4a. In this case, the particle becomes charged, and the excess charge distributes itself to satisfy the boundary conditions. This has an effect on the surrounding field, which becomes more nonuniform than in the previous case, as shown in Fig. 11.4b. The result is that it will be more difficult for the next ion of the same sign to be attracted to the particle, because the size of the region where it needs to find itself in order to be attracted is reduced.

The particles will be collected faster if they carry more charge, because then the electric force is larger. However, we have just explained that particles cannot be

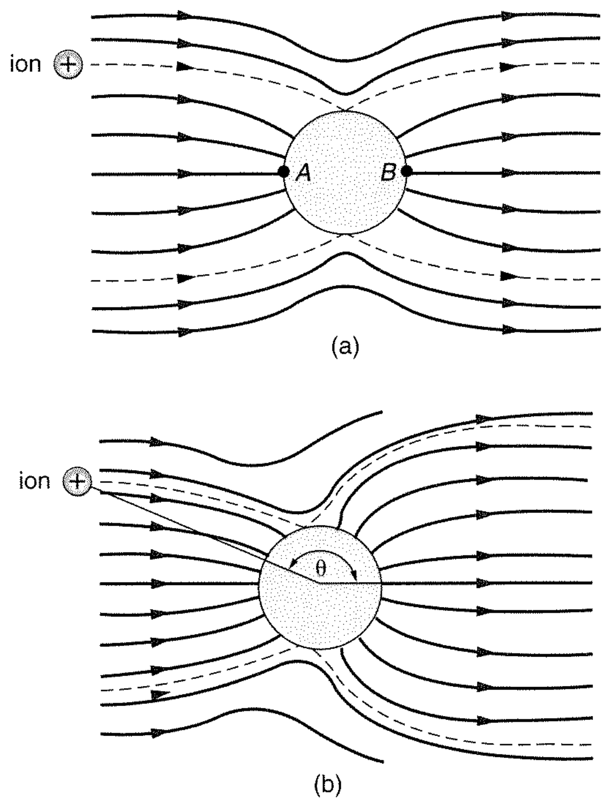


Figure 11.4 (a) An uncharged spherical dielectric pollutant particle and (b) a partly charged particle in a uniform electric field. A positive ion will be attracted to the particle in case (a) and repelled by it in case (b).

charged beyond a certain amount. From this relatively simple example, we can see that there is a limitation on how fast pollutant particles can be charged and collected inside an electrostatic filter. In order to speed up the process of collection, and therefore filter out as many pollutant particles as possible in a limited region of exhaust-gas flow inside the filter, different designs than the one shown in Fig. 11.4 have been in use. More details can be found in H.J. White, *Industrial electrostatic precipitation,* Addison-Wesley, 1963.

Example 11.3—Dielectric and conductive spherical particles in a uniform electric field. A dust particle can be approximated by a dielectric or conductive sphere in a (locally) uniform field. Consider first a uniformly polarized sphere of radius a. Let the polarization vector in the sphere be $\mathbf{P}$ (Fig. 11.5a). We assume for the moment that the sphere is situated in a vacuum, and that its polarization is the only source of the field.

The field outside and inside the sphere is the same as the field resulting only from the surface polarization charges. (Since $\mathbf{P}$ is constant, $\rho_p = \text{div}\mathbf{P} = 0$.) The density of these surface charges is given by Eq. (7.16), which in the case considered becomes

$$\sigma_p = \mathbf{P} \cdot \mathbf{n} = P \cos\theta. \tag{11.5}$$

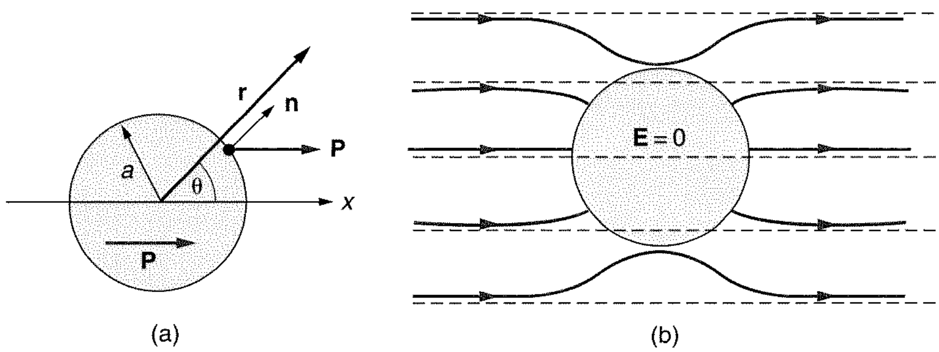


Figure 11.5 (a) A uniformly polarized dielectric sphere and (b) an uncharged conductive sphere in a uniform electric field

The field and potential of this charge distribution are not easy to find directly. However, we can use the following reasoning: the uniformly polarized dielectric sphere is equivalent to two spherical charged clouds of uniform volume densities ρ and $-\rho$ whose centers are displaced by a small distance d. We already know that the field and potential outside such a system are equal to those of a dipole. This gives us the field and potential outside the dielectric sphere. The potential of a dipole is given by

$$V(r, \theta) = \frac{p \cos\theta}{4\pi\epsilon_0 r^2},$$

so all we need to find is the equivalent dipole moment p for the two spherical clouds. Recall the definition of the polarization vector:

$$\mathbf{P} = \frac{\sum \mathrm{d}\mathbf{p}_{\text{in } \mathrm{d}v}}{\mathrm{d}v}.$$

Since all the $\mathbf{p}$ moments in $\mathrm{d}v$ are parallel, and $\mathrm{d}p = (\rho\, \mathrm{d}v)d$, we get

$$P = \rho d = \frac{Q}{4a^3\pi/3} d = \frac{p}{4a^3\pi/3}.$$

So the dipole moment of the two displaced charged spheres is

$$p = \frac{4}{3} a^3 \pi P, \tag{11.6}$$

and thus we know the potential (and hence also the field) outside the uniformly polarized sphere.

We wish now to determine the potential (and the field) inside the sphere. Consider the potential

$$V(r, \theta) = \frac{pr \cos\theta}{4\pi\epsilon_0 a^3} = \frac{px}{4\pi\epsilon_0 a^3} \qquad (r \le a). \tag{11.7}$$

It is a simple matter to prove that this potential satisfies Laplace's equation (so it is a physically possible potential). For $r = a$, it becomes identical with the potential outside the sphere. Since the potential is continuous across the sphere surface, we conclude that boundary conditions are satisfied, and that the expression in Eq. (11.7) represents the potential inside the uniformly polarized sphere.

The electric field inside the dielectric sphere has only an x component, and is given by

$$E_x = -\frac{\partial V}{\partial x} = -\frac{p}{4\pi \epsilon_0 a^3} = -\frac{P}{3\epsilon_0}. \tag{11.8}$$

Thus, *the electric field inside the uniformly polarized sphere is uniform*, and in the MINUS x direction.

Consider now a dielectric sphere in a uniform *external* electric field $E_0\mathbf{u}_x$. This field, of course, tends to uniformly polarize all dielectric bodies in it. However, polarization charges for irregular bodies will produce an irregular secondary field, and generally the polarization of the bodies will not be uniform. Only if the shape of the body is such that a uniform polarization of the body results in a uniform secondary electric field inside it will the polarization of the body in the end also be uniform. We have just demonstrated that for a dielectric sphere this is precisely the case. Thus, inside a dielectric sphere in a uniform field the total field is uniform.

To determine the polarization of the sphere and then its secondary field using the preceding equations, note that $\mathbf{P} = (\epsilon - \epsilon_0)\mathbf{E}$. In this case, $\mathbf{E}$ is the total electric field inside the sphere, equal to the sum of the external field, $E_0\mathbf{u}_x$, and the field in Eq. (11.8). It is left to the reader as an exercise to determine the polarization of the sphere, and hence the total field inside and outside the sphere.

In some cases, pollutant particles are conductive rather than insulating. For example, a dust particle can be approximated by a conductive sphere, as shown in Fig. 11.5b. When such an uncharged metal sphere is placed in a uniform electric field $\mathbf{E}$, charges are induced on its surface to cancel the electric field inside the sphere. Since the external field is uniform, the charges have to distribute themselves to produce an opposite uniform field inside the sphere. From the previous discussion of a dielectric sphere in a uniform electric field, we know that this charge distribution has to be of the form in Eq. (11.5). The electric field strength in Eq. (11.8) is E. Consequently, the induced surface charge on the sphere is of density

$$\sigma(\theta) = 3\epsilon_0 E \cos\theta. \tag{11.9}$$

Since $D_\mathrm{n} = \epsilon_0 E_\mathrm{n} = \sigma$, the largest value of the field on the surface of the ball is for $\theta = 0$ and $\theta = \pi$ (points A and B in the figure):

$$E_A = E_B = 3E. \tag{11.10}$$

At points A and B on the surface of a conducting dust particle, the electric field is *three times stronger* than the original uniform field.

Questions and problems: Q11.13 and Q11.14, P11.8

11.6 Electrostatic Imaging—Xerography

The modern photocopy machine was invented by the physicist and lawyer Chester Carlson in 1938. In his patent work he saw the need for an inexpensive and easy way to copy documents. It took him about 10 years to develop the copier, and in 1947 the Haloid Company—now Xerox Corporation—licensed the invention and began commercial production. The first copier was introduced to the market in 1950. Carlson called the process *xerography* from the Greek words *xeros* "dry," and *graphos*, "writing." Xerography uses a photosensitive material called selenium. Selenium is normally a dielectric, but when illuminated it becomes conductive. Some other materials, such as zinc oxide and anthracene, also have this property.

The copying process essentially has five steps, shown in Fig. 11.6. In the first step, a selenium-coated plate is charged evenly by sliding it under positively charged wires. An image of the document is then exposed onto the plate by a camera lens. In places where the plate is illuminated (corresponding to the white areas of the document), the selenium becomes a conductor and the charge flows away to metal contacts on the side of the plate. In other places, corresponding to the dark (printed) areas

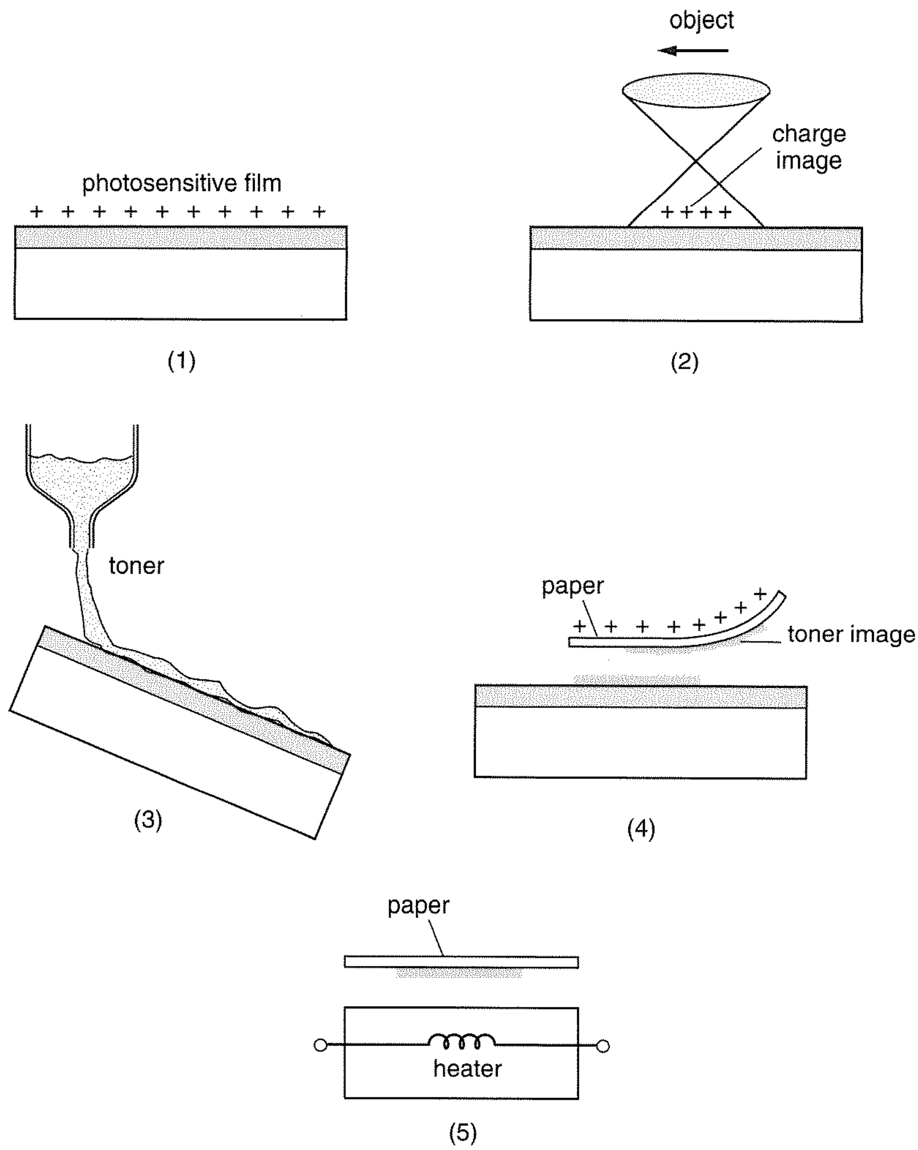


Figure 11.6 Five essential steps in the xerography process: (1) charging of the photoconductor plate; (2) charge image formation; (3) development with negatively charged toner; (4) transfer of toner image to paper; and (5) image fixing by heating

of the document, the charge remains. The plate now has an exact copy of the original in the form of a positive charge pattern. In the next step, some toner is charged negatively and the toner particles are attracted to the positively charged copy and stick to the plate. A sheet of blank paper is then placed over the plate and powder toner image. The paper is positively charged, so it attracts the toner particles onto itself. The paper is then placed on a fuser tray and the toner is baked to seal the image permanently. The entire process in the first copier took about 3 minutes. Modern copiers are faster and more sophisticated, but they operate according to the same principles.

What are some important electrostatic engineering design parameters in copiers? We said that the illuminated parts of the selenium plate become conductive and the charge flows away. But can the charge in the remaining charge image stay in place indefinitely, or do we need to time the next step in the copying process according to the temporal stability of the charge image? To answer this question, we need to know what the electric resistivity (ρ) and permittivity (ϵ) of dark selenium (selenium with no illumination) are.

Once these two properties are known, we can reason in the following way. At the surface of the film, Gauss' law gives

$$\oint_{S_0} \mathbf{E} \cdot d\mathbf{S}_0 = \frac{1}{\epsilon} \int_S \sigma \mathbf{n} \cdot d\mathbf{S}, \tag{11.11}$$

where σ is the surface charge density of the image on the film, S_0 is a thin, coinlike closed surface, with one base inside the film and the other over its very surface, and S is the intersecting surface of S_0 and the film. The continuity equation can also be written for the current density $\mathbf{J} = \mathbf{E}/\rho$ flowing through the film due to its finite resistivity ρ:

$$\oint_{S_0} \mathbf{E} \cdot d\mathbf{S}_0 = -\rho \int_S \frac{\partial \sigma}{\partial t} \mathbf{n} \cdot d\mathbf{S}. \tag{11.12}$$

Eqs. (11.11) and (11.12) must be valid for any shape of the intersecting surface S_0. This is possible only if the expressions on their right sides are equal. In that case, the two equations can be combined to give a differential equation for the surface charge density, σ:

$$\frac{d\sigma}{dt} + \frac{1}{\epsilon\rho}\sigma = 0. \tag{11.13}$$

Assuming that at $t = 0$ the surface charge density is σ_0, the solution of this equation is $\sigma = \sigma_0 e^{-t/(\epsilon\rho)}$. The quantity $\epsilon\rho$ describes how quickly the charge image on the film diffuses. This quantity is called the *charge transfer time constant*, or *dielectric relaxation constant* of dark selenium. It tells us how long it takes for $\sigma_0/e \simeq 0.368\sigma_0$ of the charge in the image to flow away ($e = 2.7182\ldots$ is the base of natural logarithms). For selenium, the resistivity varies between $10^{11}\ \Omega \cdot m$ and $10^{14}\ \Omega \cdot m$, and the relative permittivity is about 6.1. The corresponding charge transfer time constants are 5.4 s to 5400 s (1.5 hours). This means that in the former case, after the charge image is formed by illumination, the toner image needs to be formed and transferred to paper in less than a few seconds in order to create a clear image.

Note that the charge transfer time constant $\epsilon\rho$ is in some sense similar to the RC time constant of a resistor-capacitor circuit. We have seen before that the capacitance C of a dielectric-filled capacitor is proportional to ϵ, and the resistance R is proportional to the resistivity ρ.

Another practical problem in copiers is developing, i.e., making a toner image from the charge image. The toner consists of small dielectric particles, about $10\,\mu$m in diameter, which are charged to about $Q = 0.5 \cdot 10^{-14}\,$C. These particles are brought close to the photoconductive film, where a strong electric field, only a few times weaker than the air breakdown field, exists wherever there is a charge image. The electric force on a toner particle for a field strength three times smaller than the breakdown of air is then about

$$F = QE = 0.5 \cdot 10^{-14}\,\mathrm{C} \cdot 10^{6}\,\mathrm{V/m} = 0.5 \cdot 10^{-8}\,\mathrm{N}. \tag{11.14}$$

If this force were the only one present, the toner particles would move exactly along the electric field vector lines. However, as we discussed in Example 11.3, if we assume that each particle is a tiny sphere, it becomes a dipole, and there is a force in addition to the force on the toner particles we just calculated, due to the inhomogeneous electric field of the charge image. After expressing the polarization vector **P** in Example 11.3 in terms of the permittivity of the dielectric, it can be shown that this additional force due to the field nonuniformity is about three orders of magnitude smaller than the force calculated in Eq. (11.14), and can therefore be neglected. The toner particles also have mass, and therefore a gravitational force is acting on them. This force is given by $F_g = 4\pi r^3 \rho g/3 \simeq 0.5 \cdot 10^{-11}\,$N for a spherical particle $5\,\mu$m in radius and with mass density equal to that of water ($\rho = 10^3\,\mathrm{kg/m^3}$). So the gravitational force can also be neglected, and our original conclusion that the toner particles follow the selenium-film electric field lines is a good approximation.

How is the toner brought to the charge image? Of several possible processes the following one is probably the simplest. A fine dust of carbon particles (on the order of 10 μm) is electrified and blown over the charge image, covering the charged parts of the image. If the paper surface has appropriate properties, the image is transferred onto the paper efficiently. Note that the small size of the dust particles enables a very high resolution image.

One of the engineering problems in this process is the fact that the electric field distribution around the edges of the image is nonuniform, as shown in Fig. 11.7a.

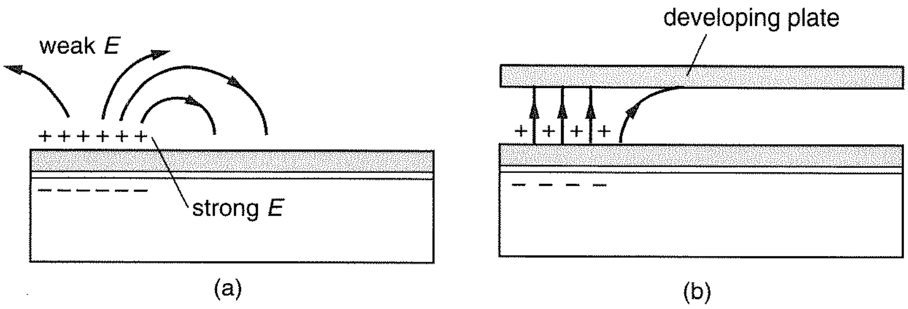


Figure 11.7 Field distribution around the edges of a charge image (a) without and (b) with the presence of a developing conductive plate

The field is the strongest around the edges, and toner particles are attracted only to the areas around the edges. In order to obtain toner coverage in the areas inside the image edges, a grounded conducting surface (called a developing electrode) is brought very near the charged film, but not touching it. The electric field strength between the photoconductive surface and the electrode is proportional to the surface charge density. The toner is introduced between the two surfaces, gets attracted to the surface charge on the film, and neutralizes this charge. The quantity of toner needed to effectively neutralize the charge at a certain spot on the film is proportional to the original surface charge density, which in turn is inversely proportional to the intensity of illumination during optical exposure. Thus the density of the developed image reproduces the continuous tones of the original optical image.

Questions and problems: Q11.15 to Q11.18, P11.9

11.7 Industrial Electrostatic Separation

An important application of electrostatic fields is separation, used in industry for purification of food, purification of ores, sorting of reusable wastes, and sizing (sorting according to size and weight). Some specific examples of electrostatic separation in the ore and mineral industry are as follows: quartz from phosphates; diamonds from silica; gold and titanium from beach sand; limestone, molybdenite, and iron ore (hematite) from silicates; and zircon from beach sand. In the food industry, peanut beans, cocoa beans, walnuts, and nut meats are separated electrostatically from shells. For grains, electrostatics is used to separate rodent excrement from barley, soybean, and rice. In the electronics industry, copper wire is electrostatically separated from its insulation for recycling purposes. It is estimated that well over 10 million of tons of products a year are processed using electrostatic separation.

The first patent in this field was issued in 1880 for a ground cereal purification process. Thomas Edison had a patent in 1892 for electrostatically concentrating gold ore, and the first commercial process used in a plant in Wisconsin in 1908 was set up to electrostatically purify zinc and lead ores. Shortly after that, flotation processes were invented for separation. However, as these processes are not suitable for arid areas, and in some cases they require chemical reagents that present water pollution problems, electrostatic separation has regained popularity. In 1965, the world's largest electrical concentration plant was installed in the Wabush Mines in Canada. This plant is used to reduce the silica content of 6 million tons of iron ore per year.

Figure 11.8 shows two basic systems of electrostatic separation (there are several others not described here). In each case, the basic components of the system are a charging mechanism, an external electric field, a device to regulate the trajectory of nonelectric particles, and a feeding and collection system. In case (a) the particles are charged by contact electrification (the triboelectric effect), and in case (b) by ion bombardment (corona discharge). Both of these effects were mentioned earlier in this chapter. In terms of regulating the trajectory of the particles, in case (a) the gravitational force is used in addition to the electric force, and in case (b) the centrifugal force acting on the particles is adjusted by a rotating cylinder.

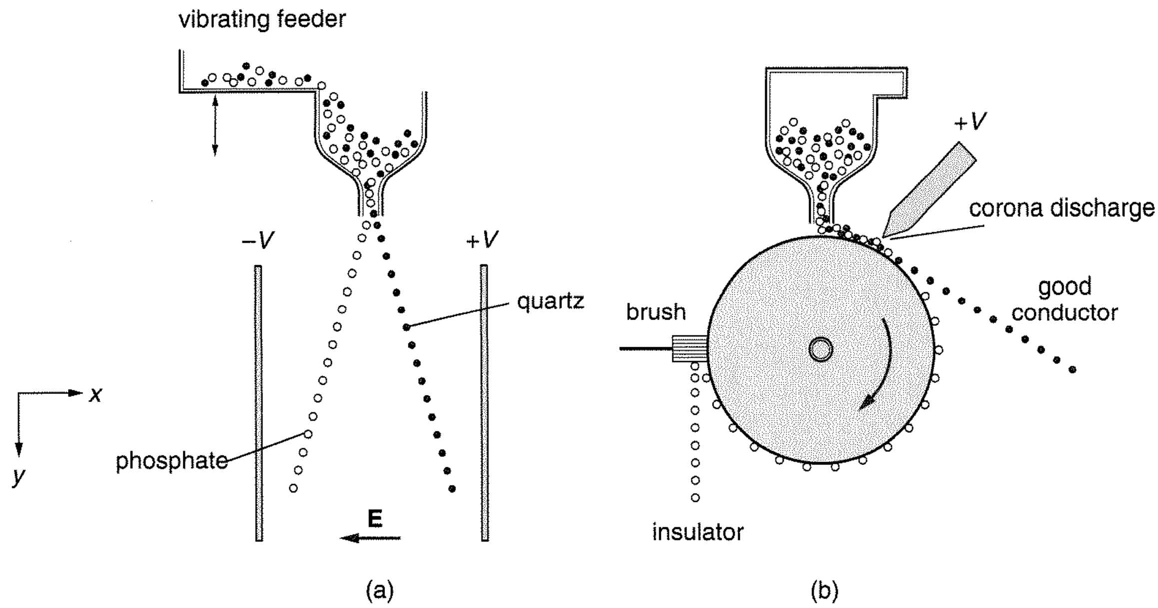


Figure 11.8 (a) Forming chute and (b) variable-speed rotating cylinder systems for electrostatic separation

In each case, physical separation of two types of particles is performed by adjusting the forces acting on the particles, as well as the time the forces act, so that different types of particles will have different trajectories. In Example 11.4, an approximate analysis of the separation process is examined for case (a) in Fig. 11.8. In case (b), used for separation of hematite from quartz or copper wire from its insulation, the particles are charged through ion bombardment. One type of particle is conductive, and the other dielectric. The particles are dropped on top of a grounded rotating cylinder. The conductor particles share their charge with the grounded rotor and are thrown from the rotor in a trajectory determined by centrifugal forces, gravity, and air resistance. The dielectric particles are held to the surface of the rotor. They either lose their charge slowly and then fall off the rotor, or are scraped off by a brush at the other end. The electric and centrifugal forces need to be roughly the same in order for the particles to stick to the rotor long enough for good separation.

Example 11.4—Separation of quartz from phosphate rock using gravitational and electric forces. An approximate analysis of the forming chute separation of quartz from phosphate rock can be done by ignoring the electric forces acting on particles due to neighboring charges. Obviously, this analysis is very rough and provides us only with an order-of-magnitude illustration.

The quartz and phosphate rock particles are washed, dried, heated to about 100°C, and finally vibrated so that the minerals make and break contact and are charged by friction with charges of opposite sign. For a uniform electric field acting along the x direction in Fig. 11.8a, Newton's first law gives us an equality between the electric force on a single particle and its mass multiplied by its acceleration,

$$\mathbf{F}_e = Q\mathbf{E} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{u}_x, \tag{11.15}$$

which can be integrated. Assuming zero initial velocity and displacement, the expression for the deflection of a charged particle due to the electric force is

$$x = \frac{1}{2}\frac{QE}{m}t^2. \tag{11.16}$$

For a 0.25-mm-diameter quartz particle, the ratio $Q/m \simeq 9 \times 10^{-6}\,\mathrm{C/kg}$, and a typical value for the electric field strength is $E = 4 \times 10^5\,\mathrm{V/m}$. Therefore, Eq. (11.16) gives $x = 1.8t^2\,\mathrm{m}$. The time required for a particle to fall a certain distance is obtained from the expression for the gravitational force

$$\mathbf{F}_g = m\mathbf{g} = -m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\mathbf{u}_y. \tag{11.17}$$

The vertical displacement of the particle due to the gravitational force is found by integration to be

$$y = -\frac{1}{2}gt^2. \tag{11.18}$$

For a falling distance of, say, 0.5 m, we can now calculate the time that the electric force has to deflect the particle in the horizontal direction as $t^2 \approx 0.1\,\mathrm{s}^2$, and the horizontal displacement as $d \approx 18\,\mathrm{cm}$. In the case of two particles that are oppositely charged, after falling 0.5 m they are separated by 36 cm, which is enough for good separation.

The engineering limitations of this process relate to the fact that the process can be used only for a certain range of particle sizes: if the particles become too large, roughly larger than 1 mm in diameter, the gravitational force becomes too large compared to the electric force. If the particles are too small, below roughly 50 μm, interparticle attractive electric forces cause the small quartz and phosphate particles to form clusters.

Questions and problems: Q11.19, P11.10

11.8 Four-Point Probe for Resistivity Measurements

Four-point probes are used in every semiconductor lab. They can be used for determining the resistivity (conductivity) of a material, or the charge concentration if the other material properties are known. First consider just two probes (two points) that are touching the surface of a material of unknown conductivity, as in Fig. 11.9. The idea behind using probes is the same as measuring the resistance of a resistor, except that some of the quantities are distributed (i.e., described at every point, not only at the two resistor terminals).

When a current source is connected to the two probes in Fig. 11.9, the current density in the material can be found using the image theorem described in Chapter 10 when we discussed grounding electrodes. The current density vector is the superposition of the current density from the point from which the current is "injected" into the material and the current density from the point out of which the current is returning to the generator (which is just the negative of the first current with respect

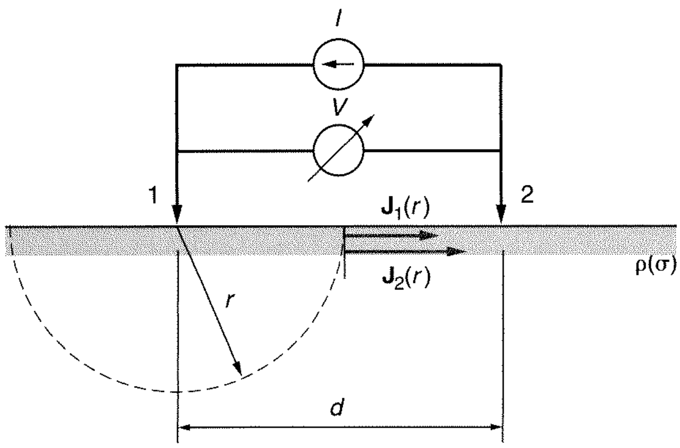


Figure 11.9 Two-point probe for resistivity measurements

to the surface of the material). As shown in the figure, this current density vector can easily be found only along the line connecting the two probes right underneath the material-air interface. Once we know the current density $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, we can use Ohm's law $\mathbf{E} = \rho\mathbf{J}$ to find the expression for the voltage drop between the two probes in terms of the source current I and the unknown resistivity ρ:

$$V = \int_{\text{probe 1}}^{\text{probe 2}} \mathbf{E} \cdot \mathbf{dl} = \int_{\text{probe 1}}^{\text{probe 2}} \rho\mathbf{J} \cdot \mathbf{dl}.$$

It is left as an exercise for the reader to write the expression for the current density and to attempt solving this integral. When solving the integral, note that the limits of integration should be the radii of the probe contacts. But because it is almost impossible to accurately know the radii of the probe contacts, we cannot accurately measure the properties of a solid material by this method.

To avoid this difficulty in measuring resistivity we use a four-point probe instead (Fig. 11.10). This time, the current is injected into the material from the outer

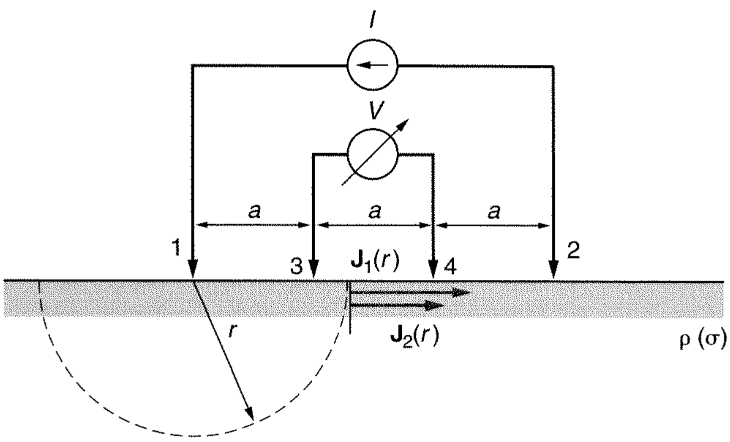


Figure 11.10 Four-point probe for resistivity measurements

two probes, but the voltage is sampled between the two inner probes. Because the two voltage probes can be pointlike, and because they are far from the current probes of radii known only approximately, this results in a much more accurate measurement of the resistivity of the material. It is left as an exercise (P11.11) to find the expression for the resistivity in this case.

Questions and problems: Q11.20, P11.11 to P11.17

11.9 Brief Overview of Other Applications

Many other applications also involve electrostatic fields, but in this section we will briefly describe only some of them.

One of the most common electrostatic applications is coating. In the car industry several coats of paint are applied to vehicles using electrostatic coating. Every washer, dryer, and refrigerator is electrostatically coated. Even such objects as golf balls and the paper you now hold in front of you have been coated for some purpose (the paper is coated in order to have a good printing surface). The basic principle of electrostatic coating is simple: the object to be coated is charged with one polarity, and the coating material with another. The coating material is sprayed into fine particles around the object and the particles are attracted to the object by electric forces and deposited upon impact. Of course, coating machines in industry are quite sophisticated because it is important to achieve uniform coats and also to use the coating material efficiently.

Imaging technology also uses an electrostatic-based device, the charge-coupled device (CCD) camera. CCD design can even be used, for example, in making the extremely sensitive cameras used in astronomy. A CCD camera consists of a large array of MOS capacitors (Fig. 11.11). We discussed MOS capacitors in Example 8.9. Incident light creates both positive and negative charge carriers inside the p semiconductor (silicon). The metal electrodes are biased positively, so that the electrons are attracted to the semiconductor-oxide interface. The number of electrons under each metal electrode is an accurate measure of the number of incident light photons. How does the camera reconstruct the image as something we can see? It needs, in effect,

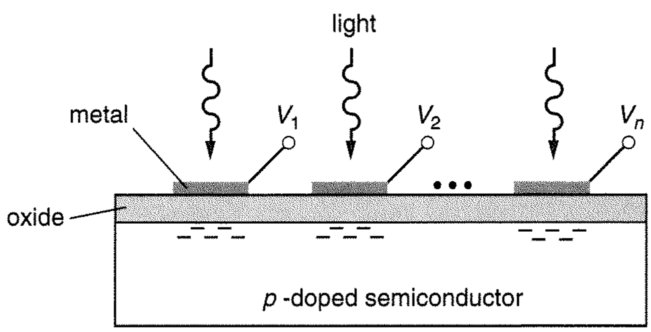


Figure 11.11 A CCD camera consists of an array of MOS capacitors

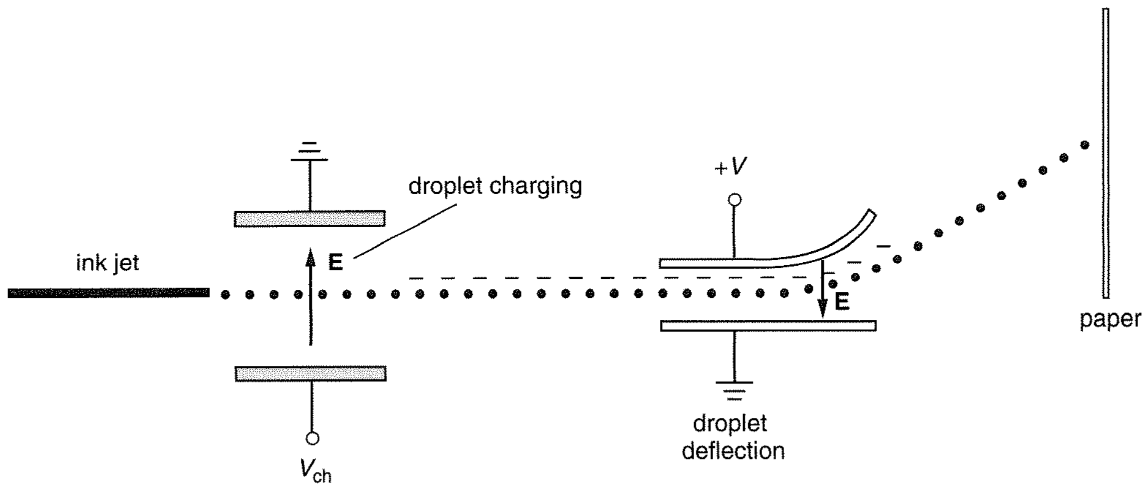


Figure 11.12 Basic components of an ink-jet printer

to measure the number of electrons in each capacitor. This is done by moving each group of electrons along a line in a serial fashion after this electron "pulsed" current is amplified in an amplifier at the end of the line. In the readout direction, each of the electrodes is biased to a progressively higher voltage, so that the electrons from the neighboring electrode are attracted to it. This transfer process can be very efficient, and in good devices only 1 out of 100,000 electrons is lost in each charge transfer step.

Another commercial application of electrostatics is nonimpact printing, for example in ink-jet printers. A basic diagram of an ink-jet printer appears in Fig. 11.12. The ink jet, on the order of 100 μm in diameter, is produced by applying pressure to an ink supply. The droplet stream, initially uncharged, passes through a cylindrical charging electrode biased at around 100 V along its axis, so that the jet and the electrode form a coaxial capacitor. The ink supply is connected to the other generator terminal, so the jet is charged by electrostatic induction. The jet is next modulated mechanically so that it turns into charged droplets, between 25 and 125 μm in diameter for 0.254-cm-high characters. The charged droplets are then deflected into the desired dot-matrix pattern by the electrostatic field between two metal plates with a voltage of 1 to 5 kV between them. The deflection in one plane is controlled by the voltage between the electrodes. The other dimension is usually controlled by the mechanical motion of the stream with respect to the paper.

Other applications of electrostatics include electrostatic motors, electrostatic generators, and electrophoresis (separation of charged colloidal particles by the electric field) used in biology (for example, to separate live yeast cells from dead ones). Electrophoresis is used in biochemistry to separate large charged molecules by placing them in an electric field. For example, in genetic research, DNA molecules of different topological forms (e.g., supercoiled and linear DNA) are separated effectively by electrophoresis, even though they are chemically identical. Hewlett-Packard has also developed an electrophoresis instrument intended for use in the drug industry. The interested reader is referred to the *Hewlett-Packard Journal*, June 1995.

Questions and problems: Q11.21

QUESTIONS

Q11.1. Describe the formation of a lightning stroke.

Q11.2. How large are currents in a lightning stroke?

Q11.3. According to which physical law does thunder occur?

Q11.4. A spherical cloud of positive charges is allowed to disperse under the influence of its own repulsive forces. Will charges follow the lines of the electric field strength vector?

Q11.5. A cloud of identical, charged particles is situated in a vacuum in the gravitational field of the earth. Is there an impressed electric field in addition to the electric field of the charges themselves? Explain.

Q11.6. Is Eq. (11.1) valid if the charge Q from time to time collides with another particle?

Q11.7. Explain why the left-hand side in Eq. (11.2) is as it is, and not $m(v_2 - v_1)^2/2$.

Q11.8. Discuss the validity of Eqs. (11.3) and (11.4) if the charge Q is negative.

Q11.9. An electron is emitted parallel to a large conducting flat plate that is uncharged. Describe qualitatively the motion of the electron.

Q11.10. If the voltage between the electrodes of an air-filled parallel-plate capacitor is increased so that corona starts on the plates, what will eventually happen without increasing the voltage further?

Q11.11. Electric charge is continually brought on the inner surface of an isolated hollow metal sphere situated in air. Explain what will happen outside the sphere.

Q11.12. Give a few examples of desirable and undesirable (1) corona and (2) spark discharges.

Q11.13. Describe how an electrostatic pollution-control filter works.

Q11.14. Sketch the field that results when an uncharged spherical conductive particle is brought into an originally uniform electric field.

Q11.15. Describe the process of making a xerographic copy.

Q11.16. Explain the physical meaning of the charge transfer time constant, or dielectric relaxation constant.

Q11.17. Derive Eq. (11.13) and solve it.

Q11.18. Describe the difference in the xerographic image with and without the developer plate.

Q11.19. Derive the equation of particle trajectory in a forming chute process for electrostatic separation.

Q11.20. Why is a four-point probe measurement more precise than a two-point probe measurement?

Q11.21. Describe how a CCD camera works.

PROBLEMS

P11.1. Calculate the voltage between the two feet of a person (0.5 m apart), standing $r = 20$ m away from a 10-kA lightning stroke, if the moderately wet homogeneous soil conductivity is 10^{-3} S/m. Do the calculation for the two cases when the person is standing in positions A and B as shown in Fig. P11.1.

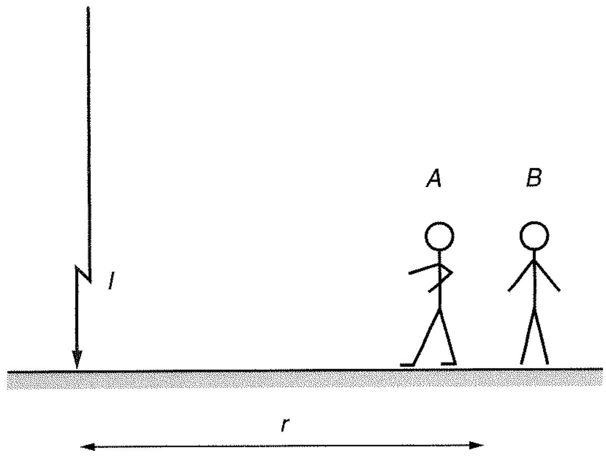


Figure P11.1 A person near a lightning stroke, standing in two positions

P11.2. Calculate the electric field strength above a tree that is $d = 1$ km away from the projection of the center of a cloud onto the earth (Fig. P11.2). Assume that because the tree is like a sharp point, the field above the tree is about 100 times that on the flat ground. As earlier, you can assume the cloud is an electric dipole above a perfectly conducting earth, with dimensions as shown in the figure, and with $Q = 4$ C of charge. (Note that the height of any tree is much smaller than the indicated height of the cloud.)

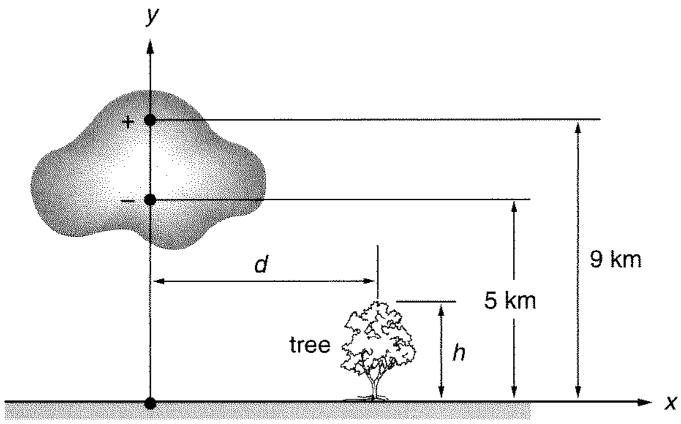


Figure P11.2 Electric field above a tree in a storm

P11.3. Derive the second equation in Example 11.1 from Eq. (11.1).

P11.4. Assuming that the initial velocity in Example 11.1 is nonzero and x-directed, solve for the velocity and the position of the charge Q as a function of time. Plot your results.

P11.5. Assuming that the initial velocity in Example 11.1 is nonzero and y-directed, solve for the velocity and the position of the charge Q as a function of time. Plot your results.

P11.6. A thin electron beam is formed with some convenient electrode system. The electrons in the beam are accelerated by a voltage V_0. The beam passes between two parallel plates, which electrostatically deflect the beam, and later falls on the screen S (Fig. P11.6). Determine and plot the deflection y_0 of the beam as a function of the

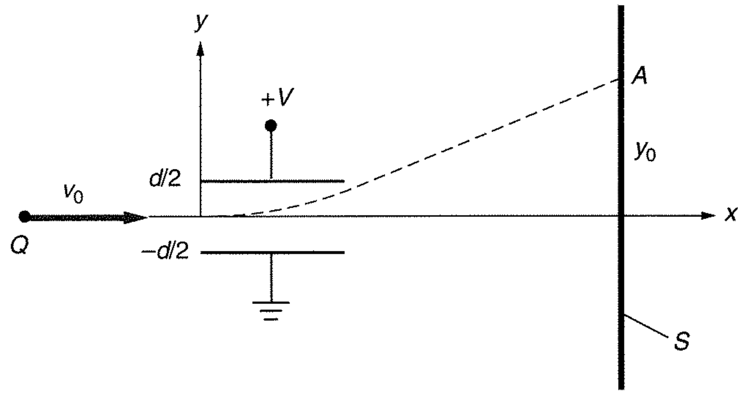


Figure P11.6 Electrostatic deflection of an electron beam

voltage V between the plates. (This method is used for electrostatic deflection of the electron beam in some cathode-ray tubes.)

P11.7. A beam of charged particles that have positive charge Q, mass m, and different velocities enters between two closely spaced curved metal plates. The distance d between the plates is much smaller than the radius R of their curvature (Fig. P11.7). Determine the velocity v_0 of the particles that are deflected by the electric field between the plates so that they leave the plates without hitting any of them. Note that this is a kind of filter for charged particles, resulting in a beam of particles of the same velocity.

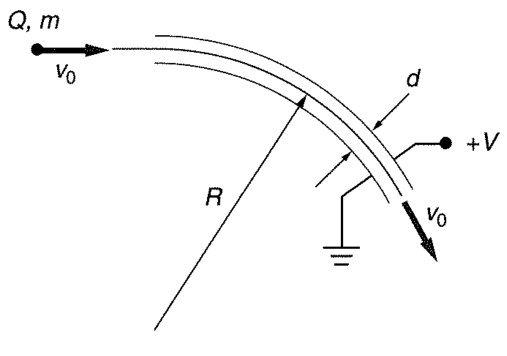


Figure P11.7 An electrostatic velocity-filter of charged particles

P11.8. A metal sphere is placed in a uniform electric field $\mathbf{E}_0$. What is the maximum value of this field that does not produce air breakdown when the metal ball is brought into it?

P11.9. Calculate the dielectric relaxation constants for selenium, n-doped silicon with carrier concentration $n = 10^{16}\,\text{cm}^{-3}$, and n-doped galium arsenide with concentration $n = 10^{16}\,\text{cm}^{-3}$. For semiconductors, such as silicon and galium arsenide, the conductivity is given by $\sigma = Q\mu n$, where Q is the electron charge. μ is a property of electrons inside a material, and it is called the mobility (defined as $v = \mu\mathbf{E}$, where v is the velocity of charges that moved by a field $\mathbf{E}$). For silicon, $\mu = 0.135\,\text{m}^2/\text{Vs}$ and $\eta_r = 12$, and for galium arsenide, $\mu = -.86\text{m}^2/\text{Vs}$ and $\epsilon_r = 11$. for selenium, $\rho = 10^{12}\ \Omega$-m and $\epsilon_r = 6.1$.

P11.10. How far do 1-mm-diameter quartz particles charged with $Q = 1\,\text{pC}$ need to fall in a field $E = 2 \cdot 10^5\,\text{V/m}$ in order to be separated by 0.5 m in a forming chute separation process? The mass density of quartz is $\rho_m = 2.2\,\text{g/cm}^3$.

P11.11. Find the expression for determining resistivity from a four-point probe measurement, as in Fig. P11.11.

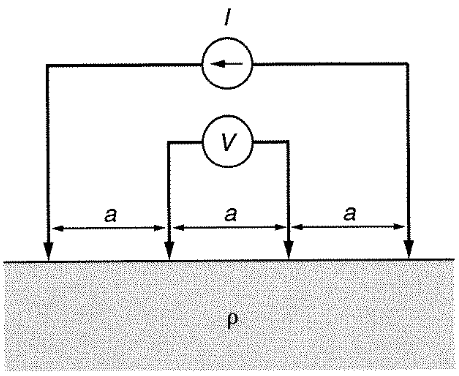


Figure P11.11 A four-point probe measurement

P11.12. Using the information given in P11.19, for a measured resistivity of 10 Ω-cm, determine the corresponding charge concentration of (1) silicon and (2) gallium arsenide.

P11.13. A Wenner array used in geology is shown in Fig. P11.13. This instrument is used for determining approximately the depth of a water layer under ground. First the electrodes are placed close together, and the resistivity of soil is determined. Then the electrodes are moved farther and farther apart, until the resistivity measurement changes due to the effect of the water layer. Assuming that the top layer of soil has a very different conductivity than the water layer, what is the approximate spacing between the probes, r, that detects a water layer at depth h under the surface? The exact analysis is complicated, so think of an approximate qualitiative solution.

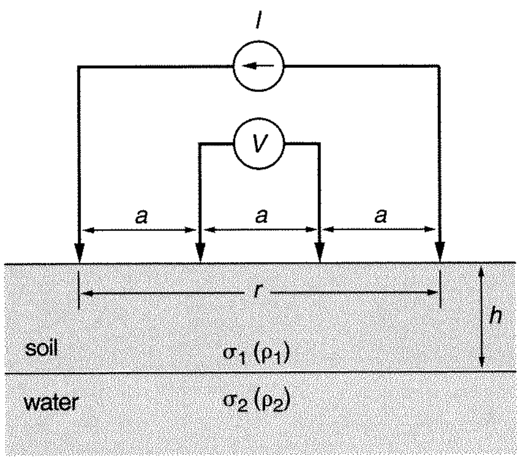


Figure P11.13 A Wenner array used in geology

P11.14. A thin film of resistive material is deposited on a perfect insulator. Using a four-point probe measurement, determine the expression for surface resistivity ρ_s of the thin film. Assume the film is very thin.

P11.15. Consider an approximate circuit equivalent of a thin resistive film as in Fig. P11.15. The mesh is infinite, and all resistors are equal and have a value of $R = 1\,\Omega$. Using a two-point probe analogy, determine the resistance between any two adjacent nodes A and B in the mesh.

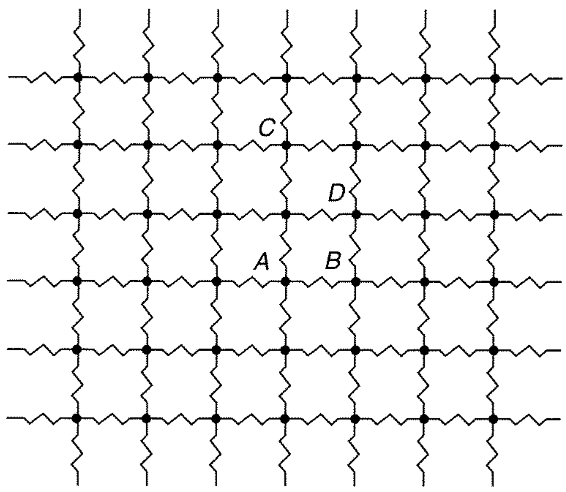


Figure P11.15 An approximate equivalent circuit of a thin resistive film

P11.16. Find the resistance between nodes (1) A and C and (2) A and D in Fig. P11.15.

P11.17. Construct an approximate equivalent circuit for a block of homogeneous resistive material. Determine the resistance between two adjacent nodes of the equivalent circuit.

PROBLEM P11.16 IS NOT SOLVABLE USING ELEMENTARY SYMMETRY TECHNIQUES.
THE GENERAL SOLUTION IS IN ZEMANIAN (1991)

12

Magnetic Field in a Vacuum

12.1 Introduction

The force between two static charges is given by Coulomb's law. Although small, this force is measurable—Coulomb established his law experimentally. If two charges are moving, it turns out that there is an additional force between them due to their motion. This force is known as the *magnetic force.*

The magnetic force between individual moving charges is extremely small when compared with the Coulomb force. Actually, it is so small that it cannot be detected experimentally if we consider just a pair of moving charges. Therefore a direct experimental derivation of the magnetic force law is not possible.

However, we can determine magnetic forces using another phenomenon—the electric current in metallic conductors—where we have an organized motion of an enormous number of electrons (practically one free electron per atom) within almost neutral substances. We can thus perform experiments in which the magnetic force can be measured with practically uncharged conductors, i.e., independently of electric forces. These experiments indicate that because of this vast number of interacting moving charges, the magnetic force between two current-carrying conductors can be much larger than the maximum obtainable electric force between them. For example, strong electromagnets can carry weights of several tons, and we know that the electric force cannot have even a fraction of that strength. Consequently, magnetic forces are used in many electrical engineering devices.

In this chapter we analyze magnetic forces in a vacuum. In many respects this chapter is of fundamental importance, just as Coulomb's law was in electrostatics. In the next chapter we will see that, in a manner somewhat analogous to that in the case of a polarized dielectric, substance in the magnetic field can be reduced to a system of electric currents situated in a vacuum.

12.2 Magnetic Force Between Two Current Elements

As explained, it is possible to measure the magnetic force between current-carrying conductors. In order to have a dc current in a conductor, we know that the conductor must be in the form of a closed loop, or *current loop.*

There are infinitely many different shapes and sizes of pairs of current loops for which we can measure magnetic forces. It is reasonable to assume that superposition applies to magnetic forces, as it did to electric forces. Therefore, in order to be able to determine the force between *any* two loops, we need to know the magnetic force between two arbitrarily oriented short segments of the loops. Such segments are known as *current elements.*

Consider two current loops, C_1 and C_2, with currents I_1 and I_2 (Fig. 12.1). We divide both loops into small vector line segments $\mathbf{dl}$, and define a current element as the product $I\,\mathbf{dl}$ (with appropriate subscripts). Note that $\mathbf{dl}$ is adopted to be *in the reference direction of the current along the loop.* From a large number of experimentally determined forces between various pairs of current loops, it is found that the magnetic force is always obtained correctly if it is assumed that the force between pairs of current elements is of the form

$$\mathbf{dF}_{12} = I_2\,\mathbf{dl}_2 \times \left(\frac{\mu_0}{4\pi}\frac{I_1\,\mathbf{dl}_1 \times \mathbf{u}_r}{r^2}\right). \tag{12.1}$$

Note that this law, like Coulomb's law, was determined experimentally. The constant of proportionality is written in the form $\mu_0/(4\pi)$ for convenience, just as in Coulomb's law it was written as $1/(4\pi\epsilon_0)$. The constant μ_0 is known as the *permeability of a vacuum* (or of *free space*). It is the second of the two basic electromagnetic constants describing what we call free space or a vacuum—the first was the permittivity, ϵ_0. In the SI system of units, it is defined to be *exactly*

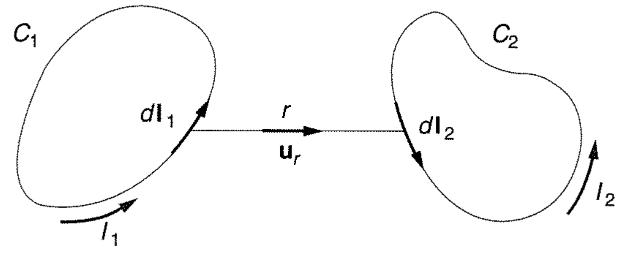


Figure 12.1 Two current loops divided into current elements

$$\mu_0 = 4\pi \cdot 10^{-7} \qquad \text{henry/m (H/m)}. \tag{12.2}$$

(Permeability of a vacuum)

The unit *henry per meter* (abbreviated H/m) comes from the unit for inductance, to be described in Chapter 15. Obviously, from Eq. (12.1), H/m is equal to N/A^2.

The force law (12.1) for two current elements is significantly more complicated than Coulomb's law. Note first that two vector cross products (see Appendix 1) are implied: we first need to determine the vector resulting from the cross product in the parentheses in Eq. (12.1) and then multiply it by the third vector in the equation. What might be confusing is that the resulting vector is oriented along the line connecting the two elements only in special cases. What might be even more perplexing is the possibility that in specific cases the magnetic force exerted by one element on another is zero, whereas the converse is not true. In all of the cases, however, the above formula has proven to be correct whenever the *total magnetic force* is calculated.

Example 12.1—Forces between pairs of current elements. Consider a few cases of the magnetic force law in Eq. (12.1). First let the current elements be parallel to each other and normal to the line joining them (Fig. 12.2a). From Eq. (12.1) we find that the force is repulsive if the currents in the elements are in *opposite* directions, and attractive if they are in the *same* direction. Note that this is formally opposite to the case of two charges, where like charges repel and unlike charges attract each other.

Consider now Fig. 12.2b. The force $\mathbf{F}_{12}$ in this case is zero, but the force $\mathbf{F}_{21}$ is not.

Finally, imagine we connect two parallel wires to batteries so that there is a current flowing through each wire (Fig. 12.3). We expect the two wires to exert magnetic forces on each other, according to Eq. (12.1). From the first example it is obvious that the wires attract when the currents are in the same direction and repel when they are in opposite directions.

We shall derive the expression for the force per unit length between two such wires in a later section. Just to get a feeling for the magnitude of magnetic forces, we mention one of the definitions of the *ampere*: the currents in two parallel, infinitely long wires a distance 1 m apart and situated in a vacuum are 1 A if the force on each of the wires is $2 \cdot 10^{-7}$ N per 1 m of length.

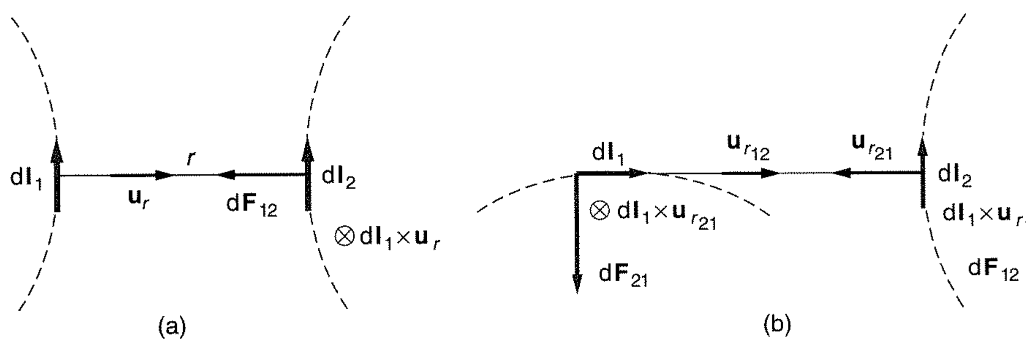


Figure 12.2 Examples of pairs of current elements

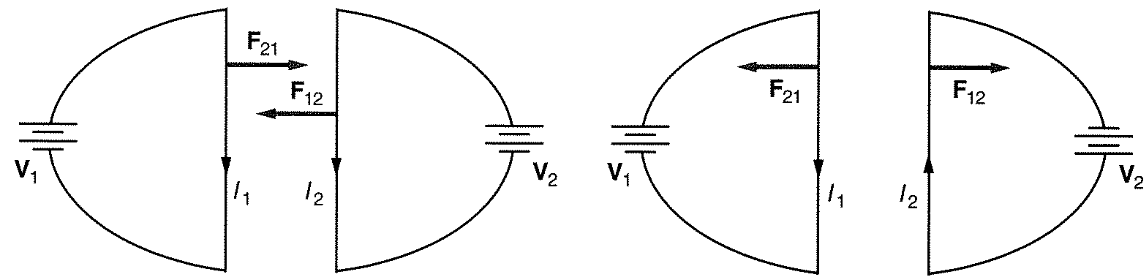


Figure 12.3 Two parallel wires can either attract or repel each other depending on the direction of the currents in them and in accordance with Eq. (12.1).

Questions and problems: Q12.1 to Q12.3

12.3 Magnetic Flux Density and the Biot-Savart Law

We have defined the electric field as a domain of space in which there is a force on a charged particle at rest. Similarly, if in a domain of space there is a force acting on a current element or on a moving charge, we say that a *magnetic field* exists in the domain.

To characterize the magnetic field (as we did the electric field), we start from Eq. (12.1). Except for the second current element, $I_2\,\mathrm{d}\mathbf{l}_2$, all the other quantities depend only on the first current element and the position and orientation of the second current element relative to the first. In analogy to the definition of the electric field intensity from Coulomb's law, we characterize the magnetic field by the *magnetic flux density vector*, $\mathrm{d}\mathbf{B}$, given by the term in parentheses in Eq. (12.1). Omitting the subscript "1," the expression for vector $\mathrm{d}\mathbf{B}$ due to a current element has the form

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}\mathbf{l} \times \mathbf{u}_r}{r^2} \quad \text{tesla (T)}. \tag{12.3}$$

(Magnetic flux density of a current element—the Biot-Savart law)

The unit vector $\mathbf{u}_r$ is adopted in the same way as in the expression for the electric field intensity of a point charge: it is directed *from the source point* (i.e., the current element) *toward the field point* (i.e., the point at which we determine $\mathrm{d}\mathbf{B}$). This is quite similar to the Coulomb force, where we used Q_2 as the "test charge" and defined the rest to be a quantity characterizing the electric field of charge Q_1.

It is important to note that the magnetic flux density vector is perpendicular to the plane of the vectors $\mathbf{u}_r$ and $\mathrm{d}\mathbf{l}$. Its orientation is determined by the right-hand rule when the vector $\mathrm{d}\mathbf{l}$ is rotated by the shortest route toward the vector $\mathbf{u}_r$ (see Appendix 1, section A1.2).

The magnetic flux density produced by the entire current loop C is found by summing the elemental flux density vectors of all the current elements of the loop:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \oint_C \frac{I\,\mathrm{d}\mathbf{l} \times \mathbf{u}_r}{r^2} \qquad \text{(T)}. \tag{12.4}$$

(Magnetic flux density of a current loop—the Biot-Savart law)

This equation and Eq. (12.3) together are referred to as the *Biot-Savart law.*

The SI unit for the magnetic flux density is the *tesla* (abbreviated T). To get a feeling for the magnitude of a tesla, let us look at a few examples. The magnetic flux density of the earth's dc magnetic field is on the order of 10^{-4} T. Around current-carrying conductors in a vacuum, the intensity of **B** is from about 10^{-6} T to about 10^{-2} T. In air gaps of electrical machines, the magnetic flux density can be on the order of 1 T. Superconducting magnets can produce flux densities of several dozen T. The unit T is named after the inventor Nikola Tesla, a Serbian immigrant in the United States. Tesla is responsible for ac power distribution, the first ac power plant on the Niagara Falls, the first radio remote control, induction motors, and other commonly used technologies. He was a somewhat peculiar person and his interesting life is described in many books, for example *Tesla, man out of time* by Margaret Cheney, Dorset Press (Prentice Hall), 1989.

From the definition of the magnetic flux density, it follows that the magnetic force on a current element $I\,\mathrm{d}\mathbf{l}$ in a magnetic field of flux density **B** is given by

$$\mathrm{d}\mathbf{F} = I\,\mathrm{d}\mathbf{l} \times \mathbf{B} \qquad \text{(N)}. \tag{12.5}$$

(Magnetic force on a current element)

Following the general definition of lines of a vector function, we define the lines of vector **B** as (generally curved) imaginary lines such that vector **B** is tangential to them at all points. For example, from Eq. (12.3) it is evident that the lines of vector **B** of a single current element are circles centered along the line of the current element and in the planes perpendicular to the element.

Example 12.2—Magnetic flux density at the center of a circular current loop. As an example of the application of Biot-Savart's law, let us find the magnetic flux density vector at the center of a circular loop with a dc current I (Fig. 12.4). The element dl is as drawn in the figure. The unit vector, $\mathbf{u}_r$, is directed from the current element to the center point. The vector d**B** is pointing into the page. The intensity of the total flux density vector is given by

$$B = \oint_C \mathrm{d}B = \frac{\mu_0}{4\pi} \oint_C \frac{I\,\mathrm{d}l\,\sin(\pi/2)}{a^2} = \frac{\mu_0 I}{4\pi}\frac{2\pi a}{a^2} = \frac{\mu_0 I}{2a}.$$

Although the most frequent cases in practice are currents in metallic wires, which allow the use of Eq. (12.4) for determining the magnetic flux density, in some

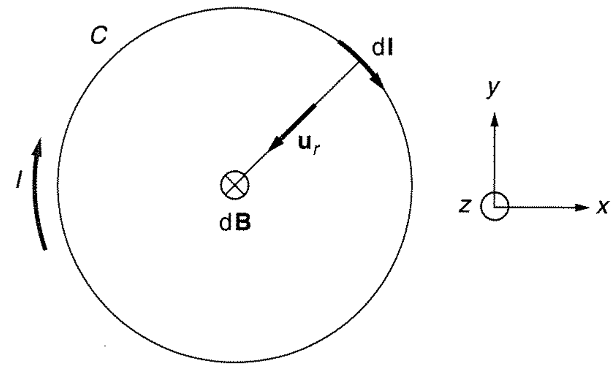


Figure 12.4 The magnetic flux density vector of a circular current loop

cases volume currents are also encountered. We know that volume currents are described by the current density vector, $\mathbf{J}$. Let ΔS be the cross-section area of the wire. Then $I\,\mathrm{d}\mathbf{l} = J\Delta S\,\mathrm{d}\mathbf{l} = \mathbf{J}\,\mathrm{d}v$ (note that $\mathbf{J}$ and $\mathrm{d}\mathbf{l}$ have the same direction), so that the Biot-Savart law for volume currents has the form

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_v \frac{\mathbf{J} \times \mathbf{u}_r\,\mathrm{d}v}{r^2} \qquad \text{(T)}. \tag{12.6}$$

(Biot-Savart law for volume currents)

At high frequencies (above about 1 MHz), currents in metallic conductors are distributed in very thin layers on conductor surfaces. These layers are so thin that they can be regarded as geometrical surfaces. It is convenient in such cases to introduce the concept of *surface current*, assuming that the current exists over the very surfaces of conductors. Of course, such a current would have infinite volume density. Therefore for its description we introduce the *surface current density*, $\mathbf{J}_s$. It is defined as current intensity, ΔI, flowing "through" a line segment of length Δl normal to the current flow, divided by the length Δl of the segment. Evidently, the unit of surface current density is ampere per meter (A/m).

To obtain the Biot-Savart law for surface currents, let $\mathrm{d}v = \mathrm{d}S\,\mathrm{d}h$, where $\mathrm{d}h$ is the thickness of the surface current. Then $\mathbf{J}\,\mathrm{d}v$ in the last equation becomes $\mathbf{J}\,\mathrm{d}v = \mathbf{J}\,\mathrm{d}S\,\mathrm{d}h = (\mathbf{J}\,\mathrm{d}h)\,\mathrm{d}S = \mathbf{J}_s\,\mathrm{d}S$, and we obtain

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{J}_s \times \mathbf{u}_r\,\mathrm{d}S}{r^2} \qquad \text{(T)}. \tag{12.7}$$

(Biot-Savart law for surface currents)

Example 12.3—The magnetic force and the moment of magnetic forces on a circular loop in a uniform magnetic field. As an application of Eq. (12.5), let us determine first the force on a circular loop of radius a situated in a uniform magnetic field of flux density $\mathbf{B}$. According to Eq. (12.5), and recalling that $\mathbf{B}$ is a constant vector,

$$\mathbf{F} = \oint_C I\,\mathrm{d}\mathbf{l} \times \mathbf{B} = I\left(\oint_C \mathrm{d}\mathbf{l}\right) \times \mathbf{B} = 0,$$

since the line integral of dl is zero. So the magnetic force on a current loop in a uniform magnetic field is zero.

Consider now the moment of magnetic forces. Let the vector **B** be normal to the loop. It is a simple matter to conclude that the magnetic forces would then tend either to stretch the loop or to compress it (depending on the direction of **B**), but the moment on the loop is zero. Thus *only that component of the vector* **B** *that is parallel to the plane of the loop may produce the moment.*

Figure 12.5 shows the loop with the parallel vector **B**. Consider the two symmetrical elements dl and dl′. It is seen that the force $\mathbf{dF} = I\,\mathbf{dl} \times \mathbf{B}$ on element dl is directed into the page, and the force on the other element is directed out of the page. These forces tend to lift the RIGHT ~~left~~ half of the loop up and to press the ~~right~~ LEFT half of the loop down. The moment of magnetic forces on the loop (we calculate only one half and therefore multiply by two) is

$$M = 2\int_{\alpha=0}^{\pi} I\,\mathrm{d}l \sin\alpha Ba \sin\alpha.$$

Noting that $\mathrm{d}l = a\,\mathrm{d}\alpha$, this becomes

$$M = 2Ia^2B\int_0^{\pi} \sin^2\alpha\,\mathrm{d}\alpha = Ia^2\pi B = ISB,$$

where S is the area of the loop. Because the product IS defines the moment of magnetic forces on the loop, it is known as the *magnetic moment of the loop* and is usually denoted by m. In addition, it is defined as a *vector*, with the surface S determined according to the right-hand rule with respect to the current in the loop (Fig. 12.5):

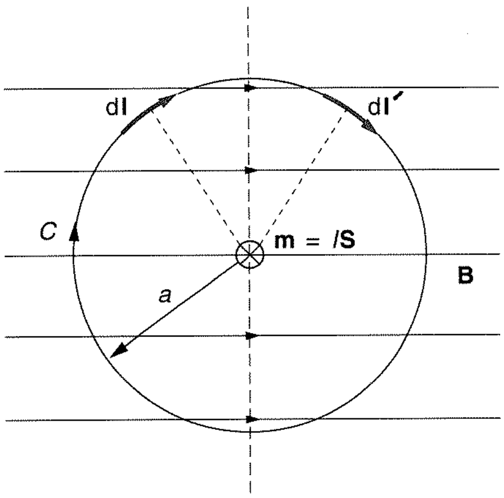


Figure 12.5 A circular current loop in a uniform magnetic field parallel to the loop plane

$$\mathbf{m} = I\mathbf{S} \qquad (\mathrm{A \cdot m^2}). \tag{12.8}$$

(Definition of the magnetic moment of a current loop)

It is a simple matter to show that the correct magnitude and direction of the moment of magnetic forces on the loop is obtained from the vector expression

$$\mathbf{M} = \mathbf{m} \times \mathbf{B} \qquad (\mathrm{N \cdot m}), \tag{12.9}$$

(Moment of magnetic forces on a current loop in a uniform magnetic field)

where **B** is the flux density vector of the uniform field *that may be in any direction*. The cross product "extracts" from **B** only the component that is parallel to the plane of the loop. Note that the moment of magnetic forces on the loop *tends to align the vectors* **m** *and* **B**.

The moment of magnetic forces is used in many applications. For example, it is used in electric motors, where current exists in a loop situated in a magnetic field.

Questions and problems: Q12.4 to Q12.13, P12.1 to P12.30

12.4 Magnetic Flux

The term *magnetic flux* implies simply the flux of vector **B** through a surface. We shall see that it plays a very important role in magnetic circuits, and a fundamental role in one of the most important electromagnetic phenomena—electromagnetic induction.

If we have a surface S in a magnetic field, the magnetic flux, Φ, through S is given by

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \qquad \text{webers (Wb)}. \tag{12.10}$$

(Definition of magnetic flux)

The unit for magnetic flux can be expressed as $\mathrm{T \cdot m^2}$. Because of the importance of magnetic flux, this unit is given the name *weber* (abbreviated Wb). So a tesla can also be expressed as a $\mathrm{Wb/m^2}$.

The magnetic flux has a very simple and important property—it is zero through *any* closed surface:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{12.11}$$

(Law of conservation of magnetic flux)

This relation is known as the *law of conservation of magnetic flux*. It follows from the expression for vector **B** of a current element in Eq. (12.3), which is proven as follows.

We know that the **B** lines of a current element are circles centered on the line that contains the current element. Along any such circle the intensity of vector **B** is constant. We can imagine the entire field of a current element divided into circular tubes formed by bunches of lines of vector **B**. The magnetic flux through any such tube is the same at any cross section of the tube. Consequently, if we imagine a closed surface in this field, the magnetic flux of each of the tubes will enter the surface at one point and leave at another, making a zero contribution of the tube to the magnetic flux through the surface. Obviously, the total magnetic flux through a closed surface in the field of a current element is therefore zero.

Any distribution of current can be represented as a large number of small current elements. Because each of these elements produces zero magnetic flux through any closed surface in the field of these currents, the total flux through the closed surface is zero, i.e., Eq. (12.11) is satisfied.

An interpretation of the law of conservation of magnetic flux is that "magnetic charges" do not exist, i.e., a south and north pole of a magnet are never found separately. The law tells us also that the lines of vector **B** do not have a beginning or an end. Sometimes this last statement is phrased more loosely: it is said that the lines of vector **B** close onto themselves.

There is an important corollary of the law of conservation of magnetic flux. If we have a closed contour C in the field and imagine any number of surfaces spanned over it, *the magnetic flux through any such surface, spanned over the same contour, is the same.* Just one condition needs to be fulfilled in order that this be true: the unit vector normal to all the surfaces must be normal with respect to the contour. It is customary to orient the contour and then to define the vector unit that is normal to any surface on it according to the right-hand rule (Fig. 12.6).

To prove this, consider two surfaces spanned over contour C, as in Fig. 12.7. They form a closed surface, to which the law of conservation of magnetic flux applies. We have, however, a specific situation concerning a vector unit normal to surface S_2 in the figure. If we consider S_2 as a part of the closed surface the vector normal should be directed outward, and if we consider it separately it should be directed according to the right-hand rule with respect to the reference contour direction, which is just opposite. So we can write

$$\begin{aligned}\oint_{S_1+S_2} \mathbf{B} \cdot d\mathbf{S} &= \int_{S_1} \mathbf{B} \cdot d\mathbf{S} + \left(\int_{S_2} \mathbf{B} \cdot d\mathbf{S}\right)_{\text{outward normal}} \\ &= \int_{S_1} \mathbf{B} \cdot d\mathbf{S} - \left(\int_{S_2} \mathbf{B} \cdot d\mathbf{S}\right)_{\text{inward normal}} = 0,\end{aligned}$$

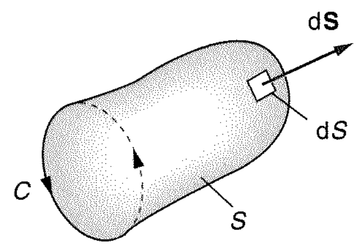


Figure 12.6 The reference direction of a vector surface element is always adopted according to the right-hand rule with respect to the reference direction of the contour defining the surface, and vice versa

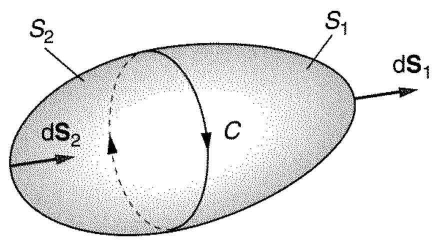


Figure 12.7 Two surfaces, S_1 and S_2, defined by a common contour, form a closed surface to which the law of conservation of magnetic flux applies

so that, indeed,

$$\int_{S_1} \mathbf{B} \cdot d\mathbf{S} = \int_{S_2} \mathbf{B} \cdot d\mathbf{S}.$$

Note that the only condition for this conclusion to be satisfied is that the flux of vector **B** is zero through any closed surface. Therefore the same conclusion holds for *any* vector that has the same property, for example the current density vector of time-invariant current fields. We shall use this conclusion in formulating Ampère's law in section 12.6.

Questions and problems: Q12.14 to Q12.17

12.5 Electromagnetic Force on a Point Charge: The Lorentz Force

Let a current element have a cross sectional area ΔS. We can then write $I\,d\mathbf{l} = J\,\Delta S\,d\mathbf{l} = NQ\mathbf{v}\,\Delta S\,dl$. Note that $\Delta S\,dl$ is the volume of the element, N the number of charge carriers per unit volume, and $\mathbf{v}$ their drift velocity. Combining this result with the expression in Eq. (12.5) for the force on a current element, we conclude that the magnetic force on a single point charge Q moving at a velocity $\mathbf{v}$ in a magnetic field of (local) flux density $\mathbf{B}$ is given by

$$\mathbf{F} = Q\mathbf{v} \times \mathbf{B} \qquad \text{(N)}. \tag{12.12}$$

(Magnetic force on a moving point charge)

If a particle is moving both in an electric and a magnetic field, the total force (often called the *Lorentz force*) on the particle is

$$\mathbf{F} = Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B} \qquad \text{(N)}. \tag{12.13}$$

(The Lorentz force—force on a point charge moving in an electric and a magnetic field)

Example 12.4—The influence of B on J. We now know that moving charges constitute a current, the current produces a magnetic field, and this magnetic field produces a force on all moving charged particles. Does this magnetic field then affect the moving charges (current) that produced it? To answer this question, we shall use the Lorentz force to examine the dependence of the current density vector $\mathbf{J}$ on both $\mathbf{E}$ and $\mathbf{B}$.

Let N be the number of free charges Q per unit volume at some point inside a wire carrying a current, and let the drift velocity of these charges be $\mathbf{v}$. We assume here that only one type of charge makes up the current (in metals, these are electrons). This means that $\mathbf{v} = \mathbf{J}/NQ$, so that the Lorentz force on each of the charges is

$$\mathbf{F} = Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B} = Q\left(\mathbf{E} + \frac{1}{NQ}\mathbf{J} \times \mathbf{B}\right) = Q\mathbf{E}_{\text{equiv}}.$$

Let the conductivity of the conductor through which the current is flowing be σ. Then

$$\mathbf{J} = \sigma\mathbf{E}_{\text{equiv}} = \sigma\left(\mathbf{E} + \frac{1}{NQ}\mathbf{J} \times \mathbf{B}\right), \quad \text{or} \quad \frac{\mathbf{J}}{\sigma} - \mathbf{J} \times \left(\frac{\mathbf{B}}{NQ}\right) = \mathbf{E}.$$

The influence of the magnetic field on the current distribution can be neglected if the second term on the left side of this equation is much smaller than the first term. Let us compare the values of $1/\sigma$ and $|\mathbf{B}/NQ|$ that occur in practice. For example, in copper, $\sigma = 57 \cdot 10^6$ S/m, $|Q| = 1.6 \cdot 10^{-19}$ C, and $N \simeq 8.47 \cdot 10^{28}$ electrons/m^3. We have already said that the intensity of the magnetic flux density vector $\mathbf{B}$ rarely exceeds 1 T, so

$$|\mathbf{B}/NQ| < 7.37 \cdot 10^{-11} \quad \ll \quad 1/\sigma = 1.75 \cdot 10^{-8} \text{ m/S},$$

and we can write, approximately,

$$\mathbf{J}(\mathbf{E}, \mathbf{B}) \simeq \sigma\mathbf{E}.$$

This means that we can consider the distribution of the dc current as virtually independent of the magnetic field that it creates, i.e., that the relation $\mathbf{J} = \sigma\mathbf{E}$ is highly accurate in most cases.

Questions and problems: Q12.18 to Q12.21, P12.31 to 12.33

12.6 Ampère's Law for Time-Invariant Currents in a Vacuum

The magnetic flux density vector **B** resulting from a time-invariant current density **J** has a very simple and important property: if we compute the line integral of **B** along any closed contour C, it will be equal to μ_0 times the total current that flows through any surface spanned over the contour. This is *Ampère's law*:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}. \tag{12.14}$$

(Ampère's law in a vacuum)

The convention in this law is that the reference direction of the vector surface elements of S is adopted according to the right-hand rule with respect to the reference direction of the contour (Fig. 12.6). In the applications of Ampère's law, it is very useful to keep in mind that the flux of the current density vector (the current intensity) is the same through all surfaces having a common boundary contour.

Ampère's law is not a new property of the magnetic field. It follows from the Biot-Savart law. Although Ampère's law itself is simple, its derivation from the Biot-Savart law is not. In addition, no physical insight is gained by this derivation, so we will not present it here. The interested reader can find it in most higher-level electromagnetics texts.

Ampère's law in Eq. (12.14) is a general law of the magnetic field of *time-invariant currents in a vacuum*. We shall see that it can be extended to cases of materials in the magnetic field, but in this form it is not valid for the magnetic field of time-varying currents.

There are two major classes of applications of Ampère's law: the determination of vector **B** in some cases with a high degree of current symmetry; and proofs of certain general properties of the magnetic field. In the examples that follow we illustrate the first kind of application of Ampère's law, similar to what we did in the case of Gauss' law.

Example 12.5—Determination of the line integral of vector B for specified current distributions. To learn how to use Ampère's law, consider Fig. 12.8, in which several current loops are shown. Let us determine the left-hand side in Ampère's law for the contours C_1, C_2, and C_3 indicated in the figure by diagonal lines.

Imagine first a surface spanned over contour C_1. It is necessary to assign a reference direction to this surface, using the right-hand rule with respect to the direction of C_1 indicated in

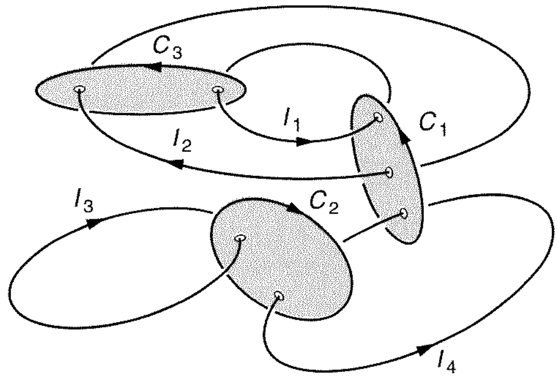


Figure 12.8 Four current loops and three contours as examples of applications of Ampère's law

Fig. 12.8. This surface is traversed in its *positive* direction by currents I_2 and I_4. It is traversed in its *negative* direction by current I_1. I_3 does not traverse it at all (or traverses it twice, once in the positive and once in the negative direction, depending on the surface we imagine). Therefore the magnetic field due to these current loops is such that the line integral of vector **B** along C_1 equals *exactly* $\mu_0(-I_1 + I_2 + I_4)$.

It is left as an exercise for the reader to prove that the magnetic field is also such that the line integral of **B** along C_2 equals $\mu_0(-I_3 - I_4)$ and along C_3 equals $\mu_0(I_1 - I_2)$.

Example 12.6—Magnetic field of a straight wire of a circular cross section. Consider now a straight, infinitely long wire of a circular cross section of radius a, as in Fig. 12.9. (A wire may be considered infinitely long if it is much longer than the shortest distance from it to the observation point.) There is a current of intensity I in the wire distributed uniformly over its cross section, and we wish to determine vector **B** inside and outside the wire. Note that from the Biot-Savart law, the lines of the magnetic flux density vector are circles centered on the wire axis, and that the magnitude of **B** depends only on the distance r from the wire axis.

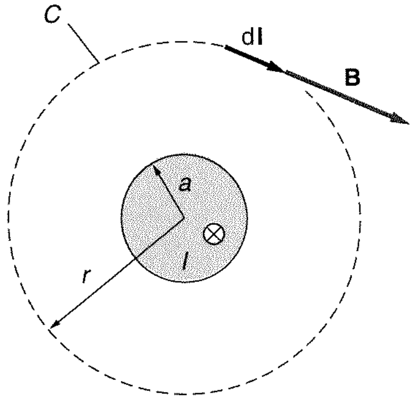


Figure 12.9 Using Ampère's law to find the magnetic flux density vector due to a current in a cylindrical wire of a circular cross section

Take a circular contour C of radius $r \geq a$ centered on the wire axis. Ampère's law gives

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \oint_C B\,\mathrm{d}l \cos 0 = B \oint_C \mathrm{d}l = 2\pi r B = \mu_0 I,$$

so that

$$B(r) = \frac{\mu_0 I}{2\pi r} \qquad (r \geq a). \tag{12.15}$$

As long as the point is outside the wire, the radius of the wire, a, is irrelevant. So this expression B outside a round wire is valid for any radius, even if the wire is infinitely thin.

The magnetic flux density inside the wire is obtained by applying Ampère's law to a circular contour of radius $r \leq a$. The line integral of $\mathbf{B}$ is simply $2\pi r B(r)$. The contour now *does not* encircle the entire current in the wire, but only a current $Jr^2\pi$, where $J = I/(a^2\pi)$. The resulting magnetic flux density inside the wire is

$$B(r) = \frac{\mu_0 I r}{2\pi a^2} \qquad (r \leq a). \tag{12.16}$$

Example 12.7—Magnetic field of a coaxial cable. Using reasoning similar to that in Example 12.6, it is a simple matter to find the magnetic flux density due to currents I and $-I$ in conductors of a coaxial cable (Fig. 12.10).

We apply Ampère's law successively to circular contours of radii $r \leq a$, $a \leq r \leq b$, $b \leq r \leq c$, and $r \geq c$. The magnetic flux density inside the inner conductor and between the conductors is the same as if the outer conductor did not exist, because the contours with radii $r \leq a$ and $a \leq r \leq b$ encircle only a part, or the total inner-conductor current. Therefore the results of the preceding example apply directly to this case, and we have

$$B(r) = \frac{\mu_0 I r}{2\pi a^2} \qquad (r \leq a). \tag{12.17}$$

$$B(r) = \frac{\mu_0 I}{2\pi r} \qquad (a \leq r \leq b). \tag{12.18}$$

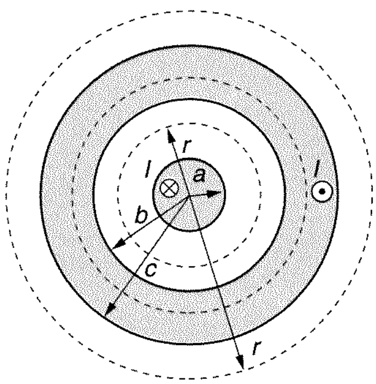


Figure 12.10 Using Ampère's law to find the magnetic flux density in a coaxial cable. The figure shows the cross section of the cable.

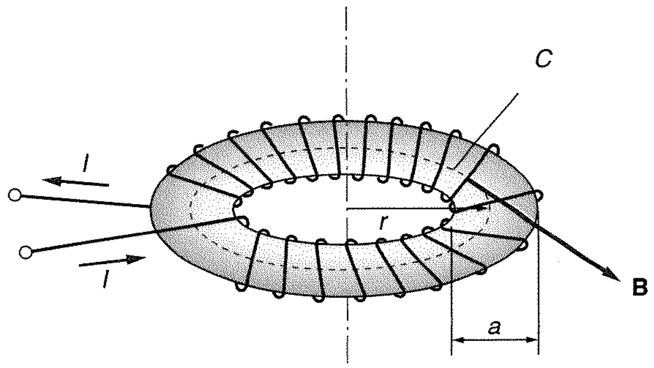


Figure 12.11 Using Ampère's law to find the magnetic flux density due to current in a toroidal coil, wound uniformly and densely with N turns of thin wire

For the contour encircling the cable ($r \geq c$), the total current encircled by the contour is zero, so there is no magnetic field outside the cable. It is left for the reader as an exercise to find the expression for the magnetic flux density inside the outer cable conductor.

Example 12.8—Magnetic field of a toroidal coil. Consider a toroidal coil as sketched in Fig. 12.11. The cross section of the toroid is arbitrary. Assume that the coil is made of N uniformly and densely wound turns with current of intensity I. From the Biot-Savart law, we know that the lines of vector **B** are circles centered at the toroid axis. Also, the magnitude of **B** depends only on the distance, r, from the axis. Applying Ampère's law yields the following expression for the magnitude, $B(r)$, of the magnetic flux density vector:

$$B = 0 \qquad \text{(outside the toroid)}, \tag{12.19}$$

and

$$B(r) = \frac{\mu_0 N I}{2\pi r} \qquad \text{(inside the toroid)}. \tag{12.20}$$

Note again that these formulas are valid for *any* shape of the toroid cross section.

As a numerical example, for $N = 1000$, $I = 2$ A, and an average toroid radius of $r = 10$ cm, we get $B = 4$ mT. This value can be larger if, for example, several layers of wire are wound on top of each other so that N is larger. Alternatively, the torus can be made of a material that increases the magnetic field, to be discussed in the next chapter.

Example 12.9—Magnetic field of a solenoid. Assume that in the preceding example the radius r of the toroid becomes very large. Then at any point inside the toroid, the toroid looks locally as if it were a cylindrical coil. Such a coil is sketched in Fig. 12.12 and is known as a solenoid. (The term *solenoid* comes from a Greek word that roughly means "tubelike.")

Outside an infinitely long solenoid the flux density vector is zero. Inside, it is given by Eq. (12.20) of the preceding example, with r very large. However, the expression $N/(2\pi r)$ represents the number of turns per unit length of the toroid, i.e., of the solenoid. If we keep the number of turns per unit length constant and equal to N', from Eq. (12.20) we obtain

$$B = \mu_0 N' I \qquad \text{(inside a solenoid)}. \tag{12.21}$$

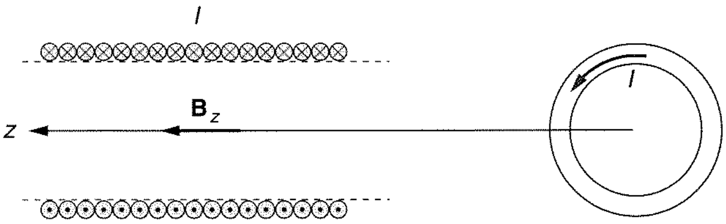


Figure 12.12 A solenoidal coil of circular cross section, wound uniformly and densely with N' turns of thin wire per unit length

Note that the field inside a very long solenoid is *uniform*, and that the expression is valid for *any* cross section of the solenoid.

As a numerical example, for $N' = 2000$ windings/m, and $I = 2$ A, we get $B \simeq 5$ mT.

Example 12.10—Magnetic field of a single current sheet and of two parallel current sheets (a strip line). Consider a large conducting sheet with constant surface current density $\mathbf{J}_s$ at all points, as in Fig. 12.13a. From the Biot-Savart law, vector $\mathbf{B}$ is parallel to the sheet and perpendicular to vector $\mathbf{J}_s$, and $\mathbf{B}$ is directed in opposite directions on the two sides of the sheet, as indicated in the figure. What we do not know is the dependence of B on ~~x~~ y. This can be determined using Ampère's law.

Let us apply Ampère's law to the rectangular contour shown in Fig. 12.13a. Along the two rectangle sides perpendicular to the sheet, the line integral of $\mathbf{B}$ is zero because $\mathbf{B}$ is perpendicular to the line elements. Along the two sides parallel to the sheet, the integral equals $2B(x)l$. The current encircled by the contour equals $J_s l$, and Ampère's law yields

$$B(x) = \mu_0 \frac{J_s}{2} \qquad \text{(current sheet)}. \tag{12.22}$$

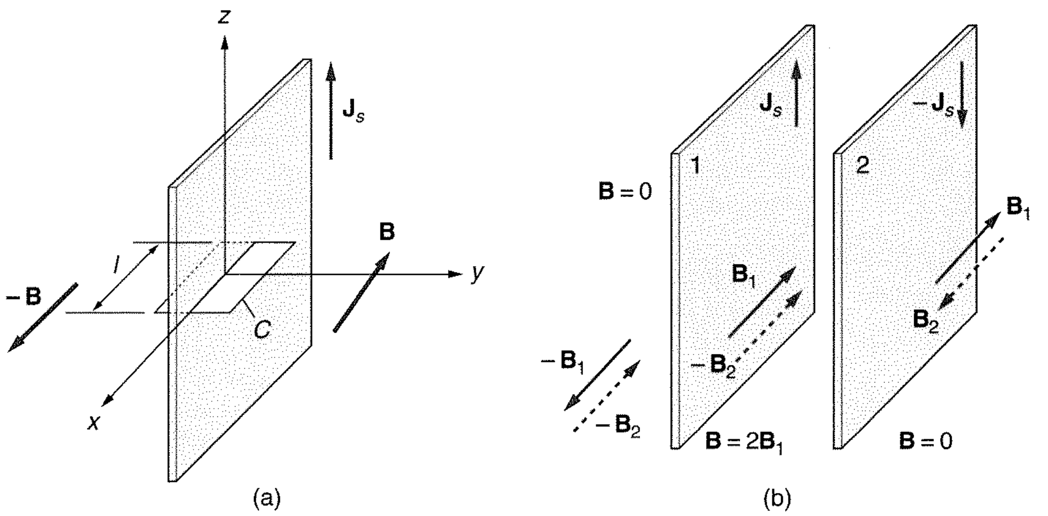


Figure 12.13 A current sheet (a) and two parallel current sheets (b)

If we have two parallel current sheets with opposite surface currents of the same magnitude (Fig. 12.13b), from the last equation and using superposition we easily find that the magnetic field outside the sheets is zero, and between the sheets

$$B = \mu_0 J_s \qquad \text{(between two parallel current sheets).} \tag{12.23}$$

This is approximately true if the sheets are not infinite but are close to each other. Such a system is called a *strip line.*

Questions and problems: Q12.22 to Q12.27, P12.34 to 12.40

12.7 Chapter Summary

1. Time-invariant electric currents produce a time-invariant magnetic field. This field acts with a force (the magnetic force) on a single *moving* charge or on a current element.
2. The magnetic field is described by the *magnetic flux density vector*, **B**. **B** that results from a known current distribution in a vacuum is determined from an expression known as the Biot-Savart law.
3. The total force on a moving point charge is a sum of the electric force, $Q\mathbf{E}$, and the magnetic force, $Q\mathbf{v} \times \mathbf{B}$. This sum is known as the Lorentz force.
4. A consequence of the Biot-Savart law is that the magnetic flux density vector satisfies a simple integral relation known as Ampère's law: the line integral of **B** along any closed contour in the magnetic field in a vacuum equals μ_0 (the permeability of a vacuum) times the total current through any surface spanned over the contour. By convention, the direction of the vector surface elements is connected with the reference direction of the contour by the right-hand rule. Ampère's law is analogous to Gauss' law in electrostatics in that it relates the field (in this case the magnetic flux density) to the source of the field (in this case currents) through an integral equation.

QUESTIONS

Q12.1. If μ_0 were defined to have a different value, e.g., $\mu_0 = 1 \cdot 10^{-7}$, what would the expression for the force between two current elements be?

Q12.2. If we would like to have the term $I_2\,\mathrm{d}\mathbf{l}_2$ in Eq. (12.1) to be at the end on the right-hand side and not at the beginning, how would the expression read?

Q12.3. Figure Q12.3 shows four current elements (the contours they belong to are not shown). Determine the magnetic force between all possible pairs of the elements (a total of twelve expressions).

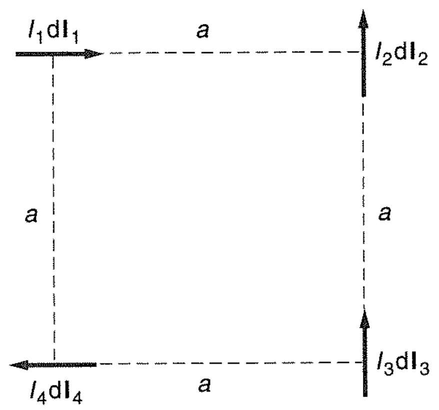


Figure Q12.3 Four current elements

Q12.4. What is the shape of the lines of vector **B** of a single current element? Is the magnitude of **B** constant along these lines? Is **B** constant along these lines?

Q12.5. Prove that $I\,d\mathbf{l}$ for line currents is equivalent to $\mathbf{J}\,dv$ for volume currents and to $\mathbf{J}_s\,dS$ for surface currents.

Q12.6. Describe an approximate solution of the vector integrals in Eqs. (12.4), (12.6), and (12.7).

Q12.7. Assume that the lines of vector **B** converge to and are directed toward a point in space. Would that be a realistic magnetic field?

Q12.8. Sketch the lines of the magnetic flux density vector for two long, parallel, straight, thin conductors with equal currents when the currents are in the (1) same and (2) opposite directions.

Q12.9. Prove that if we have N thin wire loops with currents I, connected in series and pressed onto one another, they can be represented as a single loop with a current NI. Is this conclusion valid at all points?

Q12.10. Starting from Eq. (12.5), write the expression for the magnetic force on a closed current loop C with current I.

Q12.11. Why do we always obtain two new magnets by cutting a permanent magnet? Why do we not obtain isolated "magnetic charges"?

Q12.12. Knowing that the south pole–north pole direction of a compass needle aligns itself with the local direction of the vector **B**, what is the orientation of elementary current loops in the needle?

Q12.13. In which position is a planar current loop situated in a uniform magnetic field in stable equilibrium?

Q12.14. A closed surface S encloses a small conducting loop with current I. What is the magnetic flux through S?

Q12.15. A conductor carrying a current I pierces a closed surface S. What is the magnetic flux through S?

Q12.16. A straight conductor with a current I passes through the center of a sphere of radius R. What is the magnetic flux through the spherical surface?

Q12.17. A hemispherical surface of radius R is situated in a uniform magnetic field of flux density $\mathbf{B}$. The axis of the surface makes an angle α with the vector $\mathbf{B}$. Determine the magnetic flux through the surface.

Q12.18. Discuss the possibility of changing the kinetic energy of a charged particle by a magnetic field only.

Q12.19. A charge Q is moving along the axis of a circular current-carrying contour normal to the plane of the contour. Discuss the influence of the magnetic field on the motion of the charge.

Q12.20. An electron beam passes through a region of space undeflected. Is it certain that there is no magnetic field? Explain.

Q12.21. An electron beam is deflected in passing through a region of space. Does this mean that there is a magnetic field in that region? Explain.

Q12.22. Does Ampère's law apply to a closed contour in the magnetic field of a single small charge Q moving with a velocity $\mathbf{v}$? Explain.

Q12.23. In a certain region of space the magnetic flux density vector $\mathbf{B}$ has the same direction at all points, but its magnitude is not constant in the direction perpendicular to its lines. Are there currents in that part of space? Explain.

Q12.24. An infinitely long, straight, cylindrical conductor of rectangular cross section carries a current of intensity I. Is it possible to determine the magnetic flux density inside and outside the conductor starting from Ampère's law? Explain.

Q12.25. Can the contour in Ampère's law pass through a current-carrying conductor? Explain.

Q12.26. What is the left-hand side in Ampère's law equal to for the five contours in Fig. Q12.26?

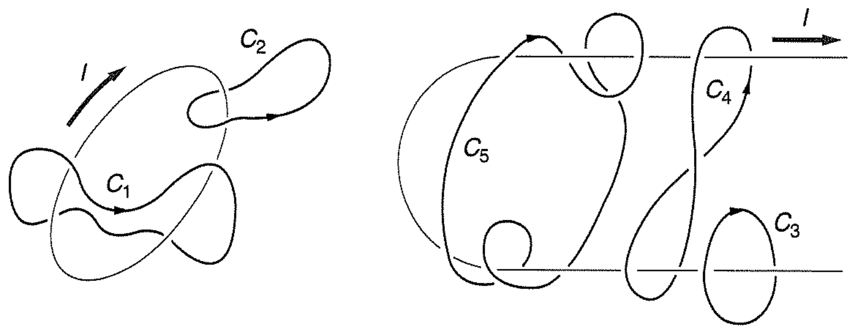


Figure Q12.26 Contours for Ampère's law

Q12.27. Compare Gauss' law and Ampère's law, and explain their differences and similarities.

PROBLEMS

P12.1. Prove that the magnetic force on a closed wire loop of any form, situated in a uniform magnetic field, is zero.

P12.2. Prove that the moment of magnetic forces on a closed planar wire loop of arbitrary shape (Fig. P12.2), of area S and with current I, situated in a uniform magnetic field of flux density $\mathbf{B}$, is $\mathbf{M} = \mathbf{m} \times \mathbf{B}$, where $\mathbf{m} = IS\mathbf{n}$, and $\mathbf{n}$ is the unit vector normal to S determined according to the right-hand rule with respect to the direction of the current in the loop.

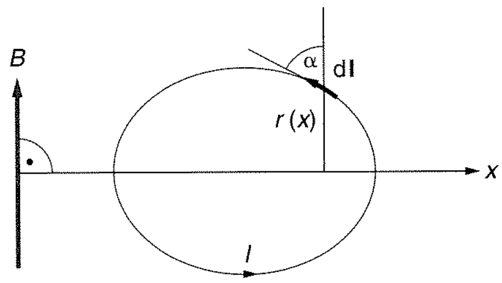


Figure P12.2 A planar current loop

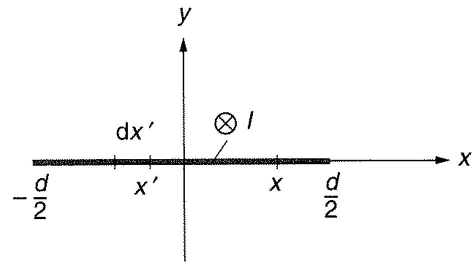


Figure P12.3 Cross section of a current strip

P12.3. Find the magnetic flux density vector at a point A in the plane of a straight current strip (Fig. P12.3). The strip is d wide and a current I flows through it. Assume that point A is x away from the center of the strip, where $x < d/2$.

P12.4. A thin dielectric disk of radius $a = 10$ cm has a surface charge density of $\sigma = 2 \cdot 10^{-6}$ C/m^2. Find the magnetic flux density at the center of the disk if the disk is rotating at $n = 15{,}000$ rpm around the axis perpendicular to its surface.

P12.5. Determine the magnetic moment of a thin triangular loop in the form of an equilateral triangle of side a with current I.

P12.6. (1) Find the magnetic flux density vector at point P in the field of two very long straight wires with equal currents I flowing through them. Point P lies in the symmetry plane between the two wires and is x away from the plane defined by the two wires. The front view of the wires is shown in Fig. P12.6a, and the top view in Fig. P12.6b. (2) What is the magnetic flux density equal to at any point in that plane if the current in one wire is I and in the other $-I$?

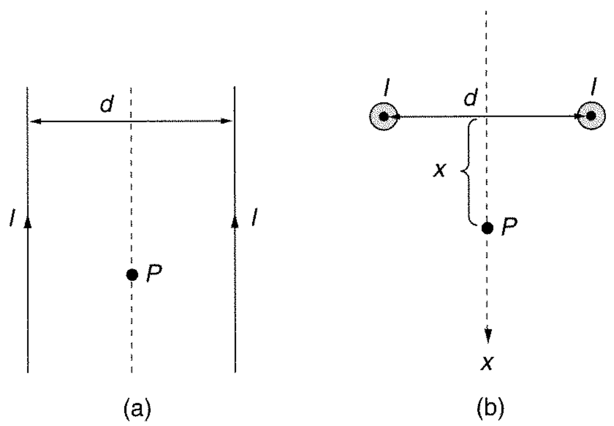


Figure P12.6 Two wires with equal currents: (a) front view, (b) top view

P12.7. Determine the magnetic flux density along the axis normal to the plane of a circular loop. The loop radius is a and current intensity in it is I.

P12.8. We know the magnetic flux density inside a very long (theoretically infinite) thin solenoid. Usually solenoids are not long enough, so we cannot assume they are infinite. Consider a solenoid of circular cross section with a radius a, b long, and having N turns with a current I flowing through them, as in Fig. P12.8. (The solenoid is actually a spiral winding, but in the figure it is shown as many closely packed circular loops.)

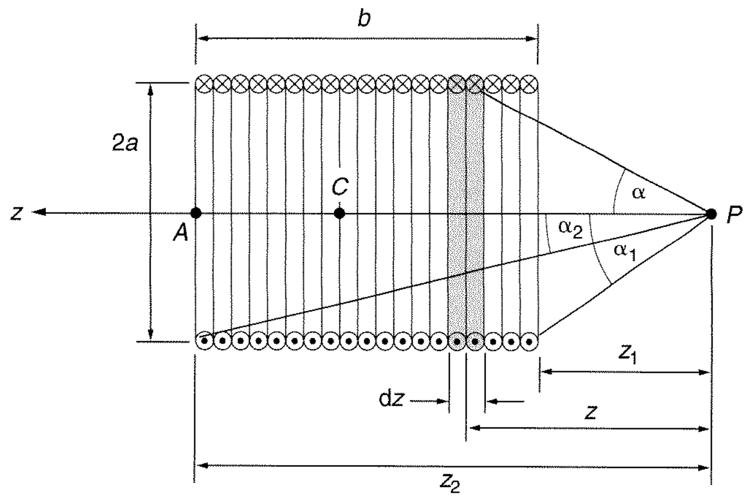


Figure P12.8 A solenoid of circular cross section

(a) How many turns are there on a length dx of the coil? We can replace this small piece of the coil with a single circular loop with a current dI. What is dI equal to?

(b) Write the expression for the magnetic flux density $d\mathbf{B}(x)$ of one of the loops with a current dI, at any point P along the axis of the solenoid.

(c) Write the expression for the total magnetic flux density at point P resulting from all of the solenoid turns. (This is an integral.)

(d) Solve the integral. It reduces to a simpler integral if you notice that $x = a/\tan\alpha$ and $a^2 + x^2 = a^2/\sin^2\alpha$.

(e) If the solenoid is thin and long, how much larger is $\mathbf{B}$ at point C at the center of the solenoid than at point A at the edge? Calculate the values of the magnetic flux density at these two points if $I = 2$ A, $b/a = 50$, $N = 1000$, and $b = 1$ m.

P12.9. A closed planar current loop carries a current of intensity I. Starting from the Biot-Savart law, derive the simplified integral expression for the magnitude of vector $\mathbf{B}$ for points in the plane of the loop.

P12.10. Derive the expression for magnitude of vector $\mathbf{B}$ of a straight current segment (Fig. P12.10). The segment is a part of a closed current loop, but only the contribution of the segment is required.

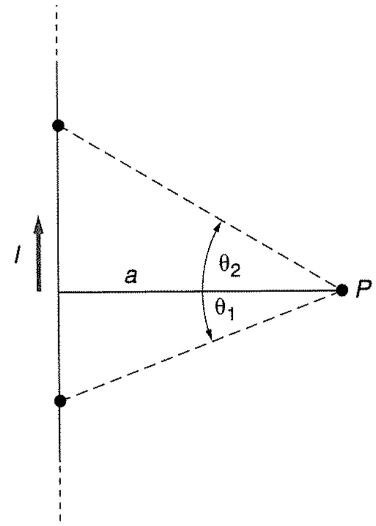


Figure P12.10 A straight current filament

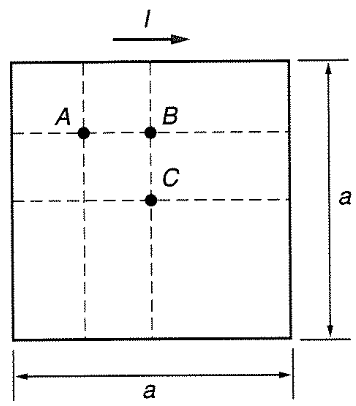


Figure P12.11 A square current loop

P12.11. Evaluate the magnetic flux density vector at points A, B, and C in the plane of a square current loop shown in Fig. P12.11.

P12.12. The lengths of wires used to make a square and a circular loop with equal current are the same. Calculate the magnetic flux density at the center of both loops. In which case it is greater?

P12.13. Evaluate the magnetic flux density at point A in Fig. P12.13.

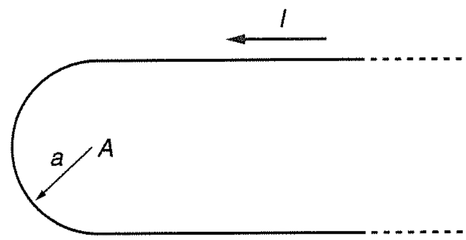


Figure P12.13 Short-circuited two-wire line

P12.14. Evaluate the magnetic flux density at point A in the plane of a straight, flat, thin strip of width d with current I. Assume that the point A is at a distance x ($x > d/2$) from the center line of the strip. Plot your result as a function of x.

P12.15. Repeat problem P12.14 for a point in a cross section of the system having coordinates (x, y). Assume the origin is at the strip center line, the x axis is normal to the strip, and the y axis is parallel to the long side of the strip cross section. Plot the magnitude of all components of the magnetic flux density vector as a function of x and y.

P12.16. A very long rectangular conductor with current I has sides a (along the x axis) and b (along the y axis). Write the integral determining the magnetic flux density at any point of the xy plane. Do *not* attempt to solve the integral (it is tricky).

P12.17. Determine and plot the magnetic flux density along the axis normal to the plane of a square loop of side a carrying a current I.

P12.18. A metal spherical shell of radius $a = 10$ cm is charged with the maximum charge that does not initiate the corona on the sphere surface. It rotates about the axis passing through its center with angular velocity $\omega = 50{,}000$ rad/min. Determine the magnetic flux density at the center of the sphere.

P12.19. A very long, straight conductor of semicircular cross section of radius a (Fig. P12.19) carries a current I. Determine the flux density at point A.

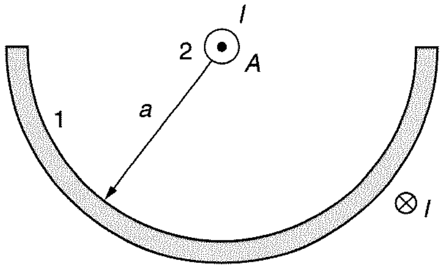


Figure P12.19 Conductor of semicircular cross section

P12.20. Assume in Fig. P12.19 that the thin wire 2 extends along the axis of conductor 1. Wire 2 carries the same current I as conductor 1, but in the opposite direction. Determine the magnetic force per unit length on conductor 2.

P12.21. Determine the magnetic force on the segment $A - A'$ of the two-wire-line short circuit shown in Fig. P12.21.

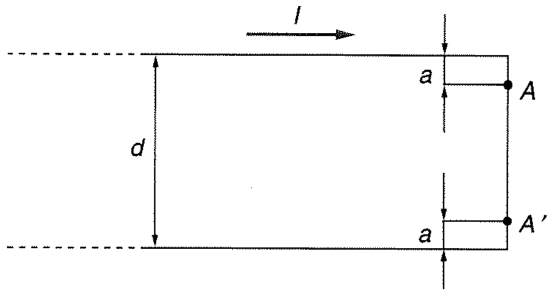


Figure P12.21 Short-circuited two-wire line

P12.22. Shown in Fig. P12.22 is a sketch of a permanent magnet used in loudspeakers. The lines of the magnetic flux density vector are radial, and at the position of the coil it has a magnitude $B = 1$ T. Determine the magnetic force on the coil (which is glued to the loudspeaker membrane) at the instant when the current in the coil is $I = 0.15$ A, in the indicated direction. The number of turns of the coil is $N = 10$, and its radius $a = 0.5$ cm.

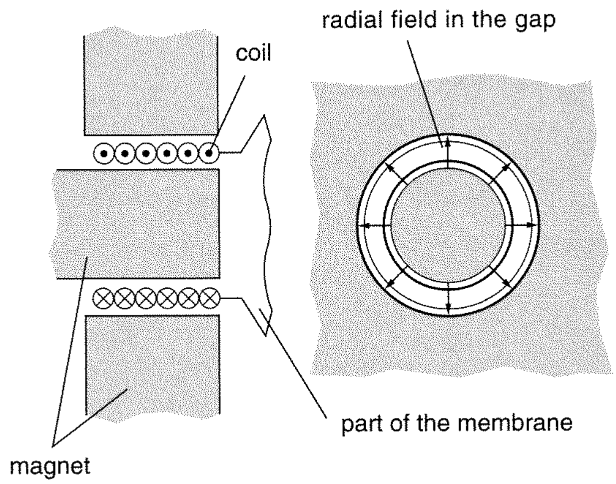


Figure P12.22 Magnet and coil of a loudspeaker

P12.23. Prove that the magnetic force on a *segment* of a closed current loop with current I, situated in a uniform magnetic field of flux density $\mathbf{B}$, does not depend on the segment shape but only on the position of its two end points.

P12.24. A circular current loop of radius a and with current I is cut into halves that are in contact. It is situated in a uniform magnetic field of flux density $\mathbf{B}$ normal to the plane of the loop. (1) What should be the direction of $\mathbf{B}$ with respect to that of the current in the loop in order that the magnetic force will press the two loop halves one onto the other? (2) What is the force on each of the loop halves? Evaluate the force for $a = 10$ cm, $I = 2$ A, and $B = 1$ T. (3) What is the direction of force on the two halves of the loop due to the current in the loop itself? (Neglect this force, but note that it always exists.)

P12.25. Three circular loops are made of three equal pieces of wire of length b, one with a single turn, one with two turns, and one with three turns of wire. If the same current I exists in the loops and they are situated in a uniform magnetic field of flux density B, determine the maximal moment of magnetic forces on the three loops. Then solve for the moments if $I = 5$ A, $b = 50$ cm, and $B = 1$ T.

P12.26. Determine the moment of magnetic forces acting on a rectangular loop of sides a and b, and with current of intensity I, situated in a uniform magnetic field of flux density $\mathbf{B}$. Side b of the loop is normal to the lines of $\mathbf{B}$, and side a parallel to them.

P12.27. Two thin, parallel, coaxial circular loops of radius a are a distance a apart. Each loop carries a current I. Prove that at the midpoint between the loops, on their common axis, the first three derivatives of the axial magnetic flux density with respect to a coordinate along the axis are zero. (This means that the field around that point is highly uniform. Two such coils are known as *Helmholtz coils*.)

P12.28. Write the expression for the vector $\mathbf{B}$ inside a long circular conductor of radius a carrying a current I. To that end, use the current density vector $\mathbf{J}$ inside the conductor, and the vector $\mathbf{r}$ representing the distance of any point inside the conductor to the conductor axis.

P12.29. A very long cylindrical conductor of circular cross section of radius a has a hole of radius b. The axis of the hole is a distance d ($d+b < a$) from the conductor axis. Using the principle of superposition and the expression for the magnetic flux density vector inside a round conductor from the preceding problem, prove that the magnetic field in the cavity is uniform.

***P12.30.** Prove that the divergence of the magnetic flux density vector given by the Biot-Savart law is zero.

P12.31. A straight, very long, thin conductor has a charge Q' per unit length. It also carries a current of intensity I. A charge Q is moving with a velocity v parallel to the wire, at a distance d from it, unaffected by the simultaneous action of both the electric and magnetic force. Determine the velocity v of the charge, assuming the necessary correct direction of the current and the sign of the charge on the conductor.

P12.32. Starting from the magnetic force between two current elements, derive the expression for the magnetic force between two moving charges.

P12.33. Assuming that the expression for the magnetic force between two moving charges from the preceding problem is true, compare the maximal possible magnetic force between the charges with the Coulomb force between them. The charges are moving with equal velocities $\mathbf{v}$ and are at a distance r.

P12.34. A copper wire of circular cross section and radius $a = 1$ mm carries a current of $I = 50$ A. This is the largest current that can flow through the wire without damaging the conductor material. Plot the magnitude of the magnetic flux density vector as a function of distance from the center of the wire. Calculate the magnetic flux density at the surface of this wire.

P12.35. A plant for aluminum electrolysis uses a dc current of 15 kA flowing through a line that consists of three metal plate electrodes, as in Fig. P12.35. All the dimensions in the figure are given in centimeters. Find the approximate magnetic flux density at points A_1, A_2, and A_3 shown in the figure.

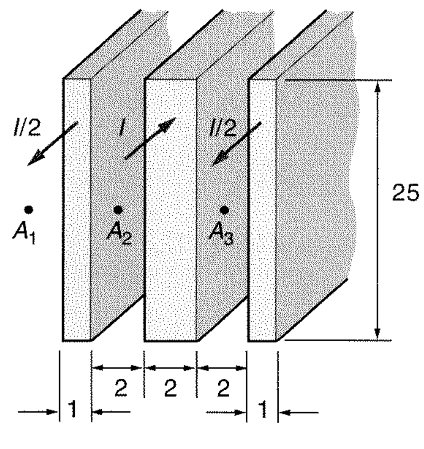


Figure P12.35 DC line for aluminum electrolysis

P12.36. A very long cylinder of radius a has a volume charge of density ρ. Find the expression for the magnetic flux density vector inside as well as outside the cylinder if the cylinder is rotating around its axis with an angular velocity ω. Plot your results.

P12.37. Find the magnetic flux density between and outside the large current sheets shown in Fig. P12.37.

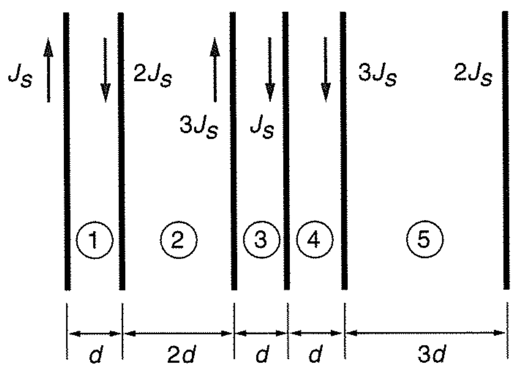


Figure P12.37 Parallel current sheets

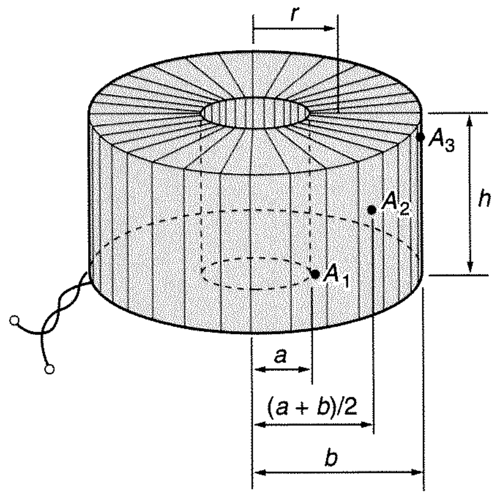


Figure P12.38 A densely wound thick toroidal coil

P12.38. A current $I = 0.5$ A flows through the torus winding shown in Fig. P12.38. Find the magnetic flux density at points A_1, A_2, and A_3 inside the torus. There are $N = 2500$ turns, $a = 5$ cm, $b = 10$ cm, and $h = 4$ cm.

P12.39. Find the dimensions and required number of turns for a torus such as the one in the previous problem so that the following design parameters are satisfied: (1) the magnetic flux density in the middle of the torus cross section is 6 mT; (2) the cross section of the core has dimensions $b - a = 10$ cm and $h = 10$ cm; and (3) the magnetic flux density does not vary by more than 3% from the value in the middle of the cross section. Assume you have at your disposal an insulated copper wire with a 1.5-mm diameter that can tolerate a maximum current of $I_{max} = 7.5$ A. If it is not possible to design the winding as a single-layer coil, design a multilayer winding. Note: many possible designs meet the criteria; choose the one that uses the least amount of wire, i.e., that has the lowest cost.

P12.40. Find and plot the magnetic flux density vector due to a current I flowing through a hollow cylindrical conductor of inner radius a and outer radius b.

13

Magnetic Fields in Materials

13.1 Introduction

The effect of the electric field on materials is related simply to the existence of charges inside the atoms, not to their motion. When a body is placed in a magnetic field, however, magnetic forces act on all *moving charges* within the atoms of the material. These moving charges make the atoms and molecules inside the material look like tiny current loops. We know that the moment of magnetic forces on a current loop is such that it tends to align vectors **m** and **B**. This means that in the presence of the field, a substance becomes a large aggregate of oriented elementary current loops. These loops produce their own magnetic field, just as dipoles in a polarized dielectric produce their own electric field. Because the rest of the body does not produce any magnetic field, it is of no importance as far as the magnetic field is concerned. Therefore, a substance in the magnetic field can be visualized as a large set of oriented elementary current loops situated in a vacuum. A material in which a magnetic force has produced such oriented elementary current loops is called a *magnetized material.*

It is also possible to find macroscopic currents (not elementary loops) that produce the same magnetic field as that of all the elementary loops in a body. Therefore, it is possible to replace a material body in a magnetic field with equivalent *macroscopic* currents *situated in a vacuum.* Because we know how to determine the field of currents in a vacuum, we are able to analyze the materials in the magnetic field as well, provided that we know how to find these equivalent currents.

Thus we can expect certain analogies between the analysis of materials in the presence of a magnetic field and the analysis of materials in the presence of an electric field. Many of the concepts are similar, so our knowledge of the electrostatic field enables a more concise discussion of materials in the magnetic field.

13.2 Substances in the Presence of a Magnetic Field: Magnetization Vector

As we have already mentioned, atoms consist of a heavy positively charged nucleus and electrons that circle around the nucleus. The number of revolutions per second of an electron around the nucleus is very large—about 10^{15} revolutions/s. Therefore, it is reasonable to say that such a rapidly revolving electron is a small, "elementary" current loop. This picture is in fact more complicated because it turns out that electrons spin about themselves as well. However, each atom can macroscopically be viewed as a complicated system of elementary current loops. Such an elementary current loop is called an *Ampère current*. It is characterized by a *magnetic moment*, $\mathbf{m} = I\mathbf{S}$, as shown in Fig. 13.1.

Similarly to the polarization vector $\mathbf{P}$ in the case of polarized dielectrics, the *magnetization vector*, $\mathbf{M}$, describes the density of the vector magnetic moments in a magnetic material at a given point:

$$\mathbf{M} = \frac{(\sum \mathbf{m})_{\text{in } dv}}{dv} \qquad (\text{A/m}). \tag{13.1}$$

(Definition of magnetization vector)

If the number of Ampère currents per unit volume at a point is N, and the magnetic moment of individual atoms or molecules of the substance at that point is $\mathbf{m}$, the magnetization vector can be written in the form

$$\mathbf{M} = N\mathbf{m} \qquad (\text{A/m}). \tag{13.2}$$

(Alternative definition of magnetization vector)

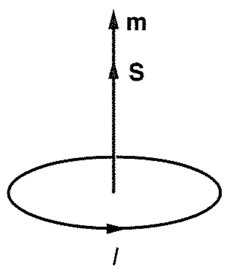


Figure 13.1 An elementary current loop is characterized by its magnetic moment, $\mathbf{m} = I\mathbf{S}$

Since the unit of N is $1/m^3$, and that of $\mathbf{m}$ is $A \cdot m^2$, the unit of the magnetization vector is A/m.

The magnetic field of a single current loop in a vacuum can be determined from the Biot-Savart law. It can be shown that the vector $\mathbf{B}$ of such a loop at large distances from the loop is proportional to the magnetic moment of the loop, $\mathbf{m}$. According to Eq. (13.1) we can subdivide magnetized materials into small volumes Δv, and represent such volumes (containing many Ampère currents) as a single larger Ampère current of moment $\mathbf{M}\,\Delta v$. Consequently, if we determine the magnetization vector at all points, we can find vector $\mathbf{B}$ by *integrating* the field of these larger Ampère currents over the magnetized material. This is much simpler than adding up the fields of individual Ampère currents, since their number is prohibitively large.

Questions and problems: Q13.1 to Q13.4, P13.1 and P13.2

13.3 Generalized Ampère's Law: Magnetic Field Intensity

We know that Ampère's law in the form in Eq. (12.14) is valid for any current distribution *in a vacuum*. We have explained, however, that a magnetized substance is just a vast number of elementary current loops in a vacuum. Therefore, we can apply Ampère's law to fields in materials, provided we find how to include these elementary currents on the right side of Eq. (12.14).

Shown in Fig. 13.2a is a surface S inside a piece of magnetized material bounded by contour C. We know that the choice of the surface S is arbitrary—the current intensity through any surface bounded by C is the same. Three classes of Ampère's currents are indicated in Fig. 13.2b: those that do not pass through S at all (e.g., the contour labeled 3); those that pass through S, but twice (contours labeled 1 and 2); and contours that encircle C, as the ones labeled 4 and 5. The first two types do not contribute to the total current through S. Contours of the third type pass through S only once, and are the only ones that contribute to the total current intensity through

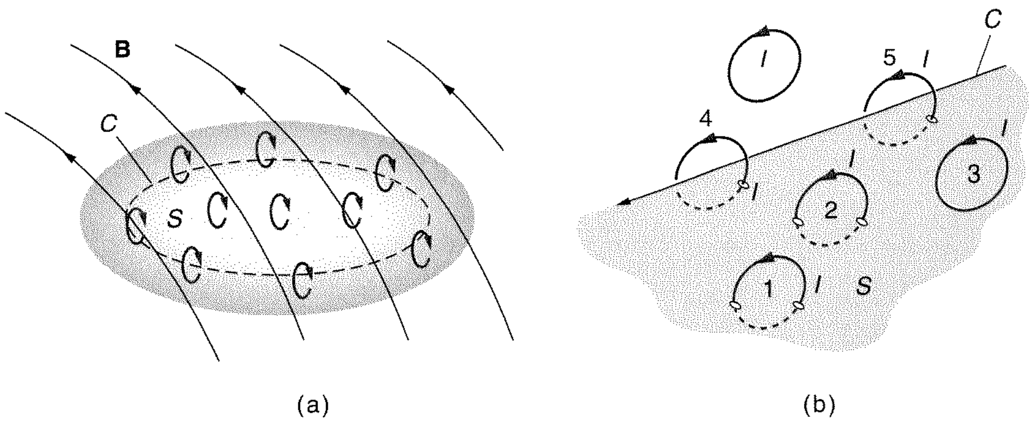


Figure 13.2 (a) A surface S in a piece of magnetized material bounded by a contour C. (b) Enlarged section of contour C and a part of surface S illustrate possible relative positions of Ampère's current loops.

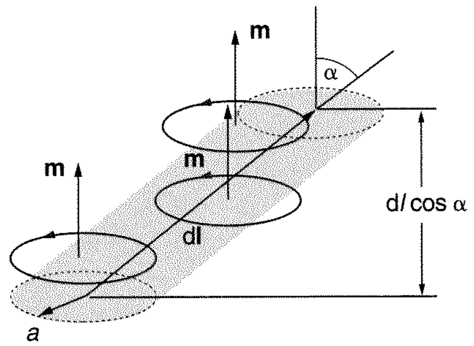


Figure 13.3 An element of contour C with neighboring current loops

S. What we have to find, therefore, is the total current of all the Ampère's currents that are "strung" along C like pearls in a necklace.

Consider Fig. 13.3, which shows an element $\mathbf{dl}$ of C with neighboring current loops. Let the radii of all the loops be a. It is clear that only those loops that are centered inside an oblique cylinder of circular base $a^2\pi$ and length $\mathbf{dl}$ fall into the third class of loops mentioned previously. Let the number of loops per unit volume be N. The number of loops encircling $\mathbf{dl}$ is then $Na^2\pi\, dl \cos\alpha$, so that the total current "strung" along $\mathbf{dl}$ is

$$dI_{\text{along } \mathbf{dl}} = NIa^2\pi\, dl \cos\alpha = Nm\, dl \cos\alpha = M\, dl \cos\alpha = \mathbf{M} \cdot \mathbf{dl}. \tag{13.3}$$

Therefore, the total current "strung" along the entire contour C, that is, the total current of all Ampère's currents through S, is given by

$$I_{\text{Ampère through } S} = \oint_C \mathbf{M} \cdot \mathbf{dl}. \tag{13.4}$$

It is now a simple matter to formulate Ampère's law valid for time-invariant currents in the presence of magnetized substance:

$$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu_0 \left(\int_S \mathbf{J} \cdot \mathbf{dS} + \oint_C \mathbf{M} \cdot \mathbf{dl} \right). \tag{13.5}$$

Because the contour C is the same for the integrals on the left and right side of the equation, this can be written as

$$\oint_C (\mathbf{B}/\mu_0 - \mathbf{M}) \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS}. \tag{13.6}$$

The combined vector, $(\mathbf{B}/\mu_0 - \mathbf{M})$, has a convenient property. Its line integral along any closed contour depends only on the *actual* current through the contour. This is the only current we can control—switch it on and off, change its intensity or direction, etc. For this reason the vector $(\mathbf{B}/\mu_0 - \mathbf{M})$ is defined as a new vector for the description of the magnetic field in the presence of materials, known as the *magnetic field intensity*, $\mathbf{H}$:

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad \text{(A/m)}. \tag{13.7}$$

(Definition of vector **H***)*

With this definition, the generalized Ampère's law takes the final form

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}. \tag{13.8}$$

(Ampère's law for time-invariant currents in the presence of substances)

As its special form, valid for currents in a vacuum, this form of Ampère's law is also valid *only for time-constant currents*. Also, just as its special form, it can be used for determining **H** resulting from highly symmetrical current distributions (e.g., a straight wire with a coaxial magnetic coating). As the procedure is essentially the same, the reader will find several examples of this application of Eq. (13.8) in the problems at the end of the chapter.

The definition of the magnetic field intensity vector in Eq. (13.7) is its most general definition, valid for any material. Most materials are those for which the magnetization vector, **M**, is a linear function of the local vector **B** (which does the actual magnetizing of the material). According to Eq. (13.7), in such cases a linear relationship exists between any two of the three vectors, **H**, **B**, and **M**. Usually vector **M** is expressed as

$$\mathbf{M} = \chi_m \mathbf{H} \qquad (\chi_{\mathrm{m}} \text{ is dimensionless; } M \text{ is in A/m}). \tag{13.9}$$

(Definition of magnetic susceptibility, χ_m)

The dimensionless factor χ_m is known as the *magnetic susceptibility* of the material. We then use Eq. (13.7) and express **B** in terms of **H**:

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H} \quad (\mu_r \text{ is dimensionless; } \mu \text{ is in H/m}). \tag{13.10}$$

(Definition of relative permeability, μ_r, and of permeability, μ)

The dimensionless factor $\mu_r = (1+\chi_m)$ is known as the *relative permeability* of the material, and μ (H/m) as the *permeability* of the material. Materials for which Eq. (13.9) holds are *linear magnetic materials*. If it does not hold, they are *nonlinear*. If at all points of the material μ is the same, the material is said to be *homogeneous;* otherwise it is *inhomogeneous*.

Linear magnetic materials can be *diamagnetic*, for which $\chi_m < 0$ (that is, $\mu_r < 1$), or *paramagnetic*, for which $\chi_m > 0$ (that is, $\mu_r > 1$). We will discuss this in more detail in section 13.6. Here it suffices to mention that for both diamagnetic and paramagnetic materials $\mu_r \simeq 1$, differing from unity by less than ± 0.001. Therefore, in almost all applications diamagnetic and paramagnetic materials can be considered to have $\mu = \mu_0$.

Like Gauss' law, Ampère's law in Eq. (13.8) can be transformed into a differential equation, i.e., its differential form. This can easily be done by applying the so-called Stokes's theorem of vector analysis (see Appendix 1, section A1.4.7). According to this theorem, a line integral around a closed contour C of a vector (e.g., vector $\mathbf{H}$) equals the integral of the curl of that vector over *any* surface spanned over C:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} \qquad \text{(Stokes's theorem)}.$$

So, instead of Eq. (13.8), we can write the equivalent equation

$$\int_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S}. \tag{13.11}$$

This equation must be valid for *any* contour C and *any* surface spanned over it. This is possible only if the integrands of the integrals on the two sides of the equation are equal, that is,

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}.} \tag{13.12}$$

(Ampère's law in differential form, for time-invariant currents)

This is the differential form of the generalized Ampère's law for magnetized materials and time-invariant currents. We shall use the differential form of Ampère's law in later chapters for solving various electromagnetic problems.

Questions and problems: Q13.5 to Q13.8, P13.3 to P13.5

13.4 Macroscopic Currents Equivalent to Ampère's Currents

Let us now determine the macroscopic currents *in a vacuum* that can replace a magnetized material. We anticipate that both volume and surface currents can be expected, similar to volume and surface polarization charges in polarized dielectrics. We will see that, in analogy to homogeneous dielectrics, there are no equivalent volume currents inside *homogeneous and linear* magnetic materials with no free currents. In such cases, equivalent surface currents in a vacuum are all that is needed.

Consider a small closed contour ΔC bounding a surface of area ΔS inside a magnetized material. The total current through ΔC (i.e., through any surface bounded by ΔC) is given in Eq. (13.4), where C should be replaced by ΔC. If we divide this by ΔS, we obtain the component of the volume current density vector due to magnetization of the material, in the direction of the normal, $\mathbf{n}$, to ΔS:

$$(J_m)_{\text{normal to } \Delta S} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_{\Delta C} \mathbf{M} \cdot d\mathbf{l}. \tag{13.13}$$

From mathematics we know that the right-hand side of this equation is precisely the component of $\nabla \times \mathbf{M}$ in the direction of the normal unit vector $\mathbf{n}$ (see Appendix 1, section A1.4.3). The total vector $\mathbf{J}_m$ at a point is equal to the vector sum of its three components, so that the volume density of magnetization current is given by

$$\mathbf{J}_m = \nabla \times \mathbf{M} \qquad (\mathrm{A/m^2}). \tag{13.14}$$

(Density of magnetization currents)

Let the material be homogeneous, of magnetic susceptibility χ_m, with no macroscopic currents in it. Then

$$\mathbf{J}_m = \nabla \times \mathbf{M} = \nabla \times (\chi_m \mathbf{H}) = \chi_m \nabla \times \mathbf{H} = 0, \tag{13.15}$$

since $\nabla \times \mathbf{H} = 0$ if $\mathbf{J} = 0$, as assumed. We have thus proven that in a homogeneous magnetized material with no macroscopic currents there is no volume distribution of magnetization currents.

In addition to the preceding relationship between current density and the magnetic field vector, it can be shown that on a boundary between two magnetized materials, as in Fig. 13.4, the surface magnetic current density is given by

$$\mathbf{J}_{\mathrm{ms}} = \mathbf{n} \times (\mathbf{M}_1 - \mathbf{M}_2) \ (\mathrm{A/m}). \tag{13.16}$$

(Density of magnetization surface currents)

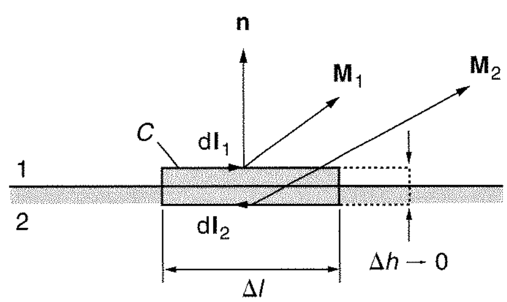


Figure 13.4 Boundary surface between two different magnetized materials

This can be proved by applying Eq. (13.4) to a small rectangular contour similar to that in Fig. 13.~~5~~4, which is left to the reader as an exercise.

Questions and problems: Q13.9 to Q13.14, P13.6 to P13.15

13.5 Boundary Conditions

Quite often it is necessary to solve magnetic problems involving inhomogeneous magnetic materials including boundaries. As in electrostatics, to be able to do this it is necessary to know the relations that must be satisfied by various magnetic quantities at two close points on the two sides of a boundary surface. We already know that all such relations are called *boundary conditions*.

We shall derive boundary conditions for the tangential components of **H** and the normal components of **B**. Assume that there are no macroscopic surface currents on the boundary surface. We use Ampère's law in Eq. (13.8) and apply it to a rectangular contour indicated in Fig. 13.5. There is no current through the contour (no macroscopic surface currents), so we find, as earlier in the case of electrostatics, that the tangential components of **H** must be equal:

$$\mathbf{H}_{1\mathrm{tang}} = \mathbf{H}_{2\mathrm{tang}}. \tag{13.17}$$

(Boundary condition for tangential components of **H**, *no surface currents on boundary)*

The condition for the normal components of **B** is obtained from the law of conservation of magnetic flux, Eq. (12.11). Let us apply Eq. (12.11) to the coinlike cylindrical surface with vanishingly small height shown in Fig. 13.5. In the same way as in electrostatics, where we derived the boundary condition for normal components of vector **D** from Gauss' law, we obtain

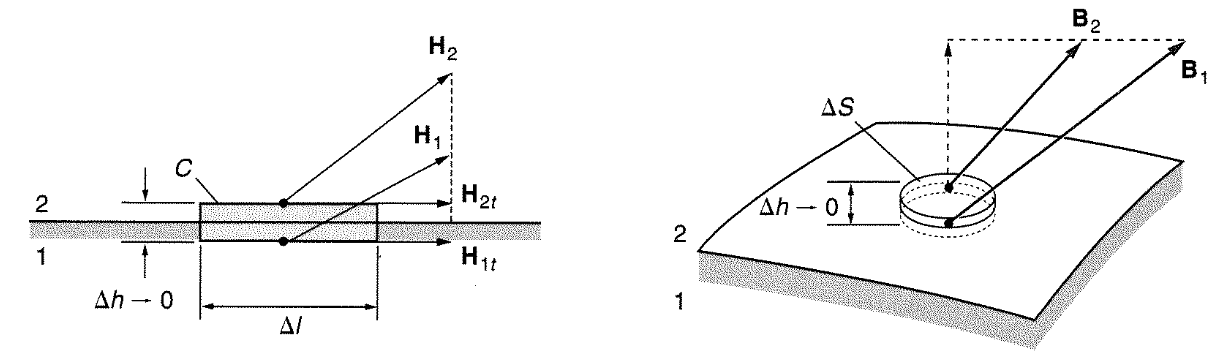


Figure 13.5 Boundary surface between two magnetic materials

$$\mathbf{B}_{1\text{norm}} = \mathbf{B}_{2\text{norm}}. \tag{13.18}$$

(Boundary condition for normal components of **B***)*

The boundary conditions in Eqs. (13.17) and (13.18) are valid for *any* media—linear or nonlinear. If the two media are linear, characterized by permeabilities μ_1 and μ_2, the two conditions can be also written in the form

$$\frac{\mathbf{B}_{1\text{tang}}}{\mu_1} = \frac{\mathbf{B}_{2\text{tang}}}{\mu_2}, \tag{13.19}$$

and

$$\mu_1 \mathbf{H}_{1\text{norm}} = \mu_2 \mathbf{H}_{2\text{norm}}. \tag{13.20}$$

Example 13.1—Law of refraction of magnetic field lines. If two media divided by a boundary surface are linear, the lines of vector **B** or **H** refract on the surface following a simple rule. This rule is obtained from boundary conditions in Eqs. (13.19) and (13.20).

As seen from Fig. 13.6,

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{H_{1\text{tang}}/H_{1\text{norm}}}{H_{2\text{tang}}/H_{2\text{norm}}} = \frac{H_{2\text{norm}}}{H_{1\text{norm}}},$$

since tangential components of **H** are equal. Using now the condition in Eq. (13.20), we obtain

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\mu_1}{\mu_2}. \tag{13.21}$$

Example 13.2—Refraction of magnetic field lines on a boundary surface between air and a material of high permeability. The most interesting practical case of refraction of magnetic field lines is on the boundary surface between air and a medium of high permeability.

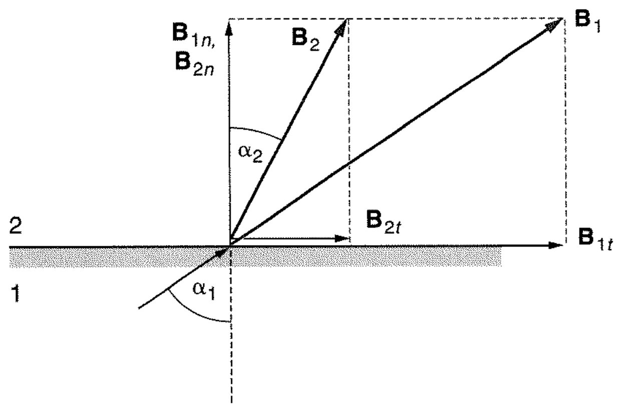


Figure 13.6 Lines of vector **B** or vector **H** refract according to Eq. (13.21)

Let air be medium 1. Then the right-hand side of Eq. (13.21) is very small. This means that $\tan\alpha_1$ must also be very small for *any* α_2 (except if $\alpha_2 = \pi/2$, that is, if the magnetic field lines in the medium of high permeability are tangential to the boundary surface). Since for small angles $\tan\alpha_1 \simeq \alpha_1$, the *magnetic field lines in air are practically normal to the surface of high permeability*. This conclusion is very important in the analysis of electrical machines with cores of high permeability.

Questions and problems: P13.16 to P13.18

13.6 Basic Magnetic Properties of Materials

In the absence of an external magnetic field, atoms and molecules of many materials have no magnetic moment. This is the first type of materials we will consider. The atoms and molecules of the second type of materials do have a magnetic moment, but with no external magnetic field these moments are distributed randomly, and no macroscopic magnetic field results.

13.6.1 DIAMAGNETIC AND PARAMAGNETIC MATERIALS

Materials of the first type are *diamagnetic materials*. When they are brought into a magnetic field, a current is induced in each atom and has the effect of *reducing* the field. (This effect is due to electromagnetic induction, to be studied in the next chapter, and exists in *all* materials. It is very small in magnitude, and in materials that are not diamagnetic it is overwhelmed by stronger effects.) Since their presence slightly *reduces* the magnetic field, diamagnetics evidently have a permeability slightly *smaller* than μ_0. Examples are water ($\mu_r = 0.9999912$), bismuth ($\mu_r = 0.99984$), and silver ($\mu_r = 0.999975$).

One class of materials of the second type is *paramagnetic* materials. With no external field present, the atoms in a paramagnetic material have their magnetic moments, but these moments are oriented randomly. When a field is applied, the Ampère currents of atoms align themselves with the field to some extent. This alignment is opposed by the thermal motion of the atoms, so alignment increases as the temperature decreases and as the applied magnetic field becomes stronger. The result of the alignment of the Ampère currents is a very small magnetic field *added* to the external field. For paramagnetic materials, therefore, μ is slightly greater than μ_0, and μ_r is slightly greater than one. Examples are air ($\mu_r = 1.00000036$) and aluminum ($\mu_r = 1.000021$).

The words *diamagnetic* and *paramagnetic* come from the first experiments performed to determine the magnetic nature of these materials. If a rod of a diamagnetic material is placed in a magnetic field, the magnetic moments of the atoms will try to oppose the field, and the rod will orient itself perpendicular to the lines of the magnetic field vector. The word *dia* in Greek means "across." In paramagnetics, the magnetic field of the atoms will tend to align with the external field, and the rod will orient itself parallel to the field lines. The word *para* in Greek means "along."

13.6.2 FERROMAGNETIC MATERIALS

The most important magnetic materials in electrical engineering are known as *ferromagnetic materials*. The name comes from the Latin word for iron, *ferrum*. They are actually paramagnetic materials, but with very strong interactions between the atoms (or molecules). As a result of these interactions, groups of atoms (10^{12} to 10^{15} atoms in a group) are formed inside the ferromagnetic material, and in these groups the magnetic moments of all the atoms are oriented in the same direction. These groups of molecules are called *Weiss' domains*. Each domain is, in fact, a small saturated magnet. Sketches of the magnetic moments in paramagnetics and ferromagnetics as shown in Fig. 13.7.

The size of a domain varies from material to material. In iron, for example, under normal conditions the linear dimensions of the domains are 10^{-5} m. In some cases they can get as large as a few millimeters, or even a few centimeters across. If a piece of a highly polished ferromagnetic material is covered with fine ferromagnetic powder, it is possible to see the outlines of the domains under a microscope. The boundary between two domains is not abrupt, and it is called a *Bloch wall*. This is a region 10^{-8} to 10^{-6} μm in width (500 to 5000 interatomic distances), in which the orientation of the atomic magnetic moments changes gradually.

The explanation of how and why domains form is beyond classical physics, and was quantum-mechanically described by Heisenberg in 1928. All ferromagnetics have a *crystal* structure. The ions of the crystal lattice have a magnetic moment that mostly comes from the electron spin. Ferromagnetic materials have very strong electric forces between electrons in adjacent ions, which align the magnetic moments of the ions in microscopical volumes. These forces are equivalent to extremely large intensities of vector **B**, on the order of 10^5 T, and they act in regions that are about 10^{-2} mm on the side. For comparison, the strongest magnetic induction obtained in a laboratory environment is about 30 T, and the strong nuclear magnetic resonance (NMR) magnetic fields used in medical diagnostics are about 2 to 4 T.

Above a certain temperature, called the *Curie temperature*, the thermal vibrations completely prevent the parallel alignment of the molecule magnetic moments, and ferromagnetic materials become paramagnetic. This temperature is 770°C for iron (for comparison, the melting temperature of iron is 1530°C).

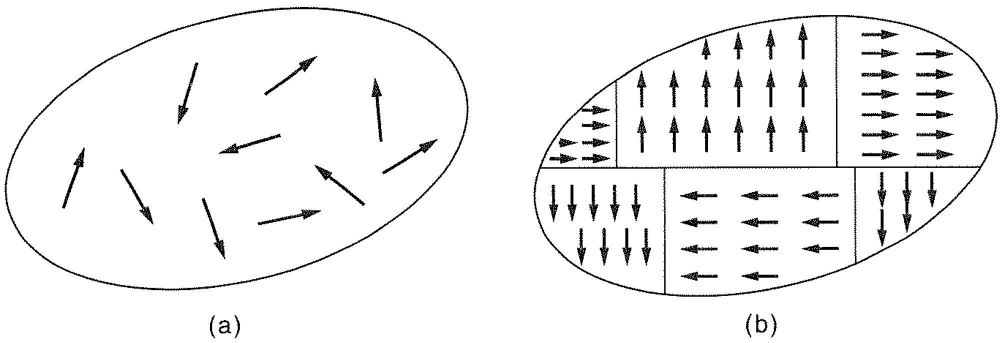


Figure 13.7 Schematic of an unmagnetized (a) paramagnetic and (b) ferromagnetic material. The arrows qualitatively show atom magnetic moments.

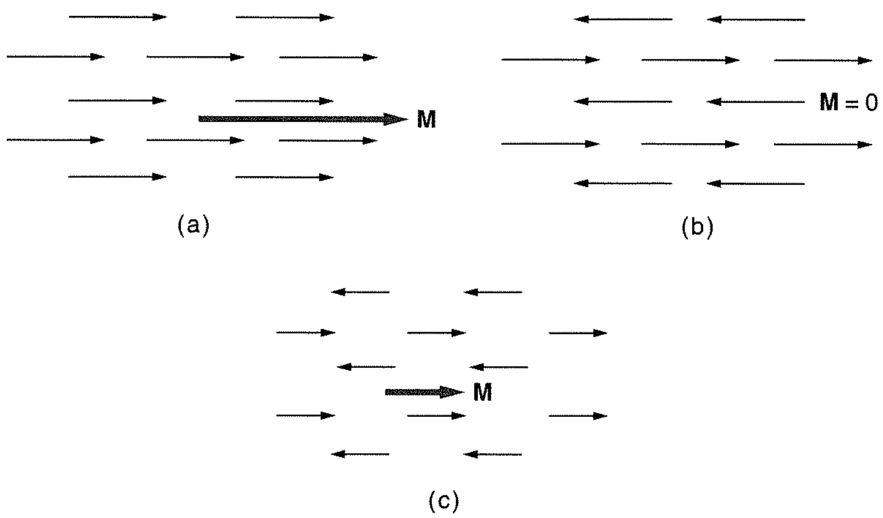


Figure 13.8 Schematic of Weiss' domains for (a) ferromagnetic, (b) antiferromagnetic, and (c) ferrite materials. The arrows represent molecular magnetic moments.

13.6.3 ANTIFERROMAGNETIC MATERIALS; FERRITES

Another class of materials is called *antiferromagnetics*. In these materials, the magnetic moments of adjacent molecules are antiparallel, so that the net magnetic moment is zero. (Examples are FeO, $CuCl_2$, and FeF_2, which are not widely used.) A subclass of antiferromagnetics called *ferrites* are widely used at radio frequencies. They also have antiparallel moments, but because of their asymmetrical structure, the net magnetic moment is not zero and Weiss' domains exist. Ferrites are weaker magnets than ferromagnetics, but they have high electrical resistivities, which makes them important for high-frequency applications, as we will see later in the text. Figure 13.8 shows a schematic comparison of the Weiss' domains for ferromagnetic, antiferromagnetic, and ferrite materials.

13.6.4 MAGNETIZATION CURVES OF FERROMAGNETIC MATERIALS

Ferromagnetic materials are *nonlinear*. This means that $\mathbf{B} = \mu\mathbf{H}$ is *not* true. How does a ferromagnetic material behave when placed in an external magnetic field? As the external magnetic field is increased from zero, the domains that are approximately aligned with the field increase in size. Up to a certain (not large) field magnitude, this process is reversible—if the field is turned off, the domains go back to their initial states. Above a certain field strength, the domains start rotating under the influence of magnetic forces, and this process is irreversible. The domains will keep rotating until they are all aligned with the local magnetic flux density vector. At this point, the ferromagnetic is *saturated*, and applying a stronger magnetic field does not increase the magnetization vector.

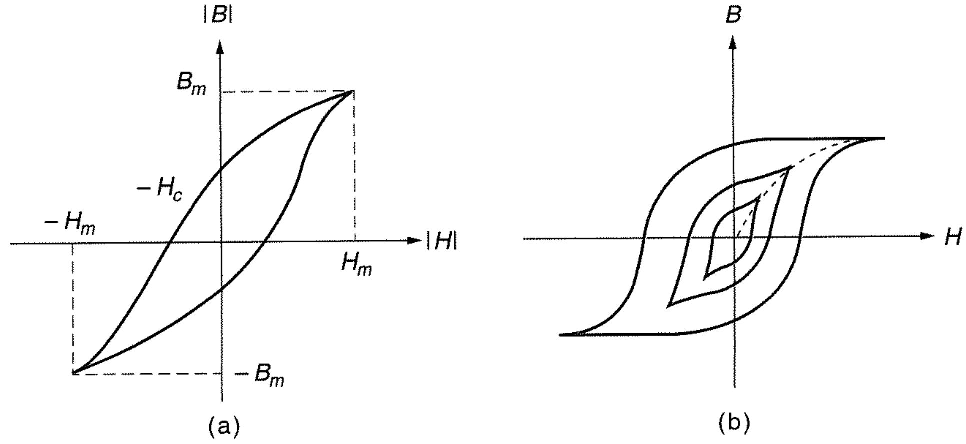


Figure 13.9 (a) Typical hysteresis loop for a ferromagnetic material. (b) The hysteresis loops for external fields of different magnitudes have different shapes. The line connecting the tips of these loops is the normal magnetization curve (shown by dashed line).

When the domains rotate there is a kind of friction between them, and this gives rise to some essential properties of ferromagnetics. If the field is turned off, the domains cannot rotate back to their original positions because they cannot overcome this friction. This means that some *permanent* magnetization is retained in the ferromagnetic material. A second consequence of friction between domains is losses to thermal energy (heat), and the third consequence is *hysteresis*, which is a term for a specific nonlinear behavior of the material. This is described by curves $B(H)$, usually measured on toroidal samples of the material. These curves are closed curves around the origin, and they are called *hysteresis loops*, shown in Fig. 13.9a. The hysteresis loops for external fields of different magnitudes have different shapes, as in Fig. 13.9b. The curve connecting the tips of these loops is known as the *normal magnetization curve*.

13.6.5 DEFINITIONS OF PERMEABILITY

In electrical engineering applications, the external magnetic field is usually sinusoidal. It needs to pass through several periods until the $B(H)$ curve stabilizes. The shape of the hysteresis loop depends on the frequency of the field, as well as its strength. For small field strengths it looks like an ellipse. It turns out that the ellipse approximation of the hysteresis loop is equivalent to a *complex permeability*. For sinusoidal time variation of the field, in complex notation we can write $\underline{\mathbf{B}} = \underline{\mu}\underline{\mathbf{H}}$. (This may look strange, but it is essentially the same as when we write, for example, that a complex voltage equals the product of complex impedance and complex current.) This approximation does not take saturation into account.

A ferromagnetic material in small external sinusoidal fields in complex notation can be characterized by the following parameters:

$$\underline{\sigma} = \sigma, \qquad \underline{\mu} = \mu' - j\mu'', \qquad \underline{\epsilon} = \epsilon' - j\epsilon''. \tag{13.22}$$

Here, σ is the conductivity of the material, and describes Joule's losses. The imaginary part, μ'', of complex permeability can be shown to describe hysteresis losses in the ferromagnetic material. If σ is small (e.g., in ferrites), in some cases it is necessary to also describe the ferromagnetic material by a complex permittivity $\underline{\epsilon}$, having the same meaning as complex permeability but with respect to a sinusoidally time-varying *electric* field. If σ is not small, it is not necessary to characterize the material by the complex permittivity.

As explained, ferrites in small sinusoidal fields need to be described by a complex permeability and a complex permittivity. They are sometimes referred to as *ceramic ferromagnetic materials*, as opposed to *metallic ferromagnetic materials* (iron, for example). The loss mechanism is different for the two. Metallic ferromagnetics have only hysteresis losses (we will see in a later chapter that they are proportional to frequency, f). In ferrites, the dielectric losses, described by ϵ'', are also present (they can even be predominant), and they are proportional to f^2.

The ratio B/H (corresponding to the permeability of linear magnetic materials) for ferromagnetic materials is not a constant. It is possible to define several "permeabilities," e.g., the one corresponding to the initial, reversible segment of the magnetization curve. This permeability is known as the *initial permeability*. The range is very large, from about $500\mu_0$ for iron to several hundreds of thousands μ_0 for some alloys.

The ratio B/H along the normal magnetization curve (Fig. 13.9b) is known as the *normal permeability*. If we magnetize a material with a dc field, and then add to this field a small sinusoidal field, a small hysteresis loop will be obtained that will have a certain ratio $\Delta B/\Delta H$. This ratio is known as the *differential permeability*.

13.6.6 MEASUREMENT OF MAGNETIZATION CURVES

The curve $B(H)$ that describes the nonlinear material is usually obtained by measurement. The way this is done is shown in Fig. 13.10a. A thin toroidal core of mean radius R, made of the material we want to measure, has N tightly wound turns of wire, and

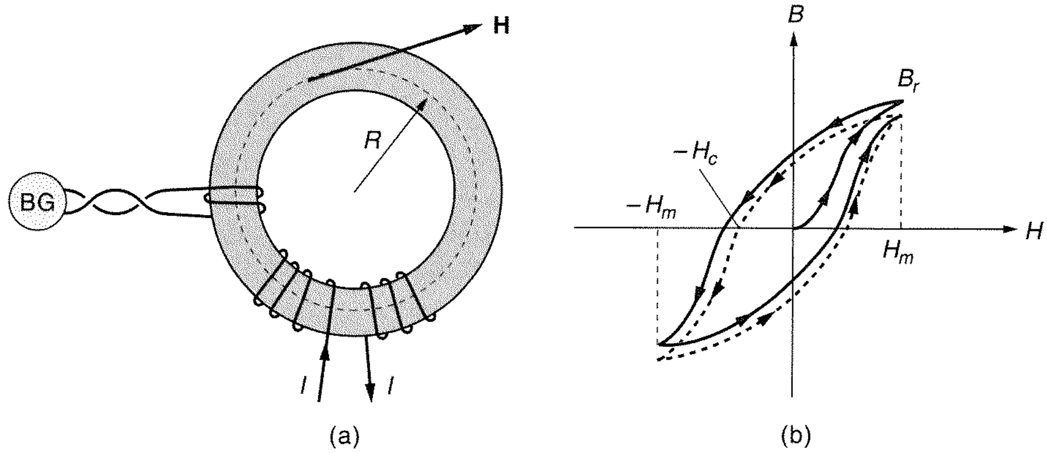


Figure 13.10 (a) The $B(H)$ curve for a nonlinear material as measured with a ballistic galvanometer. (b) An example of measured $B(H)$ shows the formation of the hysteresis loop.

a cross-sectional area S. If there is a current I through the winding, Ampère's law tells us that

$$H = \frac{NI}{2\pi R}. \tag{13.23}$$

From this formula, we can calculate the magnetic field magnitude for any given current. Around the toroidal core is a second winding connected to a ballistic galvanometer. This is an instrument that measures the charge that passes through a circuit. We will see in a later chapter that the charge that flows through the circuit is proportional to the change of the magnetic flux, $\Delta Q \propto \Delta\Phi = S\,\Delta B$, and therefore to the change of the B field as well. So by changing the current I through the first winding, we can measure the curve $B(H)$ point by point. If the field H is changing slowly during this process, the measured curves are called *static magnetization curves*. Figure 13.10b shows the magnetization curve measured on a previously nonmagnetized piece of material, with the *initial magnetization curve*.

Questions and problems: Q13.15 to Q13.27, P13.19 to P13.22

13.7 Magnetic Circuits

The most frequent and important practical applications of ferromagnetic materials involve cores for transformers, motors, generators, relays, etc. The cores have different shapes, they may or may not have air gaps, and they are magnetized by a current flowing through a coil wound around a part of the core. These problems are hard to solve strictly, but the approximate analysis is accurate enough and easy because it resembles dc circuit analysis.

Consider a coil with N turns and a current I, situated in a *linear* magnetic material. Let us look at a thin tube of small magnetic flux $\Delta\Phi$. For this case, shown in Fig. 13.11, we can write

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = NI, \qquad \Delta\Phi = B\,\Delta S, \qquad \mathbf{B} = \mu\mathbf{H}, \qquad \oint_S \mathbf{B}\cdot d\mathbf{S} = 0. \tag{13.24}$$

The first equation from the left is Ampère's law; in the second equation both ΔS and B vary along the tube; and in the last equation S is any closed surface. A completely analogous set of equations can be written for dc currents in a thin current tube:

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = e, \qquad \Delta I = J\,\Delta S, \qquad \mathbf{J} = \sigma\mathbf{E}, \qquad \oint_S \mathbf{J}\cdot d\mathbf{S} = 0. \tag{13.25}$$

In these equations, e is the electromotive force in the circuit, ΔS and J vary along the tube, σ is the conductivity, and S is any closed surface. Because the two sets of equations are analogous, the solutions must have the same form. For the second set of equations, Ohm's law and the two Kirchhoff's laws tell us that

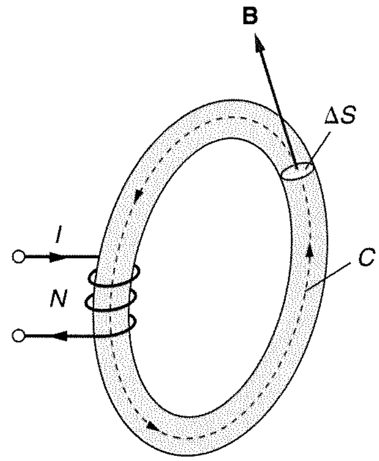


Figure 13.11 A thin magnetic circuit

$$\sum e - \sum RI = 0 \qquad \text{(any closed circuit)} \tag{13.26}$$

$$R = \int_C \frac{\mathrm{d}l}{\sigma\,\Delta S} \qquad \left(\text{or } R = \frac{l}{\sigma S} \text{ for } \Delta S \text{ constant}\right) \tag{13.27}$$

$$\sum I = 0 \qquad \text{(any node).} \tag{13.28}$$

In a magnetic circuit, the product NI plays the role of an electromotive force, and it is called the *magnetomotive force*. Permeability μ corresponds to conductivity σ. The magnetic flux corresponds to the electric current I. Therefore, we can write the following equations for the magnetic circuit:

$$\sum NI - \sum R_m \Phi_m = 0 \qquad \text{(any closed magnetic circuit)} \tag{13.29}$$

$$R_m = \int_C \frac{\mathrm{d}l}{\mu\,\Delta S} \qquad \left(\text{or} \quad R_m = \frac{l}{\mu S} \quad \text{for } \Delta S \text{ constant}\right) \tag{13.30}$$

$$\sum \Phi = 0 \qquad \text{(any node of the magnetic circuit).} \tag{13.31}$$

R_m is called the *magnetic resistance* (or sometimes *reluctance*) of the magnetic circuit. Equations (13.29) and (13.31) are known as Kirchhoff's laws for magnetic circuits.

Example 13.3—Thin toroidal coil as a magnetic circuit. Let us consider a thin toroidal coil of N turns, length l, and cross-sectional area S. Assume that the permeability of the core is μ, and that a current I is flowing through the coil. Using (13.29) and (13.30), we get

$$\Phi = \frac{NI}{R_m} = \frac{NI}{l/\mu S} = \mu\frac{NI}{l}S = \mu N' IS. \tag{13.32}$$

This is the same result we obtained when determining B for the coil using Ampère's law, and using $\Phi = BS$.

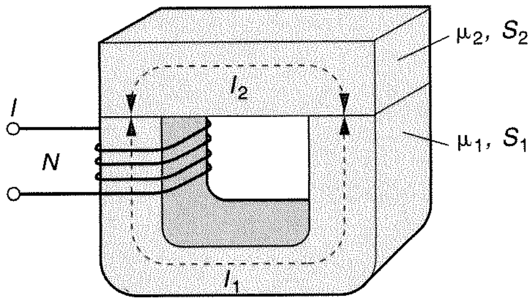


Figure 13.12 A realistic thick magnetic circuit

Example 13.4—Thick magnetic circuits and the error using thin magnetic circuit assumption. We have shown that the analysis of thin linear magnetic circuits is very simple. Unfortunately, real magnetic circuits are neither thin nor linear. However, analysis of thin linear magnetic circuits can be used as the basis for approximate analysis of actual magnetic circuits.

Consider a thick, U-shaped core of permeability μ_1, much larger than μ_0, closed by a thick bar of permeability μ_2, also much larger than μ_0, as shown in Fig. 13.12. N turns with a current I are wound on the core. The exact determination of the magnetic field in such a case is almost impossible. The first thing we can conclude is that since $\mu_1, \mu_2 \gg \mu_0$, the tangential component of the magnetic flux density **B** is much larger in the core than in the air outside it. The normal components of **B** are equal. So the magnetic flux density inside the core is generally much larger than outside. Therefore, the magnetic flux can be approximately considered to be restricted to the core. This is never exactly true, so this is the first assumption we are making.

Further, we assume that Eqs. (13.29) and (13.30) are reasonably accurate if lengths l_1 and l_2 are used as average lengths for the two circuit sections. It is interesting to show that the error in doing so is acceptable.

Consider the toroidal coil in Fig. 13.13a, the cross section of which is shown in Fig. 13.13b. The coil has N densely wound turns with a current I, so $H = (NI)/(2\pi r)$, and $B = (\mu NI)/(2\pi r)$.

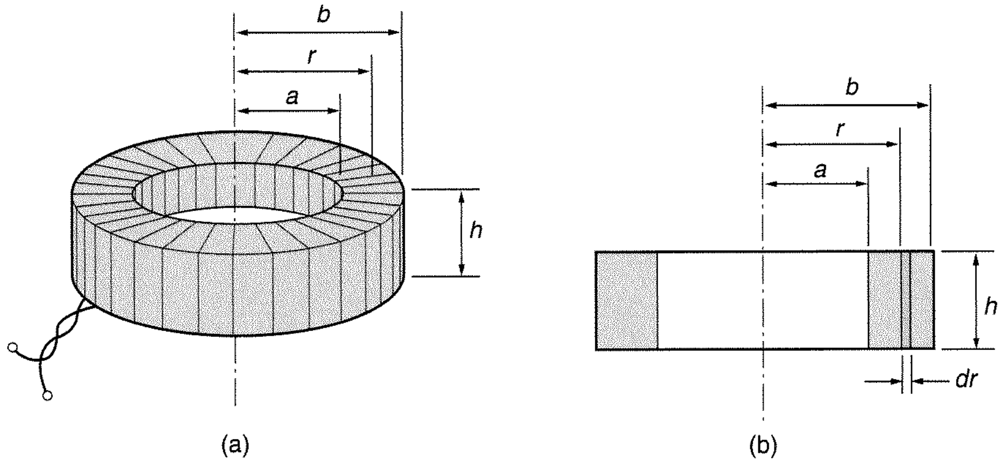


Figure 13.13 (a) A toroidal coil and (b) a cross section of the coil

The exact value of the magnetic flux through the toroid cross section is

$$\Phi_{\text{exact}} = \frac{\mu NIh}{2\pi}\int_a^b \frac{\mathrm{d}r}{r} = \frac{\mu NIh}{2\pi}\ln\frac{b}{a}. \tag{13.33}$$

According to Eqs. (13.29) and (13.30), adopting the average length of the toroidal core, the approximate flux is

$$\Phi_{\text{approx}} = \frac{NI}{R_m} = \frac{NI}{[\pi(a+b)]/[\mu(b-a)h]} = \frac{\mu NIh}{2\pi}\frac{2(b-a)}{b+a}. \tag{13.34}$$

The relative error is

$$\frac{\Phi_{\text{approx}} - \Phi_{\text{exact}}}{\Phi_{\text{exact}}} = \frac{2(b/a-1)}{b/a\ln(b/a)} - 1, \tag{13.35}$$

which is very small even for quite thick toroids. For example, if $b/a = e = 2.718\ldots$, the error is less than 8%. So the magnetic flux in the magnetic circuit in Fig. 13.12 can be determined approximately as

$$\Phi \simeq \frac{NI}{l_1/(\mu_1 S_1) + l_2/(\mu_2 S_2)}. \tag{13.36}$$

Example 13.5—A complex magnetic circuit. In the case of more complicated magnetic circuits, such as the one shown in Fig. 13.14a, finding the fluxes through the branches is analogous to solving for the currents in a dc circuit. The schematic of the magnetic circuit in terms of magnetomotive forces (analogous to batteries) and magnetic resistances (analogous to resistors) is shown in Figure 13.14b. The narrow air gap is assumed not to introduce considerable flux leakage, and its reluctance is given by $R_0 = l_0/(\mu_0 S_0)$. Since μ_0 is always much smaller than μ, even a very narrow gap has considerable influence on the distribution of flux in the circuit

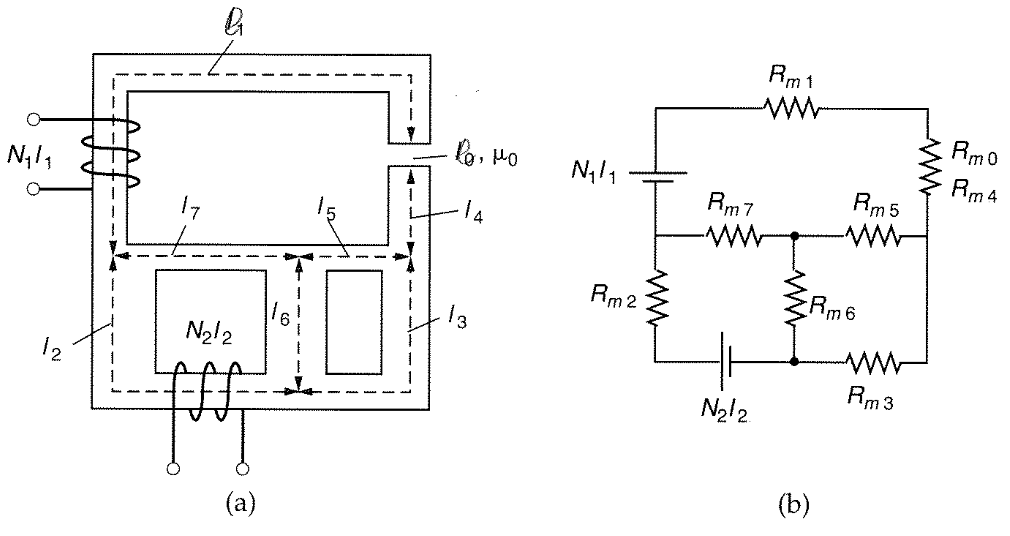
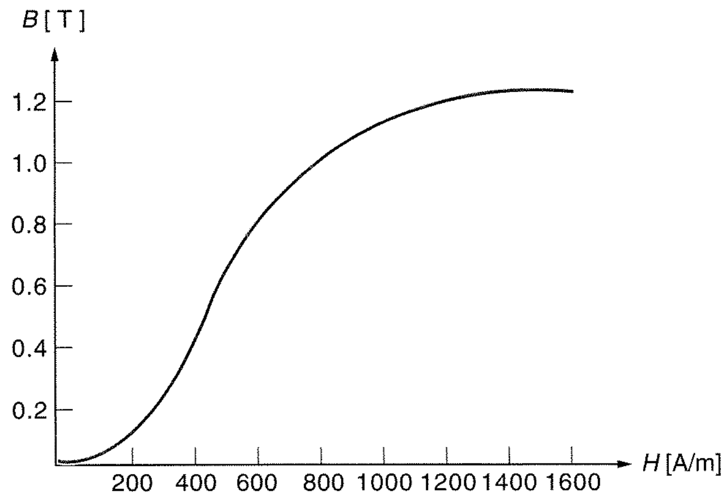


Figure 13.14 (a) A complex magnetic circuit and (b) its representation analogous to that of an electrical network

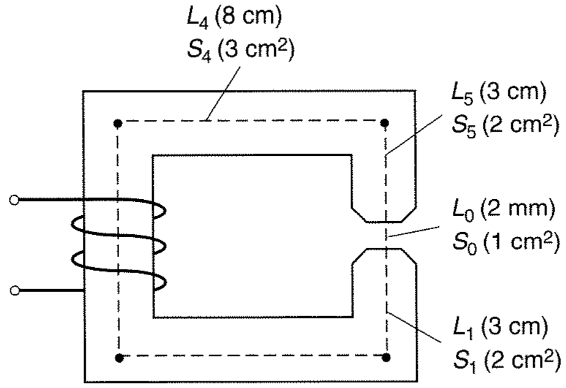
because its reluctance is very high. The nodes in the circuit are actually surfaces enclosing the entire region where three or more branches of the circuit meet.

Example 13.6—Analysis of a simple nonlinear magnetic circuit. Real magnetic circuits are made of ferromagnetic materials, so they are *always nonlinear*. If the magnetization of the core is weak, the circuit can be considered as linear. This is the case for most applications of ferromagnetic materials in electronics. In power engineering, however, cores only rarely operate in the linear region. In that case, instead of using $B = \mu H$ (as in the linear analysis of magnetic circuits), we use an experimentally obtained curve $B(H)$. We shall illustrate this procedure with a simple example.

Figure 13.15a shows the initial magnetization curve of a ferromagnetic material from which the core of the circuit in Fig. 13.15b is made. We wish to determine the current intensity I through the coil needed to produce a flux density in the air gap of $B \simeq 1$T. The core was not previously magnetized. The average lengths of the core sections and the corresponding cross-sectional areas are indicated in the figure. Ignore the leakage flux.



(a)



(b)

Figure 13.15 (a) Initial magnetization curve of a nonlinear material and (b) a nonlinear magnetic circuit made of the same material

TABLE 13.1 Summary of nonlinear magnetic circuit calculations for the circuit in Fig. 13.15. It can be seen from the rightmost column that $NI = 1717.8$. For all rows, the material is the nonlinear ferromagnetic, except for $k = 1$, when it is air in the gap.

k	L_k(cm)	S_k(cm^2)	B_k(T)	H_k(A/m)	$H_k l_k$(A)
0	0.2	1	1.0	$8 \cdot 10^5$	1600
1	3	2	0.5	540	16.2
2	8	4	0.25	380	30.4
3	6.2	4	0.25	380	19.8
4	8	3	0.33	440	35.2
5	3	2	0.5	540	16.2

The current intensity producing the desired magnetic flux intensity in the air gap is determined by the following procedure:

$\Phi = B_0 S$ current

1. From the specified B_0 in the air gap, the magnetic flux through the gap is $\Phi_0 = B_0 S_0$. Since we are ignoring the leakage flux, this is also the flux, Φ, through the whole circuit.
2. The magnetic flux density in all parts of the circuit is determined according to $B_k = \Phi/S_k = B_0 S_0/S_k$.
3. The magnetic field intensities H_k corresponding to the flux densities B_k are determined next from the magnetization curve. In the air gap, $H_0 = B_0/\mu_0$.
4. The products $H_k l_k$ are calculated for all parts of the circuit.
5. Using the (approximate) expression for Ampère's law, we find the required product $NI = \sum H_k l_k$.

Table 13.1 summarizes this procedure. Note that the flux through the circuit is $\Phi = B_0 S_0 = 10^{-4}$ Wb (webers). From the table, the required intensity of the current is $I = \sum H_k l_k / N = 17.2$ A.

Questions and problems: Q13.28 to Q13.33, P13.23 to P13.30

13.8 Chapter Summary

1. In a magnetic field, materials become magnetized, i.e., a vast number of oriented elementary current loops known as *Ampère's currents* are formed inside them. These tiny currents together produce a secondary magnetic field, which can be much larger than the field that magnetized the material.
2. With respect to their magnetic properties, all materials are divided into three basic groups: linear diamagnetic and paramagnetic materials, and nonlinear ferromagnetic materials.
3. Magnetization at a point of a magnetized material is described by the *magnetization vector*, **M**, representing the (vector) volume density of magnetic moments of Ampère's currents.

4. It is possible to generate the same magnetic field as that due to magnetization by a distribution of *macroscopic currents*, known as magnetization currents, situated in a vacuum, and derivable from the magnetization vector.
5. Ampère's law for currents in a vacuum can be extended to include magnetization effects: $\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} = \sum I$. In formulating this more general form of Ampère's law, a new vector quantity is introduced, the magnetic field intensity, $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$.
6. This extended Ampère law tells us that the line integral of $\mathbf{H}$ around any closed contour equals the total *macroscopic, controllable current* through any surface spanned over it. It is used for solving various problems, among which are problems of magnetic circuits, analogous to electric circuits.
7. The differential form of Ampère's law, $\nabla \times \mathbf{H} = \mathbf{J}$, is a partial vector differential equation in components of vector $\mathbf{H}$.

QUESTIONS

Q13.1. Are any conventions implicit in the definition of the magnetic moment of a current loop? Explain.

Q13.2. A magnetized body is introduced into a uniform magnetic field. Is there a force on the body? Is there a moment of magnetic forces on the body? Explain.

Q13.3. A small body made of soft iron is placed on a table. Also on the table is a permanent magnet. If the body is pushed toward the magnet, ultimately the magnet will pull the body toward itself, so that the body will acquire a certain kinetic energy before it hits the magnet. Where did this energy come from?

Q13.4. Prove that the units for $\mathbf{B}$ and $\mu_0\mathbf{M}$ are the same.

Q13.5. The source of the magnetic field is a permanent magnet of magnetization $\mathbf{M}$. What is the line integral of the vector $\mathbf{H}$ around a contour that passes through the magnet?

Q13.6. The magnetic core of a thin toroidal coil is magnetized to saturation, and then the current in the coil is switched off. The remanent (i.e., remaining) flux density in the core is B_r. Determine the magnetization vector and the magnetic field strength vector in the core.

Q13.7. Is there a magnetic field in the air around the core in question Q13.6? Explain.

Q13.8. Why is Eq. (13.11) valid for any contour C and any surface bounded by C?

Q13.9. Why is the reference direction of the vector $\mathbf{J}_{ms}$ in Fig. 13.4 into the page?

Q13.10. Suppose that all atomic currents contained in the page you are reading can be oriented so that $\mathbf{m}$ is toward you. What is their macroscopic resultant, and what is (qualitatively) the magnetic field of such a "magnetic sheet"?

Q13.11. What are the macroscopic resultants of the microscopic currents of a short, circular, cylindrical piece of magnetized matter with uniform magnetization $\mathbf{M}$ at all points, if (1) $\mathbf{M}$ is parallel to the cylinder axis, or (2) $\mathbf{M}$ is perpendicular to the axis?

Q13.12. Sketch roughly the lines of vectors $\mathbf{M}$, $\mathbf{B}$, and $\mathbf{H}$ in the two cases in question Q13.11.

Q13.13. A ferromagnetic cube is magnetized uniformly over its volume. The magnetization vector is perpendicular to two sides of the cube. What is this cube equivalent to in terms of the magnetic field it produces?

Q13.14. Is the north magnetic pole of the earth close to its geographical North Pole? Explain. (See Chapter 17.)

Q13.15. If a high-velocity charged elementary particle pierces a toroidal core in which there is only remanent flux density, and $\mathbf{H} = 0$, will the particle be deflected by the magnetic field? Explain.

Q13.16. Sketch the initial magnetization curve corresponding to a change of H from zero to $-H_m$.

Q13.17. Suppose that the magnetization of a thin toroidal core corresponds to the point B_r (remanent flux density). The coil around the core is removed, and the magnetic flux density is uniformly decreased to zero by some appropriate *mechanical or thermal treatment*. How does the point in the B-H plane go to zero?

Q13.18. The initial magnetization curve of a certain ferromagnetic material is determined for a thin toroidal core. Explain the process of determining the magnetic flux in a core of the same material, but of the form shown in Fig. P13.5a.

Q13.19. Make a rough sketch of the curve obtained in the B-H plane if H is increased to H_m, then decreased to zero, then again increased to H_m and decreased to zero, and so on.

Q13.20. Is it possible to obtain a higher remanent flux density than that obtained when saturation is attained and then H is reduced to zero? Explain.

Q13.21. What do you expect would happen if a thin slice is cut out of a ferromagnetic toroid with remanent flux density in it? Explain.

Q13.22. A rod of ferromagnetic material can be magnetized in various ways. If a magnetized rod attracts a lot of ferromagnetic powder (e.g., iron filings) near its ends, and very little in its middle region, how is it magnetized?

Q13.23. If a small diamagnetic body is close to a strong permanent magnet, does the magnet attract or repel it? Explain.

Q13.24. Answer question Q13.23 for a small paramagnetic body.

Q13.25. While the core in question Q13.17 is still magnetized, if just one part of the core is heated above the Curie temperature, will there be a magnetic field in the air? If you think there will be, what happens when the heated part has cooled down?

Q13.26. Assuming that you use a large number of small current loops, explain how you can make a model of (1) a paramagnetic material and (2) a ferromagnetic material.

Q13.27. If the current in the coil wound around a ferromagnetic core is sinusoidal, is the magnetic flux in the core also sinusoidal? Explain.

Q13.28. Analyze similarities and differences for Kirchhoff's laws for dc electric circuits and magnetic circuits.

Q13.29. How do you determine the direction of the magnetomotive force in a magnetic circuit?

Q13.30. A thin magnetic circuit is made of a ferromagnetic material with an initial magnetization curve that can be approximated by the expression $B(H) = B_0H/(H_0 + H)$, where B_0 and H_0 are constants. If the magnetic field strength, H, in the circuit is much smaller than H_0, can the circuit be considered as linear? What is in that case the permeability of the material? What is the physical meaning of the constant B_0?

Q13.31. Why can't we have a magnetic circuit with no leakage flux (stray field in the air surrounding the magnetic circuit)?

Q13.32. Is it possible to construct a magnetic circuit closely analogous to a dc electric circuit, if the latter is situated (1) in a vacuum, or (2) in an imperfect dielectric? Explain.

Q13.33. One half of the length of a thin toroidal coil is filled with a ferromagnetic material, and the other half with some paramagnetic material. Can the problem be analyzed as a magnetic circuit? Explain.

PROBLEMS

P13.1. The magnetic moment of the earth is about $8 \cdot 10^{22}\,\mathrm{A \cdot m^2}$. Imagine that there is a giant loop around the earth's equator. How large does the current in the loop have to be to result in the same magnetic moment? Would it be theoretically possible to cancel the magnetic field of the earth with such a current loop (1) on its surface, or (2) at far points? The radius of the earth is approximately 6370 km.

P13.2. The number of iron atoms in one cubic centimeter is approximately $8.4 \cdot 10^{22}$, and the product of μ_0 and the maximum possible magnetization (corresponding to "saturation") is $\mu_0 M_{\mathrm{sat}} = 2.15\,\mathrm{T}$. Calculate the magnetic moment of an iron atom.

P13.3. A thin toroid is uniformly magnetized along its length with a magnetization vector of magnitude M. No free currents are present. Noting that the lines of **M**, **B**, and **H** inside the toroid are circles by symmetry, determine the magnitude of **B** and prove that $\mathbf{H} = 0$.

P13.4. A straight, long copper conductor of radius a is covered with a layer of iron of thickness d. A current of intensity I exists in this composite wire. Assuming that the iron permeability is μ, determine the magnetic field, the magnetic flux density, and the magnetization in copper and iron parts of the wire. Note that the current density in the copper and iron parts of the wire is not the same.

P13.5. The ferromagnetic toroidal core sketched in Fig. P13.5a has an idealized initial magnetization curve as shown in Fig. P13.5b. Determine the magnetic field strength, the magnetic flux density, and the magnetization at all points of the core, if the core is wound uniformly with $N = 628$ turns of wire with current of intensity (1) 0.5 A, (2)

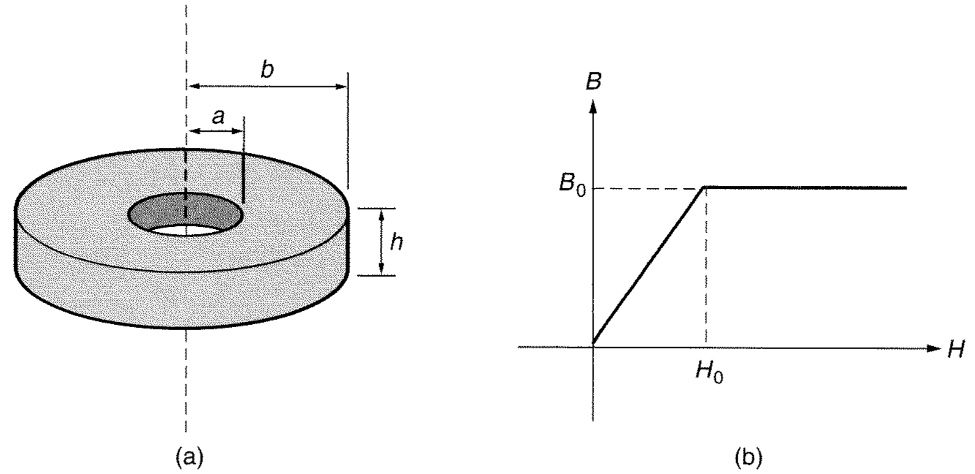


Figure P13.5 (a) A ferromagnetic core, and (b) its idealized initial magnetization curve

0.75 A, or (3) 1 A. The core dimensions are $a = 5\,\text{cm}$, $b = 10\,\text{cm}$, and $h = 5\,\text{cm}$, and the constants of the magnetization curve are $H_0 = 1000\,\text{A/m}$ and $B_0 = 2\,\text{T}$. For the three cases determine the magnetic flux through the core's cross section. Assume that the core was not magnetized prior to turning on the current in the winding.

P13.6. A straight conductor with circular cross section of radius a and permeability μ carries a current I. A coaxial conducting tube of inner radius b ($b > a$) and outer radius c, with no current, also has a permeability μ. Determine the magnetic field intensity, magnetic flux density, and magnetization at all points. Determine the volume and surface densities of macroscopic currents equivalent to Ampère currents.

P13.7. Repeat problem P13.6 if the conductor and the tube are of permeability $\mu(H) = \mu_0 H/H_0$, where H_0 is a constant.

P13.8. Repeat problems P13.6 and P13.7 assuming that the tube carries a current $-I$, so that the conductor and the tube make a coaxial cable.

P13.9. A ferromagnetic sphere of radius a is magnetized uniformly with a magnetization vector $\mathbf{M}$. Determine the density of magnetization surface currents equivalent to the magnetized sphere.

P13.10. A thin circular ferromagnetic disk of radius $a = 2\,\text{cm}$ and thickness $d = 2\,\text{mm}$ is uniformly magnetized normal to its bases. The vector $\mu_0\mathbf{M}$ is of magnitude 0.1 T. Determine the magnetic flux density vector on the disk axis normal to its bases, at a distance $r = 2\,\text{cm}$ from the center of the disk.

P13.11. A thin ferromagnetic toroidal core was magnetized to saturation, and then the current in the winding wound about the core was turned off. The remanent flux density of the core material is $B_r = 1.4\,\text{T}$. Determine the surface current density on the core equivalent to the Ampère currents. If the mean radius of the core is $R = 5\,\text{cm}$, and the winding has $N = 500$ turns of wire, find the current in the winding corresponding to this equivalent surface current. If the cross-sectional area of the core is $S = 1\,\text{cm}^2$, find the magnetic flux in the core.

P13.12. A round ferrite rod of radius $a = 0.5\,\text{cm}$ and length $b = 10\,\text{cm}$ is magnetized uniformly over its volume. The vector $\mu_0\mathbf{M}$ is in the direction of the rod axis, of magnitude 0.07 T. Determine the magnetic flux density at the center of one of the rod bases. Is it important whether the point is inside the rod, outside the rod, or on the very surface of the rod?

P13.13. Shown in Fig. P13.13 is a stripline with a ferrite dielectric. Since $a \gg d$, the magnetic field outside the strips can be neglected. Under this assumption, find the magnetic field intensity between the strips if the current in the strips is I. If the space between the strips is filled with a ferrite of relative permeability μ_r that can be considered

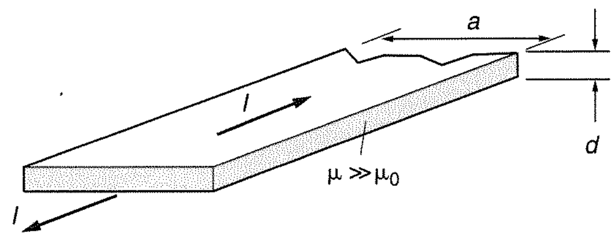


Figure P13.13 A stripline with a ferrite dielectric constant, determine the magnetic flux density and magnetization in the ferrite, and the density of surface currents equivalent to the Ampère currents in the ferrite.

P13.14. The ferromagnetic cube shown in Fig. P13.14 is magnetized in the direction of the z axis so that the magnitude of the magnetization vector is $M_z(x) = M_0 x/a$. Find the density of volume currents equivalent to the Ampère currents inside the cube, as well as the surface density of these currents over all cube sides. Follow the surface currents and note that in part they close through the magnetized material.

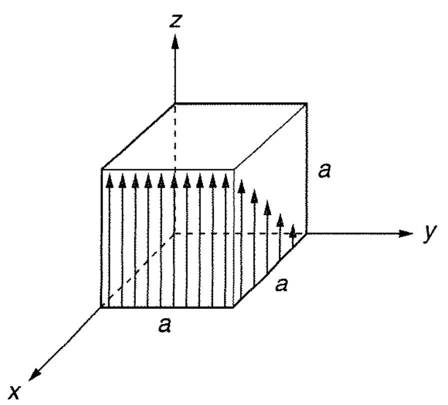


Figure P13.14 A magnetized ferromagnetic cube

P13.15. Prove that on the boundary surface of two magnetized materials the surface magnetization current is given by the expression $\mathbf{J}_{ms} = \mathbf{n} \times (\mathbf{M}_1 - \mathbf{M}_2)$. $\mathbf{M}_1$ and $\mathbf{M}_2$ are magnetization vectors in the two materials at close points on the two sides of the boundary, and $\mathbf{n}$ is the unit vector normal to the boundary, directed into medium 1.

P13.16. At a point boundary surface between air and a ferromagnetic material of permeability $\mu \gg \mu_0$ the lines of vector $\mathbf{B}$ are not normal to the boundary surface. Prove that the magnitude of the magnetic flux density vector in the ferromagnetic material is then much greater than that in the air.

P13.17. A current loop is in air above a ferromagnetic half-space. Prove that the field in the air due to the Ampère currents in the half-space is very nearly the same as that due to a loop below the boundary surface symmetrical to the original loop, carrying the current of the same intensity *and direction*, with the magnetic material removed. (This is the image method for ferromagnetic materials.)

P13.18. Inside a uniformly magnetized material of relative permeability μ_r are two cavities. One is a needlelike cavity in the direction of the vector $\mathbf{B}$. The other is a thin-disk cavity, normal to that vector. Determine the ratio of magnitudes of the magnetic flux density in the two cavities and that in the surrounding material. Using these results, estimate the greatest possible theoretical possibility of "magnetic shielding" from time-invariant external magnetic field. (We shall see that the shielding effect is greatly increased for time-varying fields.)

P13.19. Sketched in Fig. P13.19 is the normal magnetization curve of a ferromagnetic material. Using this diagram, estimate the relative normal and differential permeability, and plot their dependence on the magnetic field strength. What are the initial maximal permeabilities of the material?

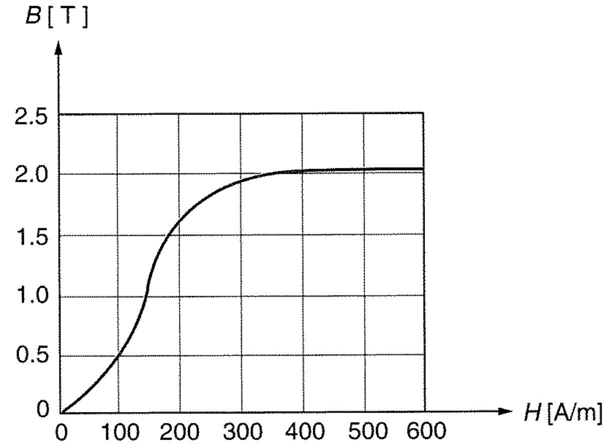


Figure P13.19 A normal magnetization curve

P13.20. Approximate the normal magnetization curve in Fig. P13.19 in the range $0 \leq H \leq 300\,\mathrm{A/m}$ by a straight line segment, and estimate the largest deviation of the normal relative permeability in this range from the relative permeability of such a hypothetical linear material.

P13.21. Figures P13.21a and b show two hysteresis loops corresponding to sinusoidal variation of the magnetic field strength in two ferromagnetic cores between $-H_m$ and $+H_m$. Plot the time dependence of the magnetic flux density in each core. Does the magnetic flux also have a sinusoidal time dependence?

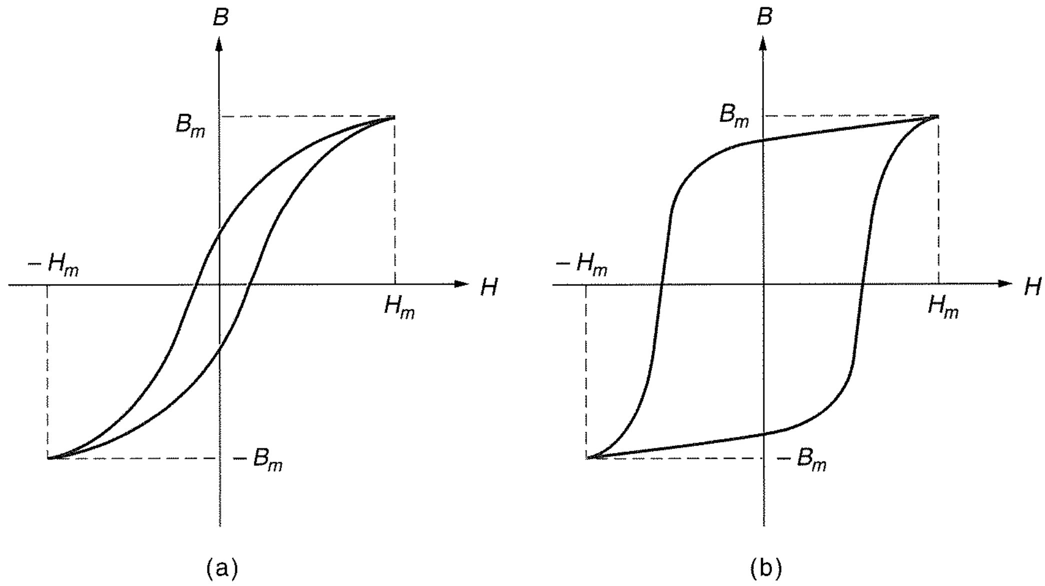


Figure P13.21 Hysteresis loops for two ferromagnetic materials

P13.22. If in Figs. P13.21a and b the magnetic flux density varies sinusoidally in time, sketch the time dependence of the magnetic field strength in the core. Is it also sinusoidal? If the hysteresis loops were obtained by measurements with sinusoidal magnetic field strength, is it absolutely correct to use such loops in this case?

P13.23. The initial magnetization curve (first part of hysteresis curve) $B(H)$ of a ferromagnetic material used for a transformer was measured and it was found that it can be approximated by a function of the form $B(H) = B_0H/(H_0 + H)$, where the coefficients are $B_0 = 1.37\,\mathrm{T}$ and $H_0 = 64\,\mathrm{A/m}$. Then a thin torus with mean radius $R = 10\,\mathrm{cm}$ and a cross section of $S = 1\,\mathrm{cm}^2$ is made out of this ferromagnetic material, and $N = 500$ turns are densely wound around it. Plot $B(H)$ and the flux through the magnetic circuit as a function of current intensity I through the winding. Find the flux for (1) $I = 0.25\,\mathrm{A}$, (2) $I = 0.5\,\mathrm{A}$, (3) $I = 0.75\,\mathrm{A}$, and (4) $I = 1\,\mathrm{A}$.

P13.24. Assume that for the ferromagnetic material in problem P13.23 you did not have a measured hysteresis curve, but you had one data point: for $H = 1000\,\mathrm{A/m}$, B was measured to be $B = 2\,\mathrm{T}$. From that, you can find an equivalent permeability and solve the circuit approximately, assuming that it is linear. Repeat the calculations from the preceding problem and calculate the error due to this approximation for the four current values given in problem P13.23.

P13.25. The thick toroidal core sketched in Fig. P13.25 is made out of the ferromagnetic material from problem P13.23. There are $N = 200$ turns wound around the core, and the core dimensions are $a = 3\,\mathrm{cm}$, $b = 6\,\mathrm{cm}$, and $h = 3\,\mathrm{cm}$. Find the magnetic flux through the core for $I = 0.2\,\mathrm{A}$ and $I = 1\,\mathrm{A}$ in two different ways: (1) using the mean radius; and (2) by dividing the core into 5 layers and finding the mean magnetic field in each of the layers.

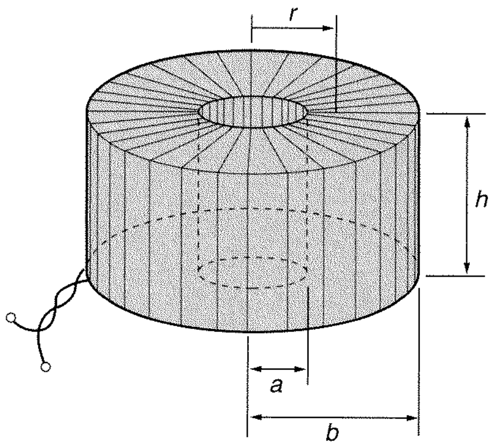


Figure P13.25 A thick toroidal coil

P13.26. Find the number of turns $N_1 = N_2 = N$ for the magnetic circuit shown in Fig. P13.26 so that the magnetic flux density in the air gap is $B_0 = 1\,\mathrm{T}$ when $I_1 = I_2 = I = 5\,\mathrm{A}$. The core is made of the same ferromagnetic material as in problem P13.23. Solve the problem in two ways: (1) taking the magnetic resistance of the core into account; and (2) neglecting the magnetic resistance of the core. Use the following values: $a = 10\,\mathrm{cm}$, $b = 6\,\mathrm{cm}$, $d_1 = d_2 = 2\,\mathrm{cm}$, $S_1 = S_2 = S = 4\,\mathrm{cm}^2$, and $l_0 = 1\,\mathrm{mm}$. What is the percentage difference between the answers in (1) and (2)?

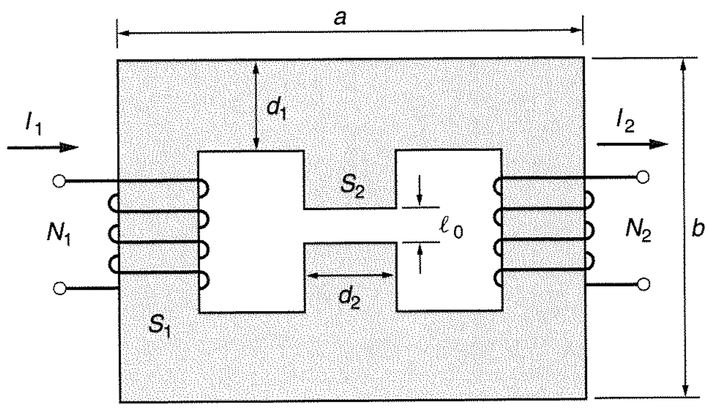


Figure P13.26 A magnetic circuit with an air gap

P13.27. The magnetization curve of a ferromagnetic material used for a magnetic circuit can be approximated by $B(H) = 2H/(400 + H)$, where B is in T and H is in A/m. The magnetic circuit has a cross-sectional area of $S = 2\,\text{cm}^2$, a mean length of $l = 50\,\text{cm}$, and $N = 200$ turns with $I = 2\,\text{A}$ flowing through them. The circuit has an air gap $l_0 = 1\,\text{mm}$ long. Find the magnetic flux density vector in the air gap.

P13.28. The magnetic circuit shown in Fig. P13.28 is made out of the same ferromagnetic material as in the previous problem. The dimensions of the circuit are $a = 6\,\text{cm}$, $b = 4\,\text{cm}$, $d = 1\,\text{cm}$, $S_1 = S_2 = S = 1\,\text{cm}^2$, $N_1 = 50$, $N_2 = 80$, and $N_3 = 40$. With $I_2 = I_3 = 0$, find the value of I_1 needed to produce a magnetic flux of $50\,\mu\text{Wb}$ in branch 3 of the circuit.

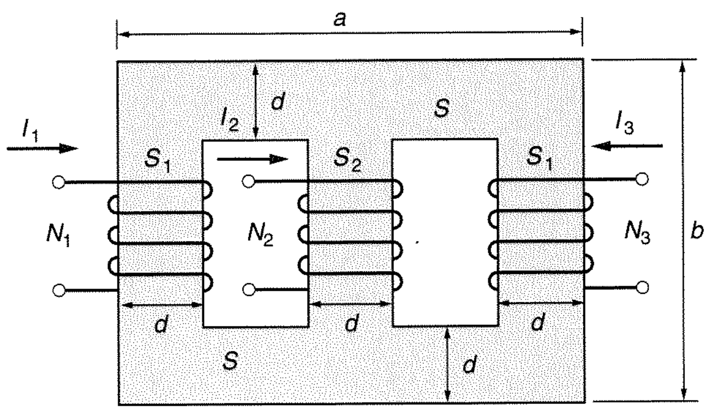


Figure P13.28 A magnetic circuit

P13.29. A linear magnetic circuit is shown in Fig. P13.29. The first winding has $N_1 = 100$ turns, and the second one $N_2 = 48$. Find the magnetic flux in all the branches of the circuit if the currents in the windings are (1) $I_1 = 10\,\text{mA}$, $I_2 = 10\,\text{mA}$; (2) $I_1 = 20\,\text{mA}$, $I_2 = 0\,\text{mA}$; (3) $I_1 = -10\,\text{mA}$, $I_2 = 10\,\text{mA}$. The magnetic material of the core has $\mu_r = 4000$, the dimensions of the core are $a = 4\,\text{cm}$, $b = 6\,\text{cm}$, $c = 1\,\text{cm}$, and the thickness of the core is $d = 1\,\text{cm}$.

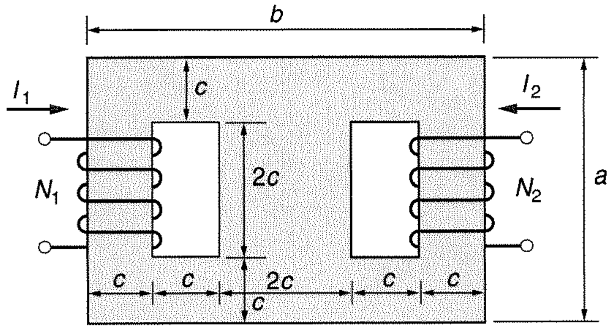


Figure P13.29 A linear magnetic circuit

P13.30. Shown in Fig. P13.30 is a single current loop on a toroidal core (indicated in dashed lines) of very high permeability. Assume that the core can be obtained by a gradual increase of the number of the Ampère currents, from zero to the final number per unit volume. Follow the process of creating the magnetic field in the core as the core becomes "denser." If your reasoning is correct, you should come to the answer to an important question: what is the physical mechanism of channeling the magnetic flux by ferromagnetic cores?

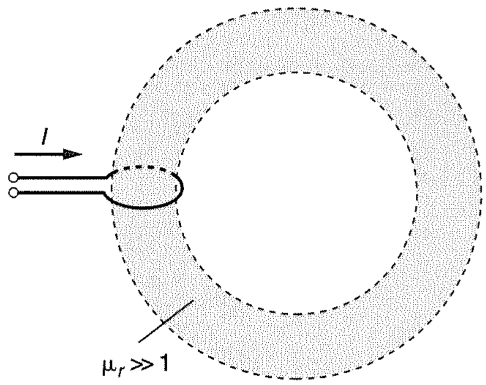


Figure P13.30 A single loop on a toroidal core

14

Electromagnetic Induction and Faraday's Law

14.1 Introduction

We mentioned earlier that in 1831 Michael Faraday performed experiments to check whether current is produced in a closed wire placed near a magnet, reciprocally to dc currents producing magnetic fields. He found no current in that case, but realized that *a time-varying current in the loop is obtained while the magnet is being moved toward or away from it.* The law he formulated is known as *Faraday's law of electromagnetic induction*. It is perhaps the most important law of electromagnetism. Without it there would be no electricity from rotating generators, no telephone, no radio and television, to mention but a few applications.

The phenomenon of electromagnetic induction has a simple physical interpretation. We now know that two charged particles at rest act on each other with a force given by Coulomb's law. We also know that two charges moving with uniform velocities act on each other with an additional force, the *magnetic force*. If a particle is *accelerated*, it turns out that it exerts yet another force on other charged particles, stationary or moving. As in the case of the magnetic force, if only a pair of charges is considered, this additional force is much smaller than Coulomb's force. However, time-varying currents in conductors involve a vast number of accelerated charges, so they produce effects significant enough to be easily measured.

This additional force is of the same *form* as the electric force ($\mathbf{F} = Q\mathbf{E}$), but the electric field vector $\mathbf{E}$ in this case has quite different properties than the electric field vector of static charges. When we wish to stress this difference, we use a slightly different name—the *induced electric field strength*. This chapter introduces the concept of the induced electric field, explains the phenomenon of electromagnetic induction, and provides a derivation of Faraday's law.

14.2 The Induced Electric Field

To understand electromagnetic induction, we need to reconsider the concepts of electric and magnetic fields.

A dc current I flowing through a stationary contour C in a coordinate system (x, y, z) produces a magnetic flux density field $\mathbf{B}$. Let us look at a charged particle Q moving at a velocity $\mathbf{v}$ with respect to contour C. We add a second coordinate system (x', y', z') that moves together with the charge Q, that is, with respect to which Q is stationary.

In our thought experiment we have two observers (electrical engineers or physicists, of course), one stationary in (x, y, z), and the other in (x', y', z'). They are interested in measuring the electric and magnetic forces acting on the charged particle, as sketched in Fig. 14.1.

Let Jack be in the first coordinate system. His instruments record a force acting on a *moving* particle. He concludes that the charge is experiencing a magnetic force $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$, since it is moving in a *time-invariant* magnetic field. If the charge stops, there is no force. Therefore, Jack's conclusion is that in his system there is no electric field.

Jill, in the second coordinate system, comes to a different conclusion. She also measures a force, proportional to Q, acting on the charge. However, for her the charge

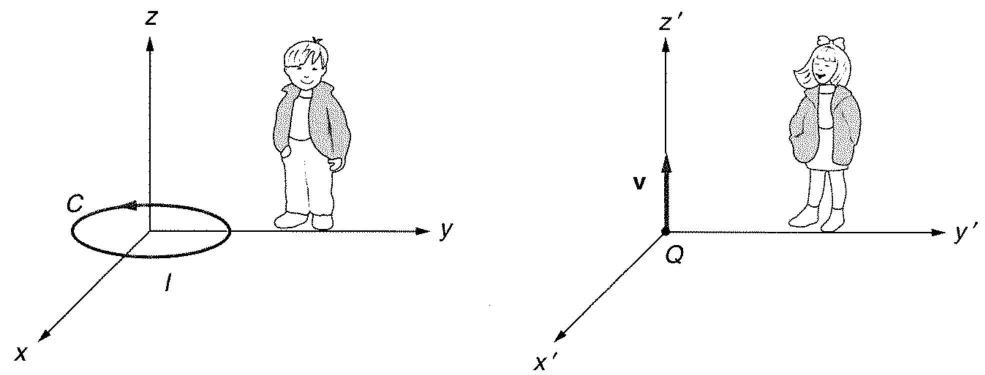


Figure 14.1 Jack and Jill are associated with different coordinate systems that are moving with respect to each other. A charged particle is at rest with respect to Jill. The two observers will explain forces measured on the charged particle in very different ways. (This picture, with the explanation given, gives a good physical insight into induced electric fields. It is worth spending some time thinking about this and understanding it.)

is *not moving*. Therefore, she concludes that the force she measured is an electric one, $\mathbf{F} = Q\mathbf{E}$. She notices, of course, that this force is time-varying. She also notices that in her system there is a time-varying magnetic field (since the source I of the magnetic field is moving with respect to her coordinate system). Thus, Jill's conclusion is that in her coordinate system both a time-varying electric field and a time-varying magnetic field exist.

This strange conclusion, that two observers moving with respect to each other explain things differently, is absolutely correct. It is based, essentially, only on the definition of the electric and magnetic fields. We say that there is an electric field in a domain if a force of the form $Q\mathbf{E}$ is acting on a stationary charge (with respect to our coordinate system). If a force of the form $Q\mathbf{v} \times \mathbf{B}$ is acting on a charge moving with a velocity $\mathbf{v}$, we say that there is a magnetic field. *There are no other definitions of the electric and magnetic fields*—alternative definitions can always be reduced to these two.

Let us rephrase the important conclusion we reached: a time-varying magnetic field is accompanied by a time-varying electric field. We found this to be true in the case of motion of the observer with respect to the source of a time-invariant magnetic field. We shall now argue that a time-varying magnetic field is always accompanied by a time-varying electric field, no matter what the cause of the variation of the field is.

Assume that the source of the magnetic field is a very long and densely wound solenoid with a current I, as in Fig. 14.2. First Jill and the charge Q do not move with respect to the solenoid. Because the charge is not moving with respect to the source of the magnetic field, and the field is constant in time, no force is acting on the charge.

Consider now the following two ways of changing the magnetic field. Let us first move the solenoid periodically between the positions 1 and 2 indicated in the figure. The charge Q and Jill are now moving with respect to the solenoid. Therefore, from the viewpoint of Jack sitting on the solenoid, a magnetic force is acting on the charge. Jill, in her coordinate system, observes a changing electric force on the charge at rest with respect to her, i.e., a changing electric field and a changing magnetic field.

Suppose that instead of moving the solenoid, we move the sliding contact in the figure back and forth between positions 1 and 2. Since this turns the current on

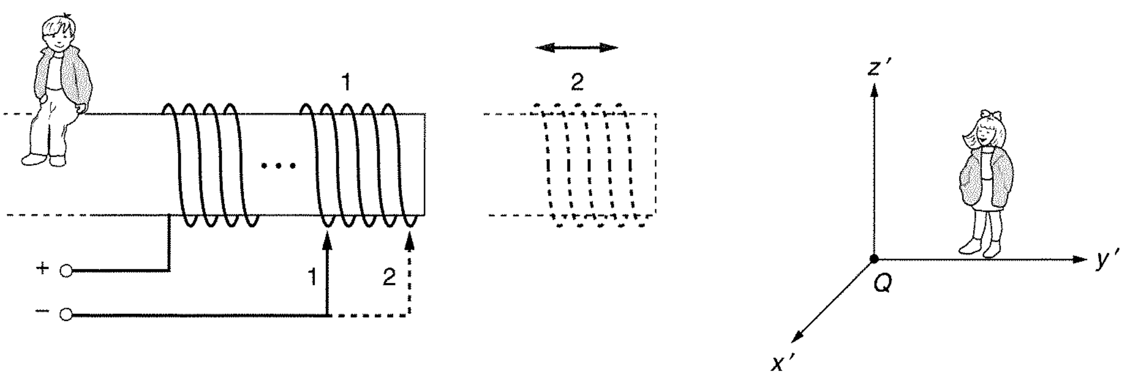


Figure 14.2 Different ways of changing the magnetic field of a solenoid lead to the same conclusion: a time-varying electric field always accompanies a time-varying magnetic field.

and off in successive solenoid windings, the magnetic field changes in the same way as before. However, the mechanism is different: we do not move the source of the magnetic field, but change the current in the source (i.e., turn it on and off in some of the windings). For Jill, who observes only the currents producing the field, there is no difference whatsoever, because the windings with the current move as before, back and forth, as far as Jill is concerned. She will find the same time-varying magnetic field as before, and the same electric force on the charge. From this simple example we infer that *no matter what the cause of the time-varying magnetic field is, a time-varying electric field is associated with it.*

This time-varying electric field is called the *induced electric field*. It is defined by the measured force on a particle $\mathbf{F} = Q\mathbf{E}$, as is the static electric field. We shall see, however, that it has quite different properties.

Of course, a charge can be situated simultaneously in both a static (Coulomb-type) and an induced field. In that case we would measure the total force

$$\mathbf{F} = Q(\mathbf{E}_{\mathrm{st}} + \mathbf{E}_{\mathrm{ind}}). \tag{14.1}$$

How can we determine the expression from which it is possible to evaluate the induced electric field strength? When a charged particle is moving with a velocity $\mathbf{v}$ with respect to the source of the magnetic field, the answer is simple: from the preceding example, the induced electric field is obtained as

$$\mathbf{E}_{\mathrm{ind}} = \mathbf{v} \times \mathbf{B} \qquad (\mathrm{V/m}). \tag{14.2}$$

(Induced electric field observed by an observer moving with velocity $\mathbf{v}$ *with respect to the source of the magnetic field)*

We concluded that time-varying currents are also sources of the induced electric field (as well as of a time-varying magnetic field). As in the case of magnetic forces, due to the small magnitude of the induced field of a single charge when compared with the Coulomb field, it is not possible to find the expression for the induced field of a single charge experimentally. However, the induced field of time-varying currents is large enough to be easily measured.

Assume we have a current distribution of density $\mathbf{J}$ (a function of time and position) in a vacuum, localized inside a volume v. The induced electric field is then found to be

$$\mathbf{E}_{\mathrm{ind}} = -\frac{\partial}{\partial t}\left(\frac{\mu_0}{4\pi}\int_v \frac{\mathbf{J}\,\mathrm{d}v}{r}\right) \qquad (\mathrm{V/m}). \tag{14.3}$$

(Induced electric field of slowly time-varying currents)

In this equation, as usual, r is the distance of the point where the induced field is being determined from the volume element $\mathrm{d}v$. In the case of currents over surfaces, $\mathbf{J}\,\mathrm{d}v$ in Eq. (14.3) should be replaced by $\mathbf{J}_s\,\mathrm{d}S$, and in the case of a thin wire by $i\,\mathrm{d}\mathbf{l}$.

If we know the distribution of time-varying currents, Eq. (14.3) enables the determination of the induced electric field at any point of interest. Most often it is not possible to obtain the induced electric field strength in analytical form, but it can always be evaluated numerically.

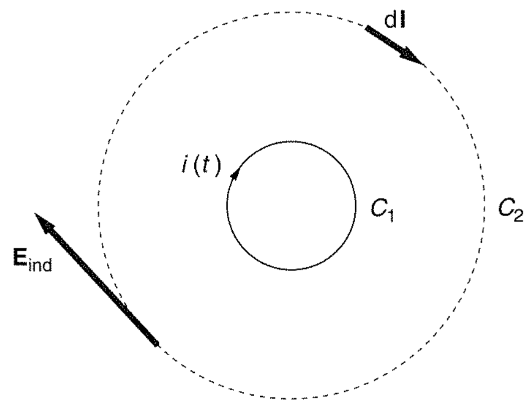


Figure 14.3 A circular loop C_1 with a time-varying current $i(t)$. The induced electric field of this current is tangential to the circular loop C_2 indicated in dashed line, so that it results in a distributed emf around the loop.

Example 14.1—The principle of magnetic coupling. Let a time-varying current $i(t)$ exist in a circular loop C_1 of radius a (Fig. 14.3). According to Eq. (14.3), lines of the induced electric field around the loop are circles, so that the line integral of the induced electric field around a circular contour C_2 indicated in the figure in dashed line *is not zero*. If the contour C_2 is a wire loop, this field will act as a distributed generator along the entire loop length, and a current will be induced in that loop.

This reasoning does not change if the loop C_2 is not circular. We have thus reached an extremely important conclusion: the induced electric field of time-varying currents in one wire loop produces a time-varying current in an adjacent closed wire loop. Note that the other loop need not (and usually does not) have any physical contact with the first loop. This means that the induced electric field enables transport of energy from one loop to the other *through a vacuum*. Although this coupling is actually obtained by means of the induced electric field, it is known as *magnetic coupling*.

Note that if the wire loop C_2 is not closed, the induced field nevertheless induces distributed generators along it. The loop behaves as an open-circuited equivalent (Thévenin) generator.

Questions and problems: Q14.1 to Q14.13, P14.1 to P14.4

14.3 Faraday's Law

Faraday's law is an equation for the *total* electromotive force (emf) induced in a closed loop due to the induced electric field. We know that this electromotive force is distributed along the loop, but we are rarely interested in this distribution. Thus Faraday's law gives us what is of importance only from the circuit-theory point of view—the emf of the Thévenin generator equivalent to all the elemental generators acting in the loop.

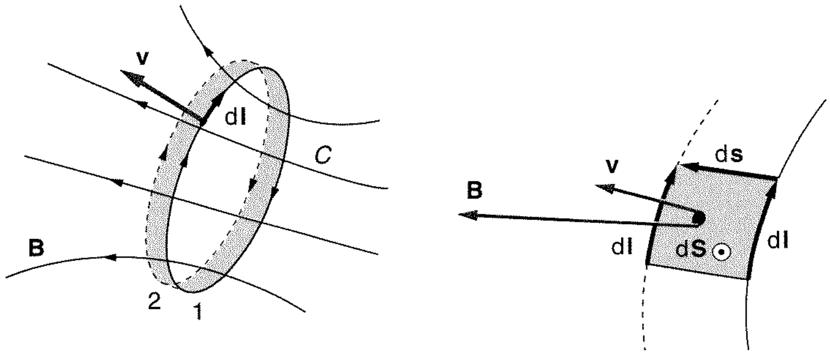


Figure 14.4 A wire loop moving in a magnetic field that is constant in time

Consider a closed conductive contour C moving arbitrarily in a time-constant magnetic field (Fig. 14.4). Let us observe the contour during a short time interval $\mathrm{d}t$. During this time, all segments $\mathbf{dl}$ of the contour move by a short distance $\mathbf{ds} = \mathbf{v}\,\mathrm{d}t$ (different for each segment), where $\mathbf{v} = \mathbf{ds}/\mathrm{d}t$ is the velocity of the segment considered.

Because the wire segments are moving in a magnetic field, there is an induced field acting along them of the form in Eq. (14.2). As a result, a segment behaves as an elemental generator of an emf $\mathrm{d}e = (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{dl}$. The emf induced in the entire contour is given by

$$e = \oint_C \mathbf{E}_{\mathrm{ind}} \cdot \mathbf{dl} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{dl} = \oint_C \left(\frac{\mathbf{ds}}{\mathrm{d}t} \times \mathbf{B} \right) \cdot \mathbf{dl} = \frac{\mathrm{d}}{\mathrm{d}t} \oint_C (\mathbf{ds} \times \mathbf{B}) \cdot \mathbf{dl}. \tag{14.4}$$

The right side of this equation can be transformed as follows. From vector algebra (see Appendix 1) we know that $(\mathbf{ds} \times \mathbf{B}) \cdot \mathbf{dl} = (\mathbf{dl} \times \mathbf{ds}) \cdot \mathbf{B}$. The vector $\mathbf{dl} \times \mathbf{ds} = \mathbf{dS}$ is a vector surface element shown in Fig. 14.4. So the integral on the right-hand side in Eq. (14.4) represents the magnetic flux through the hatched strip shown in the figure. Therefore, the emf induced in the contour can be written in the form

$$e = \frac{\mathrm{d}\Phi_{\mathrm{strip}}}{\mathrm{d}t} \qquad (\mathrm{V}). \tag{14.5}$$

From Fig. 14.4, the flux $\mathrm{d}\Phi_{\mathrm{strip}}$ can also be interpreted as the difference in fluxes through the contour from position 1 to position 2, $\mathrm{d}\Phi_{\mathrm{strip}} = \Phi_1 - \Phi_2$. On the other hand, from the definition of the increment in flux through a contour C in a time interval $\mathrm{d}t$, $\mathrm{d}\Phi_{\text{through } C \text{ in } \mathrm{d}t} = \Phi_2 - \Phi_1$. We finally have

$$\boxed{e = \oint_C \mathbf{E}_{\mathrm{ind}} \cdot \mathbf{dl} = -\frac{\mathrm{d}\Phi_{\text{through } C \text{ in } \mathrm{d}t}}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathbf{dS} \qquad (\mathrm{V}). \tag{14.6}}$$

(Faraday's law of electromagnetic induction)

This is *Faraday's law of electromagnetic induction.* Recall again that the induced emf in this equation is nothing but the voltage of the Thévenin generator equivalent to all the elemental generators of electromotive forces $\mathbf{E}_{\text{ind}} \cdot \mathbf{dl}$ acting *around the loop.*

We know that an induced electric field, which is the actual cause of the emf e, does not depend on the mechanism by which the magnetic field changed. For example, the magnetic field variation could be due to mechanical motion in the field and/or to time-variable current sources. Equation (14.6) is valid in all those cases. Note that, except for these two ways of changing the magnetic flux in time, there are no other possibilities that result in an induced emf.

Example 14.2—An electric generator based on electromagnetic induction. An important example of the application of Faraday's law is electric generators. A simplified generator is sketched in Fig. 14.5. A straight piece of wire can slide along two parallel wires 1 and 2 that are at a distance a. At one end the wires are connected by a resistor R. The entire system is in a uniform magnetic field with a magnetic flux density $\mathbf{B}$ perpendicular to and into the page. A mechanical force is tagging the piece of wire with a constant velocity $\mathbf{v}$ as shown.

Let us find the emf induced in the wire due to its motion in the magnetic field. The conductor AA' forms a closed loop with the rails and the resistor. The induced emf in the loop according to Faraday's law is

$$e = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathbf{B}\cdot \mathrm{d}\mathbf{S}(t)}{\mathrm{d}t} = -\frac{-B\,\mathrm{d}S(t)}{\mathrm{d}t} = -B\frac{av\,\mathrm{d}t}{\mathrm{d}t} = vBa.$$

Note that we can also get this from $e = (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{a}$, where $\mathbf{a}$ is defined as in the figure.

There will be a current $I = e/R = vBa/R$ in the loop $AA'RA$ due to the induced emf. The resistor power is $P = RI^2 = v^2B^2a^2/R$.

What forces act on the piece of wire sliding along the "rail"? A current $I = vBa/R$ flows through the conductor, so a magnetic force acts on elements $\mathbf{dl}$ of the conductor AA'. We know that this elementary magnetic force is obtained as

$$\mathrm{d}\mathbf{F}_m = I\ \mathrm{d}\mathbf{l} \times \mathbf{B}.$$

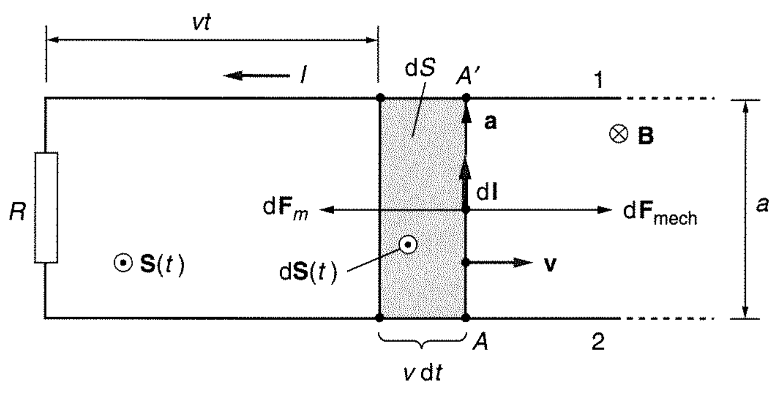


Figure 14.5 A simplified electric generator based on electromagnetic induction

Because the wire is straight and perpendicular to the magnetic flux density vector, the total magnetic force on the moving conductor is $F_m = IaB$, and its direction is as indicated in the figure. This force is *opposite* to the mechanical force that is moving the wire. Since the velocity $\mathbf{v}$ is constant, the mechanical and magnetic forces have to be equal, so $F_{\text{mech}} = IaB$. The mechanical power is $P_{\text{mech}} = F_{\text{mech}}v$, and it must be equal to the power dissipated in the resistor. Thus the velocity at which the conductor is moving is

$$v = \frac{P_{\text{mech}}}{F_{\text{mech}}} = \frac{P}{F_{\text{mech}}} = \frac{v^2B^2a^2/R}{IaB} = \frac{v^2Ba}{IR},$$

so that finally

$$v = \frac{IR}{Ba}.$$

This generator is not a practical one, but it shows in a simple way the principle of a generator based on electromagnetic induction. (The arrangement in Fig. 14.5 can be altered into a "magnetic rail gun," where the moving conductor is the bullet, if the resistor is replaced by a voltage source. Can you explain how such a gun would work?)

Example 14.3—An ac generator. Another example of Faraday's law is the ac generator sketched in Fig. 14.6a. A rectangular wire loop is rotating in a uniform magnetic field (for example, between the poles of a magnet). We can measure the induced voltage in the wire by connecting a voltmeter between contacts C_1 and C_2. $\mathbf{B}$ is perpendicular to the contour axis. The loop is rotating about this axis with an angular velocity ω. If we assume that at $t = 0$ vector $\mathbf{B}$ is parallel to vector $\mathbf{n}$ normal to the surface of the loop (see Fig. 14.6a), the flux through the contour at that time is maximal and equal to $\Phi = Bab$. As the contour rotates, the flux becomes smaller. When $\mathbf{B}$ is parallel to the contour, the flux is zero, then it becomes negative, and so on. Since $\mathbf{B} \cdot \mathbf{n} = B\cos\omega t$, we can write

$$\Phi(t) = S\mathbf{B} \cdot \mathbf{n} = Bab\cos\omega t.$$

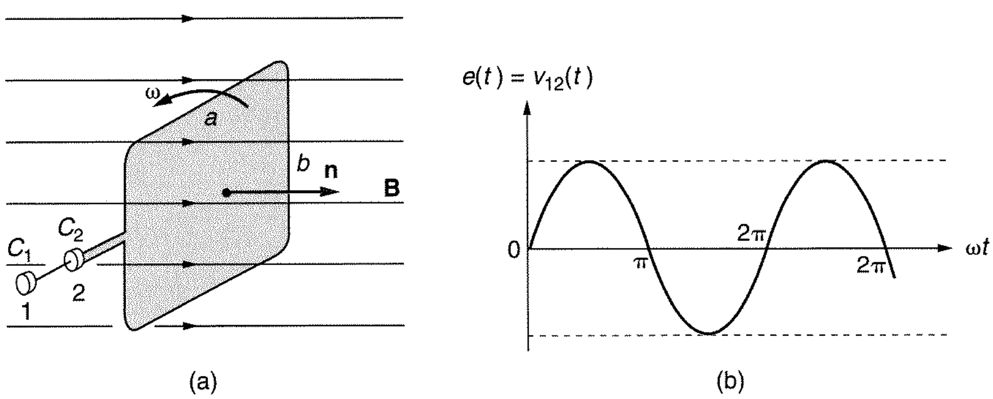


Figure 14.6 (a) A simple ac generator, and (b) the induced emf in it

The induced emf is given by

$$e(t) = -\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = \omega abB \sin \omega t = E_{\max} \sin \omega t.$$

Its shape as a function of time is sketched in Fig. 14.6b.

A real generator has a coil with many turns of wire instead of a single loop, to obtain a larger induced emf. Also, usually the coil is not rotating; instead the magnetic field is rotating around it. This avoids sliding contacts of the generator, like those in Fig. 14.6a.

Example 14.4—DC generator with a commutator. The described ac generator can be modified to a dc generator. For that purpose, two sliding ring contacts are replaced by a single sliding ring contact, cut in two mutually insulated halves, as in Fig. 14.7. Such a sliding contact is known as a *commutator.* As the contour and commutator turn, the voltage between contacts 1 and 2 will always be positive because the half of the cut ring in contact with 1 will always be at a positive potential (corresponding to the parts of the sine curve in Fig. 14.6b above the ωt axis). The induced emf is shown in Fig. 14.8a. If we wanted to make this voltage less variable in time, we could have three loops at 120 degrees between successive loops. In such a case the induced emf is of the form shown in Fig. 14.8b.

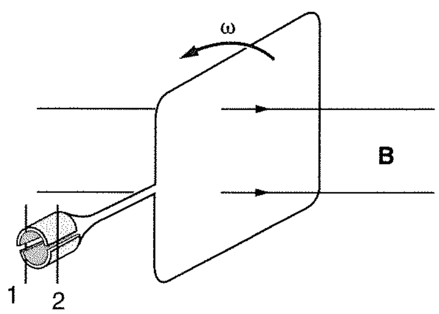


Figure 14.7 A simple dc generator with a commutator

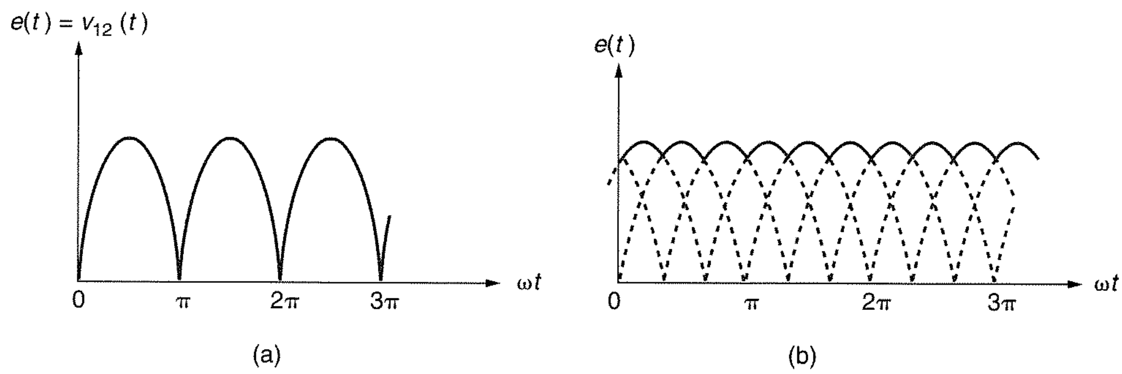


Figure 14.8 (a) The induced emf between contacts 1 and 2 of the generator from Fig. 14.7, and (b) the emf for three contours oriented at 120 degrees with respect to each other

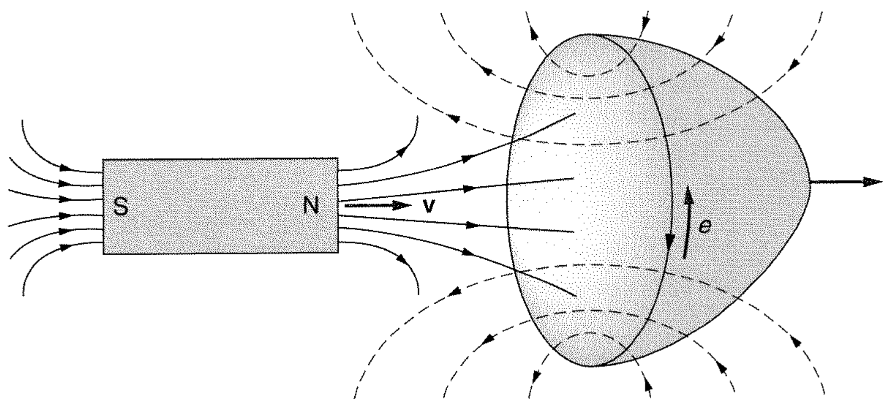


Figure 14.9 Illustration of Lentz's law

Example 14.5—Lentz's law. Consider a permanent magnet approaching a stationary loop, as shown in Fig. 14.9. The permanent magnet is equivalent to a system of macroscopic currents, and, because it is moving, the magnetic flux created by these currents through the contour varies in time. According to the reference direction of the contour shown in the figure, the change of flux is positive, $d\Phi/dt > 0$, so the induced emf is in the direction shown in the figure. The emf produces a current through the closed loop, which in turn produces its own magnetic field, shown in the figure in dashed line. As a result, the change of the magnetic flux, caused initially by the magnet motion, is reduced. This is *Lentz's law*: the induced current in a conductive contour tends to decrease the change of the magnetic flux through the contour.

Example 14.6—Eddy currents. When a wire loop finds itself in a time-varying magnetic field, a current is induced in it due to a time-varying induced electric field that always accompanies a time-varying magnetic field. A similar thing happens in solid conductors. In a metal body we can imagine many conductive loops. A current is induced throughout the body when it is situated in a time-varying magnetic field.

The induced currents inside conductive bodies that are a result of the induced electric field are called *eddy currents*. As the first consequence of eddy currents, power is lost to heat according to Joule's law. As the second consequence, there is a secondary magnetic field due to the induced currents that reduces, by Lentz's law, the magnetic field inside the body. Both of these effects are usually not desirable. For example, in a ferromagnetic core shown in Fig. 14.10, Lentz's law tells us that eddy currents tend to decrease the flux in the core, and the magnetic circuit of the core will not be used efficiently. The flux density vector is the smallest at the center of the core, because there the **B** field of all the induced currents adds up. The total magnetic field distribution in the core is thus *nonuniform*.

To reduce these two undesirable effects, ferromagnetic cores are made of mutually insulated thin sheets, as shown in Fig. 14.11. Now the flux through the sheets is encircled by much smaller loops, the emf induced in these loops is consequently much smaller, and so the eddy currents are also reduced significantly. Of course, this only works if the vector **B** is parallel to the sheets. It is left as an exercise for the reader to explain this statement.

In some instances, eddy currents are created on purpose. For example, in so-called induction furnaces for melting metals, eddy currents are used to heat solid metal pieces to high melting temperatures.

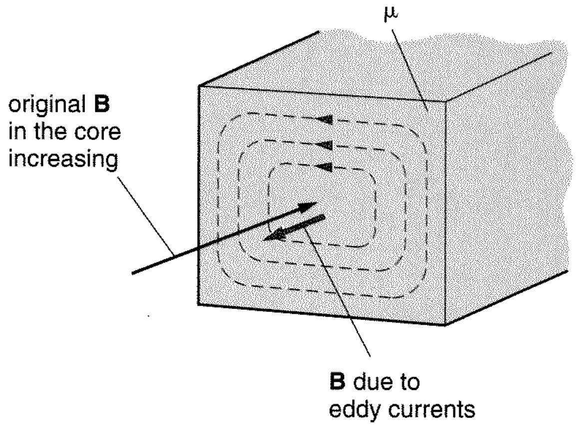


Figure 14.10 Eddy currents in a piece of ferromagnetic core. Note that the total **B** field in the core is reduced due to the opposite field created by eddy currents.

Example 14.7—Superconducting loop. Some substances have zero resistivity at very low temperatures. For example, lead has zero resistivity below about 7.3 K (just a little bit warmer than liquid helium). This phenomenon is known as *superconductivity,* and such conductors are said to be *superconductors.* Some ceramic materials (e.g., yttrium barium oxide) become superconductors at temperatures as "high" as about 70 K (corresponding to the temperature of liquid nitrogen). Superconducting loops have an interesting property: it is impossible to change the magnetic flux through such a loop by means of electromagnetic induction.

The explanation of this is simple. Consider a superconducting loop situated in a time-varying magnetic field. The Kirchhoff voltage law for such a loop has the form

$$-\frac{d\Phi}{dt} = 0,$$

since the emf in the loop is $-d\Phi/dt$, and the loop has zero resistance. From this equation, it is seen that the flux through a superconducting loop remains *constant.* Thus it is not possible

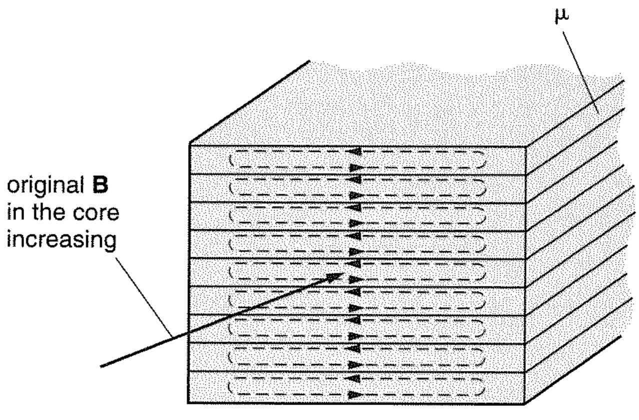


Figure 14.11 A ferromagnetic core for ac machines consists of thin insulated sheets.

to change the magnetic flux through a superconducting loop by means of electromagnetic induction.

The physical meaning of this behavior is the following: if a superconducting loop is situated in a time-varying induced electric field, the current induced in the loop must vary in time so as to produce exactly the same induced electric field in the loop, but in the opposite direction. If this were not so, infinite currents would result. We know that this induced electric field is accompanied by a time-varying magnetic field, the flux of which through the contour will be exactly the negative of the external flux.

Questions and problems: Q14.14 to Q14.38, P14.5 to P14.25

14.4 Potential Difference and Voltage in a Time-Varying Electric and Magnetic Field

In the discussion of electrostatics, we defined the voltage to be the same as the potential difference. Actually, the voltage between two points is defined as the line integral of the *total* electric field strength from one point to the other. In electrostatics, the induced electric field does not exist, and therefore voltage is identical to potential difference. We shall now show that this is *not* the case in a time-varying electric and magnetic field.

Consider arbitrary time-varying currents and charges producing a time-varying electric and magnetic field, as in Fig. 14.12. Consider two points, A and B, in this field, and two paths, a and b, between them, as indicated in the figure. The voltage between these two points is defined as

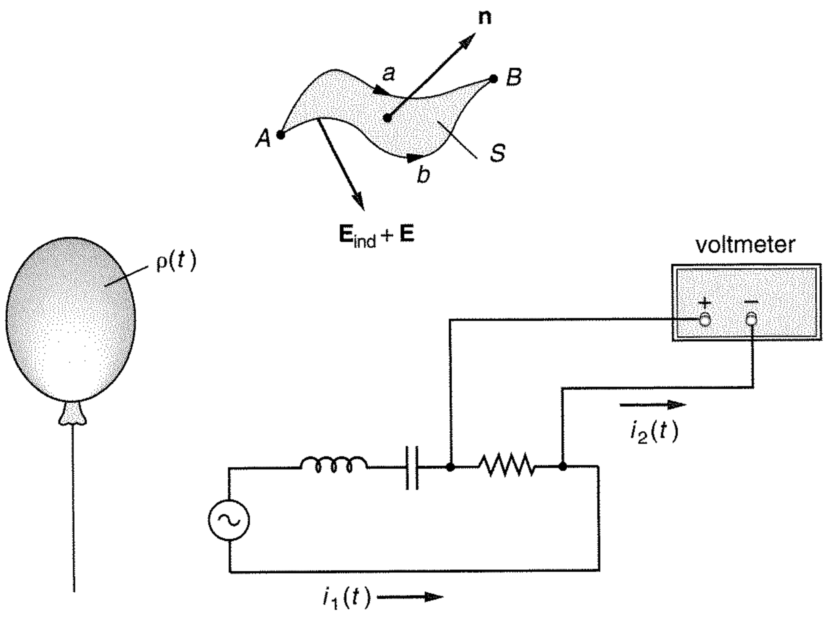


Figure 14.12 An arbitrary distribution of time-varying currents and charges

$$V_{AB} = \int_A^B \mathbf{E}_{\text{total}} \cdot \mathbf{dl}. \tag{14.7}$$

(Definition of voltage between two points)

In this definition, $\mathbf{E}_{\text{total}}$ is the total electric field strength, which means the sum of the "static" part (produced by charges) and the induced part (due to time-varying currents). We know that the integral between A and B of the static part is simply the potential difference between A and B. So we can write

$$V_{AB} = V_A - V_B + \int_A^B \mathbf{E}_{\text{ind}} \cdot \mathbf{dl}. \tag{14.8}$$

We know that the potential difference, $V_A - V_B$, does not depend on the path between A and B, but we shall now prove that the integral in this equation is different for paths a and b. These paths form a closed contour. Applying Faraday's law to that contour, we have

$$\begin{aligned} e_{\text{induced in closed contour } AaBbA} &= \oint_{AaBbA} \mathbf{E}_{\text{ind}} \cdot \mathbf{dl} \\ &= \int_{AaB} \mathbf{E}_{\text{ind}} \cdot \mathbf{dl} - \int_{AbB} \mathbf{E}_{\text{ind}} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}, \end{aligned} \tag{14.9}$$

where Φ is the magnetic flux through the surface bounded by the contour $AaBbA$. Since the right side of this equation is generally nonzero, the line integrals of $\mathbf{E}_{\text{ind}}$ from A to B along a and along b are different. Consequently, *the voltage between two points in a time-varying electric and magnetic field depends on the particular path between these two points.*

This is an important practical conclusion. We always measure the voltage by a voltmeter with leads connected to the two points between which the voltage is being measured. Circuit theory postulates that this voltage does not depend on the shape of the voltmeter leads. We now know that in the time-varying case this is not true. Because the difference in voltage for two paths depends on the rate of change of the flux, this effect is particularly pronounced at high frequencies.

Questions and problems: Q14.39, P14.26 to P14.28

14.5 Chapter Summary

1. If a closed wire loop is moving with respect to a source of a time-invariant magnetic field, or is near another loop with time-varying electric current, a distributed emf is induced along it. This phenomenon is known as *electromagnetic induction.* It is due to a component of the electric field, the *induced electric field,* which exists along the loop.

2. The induced electric field acts on a point charge Q with a force $\mathbf{F} = Q\mathbf{E}$, like the electric field due to stationary charges, but its line integral around a closed contour is not zero. Precisely this gives rise to electromagnetic induction.
3. Integrally, i.e., not taking care of the exact distribution of the induced emf (which is of interest only rarely), the induced emf equals the negative rate of time variation of the magnetic flux through the contour. This is known as Faraday's law of electromagnetic induction.
4. The voltage between two points is defined as a line integral of the *total* electric field along a line joining them. Because of the induced electric field, the voltage depends on the particular line joining the two points, although the potential difference (which is a part of this voltage) does not.

QUESTIONS

Q14.1. An observer in a coordinate system with the source of a time-invariant magnetic field observes a force $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$ on a charge Q moving with a uniform velocity $\mathbf{v}$ with respect to his coordinate system. What is the force on Q observed by an observer moving with the charge? What is his interpretation of the velocity $\mathbf{v}$?

Q14.2. Three point charges are stationary in a coordinate system of the first observer. A second observer, in his coordinate system, moves with respect to the first with a uniform velocity. What kinds of fields are observed by the first observer, and what by the second?

Q14.3. A small uncharged conducting sphere is moving in the field of a permanent magnet. Are there induced charges on the sphere surface? If they exist, how do an observer moving with the charge and an observer on the magnet explain their existence?

Q14.4. A straight metal rod moves with a constant velocity $\mathbf{v}$ in a uniform magnetic field of magnetic induction $\mathbf{B}$. The rod is normal to $\mathbf{B}$, and $\mathbf{v}$ is normal to both the rod and to $\mathbf{B}$. Sketch the distribution of the induced charges on the rod. What is the *electric* field of these charges inside the rod equal to?

Q14.5. A small dielectric sphere moves in a uniform magnetic field. Is the sphere polarized? Explain.

Q14.6. A charge Q is located close to a toroidal coil with time-varying current. Is there a force on the charge? Explain.

Q14.7. A wire of length l is situated in a magnetic field of flux density $\mathbf{B}$ parallel to the wire. Is an electromotive force induced in the wire if it is moved (1) along the lines of $\mathbf{B}$, and (2) transverse to the lines of $\mathbf{B}$? Explain.

Q14.8. Strictly speaking, do currents in branches of an ac electric circuit depend on the circuit shape? Explain.

Q14.9. Does the shape of a dc circuit influence the currents in its branches? Explain.

Q14.10. A circular metal ring carries a time-varying current, which produces a time-varying induced electric field. If the ring is set in oscillatory motion about the axis normal to its plane, will the induced electric field be changed? Explain your answer.

Q14.11. A device for accelerating electrons known as the *betatron* (Fig. Q14.11) consists of a powerful electromagnet and an evacuated tube that is bent into a circle. Electrons are accelerated in the tube when the magnetic flux in the core is forced to rise approxi-

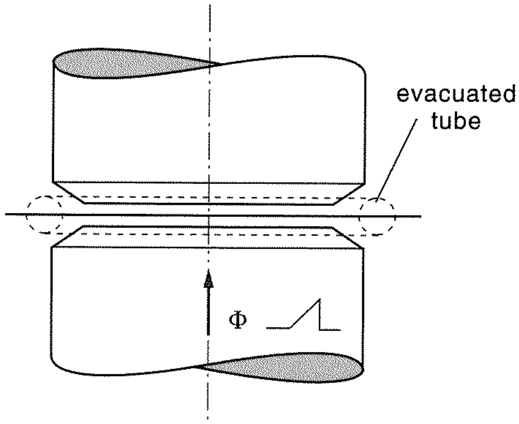


Figure Q14.11 Sketch of a betatron

mately as a linear function of time. What is the physical mechanism for accelerating the electrons in the betatron?

Q14.12. Is it physically sound to speak about a partial electromotive force induced in a segment of one loop by the current in a segment of another (or even of the same) loop? Explain.

Q14.13. A vertical conducting sheet (say, of aluminum) is permitted to fall under the action of gravity between the poles of a powerful permanent magnet. Is the motion of the sheet affected by the presence of the magnet? Explain.

Q14.14. A long solenoid wound on a Styrofoam core carries a time-varying current. It is encircled by three loops, one of copper, one of a resistive alloy, and the third of a bent moist filament (a poor conductor). In which loop is the induced electromotive force the greatest?

Q14.15. What becomes different in question Q14.14 if the solenoid is wound onto a ferromagnetic core?

Q14.16. Assume that the current $i(t)$ in circular loop 1 in Fig. Q14.16 in a certain time interval increases linearly in time. Will there be a current in the closed conducting loop 2? If you think that there will be, what is its direction?

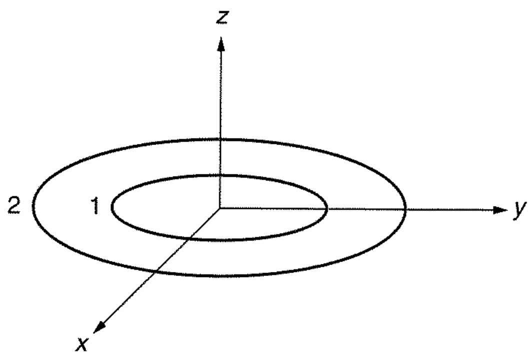


Figure Q14.16 Two coupled coils

Q14.17. What is the direction of the current induced in the loop sketched in Fig. Q14.17? Explain.

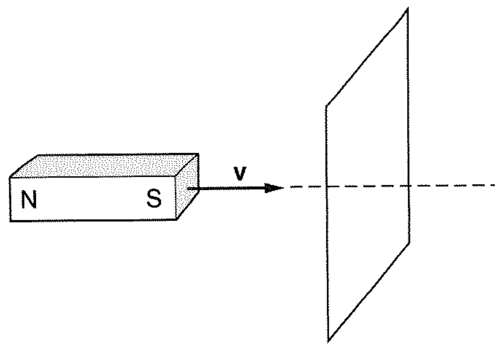


Figure Q14.17 A permanent magnet approaching a loop

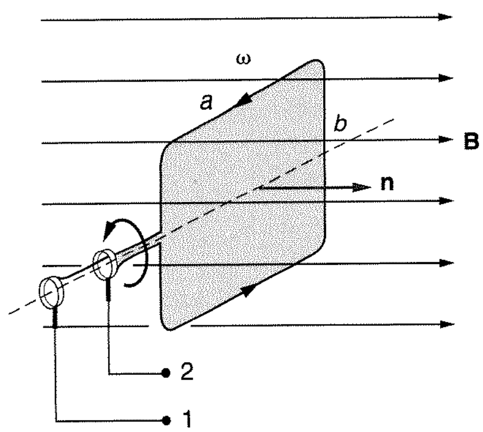


Figure Q14.18 A coil rotating in a magnetic field

Q14.18. The coil in Fig. Q14.18 consists of N densely wound turns of thin wire. What is the voltage of the generator when compared with the case of a single turn? Explain using both the concept of magnetic flux and that of the induced electric field.

Q14.19. A solenoid is wound onto a long, cylindrical permanent magnet. A voltmeter is connected to one end of the solenoid, and to a sliding contact that moves and makes a contact with a larger or smaller number of turns of the solenoid. Thus the magnetic flux in the closed loop of the voltmeter will vary in time. Does the voltmeter detect a time-varying voltage (assuming that it is sensitive enough to do so)? Explain.

Q14.20. What is the direction of the reference unit vectors normal to the three loops in Fig. Q14.20?

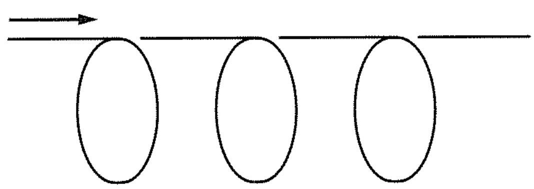


Figure Q14.20 Three series loops

Q14.21. A cylindrical permanent magnet falls without friction through a vertical evacuated metal tube. Is the fall accelerated? If not, what determines the velocity of the magnet? Explain.

Q14.22. A strong permanent magnet is brought near the end plate of the metal pendulum of a wall timepiece. Does this influence the period of the pendulum? If it does, is the pendulum accelerated or slowed down? Does this depend on the type of the magnetic pole of the magnet closer to the pendulum plate? Explain.

Q14.23. Figure Q14.23 shows a sketch of a flat strip moving between the poles of a permanent magnet. Sketch the lines of the induced current in the strip.

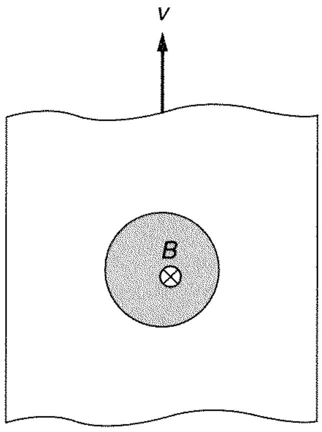


Figure Q14.23 A strip moving in a magnetic field

Q14.24. The strip in question Q14.23 has in case (1) longitudinal and in case (2) transverse slots with respect to the direction of motion. In which case are induced currents greater? Explain.

Q14.25. Explain in detail how the right-hand side in Eq. (14.4) is obtained from the middle expression in it.

Q14.26. The magnetic flux through a contour C at time t is Φ_1, and at time $t + \Delta t$ it is Φ_2. Is the time increment of the flux through C $(\Phi_2 - \Phi_1)$, or $(\Phi_1 - \Phi_2)$? Explain.

Q14.27. Is the *distribution* of the induced electromotive force around a contour seen from the right-hand side in Eq. (14.6)? Is it seen from the middle expression in that equation? Is it seen from any expression in Eq. (14.4)?

Q14.28. Imagine an electric circuit with several loops situated in a slowly time-varying magnetic (and induced electric) field. Can you analyze such a circuit by circuit-theory methods? If you think you can, explain in detail how you would do it.

Q14.29. Does it make any sense at all to speak about the electromotive force induced in an *open* loop? If you think that this makes sense, explain what happens.

Q14.30. Explain in detail what a positive and what a negative electromotive force in Eq. (14.6) mean.

Q14.31. Why is the reduction of eddy current losses possible only if the vector **B** is parallel to a thin ferromagnetic sheet?

Q14.32. What is the induced electromotive force in the loop shown in Fig. Q14.32, if the magnetic field is time-varying? If the right half of the loop is turned about the x axis by 180 degrees, what is then the induced electromotive force? Explain both in terms of the magnetic flux and of the induced electric field.

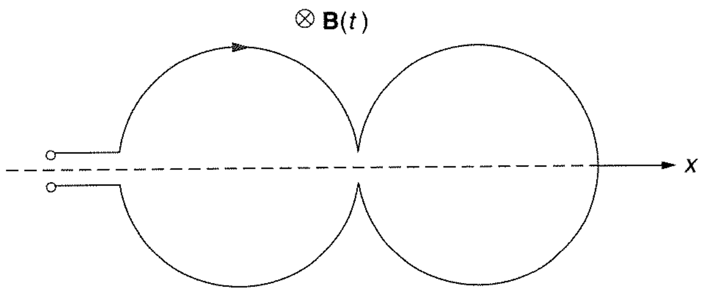


Figure Q14.32 A loop in the form of an 8

Q14.33. A solid conducting body is placed near a loop with time-varying current. Are any forces acting on free charges inside the body? Explain.

Q14.34. Is there a magnetic force between the body and the loop from the preceding example? Explain.

Q14.35. A planar insulated loop with time-varying current is placed on the surface of a plane conducting sheet. What happens in the sheet? Is the power required to drive the current in the loop different when it resides on the sheet than when it is isolated in space? Explain.

Q14.36. Of two closed conducting loops, C_1 and C_2, C_1 is connected to a generator of time-varying voltage. Is there a current in C_2? Explain.

Q14.37. An elastic metal circular ring carrying a steady current I is periodically deformed to a flat ellipse, and then released to retain its original circular shape. Is an electromotive force induced in the loop? Explain.

Q14.38. In Example 14.7 it was demonstrated that the flux through a superconducting loop cannot be changed. Does this mean that the flux through a superconducting loop cannot be changed by *any* means?

Q14.39. Why does the shape of voltmeter leads influence the time-varying voltage the voltmeter measures? Why does this influence increase with frequency?

PROBLEMS

P14.1. Starting from Eq. (14.3), prove that the lines of the induced electric field vector of a circular current loop with a time-varying current are circles centered at the loop axis.

P14.2. Assume that you know the induced electric field $\mathbf{E}_{\text{ind}}(t)$ along a circular line C of radius a in problem P14.1. A wire loop of radius a coincides with C. Evaluate the total electromotive force induced in the loop. Prove that this is, actually, the voltage of the Thévenin generator equivalent to the distributed infinitesimal generators around the loop.

P14.3. Two coaxial solenoids shown in Fig. P14.3 are connected in series. A current $i(t) = 1.5 \sin 1000t$ A, where time is in seconds, flows through the solenoids. The dimensions are $a = 1$ cm, $b = 2$ cm, and $L = 50$ cm. The number of turns in both is the same, $N_1 = N_2 = 1000$. Find the approximate induced electric field at points A_1 (surface of the inner solenoid), A_2 (halfway between the two solenoids), and A_3 (right outside the outer solenoid).

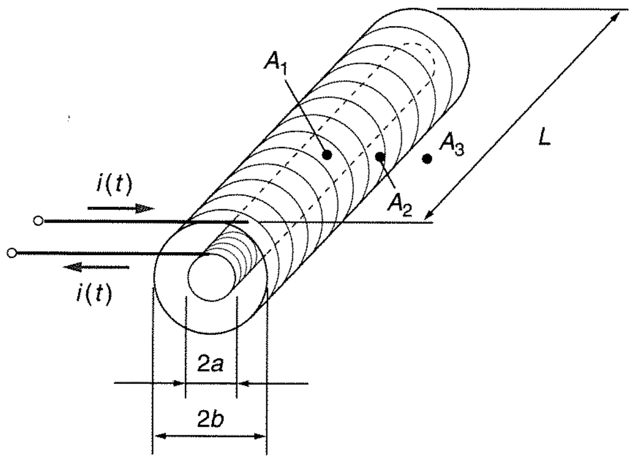


Figure P14.3 Two coaxial solenoids

P14.4. A circular loop of radius a, with a current I, rotates about the axis normal to its plane with an angular frequency ω. A small charge Q is fastened to a loop radius and rotates with the loop, as in Fig. P14.4. Is there a force on the charge? If it exists, determine its direction.

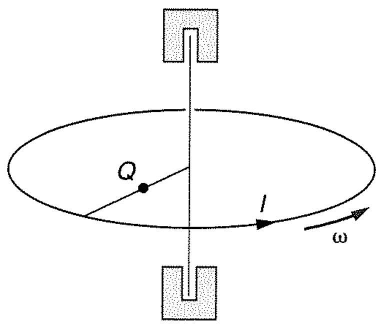


Figure P14.4 Rotating current loop and charge

P14.5. A side of a rectangular wire loop is partly shielded from the magnetic field normal to the loop plane with a hollow ferromagnetic cylinder, as in Fig. P14.5. Therefore the sides ad and bc are situated in different magnetic fields. If the loop, together with the cylinder, moves in the indicated direction with a velocity $\mathbf{v}$, will there be a current in

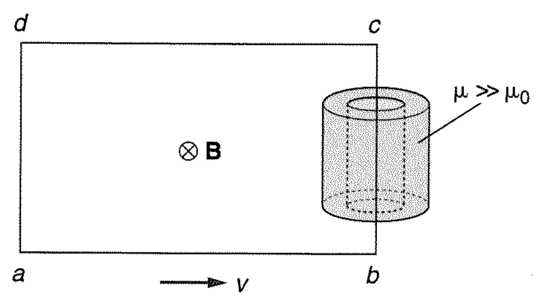


Figure P14.5 A partly shielded wire loop the loop? If the answer is yes, could this serve for measuring the velocity with respect to the earth's magnetic field? Explain.

P14.6. A current $i(t) = I_m \sin(2\pi f t) = 2.5\sin 314t$ A is flowing through the solenoid in Fig. P14.6, where frequency is in hertz and time is in seconds. The solenoid has $N_1 = 50$ turns of wire, and the coil K shown in the figure has $N_2 = 3$ turns. Calculate the emf induced in the coil, as well as the amplitude of the induced electric field along the coil turns. The dimensions indicated in the figure are $a = 0.5$ cm, $b = 1$ cm, and $L = 10$ cm. Plot the induced emf as a function of N_1, N_2, and f.

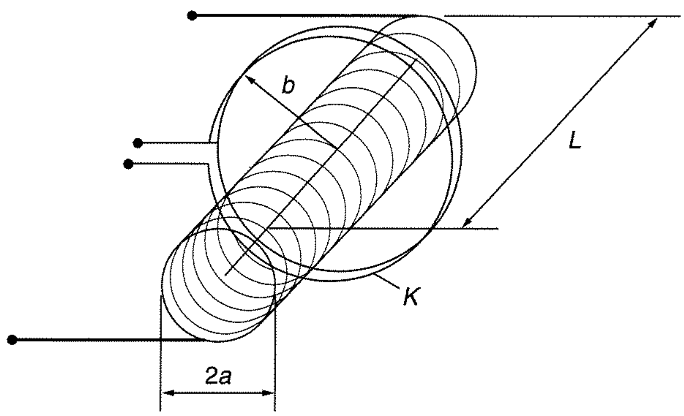


Figure P14.6 A solenoid and a coil

P14.7. If the conductor AA' in Fig. P14.7 is rotating at a constant angular velocity and makes n turns per second, find the voltage $V_{AA'}$ as a function of time. Assume that at $t = 0$ the conductor is in the position shown in the figure.

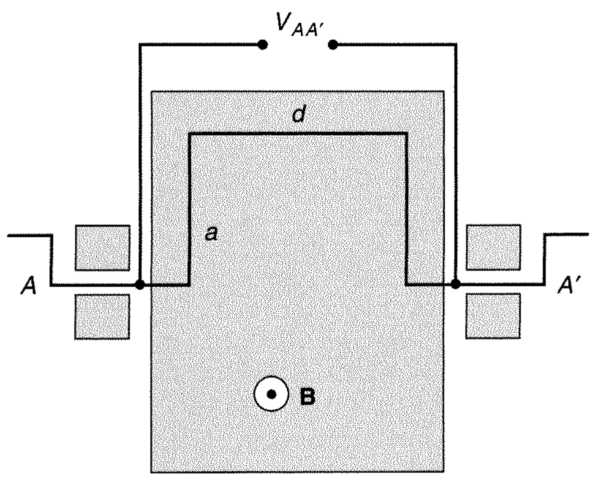


Figure P14.7 A rotating conductor

P14.8. A two-wire line is parallel to a long straight conductor with a dc current I (Fig. P14.8). The two-wire line is open at both ends, and a conductive bar is sliding along it with a

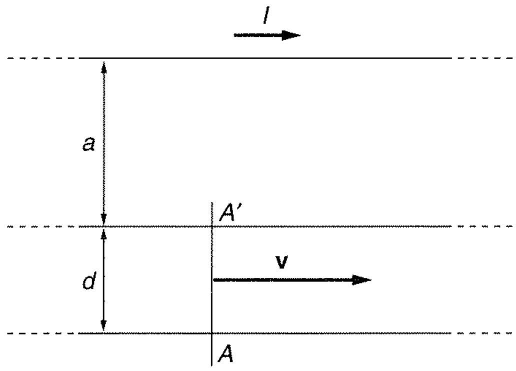


Figure P14.8 A conductor and two-wire line

uniform velocity v, as shown in the figure. Find the potential difference between the two line conductors.

P14.9. A rectangular wire loop with sides of lengths a and b is moving away from a straight wire with a current I (Fig. P14.9). The velocity of the loop, v, is constant. Find the induced emf in the loop. The reference direction of the loop is shown in the figure. Assume that at $t = 0$ the position of the loop is defined by $x = a$.

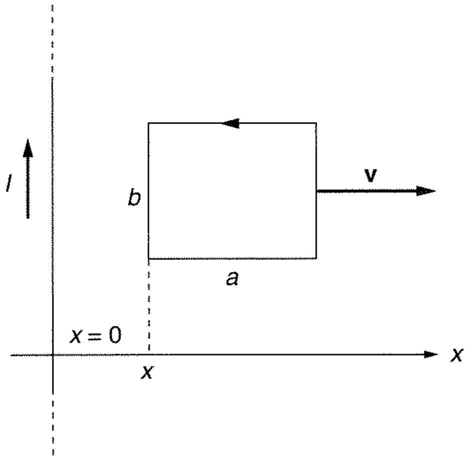


Figure P14.9 A moving frame in a magnetic field

P14.10. The current flowing through the straight wire from the preceding problem is now $i(t)$ (a function of time). Find the induced emf in the loop, which is moving away from the wire as in the preceding problem. What happens to your expression for the emf when (1) the frame stops moving, or (2) when $i(t)$ becomes a dc current, I?

P14.11. A liquid with a small but finite conductivity is flowing through a flat insulating pipe with an unknown velocity v. The velocity of the fluid is roughly uniform over the cross section of the pipe. To measure the fluid velocity, the pipe is in a magnetic field with a flux density vector **B** normal to the pipe, as shown in Fig. P14.11. Two small electrodes are in contact with the fluid at the two ends of the pipe. A voltmeter with large input impedance shows a voltage V when connected to the electrodes. Find the velocity of the fluid.

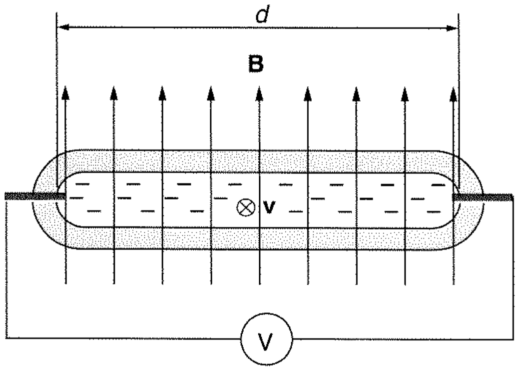


Figure P14.11 Measurement of fluid velocity

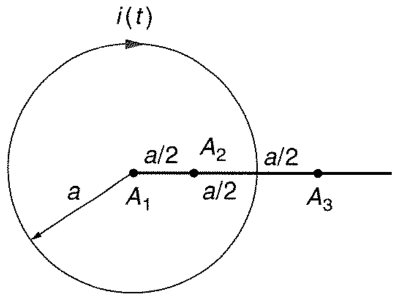


Figure P14.12 Cross section of a solenoid

P14.12. Shown in Fig. P14.12 is the cross section of a very long solenoid of radius $a = 1$ cm, with $N' = 2000$ turns/m. In the time interval $0 \leq t \leq 1$ s, a current $i(t) = 50t$ A flows through the solenoid. Determine the acceleration of an electron at points A_1, A_2, and A_3 indicated in the figure. (Note: the acceleration, **a**, is found from the relation $\mathbf{F} = m\mathbf{a}$, where **F** is the force on the electron.)

P14.13. Determine approximately the induced electric field strength inside the tubular coil sketched in Fig. P14.13. The current intensity in the coil is $I = 0.02 \cos 10^6 t$ A, the number of turns is $N = 100$, and the coil dimensions are $a = 1$ cm, $b = 1.5$ cm, and $L = 10$ cm.

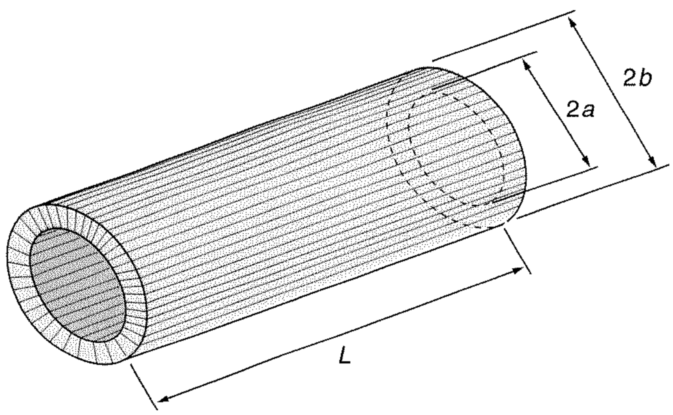


Figure P14.13 A tubular coil

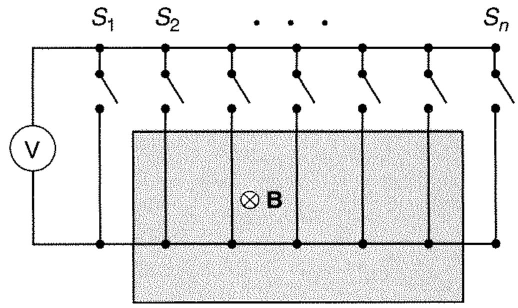


Figure P14.14 A test of electromagnetic induction

P14.14. Sketched in Fig. P14.14 is an experimental setup for the analysis of electromagnetic induction. By closing sequentially the switches $S_1, \ldots, S_n$, it is possible to change the magnetic flux through the closed contour shown from zero to a maximal value. Will the voltmeter indicate an emf induced in the circuit? Explain.

P14.15. Find the angular velocity of the rotor of an idealized electric motor shown in Fig. P14.15 for the case when no load is connected to it. Does the value of R influence the angular velocity? The rotor is in the form of a metal wheel with four spokes, situated in a uniform magnetic field of magnetic flux density $\mathbf{B}$, as shown in the figure. What is the direction of rotation of the rotor?

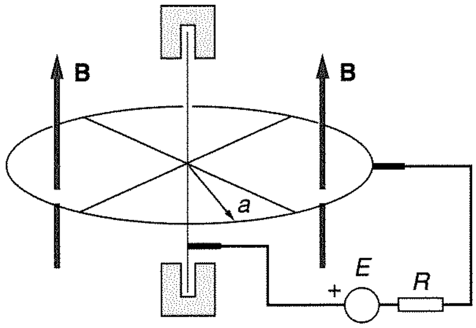


Figure P14.15 An idealized electric motor

P14.16. A circular loop of radius a rotates with an angular velocity ω about the axis lying in its plane and containing the center of the loop. It is situated in a uniform magnetic field of flux density $\mathbf{B}(t)$ normal to the axis of rotation. Determine the induced emf in the loop. At $t = 0$, the position of the loop is such that $\mathbf{B}$ is normal to its plane.

P14.17. A winding of $N = 1000$ turns of wire with sinusoidal current of amplitude $I_m = 200\,\text{mA}$ is wound on a thin toroidal ferromagnetic core of mean radius $a = 10\,\text{cm}$. Figure P14.17 shows the idealized hysteresis loop of the core corresponding to the sinusoidal magnetization of the core to saturation in both directions. Also wound on the toroid are several turns of wire over the first winding. Plot the emf induced in the second winding during one period of the sinusoidal current in the first winding.

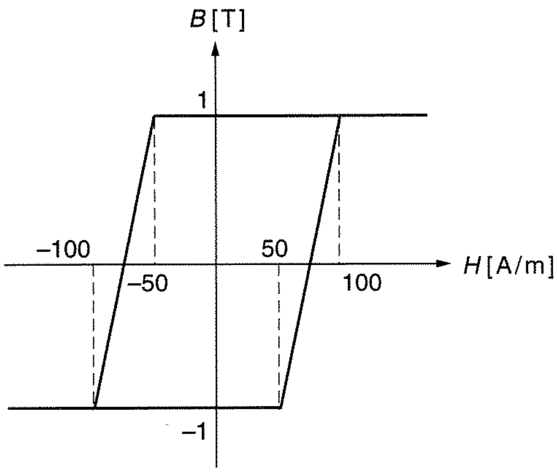


Figure P14.17 An idealized hysteresis loop

P14.18. Shown in Fig. P14.18 is a rectangular loop encircling a very long solenoid of radius R and with N' turns of wire per unit length. The amplitude of current in the winding is

I_m, and its angular frequency is ω. Determine the emf induced in the entire rectangular loop, as well as in its sides a and b separately.

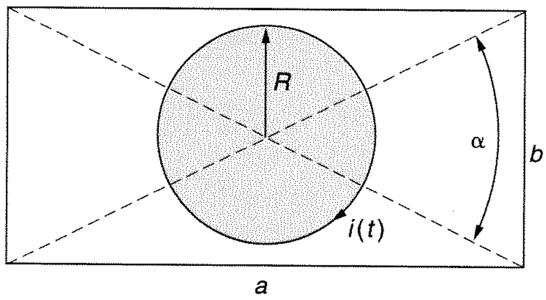


Figure P14.18 A rectangular loop encircling a solenoid

P14.19. The cross section of a thick coil with a large number, N, of turns of thin wire, is shown in Fig. P14.19. The coil is situated in a time-varying magnetic field of flux density $\mathbf{B}(t)$, in the indicated direction. Determine the emf induced in the coil.

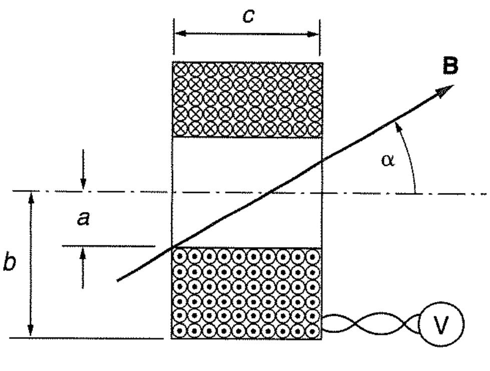


Figure P14.19 A coil of rectangular cross section

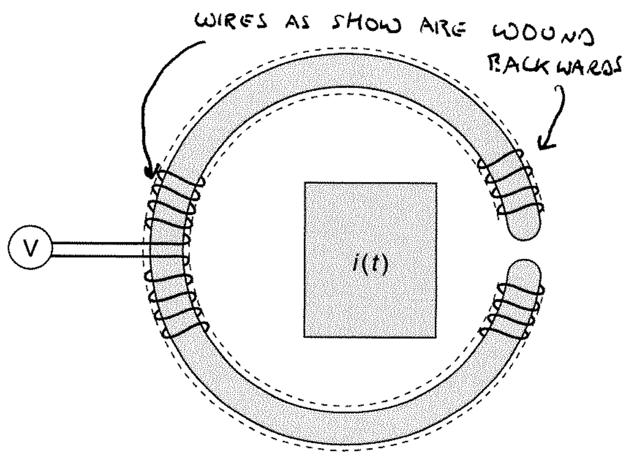


Figure P14.20 A conductor encircled by a coil

P14.20. The conductor whose cross section is shown shaded in Fig. P14.20 carries a sinusoidal current of amplitude I_m and angular frequency ω. The conductor is encircled by a flexible thin rubber strip of cross-sectional area S, densely wound along its length with N' turns of wire per unit length. The measured amplitude of the voltage between the terminals of the strip winding is V_m. Determine I_m.

P14.21. Wire is being wound from a drum D_1 onto a drum D, at a rate of N' turns per unit time (Fig. P14.21). The end of the wire on drum D is fastened to the ring R, which has a sliding contact F. A voltmeter is connected between the contact F and another contact G. Through the drum D there is a constant flux Φ, as indicated. What is the electromotive force measured by the voltmeter?

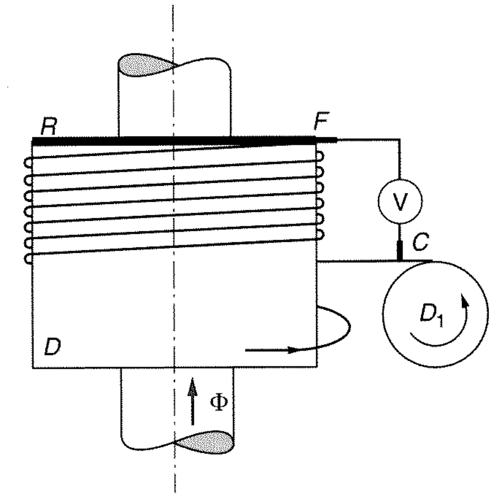


Figure P14.21 A test of electromagnetic induction

P14.22. A cylindrical coil is tightly wound around a ferromagnetic core with time-varying magnetic flux $\Phi(t) = \Phi_m \cos \omega t$, as shown in Fig. P14.22. The length of the coil is L, and the number of turns in the coil is N. If a sliding contact K moves along the coil according to the law $x = L(1 + \cos \omega_1 t)/2$, what is the time dependence of the electromotive force between contacts A and K? Plot your result.

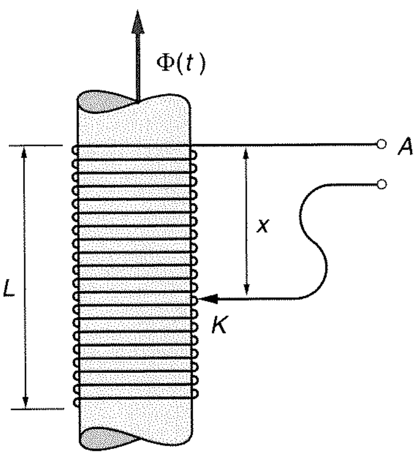


Figure P14.22 A coil with a sliding contact

P14.23. A cylindrical conducting magnet of circular cross section rotates about its axis with a uniform angular velocity. A galvanometer G is connected to the equator of the magnet and to the center of one of its bases by means of sliding contacts, as shown in Fig. P14.23. If such an experiment is performed, the galvanometer indicates a certain current through the circuit (Faraday, 1832). Where is the electromotive force induced: in the stationary conductors connecting the sliding contacts with the galvanometer, or in the magnet itself?

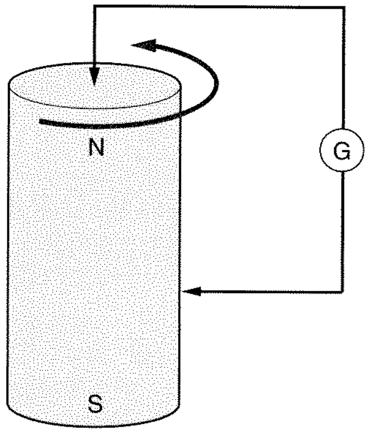


Figure P14.23 A rotating magnet with sliding contacts

P14.24. A ferromagnetic toroid with no air gap is magnetized so that no magnetic field exists outside it. The toroid is encircled by an elastic metal loop, as in Fig. P14.24a. The loop is now taken from the toroid in such a way that during the process, the loop is always electrically closed through the conducting material of the toroid, as in Figs. P14.24b and c. The magnetic flux through the contour was obviously changed from a value Φ, the flux through the toroid, to zero. A formal application of Faraday's law leads to the conclusion that a certain charge will flow through the circuit during the process, but in this case it is not possible to detect any current (Herring, 1908). Explain the negative result of the experiment.

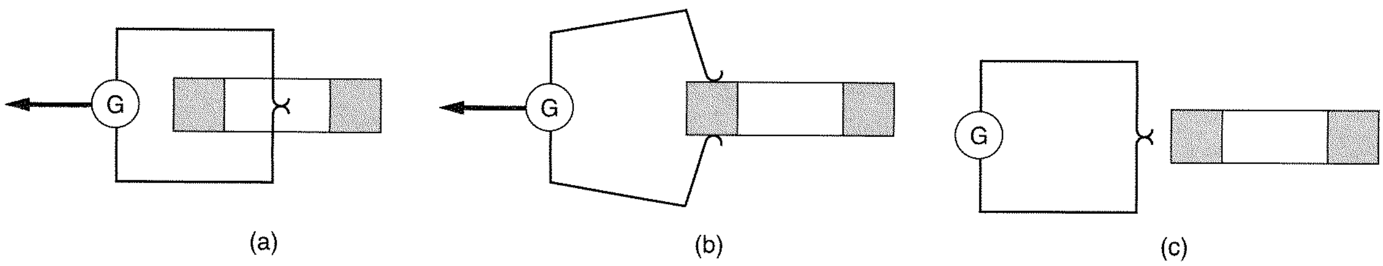


Figure P14.24 (a) A moving elastic loop encircles a magnetized toroid. (b) The loop is moving, but is closed by the conducting toroid body. (c) The loop does not move and does not encircle the toroid.

P14.25. Discuss the possibility of constructing a generator of electromotive force constant in time, operating on the basis of electromagnetic induction.

P14.26. In a straight copper wire of radius $a = 1\,\text{mm}$ there is a sinusoidal current $i(t) = 1 \cos \omega t$ A. A voltmeter is connected between points 1 and 2, with leads of the shape shown in Fig. P14.26. If $b = 50\,\text{cm}$ and $c = 20\,\text{cm}$, evaluate the voltage measured by the voltmeter for (1) $\omega = 314\,\text{rad/s}$, (2) $\omega = 10^4\,\text{rad/s}$, and (3) $\omega = 10^6\,\text{rad/s}$. Assume that the resistance of the copper conductor per unit length, R', is approximately that for a dc current (which actually is *not* the case, due to the so-called skin effect), and evaluate for the three cases the difference between the potential difference $V_1 - V_2 = R'bi(t)$ and the voltage induced in the leads of the voltmeter.

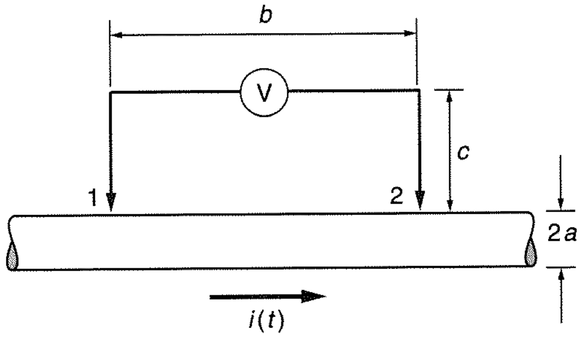


Figure P14.26 Measurement of ac voltage

P14.27. A circular metal loop of radius R, conductivity σ, and cross-sectional area S encircles a long solenoid with a time-varying current $i(t)$ (Fig. P14.27). The solenoid has N' turns of wire per unit length, and its radius is r. Determine the current in the loop, and the voltage between points A and B of the loop along paths a, b, c, and d. Neglect the induced electric field in the loop due to the loop current itself. Determine also the voltage between points A and C, along the path AcC, and along the path $AbBC$.

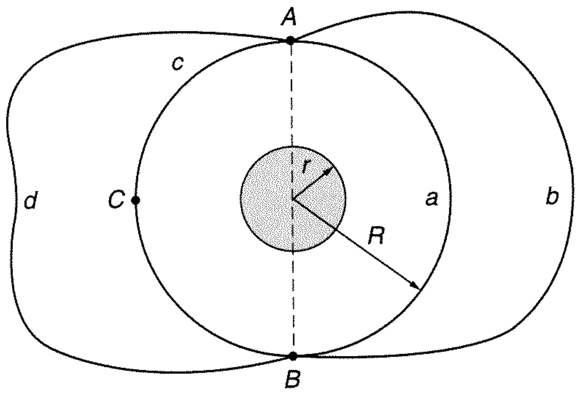


Figure P14.27 A solenoid encircled by a circular loop

P14.28. Repeat problem P14.27 assuming that the two halves of the loop have different conductivites, σ_1 and σ_2, and they meet at points A and B.

15

Inductance

15.1 Introduction

We now know that a time-varying current in one wire loop induces an emf in another loop. We do not know, however, how to compute that emf. In linear media, an electromagnetic parameter that enables simple determination of this emf is the *mutual inductance.*

A wire loop with time-varying current creates a time-varying induced electric field not only in the space around it but also along the loop itself. As a consequence, we have a kind of feedback—the current produces an effect that affects itself. The parameter known as inductance, or *self-inductance*, of the loop enables simple evaluation of this effect.

Mutual inductance and self-inductance are familiar because they are used in circuit theory for describing magnetic coupling. Many manifestations of magnetic coupling are not as familiar, however, although they are not unimportant. For example, what we call magnetic coupling can exist between a 60-Hz power line and a human body, or between parallel printed strips of a computer bus. With a knowledge of the induced electric field, these and related phenomena can be easily understood.

15.2 Mutual Inductance

Consider two stationary thin conductive contours C_1 and C_2 in a linear medium (e.g., air), shown in Fig. 15.1. When a time-varying current $i_1(t)$ flows through the first

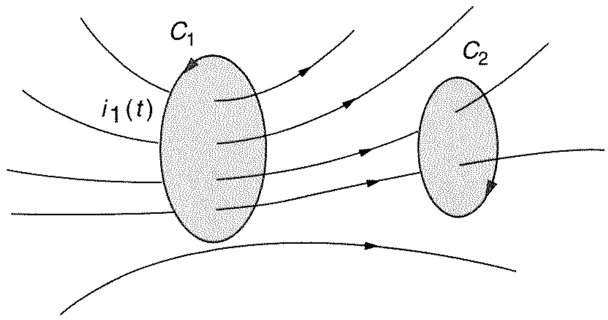


Figure 15.1 Two coupled conductive contours

contour, it creates a time-varying magnetic field as well as a time-varying induced electric field, $\mathbf{E}_{1\text{ind}}$. The latter produces an emf $e_{12}(t)$ in the second contour, given by

$$e_{12}(t) = \oint_{C_2} \mathbf{E}_{1\text{ind}} \cdot \mathrm{d}\mathbf{l}_2, \tag{15.1}$$

(Calculation of distributed emf along a loop)

where the first index denotes the source of the field (contour 1 in this case).

As mentioned earlier, it is usually much easier to find the induced emf using Faraday's law than in any other way. The magnetic flux density vector in linear media is proportional to the current that causes the magnetic field. It follows that the flux $\Phi_{12}(t)$ through C_2 caused by the current $i_1(t)$ in C_1 is also proportional to $i_1(t)$:

$$\Phi_{12}(t) = L_{12} i_1(t). \tag{15.2}$$

The proportionality constant, L_{12}, is called the *mutual inductance* between the two contours. This constant depends only on the geometry of the system and the properties of the (linear) medium surrounding the current contours. Mutual inductance is denoted by both L_{12} (or whatever subscripts are chosen) and—particularly in circuit theory—by M.

Because the variation of $i_1(t)$ can be arbitrary, the same expression holds when the current through C_1 is a dc current:

$$\Phi_{12} = L_{12} I_1. \tag{15.3}$$

(Flux definition of mutual inductance)

Although mutual inductance has no practical meaning for the case of dc currents, this definition is frequently used for the determination of mutual inductance.

According to Faraday's law, the emf can alternatively be written as

$$e_{12}(t) = -\frac{\mathrm{d}\Phi_{12}(t)}{\mathrm{d}t} = -L_{12}\frac{\mathrm{d}i_1(t)}{\mathrm{d}t}. \tag{15.4}$$

(EMF definition of mutual inductance)

The unit for inductance, equal to a Wb/A, is called a henry (H). (Joseph Henry was an American physicist who independently discovered electromagnetic induction at almost the same time as Faraday did.) One henry is quite a large unit. Most frequent values of mutual inductance are on the order of a mH, μH, or even nH.

If we now assume that a current $i_2(t)$ in C_2 causes an induced emf in C_1, we talk about a mutual inductance L_{21}. It turns out that $L_{12} = L_{21}$ always. [This follows from the expression for the induced electric field in Eq. (~~15~~.3) and Eqs. (15.1) and (15.4), but the proof will not be given here.] So we can write 14

$$\boxed{L_{12} = \frac{\Phi_{12}}{I_1} = L_{21} = \frac{\Phi_{21}}{I_2} \qquad \text{(H)}. \qquad (15.5)}$$

These equations show that we need to calculate either Φ_{12} or Φ_{21} to determine the mutual inductance. In some instances one of these is much simpler to calculate, as the following example shows.

Example 15.1—Mutual inductance of a toroidal coil and a wire loop. Let us find the mutual inductance between a contour C_1 and a toroidal coil C_2 with N turns, as in Fig. 15.2. If we try to imagine how to determine L_{12}, it is not at all obvious because the surface of a toroidal coil is complicated. However, $L_{21} = \Phi_{21}/I_2$ is quite simple to find. The flux $\mathrm{d}\Phi$ through the surface $\mathrm{d}S = h\,\mathrm{d}r$ in the figure is given by

$$\mathrm{d}\Phi_{21}(r) = B(r)\,\mathrm{d}S = \frac{\mu_0 N I_2}{2\pi r} h\,\mathrm{d}r.$$

To obtain the total flux through C_1, we integrate over the cross section of the torus,

$$\Phi_{21} = \frac{\mu_0 N I_2 h}{2\pi} \int_a^b \frac{\mathrm{d}r}{r} = \frac{\mu_0 N I_2 h}{2\pi} \ln\frac{b}{a},$$

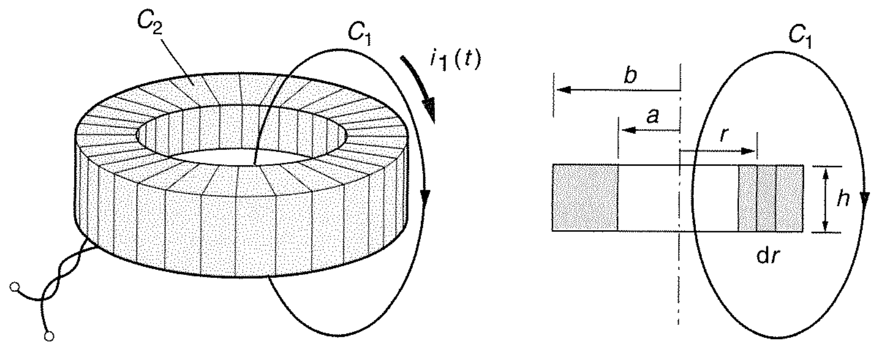


Figure 15.2 A toroidal coil and a single wire loop encircling the toroid

or

$$L_{12} = L_{21} = \frac{\mu_0 N h}{2\pi} \ln \frac{b}{a}.$$

Note that mutual inductance in this case does not depend at all on the shape of the wire loop. Also, if we need a larger mutual inductance (and thus larger induced emf), we can simply wind the loop two or more times around the toroid to obtain two or more times larger inductance. This is the principle of operation of transformers.

Example 15.2—Mutual inductance of two coils wound on a toroidal core. As another example, let us find the mutual inductance between two toroidal coils tightly wound one on top of the other on a core of the form shown in Fig. 15.2. Assume that one coil has N_1 turns and the other N_2 turns. If a current I_2 flows through coil 2, the flux through coil 1 is just N_1 times the flux Φ_{21} from the preceding example, where N should be substituted by N_2. So

$$L_{12} = L_{21} = \frac{\mu_0 N_1 N_2 h}{2\pi} \ln \frac{b}{a}.$$

Questions and problems: Q15.1 to Q15.7, P15.1 to P15.8

15.3 Self-Inductance

As mentioned in the introduction to this chapter, when a current in a contour varies in time, the induced electric field exists everywhere around it and therefore also along its entire length. Consequently there is an induced emf in the contour itself. This process is known as *self-induction.*

The simplest (but not physically the clearest) way of expressing this emf is to use Faraday's law:

$$e(t) = -\frac{d\Phi_{\text{self}}(t)}{dt}. \tag{15.6}$$

If the contour is in a linear medium (i.e., the flux through the contour is proportional to the current), we define the self-inductance of the contour as the ratio of the flux through the contour due to current $i(t)$ in it, and $i(t)$,

$$\boxed{L = \frac{\Phi_{\text{self}}(t)}{i(t)} \qquad (\text{H}).} \tag{15.7}$$

Using this definition, the induced emf can be written as

$$\boxed{e(t) = -L\frac{di}{dt}.} \tag{15.8}$$

The constant L depends only on the geometry of the system and the properties of the medium, and its unit is again a henry (H). In the case of a dc current, $L = \Phi/I$, which can be used for determining the self-inductance in some cases in a simple manner.

How do self-inductances of two contours compare with their mutual inductance? This is easy to answer for two simple loops. In that case, it is evident that the largest possible flux due to a current i_1 through a contour C_1 is the flux through the contour itself (the contour cannot be closer to any other contour than to itself). Therefore

$$\Phi_{11} \geq \Phi_{12} \qquad \text{and} \qquad \Phi_{22} \geq \Phi_{21}. \tag{15.9}$$

When we multiply these inequalities together and divide by $I_1 I_2$, we obtain

$$L_{11} L_{22} \geq L_{12}^2. \tag{15.10}$$

Therefore the largest possible value of mutual inductance is the geometric mean of the self-inductances. Although Eq. (15.10) is derived for a somewhat special case (two simple loops), it can be shown to be valid in general (see Example 16.2 in the following chapter).

Frequently, Eq. (15.10) is written as

$$L_{12} = k\sqrt{L_{11}L_{22}} \qquad -1 \leq k \leq 1. \tag{15.11}$$

The coefficient k is called the *coupling coefficient*.

Example 15.3—Self-inductance of a toroidal coil. Consider again the toroidal coil in Fig. 15.2. If the coil has N turns, what is its self-inductance?

In Example 15.1 we found the flux the coil produces through a cross section of the core. This flux exists through all the N turns of the coil, so that the flux the coil produces through itself is simply N times what we found in Example 15.1. The self-inductance of the coil in Fig. 15.2 is therefore

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\frac{b}{a}.$$

Example 15.4—Self-inductance of a thin two-wire line. Let us find the self-inductance per unit length of a thin two-wire line (Fig. 15.3). We can imagine that the line is actually a very long rectangular contour (closed with a load at one end and a generator at the other end), and that we are looking at only one part of it, hatched in the figure. At a distance r from conductor 1, the current in it produces a magnetic flux density of intensity $B_1(r) = \mu_0 I/(2\pi r)$, and the current in conductor 2 a magnetic flux density $B_2(r) = \mu_0 I/[2\pi(d-r)]$. The total flux through a strip of width $\mathrm{d}r$ and length h shown in the figure is therefore

$$\Phi = \int_a^{d-a} [B_1(r) + B_2(r)]h\,\mathrm{d}r = \frac{\mu_0 I h}{\pi}\ln\frac{d-a}{a} \simeq \frac{\mu_0 I h}{\pi}\ln\frac{d}{a},$$

since $d \gg a$. The inductance per unit length of the two-wire line is therefore

$$L' = \frac{\mu_0}{\pi}\ln\frac{d}{a}. \tag{15.12}$$

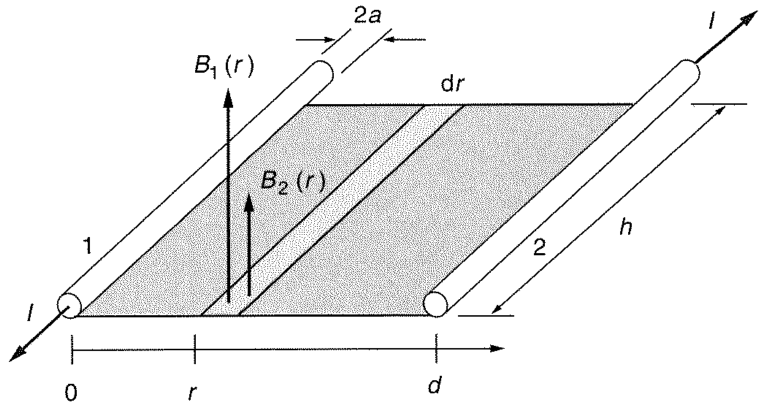


Figure 15.3 Calculating the self-inductance of a thin two-wire line

As a numerical example, for $d/a = 200$, $L' = 2.12\,\mu\text{H/m}$. We have only calculated the flux through the surface outside of the conductors. The expression for L in Eq. (15.12) is therefore called the *external self-inductance* of the line. There is also an *internal self-inductance*, due to the flux through the wires themselves. We will introduce the concept of the internal inductance in terms of energy in the next chapter.

Example 15.5—Self-inductance of a coaxial cable. Let us find the external self-inductance per unit length of a coaxial cable. We first need to figure out through which surface to find the flux. If we imagine that the cable is connected to a generator at one end and to a load at the other, the current flows "in" through the inner conductor and flows back through the outer conductor. The flux through such a contour, for a cable of length h, is the flux through the rectangular surface in Fig. 15.4,

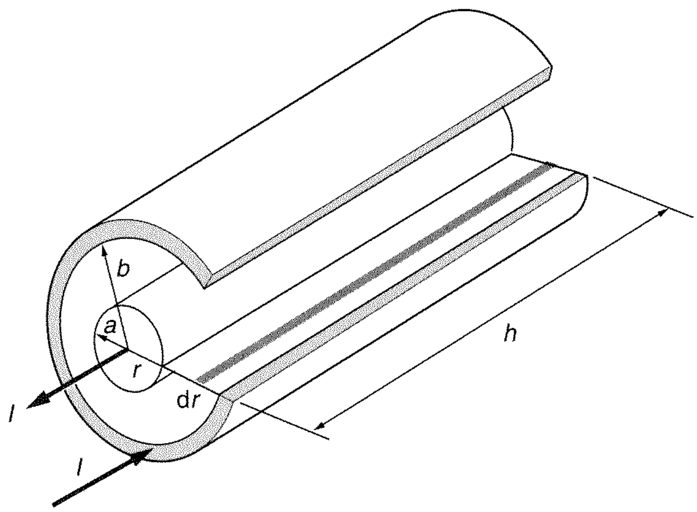


Figure 15.4 Calculating the self-inductance of a coaxial cable

$$\Phi = \int_a^b B(r)h\,\mathrm{d}r = \frac{\mu_0 I h}{2\pi}\ln\frac{b}{a}.$$

The external self-inductance per unit length of the cable is

$$L' = \frac{\mu_0}{2\pi}\ln\frac{b}{a}. \tag{15.13}$$

As a numerical example, for $b/a = \mathrm{e} = 2.71828\ldots$, $L' = 0.2\,\mu\mathrm{H/m}$. It is left as an exercise for the reader to calculate the inductance per unit length of the RG-55/U high-frequency coaxial cable from Example 8.7.

Questions and problems: Q15.8 to Q15.20, P15.9 to P15.21

15.4 Chapter Summary

1. The coupling between two loops by means of the induced electric field is usually termed *magnetic coupling*.
2. The level of coupling between two loops is described by mutual inductance between the loops.
3. Mutual inductance can be calculated as $L_{12} = \Phi_{12}/I_1$, where Φ_{12} is the flux through contour 2 due to a current I_1 in contour 1.
4. A loop with a time-varying current produces an induced electric field also along the loop, which affects the current in the loop. This is known as self-induction, and the parameter describing it is self-inductance.
5. Self-inductance can be evaluated as the ratio of the flux through the contour due to a current in it, divided by that current ($L = \Phi/I$).

QUESTIONS

Q15.1. What does the expression in Eq. (15.1) for the emf induced in a wire loop actually represent?

Q15.2. Why does mutual (and self) inductance have no practical meaning in the dc case?

Q15.3. Explain why mutual inductance for a toroidal coil and a wire loop encircling it (e.g., see Fig. P15.1) does not depend on the shape of the wire loop.

Q15.4. Explain in terms of the induced electric field why the emf induced in a coil encircling a toroidal coil and consisting of several loops (e.g., see Fig. P15.1) is proportional to the number of turns of the loop.

Q15.5. Can mutual inductance be negative as well as positive? Explain by considering reference directions of the loops.

Q15.6. Mutual inductance of two simple loops is L_{12}. We replace the two loops by two very thin coils of the same shapes, with N_1 and N_2 turns of very thin wire. What is the mutual inductance between the coils? Explain in terms of the induced electric field.

Q15.7. A two-wire line crosses another two-wire line at a distance d. The two lines are normal. Prove that the mutual inductance is zero, starting from the induced electric field.

Q15.8. In Example 15.3 we found that the self-inductance of a toroidal coil is proportional to the *square* of the number of turns of the coil. Explain this in terms of the induced electric field and induced voltage in the coil due to the current in the coil.

Q15.9. A thin coil is made of N turns of very thin wire pressed tightly together. If the self-inductance of a single turn of wire is L, what do you expect is the self-inductance of the coil? Explain in terms of the induced electric field.

Q15.10. Explain in your own words what the meaning of self-inductance of a coaxial cable is.

Q15.11. Is it physically sound to speak about the mutual inductance between two wire segments belonging either to two loops or to a single loop? Explain.

Q15.12. Is it physically sound to speak about the self-inductance of a segment of a closed loop? Explain.

Q15.13. To obtain a resistive wire with the smallest self-inductance possible, the wire is sharply bent in the middle and the two mutually insulated halves are pressed tightly together, as shown in Fig. Q15.13. Explain why the self-inductance is minimal in terms of the induced electric field and in terms of the magnetic flux through the loop.

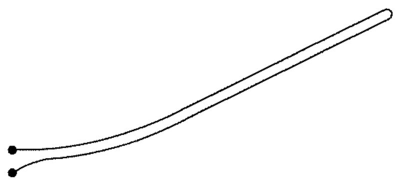


Figure Q15.13 A loop with small self-inductance

Q15.14. Can self-inductance be negative as well as positive? Explain in terms of the flux.

Q15.15. The self-inductance of two identical loops is L. What is approximately the mutual inductance between them if they are pressed together? Explain in terms of the induced electric field and in terms of the magnetic flux.

Q15.16. Two coils are connected in series. Does the total (equivalent) inductance of the connection depend on their mutual position? Explain.

Q15.17. Pressed onto a thin conducting loop is an identical thin *superconducting* loop. What is the self-inductance of the conducting loop? Explain.

Q15.18. A loop is connected to a source of voltage $v(t)$. As a consequence, a current $i(t)$ exists in the loop. Another conducting loop with no source is brought near the first loop. Will the current in the first loop be changed? Explain.

Q15.19. Answer question Q15.18 assuming that the source in the first loop is a dc source. Explain.

Q15.20. A thin, flat loop of self-inductance L is placed over a flat surface of very high permeability. What is the new self-inductance of the loop?

PROBLEMS

P15.1. Find the mutual inductance between an arbitrary loop and the toroidal coil in Fig. P15.1. There are N turns around the torus, and the permeability of the core is μ.

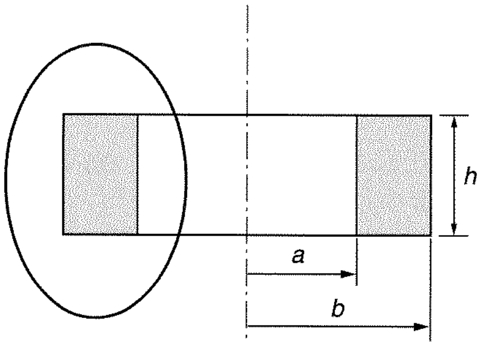


Figure P15.1 A toroidal coil and wire loop

P15.2. Find the mutual inductance of two two-wire lines running parallel to each other. The cross section of the lines is shown in Fig. P15.2.

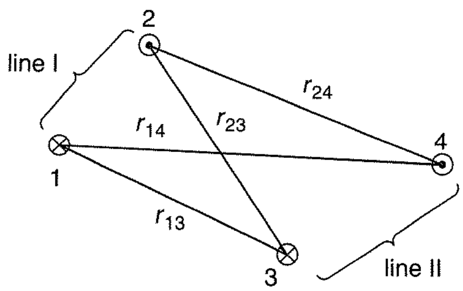


Figure P15.2 Two parallel two-wire lines

P15.3. A cable-car track runs parallel to a two-wire phone line, as in Fig. P15.3. The cable-car power line and track form a two-wire line. The amplitude of the sinusoidal current through the cable-car wire is I_m and its angular frequency is ω. All conductors are very

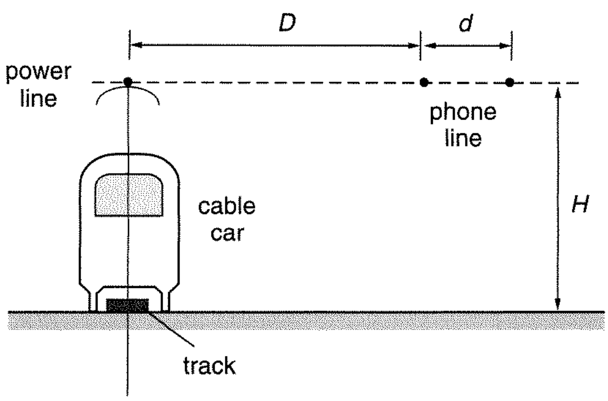


Figure P15.3 Cable-car track parallel to phone line thin compared to the distances between them. Find the amplitude of the induced emf in a section of the phone line b long.

P15.4. Parallel to a thin two-wire symmetrical power line along a distance h is a thin two-wire telephone line, as shown in Fig. P15.4. (1) Find the mutual inductance between the two lines. (2) Find the amplitude of the emf induced in the telephone line when there is a sinusoidal current with amplitude I_m and frequency f in the power line. As a numerical example, assume the following: $f = 100\,\text{Hz}$, $I_m = 100\,\text{A}$, $h = 50\,\text{m}$, $d = 10\,\text{m}$, $a = 50\,\text{cm}$, and $b = 25\,\text{cm}$.

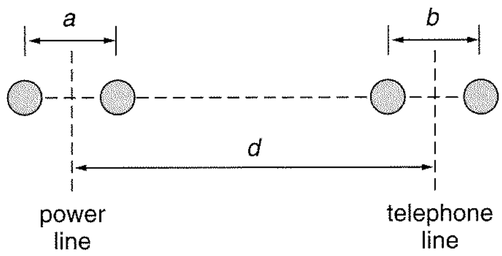


Figure P15.4 Parallel power and phone lines

P15.5. Two coaxial thin circular loops of radii a and b are in the same plane. Assuming that $a \gg b$ and that the medium is air, determine approximately the mutual inductance of the loops. As a numerical example, evaluate the mutual inductance if $a = 10\,\text{cm}$ and $b = 1\,\text{cm}$.

P15.6. Two coaxial thin circular loops of radii a and b are in air a distance d ($d \gg a, b$) apart. Determine approximately the mutual inductance of the loops. As a numerical example, evaluate the mutual inductance if $a = b = 1\,\text{cm}$ and $d = 10\,\text{cm}$.

P15.7. Inside a very long solenoid wound with N' turns per unit length is a small flat loop of surface area S. The plane of the loop makes an angle θ with the solenoid axis. Determine and plot the mutual inductance between the solenoid and the loop as a function of θ. The medium is air.

P15.8. Assume that within a certain time interval the current in circuit 1 in Fig. P15.8 grows linearly, $i_1(t) = I_0 + It/t_1$. Will there be any current in circuit 2 during this time? If yes, what is the direction and magnitude of the current? The number of turns of the two coils is the same.

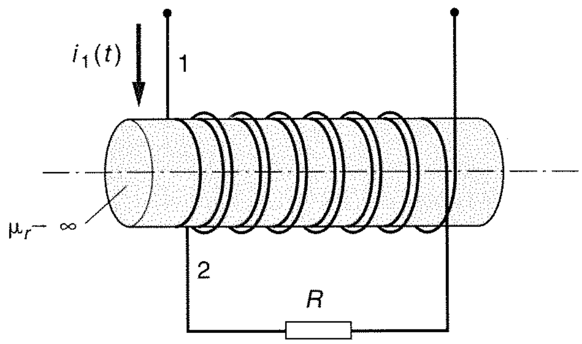


Figure P15.8 Two coupled circuits

P15.9. Three coupled closed circuits have self-inductances equal to L_1, L_2, and L_3, resistances R_1, R_2, and R_3, and mutual inductances L_{12}, L_{13}, and L_{23}. Write the equations for the currents in all three circuits if a voltage $v_1(t)$ is connected to circuit 1 only. Then write the equations for the case when three sources of voltages $v_1(t)$, $v_2(t)$, and $v_3(t)$ are connected to circuits 1, 2, and 3, respectively.

P15.10. A coaxial cable has conductors of radii a and b. The inner conductor is coated with a layer of ferrite d thick ($d < b - a$) and of permeability μ. The rest of the cable is air-filled. Find the external self-inductance per unit length of the cable. What should your expression reduce to (1) when $d = 0$ and (2) when $d = b - a$?

P15.11. The conductor radii of a two-wire line are a and the distance between them is d ($d \gg a$). Both conductors are coated with a thin layer of ferrite b thick ($b \ll d$) and of permeability μ. The ferrite is an insulator. Calculate the external self-inductance per unit length of the line.

P15.12. The core of a toroidal coil of N turns consists of two materials of respective permeabilities μ_1 and μ_2, as in each part of Fig. P15.12. Find the self-inductance of the toroidal coil and the mutual inductance between the coil and the loop positioned as in Fig. P15.1 if (1) the ferrite layers are of equal thicknesses, $h/2$, in Fig. P15.12a, and (2) the ferrite layers are of equal heights h and the radius of the surface between them is c ($a < c < b$), in Fig. P15.12b.

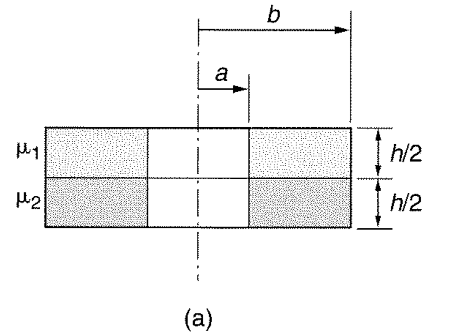


Figure P15.12 Two toroidal coils with inhomogeneous cores

P15.13. Three toroidal coils are wound in such a way that the coils 2 and 3 are inside coil 1, as in the cross section shown in Fig. P15.13. The medium is air. Find the self-inductances

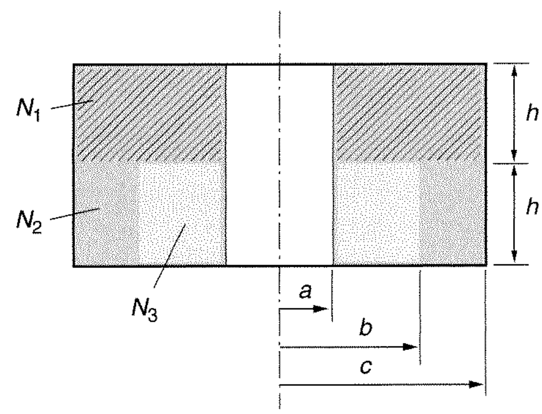


Figure P15.13 Three toroidal coils

L_1, L_2, and L_3 and mutual inductances L_{12}, L_{13}, and L_{23}. What are the different values of inductance that can be obtained by connecting the three windings in series in different ways?

P15.14. The width of the strips of a long, straight strip line is a and their distance is d (Fig. P15.14 for $d_2 = 0$). Between the strips is a ferrite of permeability μ. Neglecting edge effects, find the inductance of the line per unit length.

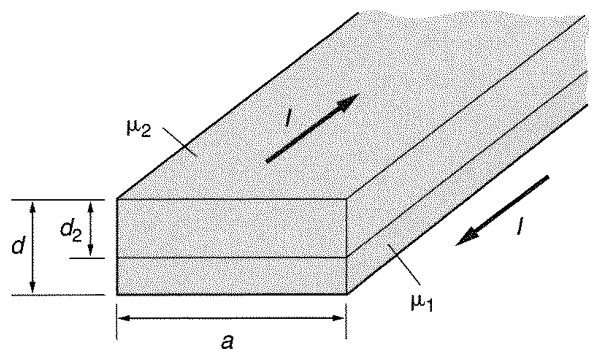


Figure P15.14 A strip line with a two-layer dielectric

P15.15. The width of the strips of a strip line is a and their distance is d. Between the strips are two ferrite layers of permeabilities μ_1 and μ_2, and the latter is d_2 thick, as in Fig. P15.14. Neglecting edge effects, find the inductance of the line per unit length.

P15.16. A long thin solenoid of length b and cross-sectional area S is situated in air and has N tightly wound turns of thin wire. Neglecting edge effects, determine the inductance of the solenoid.

P15.17. A thin toroidal core of permeability μ, mean radius R, and cross-sectional area S is densely wound with two coils of thin wire with N_1 and N_2 turns, respectively. The windings are wound one over the other. Determine the self- and mutual inductances of the coils and the coefficient of coupling between them.

P15.18. A thin toroidal ferromagnetic core of mean radius R and cross-sectional area S is densely wound with N turns of thin wire. A current $i(t) = I_0 + I_m \cos \omega t$, where I_0 and I_m are constants and $I_0 \gg I_m$, is flowing through the coil. Which permeability would you adopt in approximately determining the coil self-inductance? Assuming that this permeability is μ, determine the self-inductance of the coil. Does it depend on I_0?

P15.19. A thin solenoid is made of a large number of turns of very thin wire tightly wound in several layers. The radius of the innermost layer is a, of the outermost layer b, and the solenoid length is d ($d \gg a, b$). The total number of turns is N, and the solenoid core is made out of cardboard. Neglecting edge effects, determine approximately the solenoid self-inductance. Note that the magnetic flux through the turns differs from one layer to the next. Plot this flux as a function of radius, assuming the layers of wire are very thin.

P15.20. Repeat problem P15.19 for a thin toroidal core. Assume that the mean toroid radius is R, the total number of turns N, the radius of the innermost layer a, and that of the outermost layer b, with $R \gg a, b$.

P15.21. The current intensity in a circuit of self-inductance L and negligible resistance was kept constant during a period of time at a level I_0. Then during a short time interval Δt, the current was linearly reduced to zero. Determine the emf induced in the circuit during this time interval. Does this have any connection with a spark you have probably seen inside a switch you turned off in the dark? Explain.

16

Energy and Forces in the Magnetic Field

16.1 Introduction

Many devices make use of electric or magnetic forces. Most can be made in an electric version or a magnetic version. We shall see that magnetic forces are several orders of magnitude stronger than electric forces. Consequently, devices based on magnetic forces are much smaller and are used more often: for example, electric motors, large cranes for lifting ferromagnetic objects, doorbells, and electromagnetic relays.

This chapter derives the expressions for calculating magnetic energy, forces, and pressures. To a large extent, it parallels the chapter on electric energy, forces, and pressures, so the discussion is fairly brief.

16.2 Energy in the Magnetic Field

We did not mention energy when we discussed time-invariant magnetic fields because while establishing a dc current the current through a contour has to change from zero to its final dc value. During this process, there is a changing magnetic flux through the contour due to the changing current, and an emf is induced in the contour. This emf opposes the change of flux, according to Lentz's law. To establish the final static magnetic field, the sources have to overcome this emf. Therefore, we could not talk about energy in the field without knowing about electromagnetic induction.

Let n contours, with currents $i_1(t),\ i_2(t), \ldots, i_n(t)$, be the sources of a magnetic field. Assume that the contours have resistances $R_1, R_2, \ldots, R_n$ and are connected to generators of electromotive forces $e_1(t),\ e_2(t), \ldots, e_n(t)$. Finally, let the contours be stationary and rigid (i.e., they cannot be deformed), with *total* fluxes $\Phi_1(t),\ \Phi_2(t), \ldots, \Phi_n(t)$.

The work done by the voltage generators in *all* the contours during a short time interval $\mathrm{d}t$ is partly converted into Joule's losses and partly used to change the magnetic field:

$$\mathrm{d}A_g = \mathrm{d}A_J + \mathrm{d}A_m. \tag{16.1}$$

Generators in individual contours $k, k = 1, 2, \ldots, n$ do the following work:

$$(\mathrm{d}A_g)_k = e_k(t) i_k(t)\,\mathrm{d}t \qquad k = 1, 2, \ldots, n. \tag{16.2}$$

We also know that the total emf's in the loops are $e_k(t) - \mathrm{d}\Phi_k(t)/\mathrm{d}t$, so that $e_k(t) = R_k i_k(t) + \mathrm{d}\Phi_k(t)/\mathrm{d}t$. The last equation thus becomes

$$(\mathrm{d}A_g)_k = R_k i_k^2(t)\,\mathrm{d}t + i_k(t)\,\mathrm{d}\Phi_k(t) \qquad k = 1, 2, \ldots, n. \tag{16.3}$$

The work of all the generators in the system is hence

$$\mathrm{d}A_g = \sum_{k=1}^{n} R_k i_k^2(t)\,\mathrm{d}t + \sum_{k=1}^{n} i_k(t)\,\mathrm{d}\Phi_k(t). \tag{16.4}$$

The first term on the right-hand side is equal to the Joule's losses, $\mathrm{d}A_J$, during time interval $\mathrm{d}t$. Therefore, according to Eq. (16.1), the second term on the right-hand side is equal to $\mathrm{d}A_m$ (the energy used to change the magnetic field):

$$\mathrm{d}A_m = \mathrm{d}A_g - \mathrm{d}A_J = \sum_{k=1}^{n} i_k(t)\,\mathrm{d}\Phi_k(t). \tag{16.5}$$

This equation expresses the law of conservation of energy for n stationary current contours. $\mathrm{d}A_m$ is the work necessary to change the fluxes through the n contours by $\mathrm{d}\Phi_1, \mathrm{d}\Phi_2, \ldots, \mathrm{d}\Phi_n$.

Let us now find the total work A_m needed to establish dc currents $I_1, I_2, \ldots, I_n$ for which the fluxes through the contours are $\Phi_1, \Phi_2, \ldots, \Phi_n$. This is obtained by integrating the last equation from zero fluxes through the contours to fluxes $\Phi_1, \Phi_2, \ldots, \Phi_n$:

$$(A_m)_{\text{in establishing currents}} = \sum_{k=1}^{n} \int_0^{\Phi_k} i_k(t)\,\mathrm{d}\Phi_k(t). \tag{16.6}$$

During the time needed to establish the fluxes Φ_k, $k = 1, 2, \ldots, n$, the currents in the contours could have varied from zero to their final values in an infinite number of ways. From the law of conservation of energy, no matter how they have changed the final energy would have to be the same. If this is so, assume simply that all of the currents changed linearly with time and that it took a time T to establish the final dc currents. So we assume that $i_k(t) = I_k t/T$. Obviously the fluxes through the contours then also change linearly, $\Phi_k(t) = \Phi_k t/T$, so we have

$$(A_m)_{\text{in establishing currents}} = \sum_{k=1}^{n} \int_0^T I_k \frac{t}{T} \Phi_k \frac{\mathrm{d}t}{T} = \sum_{k=1}^{n} \frac{1}{2} I_k \Phi_k. \tag{16.7}$$

This is valid only for linear media because we assumed that no work was spent on magnetizing any ferromagnetic body. The energy equal to this work is now stored in the magnetic field, and if the currents are reduced to zero this amount of energy is obtained from the system. Therefore we know that there is energy in a static magnetic field equal to

$$W_m = \frac{1}{2} \sum_{k=1}^{n} I_k \Phi_k. \tag{16.8}$$

(Magnetic energy of n current contours)

This can also be expressed in terms of self- and mutual inductances of the contours and currents in them. First, the *total* flux through the k-th contour (due to the current in itself and in all the other contours) can be expressed as

$$\begin{aligned}\Phi_k &= \Phi_{1k} + \Phi_{2k} + \cdots + \Phi_{kk} + \cdots + \Phi_{nk} \\ &= L_{1k} I_1 + L_{2k} I_2 + \cdots + L_{kk} I_k + \cdots + L_{nk} I_n.\end{aligned} \tag{16.9}$$

This can be written in the form

$$\Phi_k = \sum_{j=1}^{n} L_{jk} I_j. \tag{16.10}$$

Thus the magnetic energy in Eq. (16.8) of n contours with currents $I_1, I_2, \ldots, I_n$ can also be written as

$$W_m = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} L_{jk} I_j I_k. \tag{16.11}$$

(Magnetic energy of n current contours)

The important case is that of a single contour:

$$W_m = \frac{1}{2} I \Phi = \frac{1}{2} L I^2. \tag{16.12}$$

(Magnetic energy of a single current contour)

Example 16.1—Magnetic energy of two magnetically coupled contours. In the case of two contours ($n = 2$), Eqs. (16.8) and (16.11) for the magnetic energy of n contours become

$$W_m = \frac{1}{2}(I_1\Phi_1 + I_2\Phi_2)$$

and

$$W_m = \frac{1}{2}L_{11}I_1^2 + \frac{1}{2}L_{22}I_2^2 + L_{12}I_1I_2.$$

This energy can be smaller or larger than the sum of energies of the two contours when isolated because L_{12} can be positive or negative.

Example 16.2—General proof that $|L_{12}| \leq \sqrt{L_{11}L_{22}}$. We have proved the inequality $|L_{12}| \leq \sqrt{L_{11}L_{22}}$ considering two simple loops only. We now prove that this is true for any two contours.

If I_1 is kept constant, the preceding equation can be rewritten as

$$W_m = I_1^2\left(\frac{1}{2}L_{11} + \frac{1}{2}L_{22}x^2 + L_{12}x\right), \qquad \text{where} \qquad x = I_2/I_1.$$

The magnetic energy W_m is always larger than zero. Its minimum is found from $\mathrm{d}W_m/\mathrm{d}x = 0$:

$$\frac{\mathrm{d}W_m}{\mathrm{d}x} = I_1^2(L_{22}x + L_{12}) = 0.$$

The latter is true for $x = -L_{12}/L_{22}$. For this value of x, the expression for the magnetic energy becomes

$$(W_m)_{\min} = \frac{1}{2}I_1^2(L_{11} - L_{12}^2/L_{22}).$$

Because $W_{\mathrm{m}} \geq 0$, we see that for any two coupled contours, $L_{12}^2 \leq L_{11}L_{22}$.

Questions and problems: Q16.1 to Q16.15, P16.1

16.3 Distribution of Energy in the Magnetic Field

We saw earlier that in the electrostatic field we could find the energy in two ways: as a potential energy of a system of charges or as energy distributed in the entire field with a certain density. We shall now show that an expression of the form in Eq. (9.7) can also be derived for the energy of a magnetic field.

Consider first a simple example, a thin torus of radius R and core cross-sectional area S with N turns carrying a current $i(t)$. The core can be of any homogeneous magnetic material. The magnetic field in the torus is

$$H(t) = \frac{Ni(t)}{2\pi R}, \tag{16.13}$$

from which $i(t) = 2\pi RH(t)/N$. Let $\mathrm{d}\Phi(t) = S\,\mathrm{d}B(t)$ be the increase in the flux in the core of the torus during a short time interval $\mathrm{d}t$. Then the increase in the flux in all the N turns is $NS\,\mathrm{d}B(t)$. According to the formula in Eq. (16.6), the work done by the sources to change the flux through the torus from Φ_1 to Φ_2 is

$$(A_m)_{\text{from } \Phi_1 \text{ to } \Phi_2} = \int_{\Phi_1}^{\Phi_2} i(t)\,\mathrm{d}\Phi(t) = 2\pi RS \int_{B_1}^{B_2} H(t)\,\mathrm{d}B(t), \tag{16.14}$$

where B_1 is the initial magnetic flux density and B_2 the final flux density. Because $2\pi RS$ is the volume of the torus, we see that the volume energy density spent in order to change the magnetic flux density vector from B_1 to B_2 is equal to

$$\frac{\mathrm{d}A_m}{\mathrm{d}v} = \int_{B_1}^{B_2} H(t)\,\mathrm{d}B(t). \tag{16.15}$$

(Density of work that needs to be done to change B from B_1 to B_2 at a point)

This formula was derived for a special case of a toroidal coil. It can be shown that it remains valid for an arbitrary magnetic field (in a manner analogous to that which we used for the electric energy, when we generalized the proof obtained for a parallel-plate capacitor to the general case). In the present case, we imagine the entire field divided into elemental tubes of flux of vector **B**. The proof, which is somewhat more complicated than in the electrostatic case but quite analogous, is left to the interested reader as an exercise.

In the case of linear media, energy used for changing the magnetic field is stored in the field, that is, $\mathrm{d}A_m = \mathrm{d}W_m$. Assuming that the B field changed from zero to some value B, we have

$$\frac{\mathrm{d}W_m}{\mathrm{d}v} = \int_0^B \frac{B}{\mu}\mathrm{d}B = \frac{1}{2}\frac{B^2}{\mu} = \frac{1}{2}\mu H^2 = \frac{1}{2}BH. \tag{16.16}$$

(Density of energy in magnetic field—linear media only)

The energy *in a linear medium* can now be found by integrating over the entire volume of the field:

$$W_m = \int_v \frac{1}{2}\mu H^2 \mathrm{d}v. \tag{16.17}$$

(Magnetic energy distributed over the entire field—linear media only)

Example 16.3—Losses in ferromagnetics due to hysteresis. Let us observe what happens to energy spent in maintaining a sinusoidal magnetic field in a piece of ferromagnetic material. The hysteresis curve of the material is shown in Fig. 16.1, and the arrows show the direction in which the point describing the curve is moving in the course of time. According

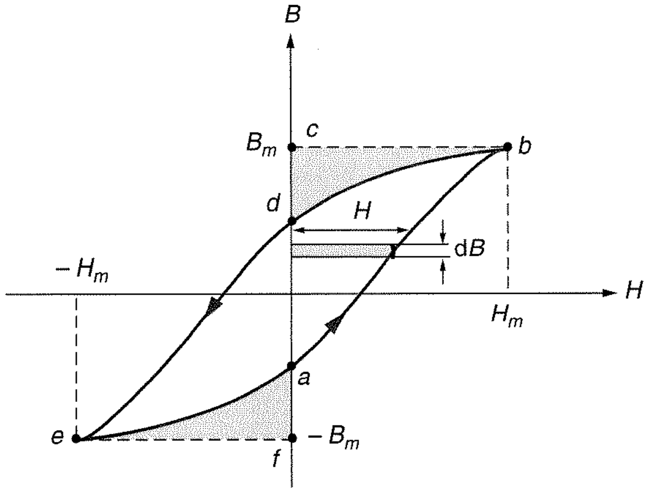


Figure 16.1 Hysteresis curve of a ferromagnetic material

to Eq. (16.15), the energy density that needs to be spent at a point where the magnetic field is H, in order to change the magnetic flux density by $\mathrm{d}B$, is equal to $H\,\mathrm{d}B$. In the diagram in Fig. 16.1, this is proportional to the area of the little shaded rectangle. So the integral of $H\,\mathrm{d}B$ is proportional to the sum of all such rectangles as the point moves around the hysteresis curve.

Let us start from point a in the figure. From a to b, the magnetic field H is positive. The increase $\mathrm{d}B$ is also positive, so $H\,\mathrm{d}B$ is positive and the energy density spent moving from point a to b is proportional to the area of the curved triangle abc in the figure.

From b to d, H is positive but B is decreasing, so that $\mathrm{d}B$ is negative. Therefore the product $H\,\mathrm{d}B$ is negative, which means that in this region the energy spent on maintaining the field is negative. This in turn means that this portion of the energy is returned from the field to the sources. The density of this returned energy is proportional to the area of the curved triangle bdc.

From d to e the product $H\mathrm{d}B$ is positive, so this energy is spent on maintaining the field, and from e to a the product is negative, so this energy is returned to the sources. So only the energy density proportional to the area of the curved triangles bcd and efa is returned to the sources. All the rest, which is proportional to the area formed by the hysteresis loop, is lost to heat in the ferromagnetic material. These losses are known as *hysteresis losses.* If the frequency of the field is f, the loop is circumscribed f times per second. Consequently, *the power of hysteresis losses is proportional to frequency* (and to the volume of the ferromagnetic material if the field is uniform).

Example 16.4—Internal inductance of a straight wire. The energy of a wire with a current i is distributed around the wire as well as inside the wire because there is a magnetic field both outside and inside the wire. From the energy expression $W_m = \frac{1}{2}Li^2$ for a single current contour, we can write

$$L_{\text{internal}} = \frac{2(W_m)_{\text{inside conductor}}}{i^2}$$

and

$$L_{\text{external}} = \frac{2(W_m)_{\text{outside conductor}}}{i^2}.$$

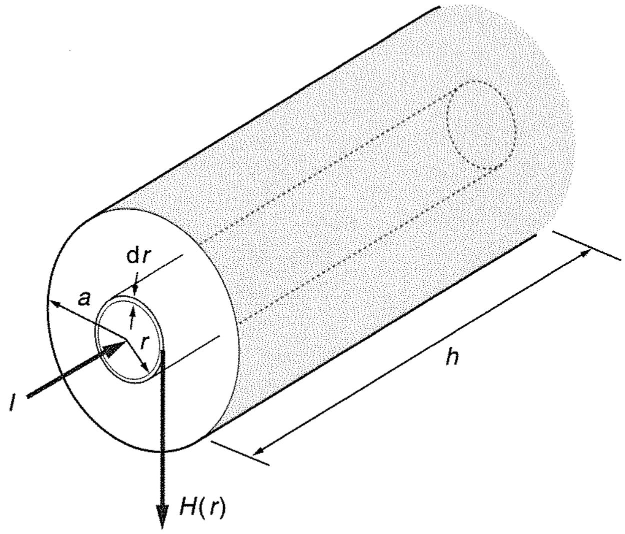


Figure 16.2 A long straight wire of circular cross section

A long straight wire of circular cross section is shown in Fig. 16.2. According to Ampère's law, the magnetic field inside the wire is equal to $H(r) = (Ir)/(2\pi a^2)$, so the energy density in the wire is

$$\frac{\mathrm{d}W_m}{\mathrm{d}v} = \frac{1}{2}\mu H^2 = \frac{1}{2}\frac{\mu I^2 r^2}{(2\pi a^2)^2}.$$

The magnetic energy stored in a length h of the wire is obtained by integrating the last expression over the volume of the wire segment. The integration is easily done if the volume element is chosen to be a thin tube shown in the figure. The volume of the tube is $\mathrm{d}v = 2\pi rh\,\mathrm{d}r$. We thus find

$$(W_m)_{\text{inside wire}} = \int_0^a \frac{1}{2}\mu H^2(r) 2\pi rh\,\mathrm{d}r = \frac{\mu I^2 h}{4\pi a^4}\int_0^a r^3 \mathrm{d}r = \frac{\mu I^2 h}{16\pi}.$$

The internal self-inductance per unit length of a wire is hence

$$L'_{\text{internal}} = \frac{\mu}{8\pi}. \tag{16.18}$$

Note that the internal inductance does not depend on the radius of the wire. (Also note that it is very difficult to find internal inductance using the methods from Chapter 15 for external self- and mutual inductance. Why?)

Example 16.5—Total inductance of a thin two-wire line. Let us find the total self-inductance per unit length of a thin two-wire line with wires made of a material with permeability μ. Assume that the wire radius is a and the distance between the wire axes d. We found the external inductance in Example 15.4, so

$$L' = L'_{\text{external}} + L'_{\text{internal}} = \frac{\mu_0}{\pi}\ln\frac{d}{a} + 2\frac{\mu}{8\pi}. \tag{16.19}$$

We multiplied the expression for the internal inductance of a wire by 2 because there are two wires in the line. As a numerical example, if $\mu = \mu_0$ and $d/a = 100$, we get $L'_{\text{external}} = 1.84\,\mu\text{H/m}$ and $L'_{\text{internal}} = 0.1\,\mu\text{H/m}$. In this example, the external inductance is much larger than the internal inductance. This is usually the case.

Questions and problems: Q16.16 to Q16.29, P16.2 to P16.15

16.4 Magnetic Forces

Suppose we know the distribution of currents, and that the currents exist in a magnetically homogeneous medium. In this case, the Biot-Savart law can be used for determining the magnetic flux density. Combined with the relation $d\mathbf{F}_m = I\,d\mathbf{l} \times \mathbf{B}$, we can find the magnetic force on any part of the current distribution. In many cases, however, this is quite complicated.

Similarly to finding the electric force from a change in energy, we can find the magnetic force as a derivative of the magnetic energy. This can be done assuming either (1) the fluxes through all the contours are kept constant or (2) the currents in all the contours are kept constant.

Assume first that during a displacement dx of a body in the magnetic field along the x axis we keep the fluxes through all the contours constant. This, of course, can be done by varying the currents in the contours appropriately. According to Eq. (16.5), during such a displacement the sources do not perform any work. (In fact, this situation corresponds to all the loops being superconducting, when no change of flux is possible—see Example 14.7.) Therefore, the work by the magnetic force in moving the body was done at the expense of the magnetic energy of the system:

$$F_x = -\left(\frac{dW_m}{dx}\right)_{\Phi=\text{constant}}. \tag{16.20}$$

In the second case, when the currents are kept constant, the fluxes can change. Therefore the sources have to do some work during the displacement, and it can be shown that

$$F_x = +\left(\frac{dW_m}{dx}\right)_{I=\text{constant}}. \tag{16.21}$$

Example 16.6—Lifting force of an electromagnet. As an example of the first formula let us find the attractive force of an electromagnet, sketched in Fig. 16.3. The electromagnet is in the shape of a horseshoe and its magnetic force is lifting a weight W, shown in the figure. This is a magnetic circuit. Let us assume that when the weight W moves by a small amount dx upward, the flux in the magnetic circuit does not change. That means that when the weight is moved upward the only change in magnetic energy is the reduction in energy contained in the two air gaps due to their decreased length. This energy reduction is

$$-dW_m = \frac{1}{2}\frac{B^2}{\mu_0}2S\,dx,$$

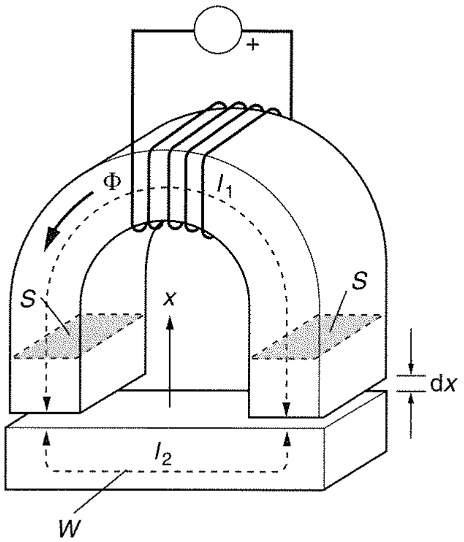


Figure 16.3 An electromagnet

so that

$$F_x = \frac{1}{2}\frac{B^2}{\mu_0}2S = \frac{\Phi^2}{\mu_0 S}.$$

As a numerical example, let $B = 1\,\mathrm{T}$ and $S = 1000\,\mathrm{cm}^2$. For this case, $F_x = 7.96 \cdot 10^4\,\mathrm{N}$, which means that this electromagnet can lift a weight of about 8 tons! Such electromagnets are used in cranes for lifting large pieces of iron, for example.

Example 16.7—Magnetic force acting on a rectangular loop in the field of a straight wire with current. As an example of the second formula for calculating the magnetic forces, consider a rectangular contour with a current I_2 that is y away from a long straight wire with current I_1, as shown in Fig. 16.4. Let us find all three components of the force, F_x, F_y, and F_z.

There is obviously no flux change through the contour if it is moved in the x or z directions. So $F_x = F_z = 0$, and we need to find only F_y. When the currents through the wires are kept constant, according to the last equation in Example 16.1 we can write

$$F_y = \frac{\mathrm{d}W_\mathrm{m}}{\mathrm{d}y} = \frac{\mathrm{d}(L_{12}I_1I_2)}{\mathrm{d}y} = I_2\frac{\mathrm{d}\Phi_{12}}{\mathrm{d}y}, \qquad I_2 \text{ is constant.}$$

The change of flux through the rectangular contour is only due to the current in the straight wire,

$$\Phi_{12}(y) = \frac{\mu_0 I_1 b}{2\pi}\ln\frac{y+a}{y},$$

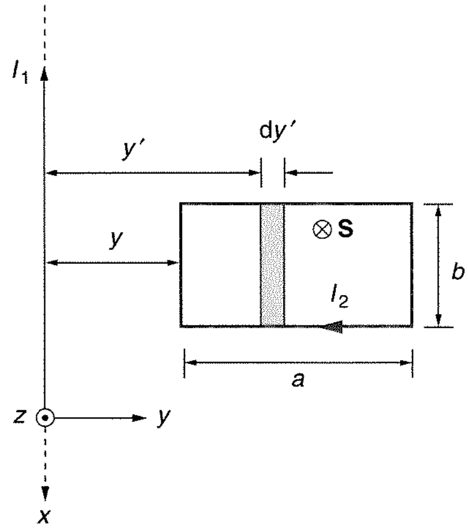


Figure 16.4 An example of the magnetic force calculation

so that

$$F_y = I_2 \frac{d\Phi_{12}}{dy} = -\mu_0 \frac{I_1 I_2 b}{2\pi} \frac{a}{y(y+a)}.$$

The negative sign means that the force is in the $-y$ direction, i.e., it is attractive.

Example 16.8—An ammeter. A possible way to build a simple ammeter using magnetic forces is shown in Fig. 16.5. A piece of ferromagnetic material, for example an iron nail (of cross-sectional radius a), is inserted partway into a solenoid and hangs off a spring. The relative

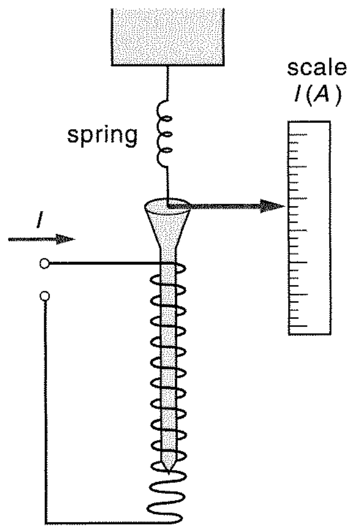


Figure 16.5 A simple ammeter

position of the nail in the vertical direction can be measured against a scale. When no current is flowing through the solenoid, the nail position is at zero. When a current flows through the coil, there is a force acting on the nail in addition to the gravitational force, the nail moves downward, and the new position of the nail is a direct (but not linear) measure of the current intensity in the coil. As an exercise (see problem P16.29), plot the "scale calibration curve" of such an ammeter. For what current levels is it useful given the dimensions in P16.29?

Example 16.9—Comparison of electric and magnetic pressure. We derived the expressions for the pressure of electric forces starting from the formulas analogous to Eqs. (16.20) and (16.21). The derivation of pressure in this case is therefore completely the same and will not be repeated here (although it is suggested to the reader to repeat the derivation as an exercise). For two magnetic media of permeabilities μ_1 and μ_2, the pressure on the interface, assumed to be directed into medium 1, is given by

$$p = \frac{1}{2}(\mu_2 - \mu_1)\left(H_{\text{tang}}^2 + \frac{B_{\text{norm}}^2}{\mu_1\mu_2}\right) \qquad \text{(reference direction of pressure into medium 1).} \tag{16.22}$$

We know that magnetic flux density of about 1 T is quite large and not easily attainable. Therefore, according to the expression derived in Example 16.6, the maximal magnetic pressure that can be obtained is on the order of

$$(p_m)_{\max} = \frac{1^2}{2 \cdot 4\pi \cdot 10^{-7}} \simeq 400{,}000 \frac{\text{N}}{\text{m}^2}.$$

The electric pressure on a metallic conductor in a vacuum is given in Eq. (9.18), which can be rewritten as $p_e = \frac{1}{2}\epsilon_0 E^2$. We know that the electric strength of air is about $3 \cdot 10^6$ V/m. This means that the largest electric pressure in air is approximately

$$(p_e)_{\max} = 0.5 \cdot 8.86 \cdot 10^{-12} \cdot (3 \cdot 10^6)^2 \simeq 40 \frac{\text{N}}{\text{m}^2}.$$

Consequently, the ratio of the maximal magnetic and maximal electric pressure is approximately

$$\frac{(p_m)_{\max}}{(p_e)_{\max}} = 10{,}000.$$

This is an extremely important conclusion. Although we can have electric and magnetic versions of almost any device using electric and magnetic forces, the magnetic version will require much less space for the same amount of power.

Questions and problems: Q16.30 and Q16.31, P16.16 to P16.30

16.5 Chapter Summary

1. The energy necessary for establishing a magnetic field can be calculated in two ways: in terms of currents in wire loops or as an integral of energy density over the entire field.

2. If there are no losses (such as hysteresis losses), the energy used for creating the magnetic field can be retrieved when the field is switched off, so it represents the energy of the magnetic field.
3. The power of hysteresis losses is proportional to the area of the hysteresis loop and to frequency.
4. The energy concept of self-inductance indicates that it can be represented as a sum of the energy associated with the field external to the region with the current (the external inductance) and that associated with the field inside the current region (the internal inductance).
5. For current loops in a vacuum, the magnetic force on any loop can always be calculated, but this may be difficult. The energy-based approach to calculating magnetic forces in such cases might be simpler. In particular, if magnetic materials are present, magnetic forces can be evaluated by formulas based on the law of conservation of energy in the magnetic field.

QUESTIONS

Q16.1. What does Eq. (16.1) actually represent?

Q16.2. Explain why the expression $dA_g = e(t)\, i(t)\, dt$ is the work done by a generator.

Q16.3. For a simple circuit of resistance R, with an emf $e(t)$, $e(t) = Ri(t) + d\Phi(t)/dt$. Explain the physical meaning of the last term.

Q16.4. Why does the energy of a system of current loops not depend on how the currents in the loops attained their final values?

Q16.5. Is Eq. (16.11) valid for nonlinear magnetic media? Explain.

Q16.6. The current in a thin loop 1 is increased from zero to a constant value I. A thin resistive loop 2 has no generators in it, but is in the magnetic field of the current in loop 1. Both loops are made of a linear magnetic material. Are the power $p_g(t)$ of the generator in loop 1 and the final value W_m of the energy *stored* in the system affected by the presence of loop 2?

Q16.7. Repeat question Q16.6 with loop 2 open-circuited.

Q16.8. A body of a linear magnetic material is placed in the vicinity of loop 1 of question Q16.6. Is some energy associated with the magnetization of the body?

Q16.9. Equation (16.8) was derived by assuming that the currents were increased inside *stationary* conductors. Using the law of conservation of energy as an argument, prove that this expression must be valid for the magnetic energy of the system considered, irrespective of the process by which the current system is obtained.

Q16.10. Using a sound physical argument, explain why the work in Eq. (16.6) done by the generators in establishing a given time-constant magnetic field is a function of the process by which the system of currents is established when ferromagnetic materials are present in the field.

Q16.11. Will the magnetic energy of a system of fixed quasi-filamentary dc current loops be changed if a closed conducting loop with *no* current is introduced into the system? Explain.

Q16.12. Is it possible to determine theoretically the self-inductances and mutual inductances in a system of current loops by starting from Eq. (16.11) if W_m is known? Explain.

Q16.13. Two equal thin loops of self-inductance L are pressed onto each other so that $|L_{12}| \simeq L$. If the currents in the loops are I_1 and I_2, what is the magnetic energy of the system? Answer the question if the two currents are (1) in the same direction and (2) in opposite directions.

Q16.14. How would you make an electric version of a generator of sinusoidal emf?

Q16.15. How would you make an electric version of a generator of "rectified" sinusoidal emf?

Q16.16. Imagine somebody came to you with a piece of a ferromagnetic material he developed, and stated that the working point moves along the hysteresis loop in the clockwise direction. Would you believe him? Explain.

Q16.17. Is it possible to derive Eq. (16.15) from Eq. (16.16)? Explain.

Q16.18. If a hysteresis loop was obtained by a sinusoidally varying $H(t)$, will the hysteresis losses be exactly equal to the area of this loop if $H(t)$ varies as a triangular function of time (i.e., varies linearly from $-H_m$ to H_m, then back to $-H_m$, and so on)? Explain.

Q16.19. If the frequency of the alternating current producing a magnetic field is f (cycles per second), what is the power per unit volume necessary to maintain the field in a piece of ferromagnetic substance?

Q16.20. According to the expression in Eq. (16.16), the volume density of magnetic energy is always greater in a vacuum than in a paramagnetic or idealized linear ferromagnetic material for the same B. Using a sound physical argument, explain this result.

Q16.21. The magnetization curve of a real ferromagnetic material is approximated by a nonlinear, but single-valued, function $B(H)$ (not by a hysteresis loop). Is it possible to speak about the energy density of the magnetic field inside the material? If you think it is, what is the energy density equal to?

Q16.22. A thin toroidal ferromagnetic core is magnetized to saturation and the current in the excitation coil is reduced to zero, so that the operating point in the H-B plane is $H = 0$, $B = B_r$. Is it possible to speak in that case about the energy of the magnetic field stored in the core? Is it possible to speak about the energy used to create the field? Explain.

Q16.23. The first part of the magnetization curve can be approximated as $B(H) = CH^2$, where C is a constant. How can you evaluate in that case the energy density necessary for the magnetization of the material? Is that also the energy density of the magnetic field?

Q16.24. Is it possible for the initial magnetization curve to be partly decreasing in B as H increases? What would that mean?

Q16.25. Evaluate approximately the density of hysteresis losses per cycle for the hysteresis loop in Fig. Q16.25 if $H_m = 200\,\mathrm{A/m}$ and $B_m = 0.5\,\mathrm{T}$.

Q16.26. Why are hysteresis losses linearly proportional to frequency?

Q16.27. Is the volume density of hysteresis losses in a thick toroidal ferromagnetic core with a coil carrying a sinusoidal current the same at all points of the core? Is the answer the same if the current intensity is such that at all points of the core saturation is attained, and if it is not?

Q16.28. Why can the self-inductance of a thick conductor not be defined naturally in terms of the induced emf or flux through the conductor?

Q16.29. Why is it very difficult to obtain the internal inductance using the methods from Chapter 15 for mutual inductance and self-inductance? To answer the question, con-

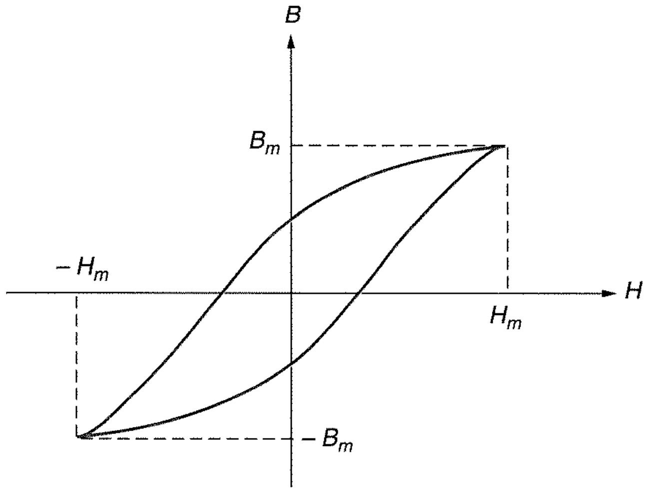


Figure Q16.25 A hysteresis loop

sider two wires, one thin and the other thick, with the same current I flowing through them.

Q16.30. Direct current due to a lightning stroke on a three-phase line propagates along the three conductors. Will the force repel or attract the conductors?

Q16.31. If a lightning stroke hits a transformer, in some cases it may be noticed that the transformer "swells" (increases in volume). Explain why.

PROBLEMS

P16.1. Write the explicit expression for the magnetic energy of three current loops with currents I_1, I_2, and I_3. Assume that the self-inductances and mutual inductances of the loops are known.

P16.2. Find the magnetic energy per unit length in the dielectric of a coaxial cable of inner conductor radius a and outer conductor radius b, carrying a current I. The permeability of the dielectric is μ_0. Show that $W'_m = L'_{\text{external}} I^2/2$.

P16.3. Find the total inductance per unit length of a coaxial cable that has an inner conductor of radius a and an outer conductor with inner radius b and outer radius c. The permeability of the conductors and the dielectric is μ_0, and current is distributed uniformly over the cross sections of the two conductors.

P16.4. A thin ferromagnetic toroidal core is made of a material that can be characterized approximately by a constant permeability $\mu = 4000\mu_0$. The mean radius of the core is $R = 10\,\text{cm}$ and the core cross-sectional area is $S = 1\,\text{cm}^2$. A current of $I = 0.1\,\text{A}$ is flowing through $N = 500$ turns wound around the core. Find the energy spent on magnetizing the core. Is this equal to the energy contained in the magnetic field in the core?

P16.5. In the toroidal core of the preceding problem, a small part of length $l_0 = 2\,\text{mm}$ is cut out so that now there is a small air gap in the core. The current in the coil is kept constant while the piece is being cut out. Find the energy contained in the magnetic field in this case.

P16.6. Show that the same expression for the self-inductance of the toroidal coil in Fig. P16.6, as calculated in Example 15.3, is obtained from the expression $2W_m = LI^2$.

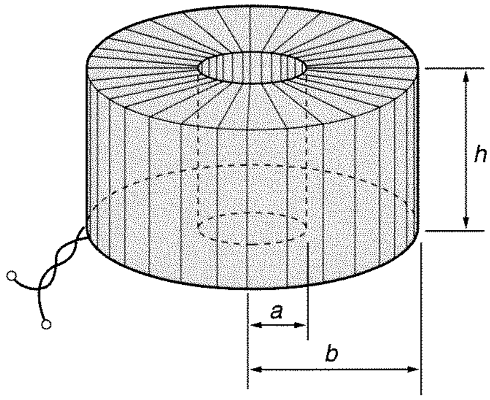


Figure P16.6 A thick toroidal coil

P16.7. On a thin ferromagnetic toroidal core of cross-sectional area $S = 1\,\text{cm}^2$, $N = 1000$ turns of thin wire are tightly wound. The mean radius of the core is $R = 16\,\text{cm}$. It may be assumed that the magnetic field is uniform over the cross section of the toroid. The idealized initial magnetization curve is shown in Fig. P16.7. Determine the work A_m done in establishing the magnetic field inside the toroid if the intensity of the current through the coil is $I = 2\,\text{A}$.

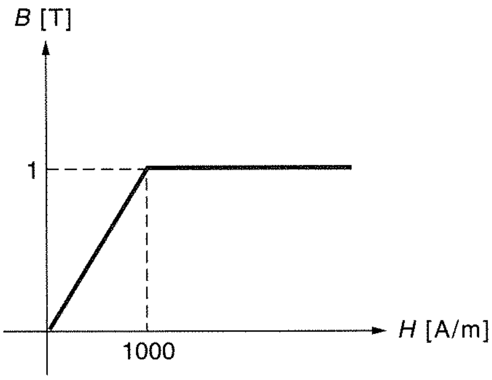


Figure P16.7 An idealized magnetization curve

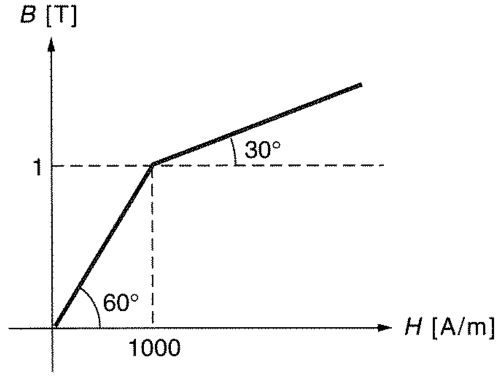


Figure P16.8 An idealized magnetization curve

P16.8. Repeat problem P16.7 if the idealized initial magnetization curve is as shown in Fig. P16.8.

P16.9. On the toroidal core shown in Fig. P16.6, $N = 650$ turns of thin wire are tightly wound. The intensity of the time-constant current in the coil is $I = 2\,\text{A}$ and the idealized initial magnetization curve of the core is as shown in Fig. P16.7. Determine the work done in establishing the magnetic field in the core if $a = 5\,\text{cm}$, $b = 15\,\text{cm}$, and $h = 10\,\text{cm}$.

P16.10. Repeat the preceding problem for intensities of current through the coil of (1) 0.5 A and (2) 1 A.

P16.11. The initial magnetization curve of a ferromagnetic material can be approximated by $B(H) = B_0H/(H_0 + H)$, where B_0 and H_0 are constants. Determine the work done per unit volume in magnetizing this material from zero to a magnetic field intensity H.

P16.12. The idealized hysteresis loops of the ferromagnetic core in Fig. P16.6 are as in Fig. P16.12. Determine the power of hysteresis losses in the core if it is wound with N turns of wire with sinusoidal current of amplitude I_m and frequency f. Assume that saturation is not reached at any point, and neglect the field of eddy currents.

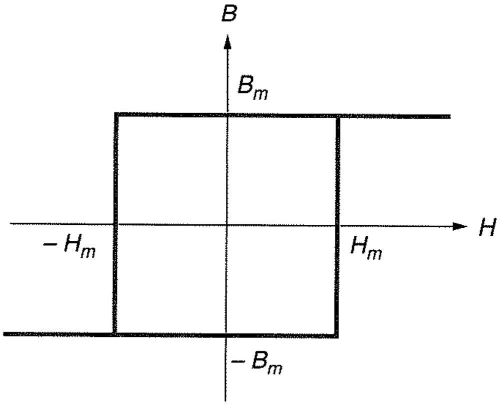


Figure P16.12 An idealized hysteresis loop

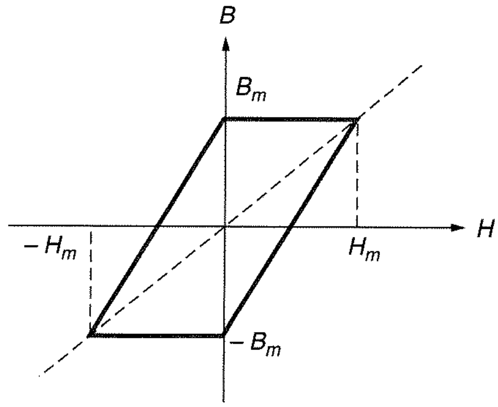


Figure P16.13 An idealized hysteresis loop

P16.13. Repeat problem P16.12 for idealized hysteresis loops shown in Fig. P16.13, assuming B_m/H_m for all the loops is the same and that saturation is not reached at any point. Ignore the field of eddy currents.

P16.14. A ferromagnetic core of a solenoid is made of thin, mutually insulated sheets. To estimate the eddy current and hysteresis losses, the total power losses were measured at two frequencies, f_1 and f_2, for the same amplitude of the magnetic flux density. The total power losses were found to be P_1 and P_2, respectively. Determine the power of hysteresis losses and of eddy current losses at both frequencies.

***P16.15.** Prove that Eq. (16.15) is valid for any magnetic field, not necessarily uniform.

P16.16. Two coaxial solenoids of radii a and b, lengths l_1 and l_2, and number of turns N_1 and N_2 have the same current I flowing through them. Find the axial force that the solenoids exert on each other if the thinner solenoid is pulled by a length x ($x < l_1, l_2$) into the other solenoid. Neglect edge effects and assume that the medium is air.

P16.17. An electromagnet and the weight it is supposed to lift are shown in Fig. P16.17. The dimensions are $S = 100\,\text{cm}^2$, $l_1 = 50\,\text{cm}$, $l_2 = 20\,\text{cm}$. Find the current through the winding of the electromagnet and the number of turns in the winding so that it can lift a load that is $W = 300$ kiloponds (a kp is 9.81 N) heavy. The electromagnet is made of a material whose magnetization curve can be approximated by $B(H) = 2H/(400 + H)$, where B is in T and H is in A/m.

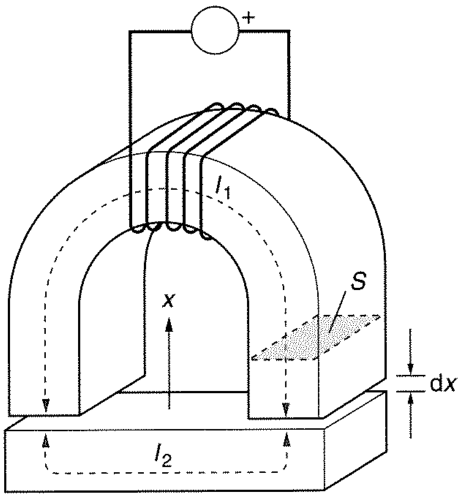


Figure P16.17 An electromagnet

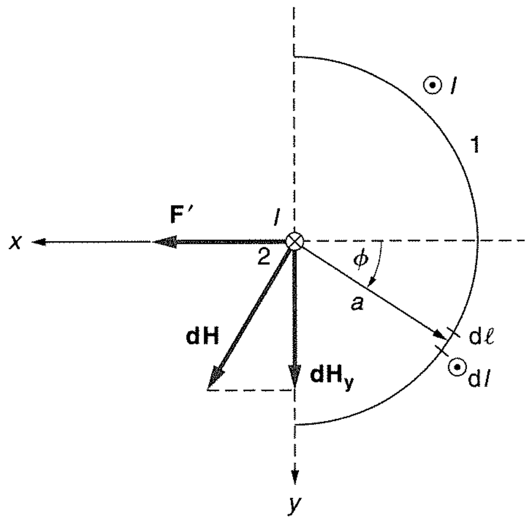


Figure P16.18 Cross section of a two-conductor line

P16.18. One of the conductors of a two-conductor line is in the form of one half of a thin circular cylinder. The other conductor is a thin wire running along the axis of the first (Fig. P16.18). If a current I flows through the two conductors in opposite directions, determine the force per unit length on the conductors.

P16.19. A thin conductor 2 runs parallel to a thin metal strip 1 (Fig. P16.19). Both a and b are much larger than the thickness of the strip. Determine the force per unit length on the two conductors for a current I flowing through them in opposite directions.

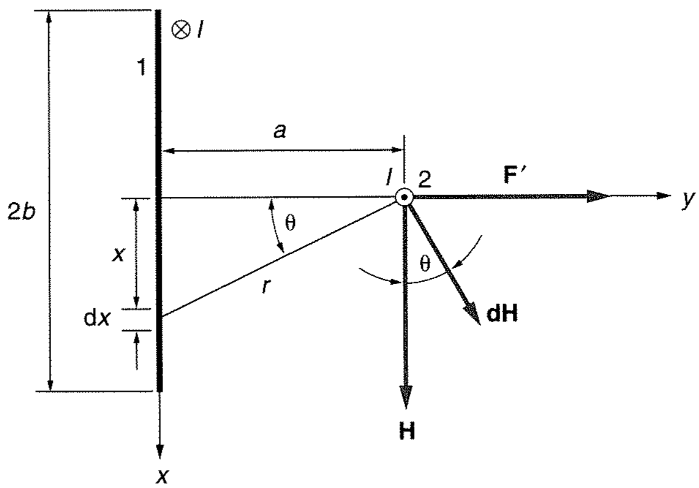


Figure P16.19 Cross section of a two-conductor line

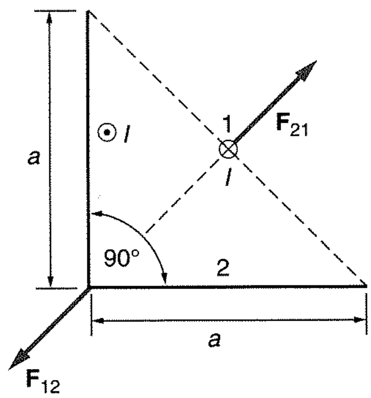


Figure P16.20 Cross section of a two-conductor line

P16.20. Determine the force per unit length on the conductors of the line with a cross section as shown in Fig. P16.20. The current in the conductors is I and the medium is air.

***P16.21.** Determine the force per unit length on the conductors of the stripline with a cross section as shown in Fig. P16.21. The current in the conductors is I, in opposite directions.

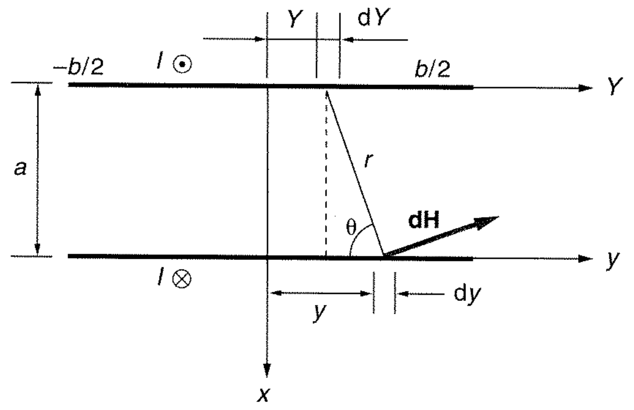


Figure P16.21 Cross section of a stripline

P16.22. A thin two-wire line has conductors of circular cross section of radius a and the distance between their axes d and is short-circuited by a straight conducting bar, as shown in Fig. P16.22. If a current I flows through the line, what is the force on the bar? Assume that the section of the line to the left of the bar is very long, and that the medium is air.

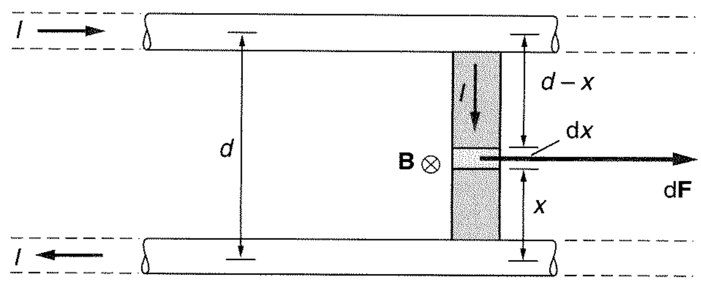


Figure P16.22 Short-circuited two-wire line

P16.23. A long air-filled coaxial cable is short-circuited at its end by a thin conducting plate, as shown in Fig. P16.23. Determine the force on the end plate corresponding to a current of intensity I through the cable.

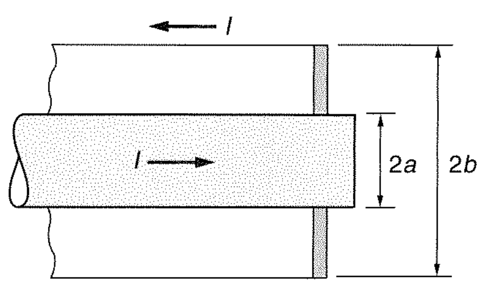


Figure P16.23 Short-circuited coaxial cable

P16.24. Determine approximately the force between two parallel coaxial circular loops with currents I_1 and I_2. The radii of the loops are a and b, respectively, with $a \gg b$ and the distance between their centers z.

P16.25. A metal strip of conductivity σ and of small thickness b moves with a uniform velocity v between the round poles of a permanent magnet. The radius of the poles of the magnet is a and the width of the strip is much larger than a (Fig. P16.25). The flux density $\mathbf{B}$ is very nearly constant over the circle shown hatched and practically zero outside it. Assuming that the induced current density in that circle is given by $\mathbf{J} = \sigma\mathbf{v} \times \mathbf{B}/2$, determine the force on the strip.

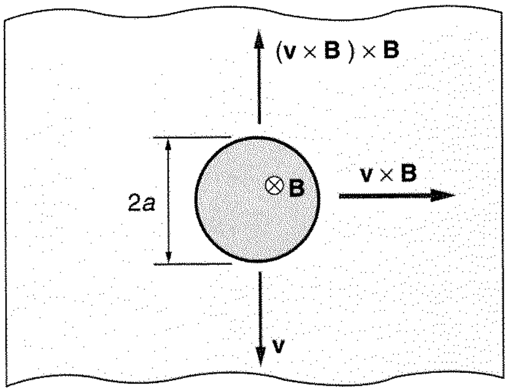


Figure P16.25 A strip moving in a magnetic field

P16.26. A thin metal plate is falling between the poles of a permanent magnet (Fig. P16.26) under the action of the gravitational field. The pole radius is $a = 2\,\text{cm}$ and the flux density in the gap is $B = 1\,\text{T}$. Determine approximately the velocity of the plate if its thickness is $b = 0.5\,\text{mm}$, its surface area $S = 100\,\text{cm}^2$, its conductivity $\sigma = 36 \cdot 10^6\,\text{S/m}$ (aluminum), and its mass density $\rho_m = 2.7\,\text{g/cm}^3$ (aluminum).

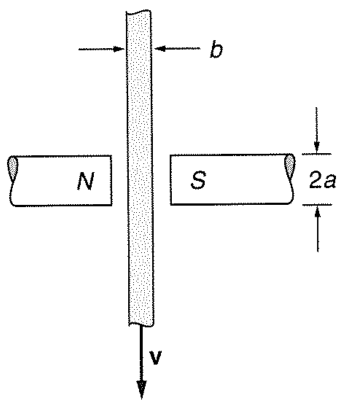


Figure P16.26 A plate falling in a magnetic field

P16.27. A metal ring K of negligible resistance is placed above a short cylindrical electromagnet, as shown in Fig. P16.27. Determine qualitatively the time dependence of the total force on the ring if the current through the electromagnet coil is of the form $i(t) = I_m \cos \omega t$.

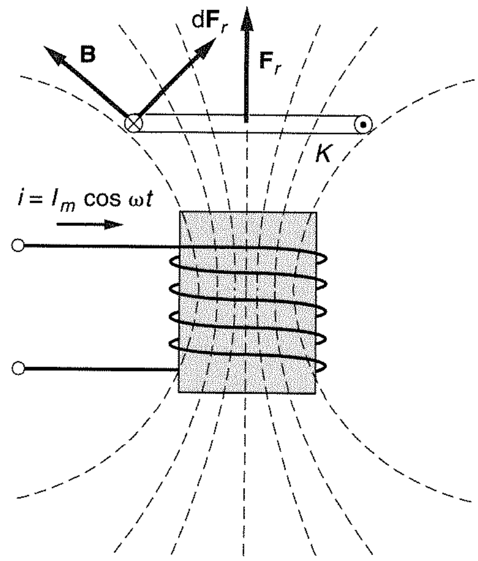


Figure P16.27 A ring in a magnetic field

P16.28. A U-shaped glass tube is filled with a paramagnetic liquid of unknown magnetic susceptibility χ_m. A part of the tube inside the dashed square in Fig. P16.28 is exposed to a uniform magnetic field of intensity H. Under the influence of magnetic forces, a difference h of the levels of the liquid in the two sections of the tube is observed. Given that the mass density of the liquid is ρ_m and that the medium above the liquid is air, determine χ_m.

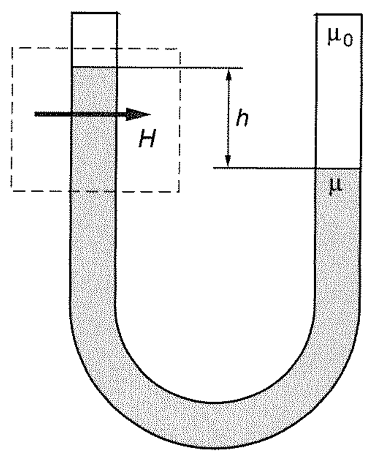


Figure P16.28 A U-shaped tube in a magnetic field

P16.29. Plot the scale calibration curve $F_{\text{tot}}(I)$ for the ammeter sketched in Fig. P16.29. $F_{\text{tot}}(I)$ is the total force acting on the iron nail for a given current I in the coil. Given are

$a = 1\,\text{mm}$, $l = 5\,\text{cm}$, $N' = 10$ turns/cm (you need to look up the relative permeability for iron and its mass density).

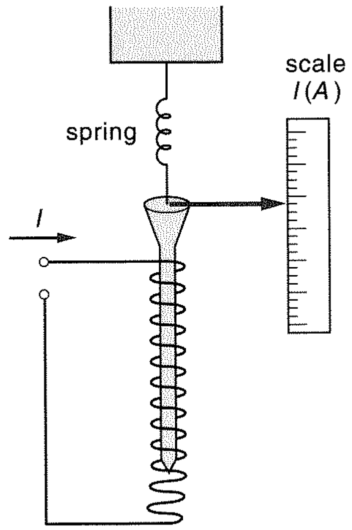


Figure P16.29 Sketch of a simple ammeter

***P16.30.** Derive the general expression for pressure of magnetic forces, Eq. (16.22).

17

Some Examples and Applications of Time-Invariant and Slowly Time-Varying Magnetic Fields

17.1 Introduction

Magnetic fields are present in many practical applications, as well as in the natural world. For example, we are continuously situated in the relatively strong time-invariant (or extremely slowly variant) magnetic field of the earth; this field may, for example, affect the way your computer monitor works depending on whether you happen to turn it on in the Northern or Southern Hemisphere. We also often find ourselves in magnetic and electric fields existing around high-voltage and high-current power lines, and it is interesting to calculate the order of magnitude of voltages induced in our body. Most electrical home appliances contain devices that use magnetic forces and moments of magnetic forces, and our computers, tape recorders, and video recorders use magnetic storage devices. The aim of this chapter is to review some of the more important and interesting applications of magnetic (along with electric) fields.

17.2 The Magnetic Field of the Earth

The earth behaves like a large permanent magnet whose magnetic field is similar to the field of a giant current loop with an axis declined 11 degrees with respect to the earth's axis of rotation (Fig. 17.1a). (Geologists believe that the magnetic field is created by the difference in the speed of rotation of the earth's liquid core and its solid mantle.) The planet's geographic North Pole is approximately the *south* magnetic pole (this is why the north pole of the magnetic needle of a compass always points to the north). The magnitude of the earth's magnetic flux density is about 50 μT at our latitude and about 20% stronger at the poles.

About 90% of the magnetic field measured at the earth's surface is due to the field originating inside the planet. The rest is due to the currents produced by charged particles coming from the sun, and to the magnetism of the rocks in the crust. The region in which the earth's magnetic field can be detected is called the *magnetosphere*. It is not symmetrical, but rather has the shape of a teardrop. This is due to the charged particles streaming from the sun that are deflected by the earth's magnetic field; the earth forms a "shadow" for charges, which has the effect of elongating the magnetosphere.

Because the magnetic field everywhere on the surface of the earth is partly due to magnetization of rocks at that point, magnetometers can be used in geology for de-

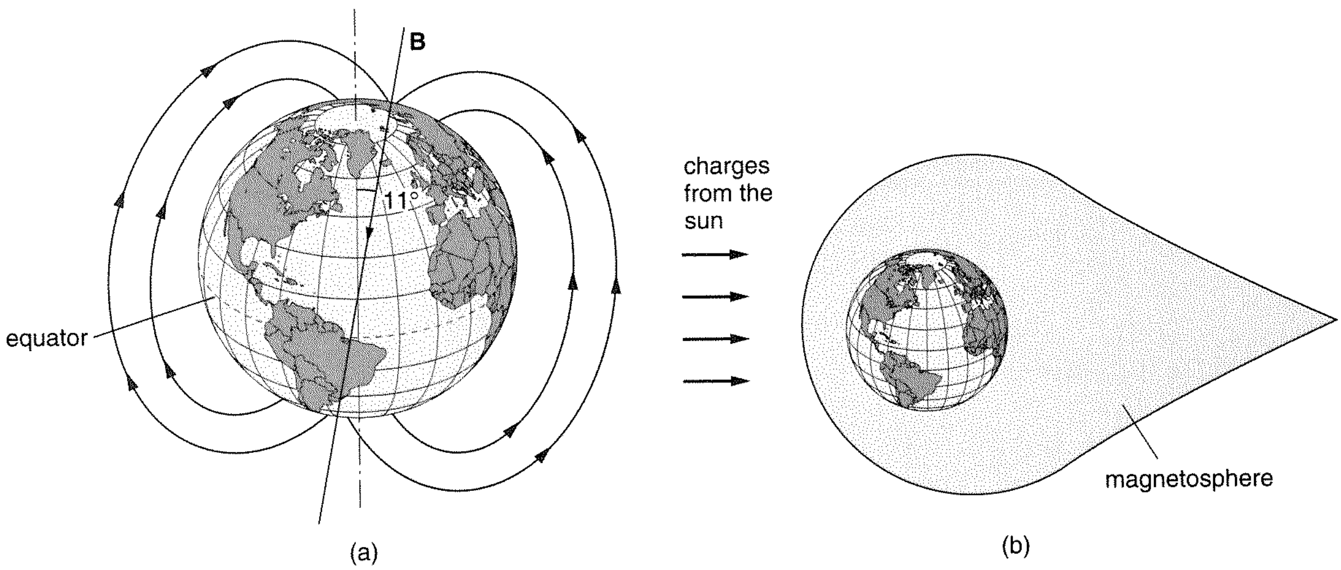


Figure 17.1 (a) The earth produces a dc magnetic field roughly equivalent to the magnetic field of a very large current loop. The plane of the loop is declined with respect to the earth's axis of rotation. (b) The region in which the earth's magnetic field can be detected is called the magnetosphere and is asymmetrical due to the charges emitted from the sun.

tecting different types of ores. Measuring magnetization of rocks also gives us insight into the earth's magnetic history. Rocks become magnetized when they are formed, or when they remelt and recool at some later time. When rocks are heated they lose their magnetization and are remagnetized by the earth's magnetic field as they cool. Therefore, they carry a permanent record of what the earth's magnetic field was like at the time of the rock formation. Measurements of rock magnetization show that the earth's magnetic poles have wandered. Some rocks that formed over short time intervals show fossil magnetic polarities 180 degrees apart, which cannot be explained by a 180-degree rotation of a continent (the time of reversal was too short for this to be possible). The conclusion is that the earth's magnetic field switched polarity, similarly to the field of a loop in which the current changes direction. These field reversals occurred many times during our planet's geological history, and about 10 times in the last 4 million years. Rock magnetization indicates that the polarity does not flip instantly: it first slowly decreases and then increases in the opposite direction.

Questions and problems: Q17.1 to Q17.3

17.3 Applications Related to Motion of Charged Particles in Electric and Magnetic Fields

Charged particles in both electric and magnetic fields always move. In many instances one of the fields is of much less influence than the other. For example, we have seen that both an electric and a magnetic field act on moving charges that form an electric current in a conductor, but that the influence of the magnetic field is negligible. There are examples of the other kind, where electric forces exist but are negligible. In some cases, the effects of electric and magnetic fields on a moving particle are of the same order of magnitude and must both be taken into account.

The motion of charged particles in electric and magnetic fields may be in a vacuum (or very rarefied gas), in gases, and in solid or liquid conductors. This brief section is aimed at explaining the principles of motion of charged particles in electric and magnetic fields and at presenting examples of how some engineering applications take advantage of this motion.

We know that the force on a charge Q moving in an electric and a magnetic field with a velocity $\mathbf{v}$ is the Lorentz force, Eq. (12.13), which is repeated here for convenience:

$$\mathbf{F} = Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B}. \tag{17.1}$$

If the electric field can be neglected, we omit the first term of the Lorentz force. If the magnetic field can be neglected, we omit the second term.

If a charge moves in a vacuum, then this force in any instant must be equal in magnitude and opposite in direction to the inertial force. If the mass of the charge is m, the equation of motion therefore has the form

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = Q\mathbf{E} + Q\mathbf{v} \times \mathbf{B}. \tag{17.2}$$

In this equation, **E** and **B** in general are functions of space coordinates and of time. Except in rare cases, it is impossible to solve such a general equation for the velocity of the charge analytically, but it can always be solved numerically.

If a charge moves in a material (a gas, a liquid, or a solid), collisions influence the (macroscopic) charge motion to a great extent. For example, we have seen that in solid and liquid conductors the motion of free charges is always along the lines of vector **E**. An equation like (17.2) is not valid for average (drift) velocity.

We now discuss a few examples of the motion of charged particles in an electric and a magnetic field.

Example 17.1—Motion of a charged particle in a uniform electric field. In Example 11.1 we analyzed the simplest case of motion of a charged particle in a uniform electric field. Let us now consider a more general case, when a charge Q ($Q > 0$) moves in a uniform field with arbitrary initial velocity $\mathbf{v}_0 = \mathbf{v}_{0x} + \mathbf{v}_{0y}$, as in Fig. 17.2a.

The equation of motion (17.2) becomes $m(\mathrm{d}\mathbf{v}/\mathrm{d}t) = Q\mathbf{E}$. Integrating the scalar x and y components of this equation twice, with respect to the position of the charge as a function of time, we obtain

$$x(t) = \frac{QE}{2m}t^2 + v_{0x}t + x_0 \quad \text{and} \quad y(t) = v_{0y}t + y_0,$$

where x_0 and y_0 are the initial x and y coordinates of the charge. Consequently, the charge will move along a parabola. This is the same as when we throw a stone at an angle (other than 90 degrees) with respect to the earth's surface.

Example 17.2—Deflection of an electron stream by a charged capacitor. Imagine that we shoot an electron between the plates of a charged parallel-plate capacitor, perpendicularly to the electric field. We now know that the electron trajectory will curve toward the positive electrode. So if we put a screen behind the capacitor, with no voltage on the electrodes the

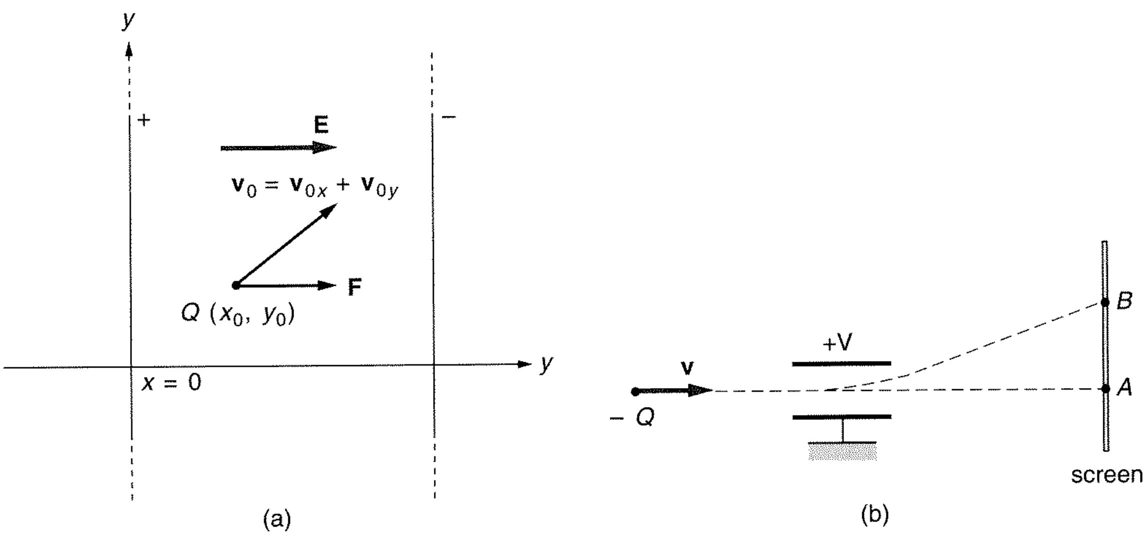


Figure 17.2 (a) Charged particle in a uniform electric field, revisited; (b) deflection of a charged particle by a charged capacitor

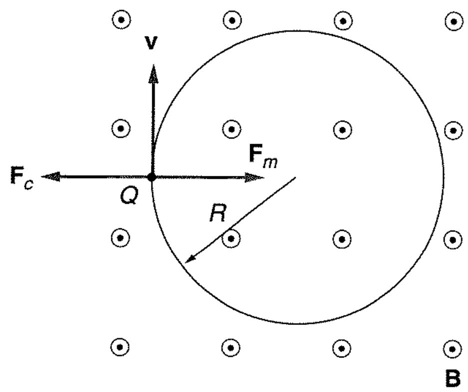


Figure 17.3 Charged particle in a uniform magnetic field

electron hits point A in Fig. 17.2b. When a voltage is applied, the electron is deflected and hits point B on the screen in Fig. 17.2b. Some cathode-ray tubes use this principle for deflecting the electron beam, although the practical deflections are rather small.

Example 17.3—Motion of a charged particle in a uniform magnetic field. Consider a charged particle Q ($Q > 0$) moving in a magnetic field of flux density $\mathbf{B}$ with a velocity $\mathbf{v}$ normal to the lines of vector $\mathbf{B}$, as in Fig. 17.3.

Since the magnetic force on the charge is $\mathbf{F}_m = Q\mathbf{v} \times \mathbf{B}$, it is *always perpendicular to the direction of motion.* This means that a magnetic field cannot change the magnitude of the velocity (i.e., it cannot speed up or slow down charged bodies); it can only change the direction of the charged particle motion. In other words, magnetic forces cannot change the kinetic energy of moving charges.

In the case considered in Fig. 17.3, there is a magnetic force on the charged particle directed as indicated, tending to curve the particle trajectory. Since $\mathbf{v}$ is normal to $\mathbf{B}$, the force magnitude is simply QvB. It is opposed by the centrifugal force, mv^2/R, where R is the radius of curvature of the trajectory. Therefore, we have

$$QvB = \frac{mv^2}{R},$$

so that the radius of curvature is constant, $R = (mv)/(QB)$. Thus the particle moves in a circle. It makes a full circle in

$$t = T = \frac{2\pi R}{v} = \frac{2\pi m}{QB},$$

which means that the frequency of rotation of the particle is equal to $f = 1/T = (QB)/(2\pi m)$. Note that f does not depend on v. Consequently, all particles that have the same charge and mass make the same number of revolutions per second. This frequency is called the *cyclotron frequency*.

Example 17.4—The cyclotron. The cyclotron is a device used for accelerating charged particles. It is sketched in Fig. 17.4. The main part of the cyclotron is a flat metal cylinder, cut along its middle. The two halves of the cylinder are connected to the terminals of an oscillator

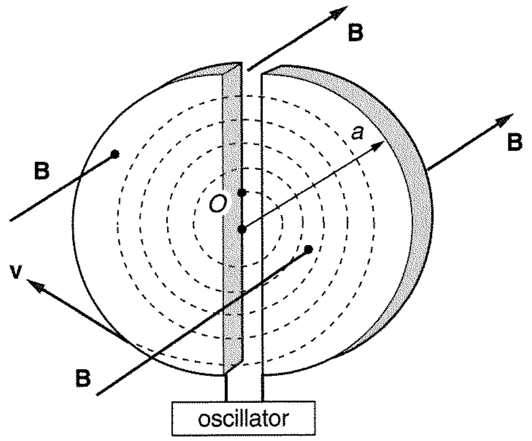


Figure 17.4 A cyclotron

(a source of periodically changing voltage). The whole system is in a uniform magnetic field normal to the bases of the cylinder, and inside the cylinder is a vacuum (i.e., highly rarefied air).

A charged particle from source O finds itself in an electric field that exists between the halves of the cylinder, and it accelerates toward the other half of the cylinder. While outside of the space between the two cylinder halves, the charge finds itself only in a magnetic field, and it circles around with a radius of curvature found as in the preceding example. We saw that the time the charge takes to go around a semicircle does not depend on its velocity. That means that the charge will always take the same amount of time to again reach the gap between the two cylinders. If the electric field variation in this region is adjusted in such a way that the charge is always accelerated, the charge will circle around larger and larger circles, with increasingly larger velocity, until it finally shoots out of the cyclotron. The velocity of the charge when it gets out of the cyclotron is $v = (QBa)/m$. This equation is valid only for velocities not close to the speed of light. If this is not the case, the relativistic effects increase the mass, i.e., the mass is not constant.

As a numerical example, for $B = 1\,\mathrm{T}$, $Q = e$, $a = 0.5\,\mathrm{m}$, $m = 1.672 \cdot 10^{-27}\,\mathrm{kg}$ (a proton), we get $v = 47.9 \cdot 10^6\,\mathrm{m/s}$.

Example 17.5—Cathode-ray tube. Cathode-ray tubes (CRTs), used in some TVs and computer monitors, have controlled electron beams that show traces on a screen. One system for deflecting electron streams in CRTs is sketched in Fig. 17.2b. Basically, there are two mutually orthogonal parallel-plate capacitors, which can deflect the stream in two orthogonal directions. In this way the electron stream can hit any point of the screen, and precisely where it hits is controlled by appropriate voltages between the electrodes of the two capacitors.

We have already mentioned that the electric field can deflect electron streams only by relatively small distances. When a large deflection is required, as in television receivers, a magnetic field is used, as sketched in Fig. 17.5. The design of the magnetic deflecting system (a coil of a specific geometry and with many turns of wire) is rather complicated and is usually done experimentally. Part of the experimental adjustment is due to the effect of the earth's magnetic field on charged particles at any point on the planet.

If we think of the earth as of an equivalent current loop, as described in section 17.2, the horizontal component of the magnetic flux density vector is oriented along the north-south direction, and the vertical component is oriented downward in the Northern Hemisphere and

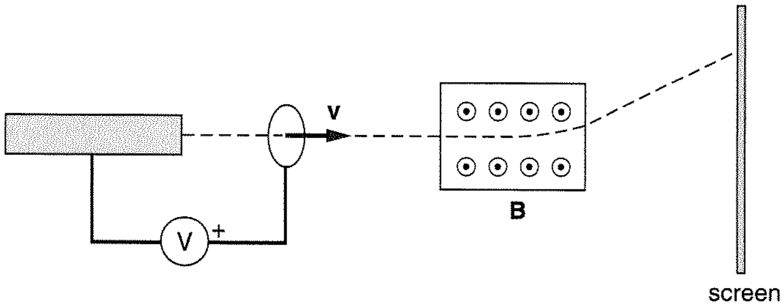


Figure 17.5 A system for forming a stream of electrons and then deflecting it with a magnetic field

upward in the Southern Hemisphere. Therefore, CRTs that use magnetic field deflection have to be tuned to take this external field into account. It is likely that your computer monitor (if a CRT) will not work exactly the same way if you turn it sideways (it might slightly change colors or shift the beam by a couple of millimeters), or if you use it in the other hemisphere of the globe.

Example 17.6—The Hall effect. In 1879, Edwin Hall thought of a clever way of determining the sign of free charges in conductors. A ribbon made of the conductor we are interested in has a width d and is in a uniform magnetic field of flux density $\mathbf{B}$ perpendicular to the ribbon (Fig. 17.6). A current of density $\mathbf{J}$ flows through the ribbon. The free charges can in principle be positive, as in Fig. 17.6a, or negative, as in Fig. 17.6b. The charges that form the current are moving in a magnetic field, and therefore a magnetic force $\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$ is acting on them. Due to this force, positive charges accumulate on one side of the ribbon, and negative ones on the other side. These accumulated charges produce an electric field E_H. This electric field, in turn, acts on the free charges with a force that is in the opposite direction to the magnetic force. The charges will stop accumulating when the electric force is equal in magnitude

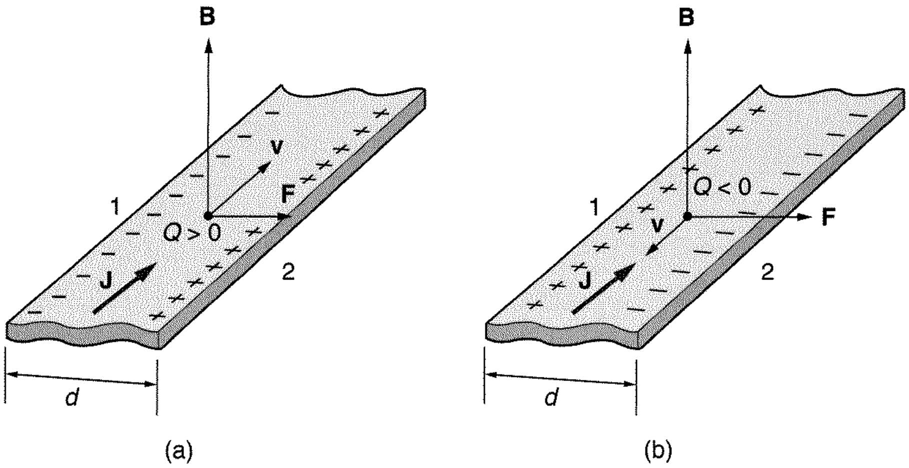


Figure 17.6 The Hall effect in case of (a) positive free charge carriers, and (b) negative free charge carriers

to the magnetic force acting on each of the charges. So in the steady state,

$$QvB = QE_H, \qquad \text{or} \qquad E_H = vB.$$

Between the left and right edge of the ribbon, we can measure a voltage equal to

$$|V_{12}| = E_H d = vBd.$$

In the case shown in Fig. 17.6a this voltage is negative, and in Fig. 17.6b it is positive. So the sign of the voltage tells us the sign of free charge carriers, and a voltmeter can be used to determine this sign.

Since $J = NQv$, where N is the number of free charges per unit volume, we can write

$$|V_{12}| = \frac{Jd}{NQ}B.$$

Thus if we determine the coefficient Jd/NQ for either ribbon sketched in Fig. 17.6 (which is usually done experimentally), by measuring V_{12} we can measure **B**. This ribbon has four terminals—two for the connection to a source producing current in the ribbon, and two for the measurement of voltage across it. Such a ribbon is called a *Hall element*.

For single valence metals, e.g., copper, if we assume that there is one free electron per atom, the charge concentration is given by

$$N = \frac{N_A \rho_m}{M},$$

where N_A is Avogadro's number ($6.02 \cdot 10^{23}$ atoms/mole), ρ_m is the mass density of the metal, and M is the atomic mass.

Questions and problems: Q17.4 to Q17.7, P17.1 to P17.5

17.4 Magnetic Storage

Magnetic materials have been used for storing data since the very first computers. The first computer memories consisted of small toroidal ferromagnetic cores arranged in two-dimensional arrays, in which digital information was stored in the form of magnetization. These memories are bulky and slow, as can be concluded from their description in Example 17.7. Today's memories are essentially electrostatic: capacitances inside transistors are used for storing bits of information in the form of charges.

The hard disk in every computer is also a magnetic memory. We can write to the disk by magnetizing a small piece of the disk surface, and we can read from the disk by inducing a voltage in a small loop that is moving in close proximity to the magnetized disk surface element. As technology has improved, the amount of information that can be stored on a standard-size hard disk has grown rapidly. Between 1995 and ~~1997~~ 2020 the standard capacity of hard disks on new personal computers shot from a few hundred megabytes to more than ~~2~~ 20 gigabytes. The development is in the direction not only of increasing disk capacity but also of increasing speed (or reducing the access time). As we will see in Example 17.8, these two requirements compete with each other, and the engineering solution, as is usually the case, needs to be a compromise.

Example 17.7—History: magnetic core memories. Magnetic core memories were used in computers around 1970 but are now completely obsolete. The principle of their operation, however, is clever.

A magnetic core memory uses the hysteresis properties of ferromagnetics. One "bit" of the memory is a small ferromagnetic torus, shown in Fig. 17.7a. Two wires, in circuits 1 and 2, pass through the torus. Circuit 1 is used for writing and reading, and circuit 2 is used only for reading. To write, a positive ("1") or negative ("0") current pulse is passed through circuit 1 in the figure. When a positive current pulse is sent through the circuit, the core is magnetized to the point labeled B_r on the hysteresis curve in Fig. 17.7b. When a negative current pulse is passed through the circuit, the core is magnetized to the point labeled $-B_r$. So the point B_r corresponds to a "1," and $-B_r$ to a "0."

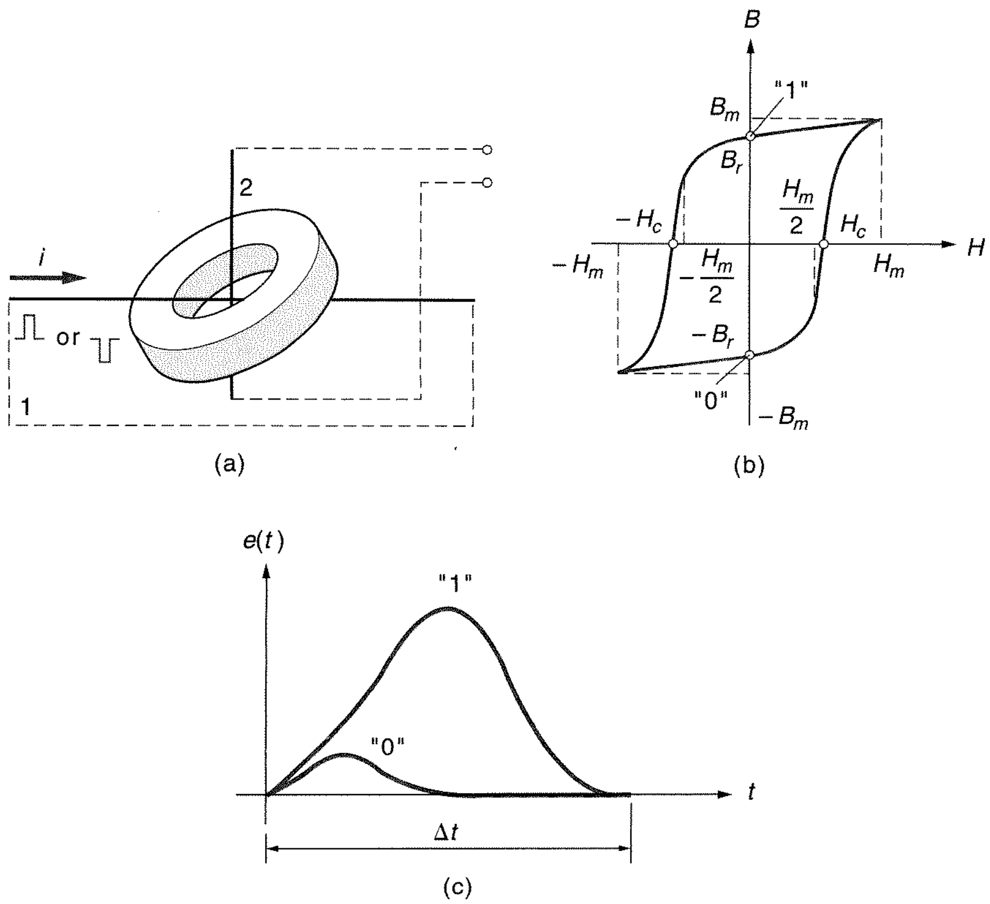


Figure 17.7 (a) One bit of a magnetic core memory. (b) Hysteresis loop of the core. (c) The induced emf pulses produced in circuit 2 while reading out the binary value written in the core

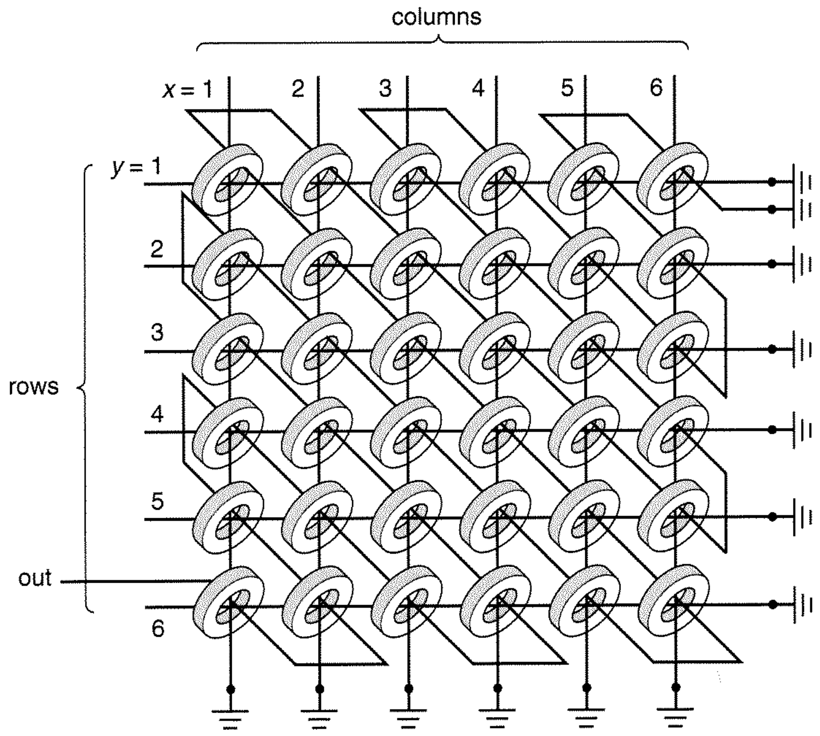


Figure 17.8 A sketch of a magnetic memory

How is the reading performed? A negative current pulse is passed through circuit 1. If the core is at a "1," the negative current pulse will bring the operating point to $-B_m$ (the negative tip of the loop), and after the pulse is over the point will move to $-B_r$ on the hysteresis loop. If the core is at a "0," the negative pulse will make the point go to the negative tip of the loop, and then return to $-B_r$.

While this is done, an emf is induced in circuit 2, resulting in one of the two possible readings shown in Fig. 17.7c. These two "pulses" correspond to a "1" and a "0." The speed at which this is done is about $\Delta t = 0.5 - 5\,\mu s$. The dimensions of the torus are small: the outer diameter is 0.55 to 2 mm, the inner diameter is 0.3 to 1.3 mm, and the thickness is 0.12 to 0.56 mm.

Elements of an entire memory are arranged in matrices, as shown in Fig. 17.8. Two wires pass through each torus, as in Fig. 17.7a. The current passing through each row or column is only half the current needed to saturate the torus, so both the row and column of a specific core need to be addressed.

Example 17.8—Computer hard disks. The hard disk in every computer has information written to and read from it. We will describe how both processes work in modern hard disks and discuss some of the engineering parameters important for hard disk design. The hard disk itself is coated with a thin coating of ferromagnetic material such as Fe_2O_3. The disk is organized in sectors and tracks, as shown in Fig. 17.9.

The device that writes data to the disk and reads data from it is called a *magnetic head*. Magnetic heads are made in many different shapes, but all operate according to the same principle. We will describe the read-write process for a simplified head, shown in Fig. 17.10. It is a magnetic circuit with a gap. The gap is in close proximity to the tracks, so there is some leakage flux between the head and the ferromagnetic track.

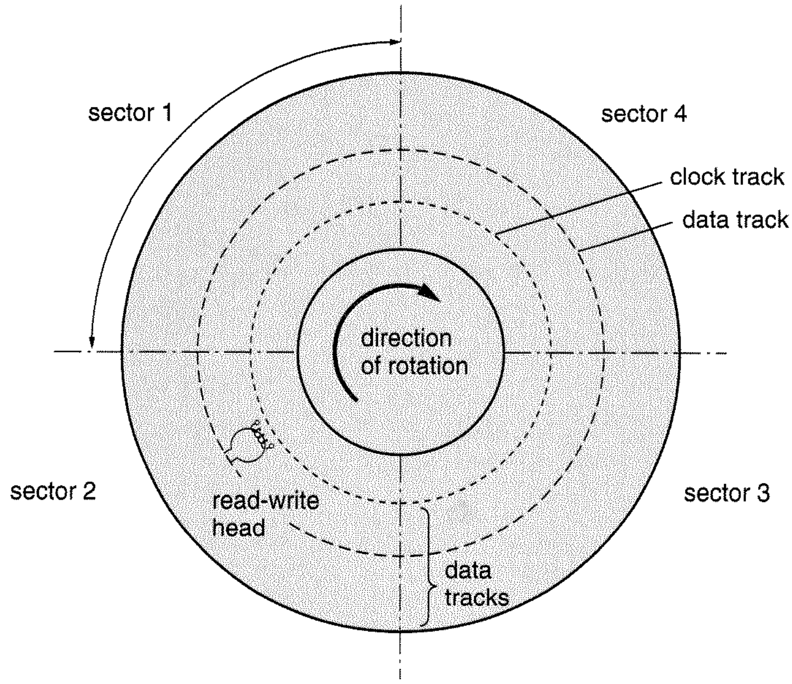


Figure 17.9 Hard disk tracks

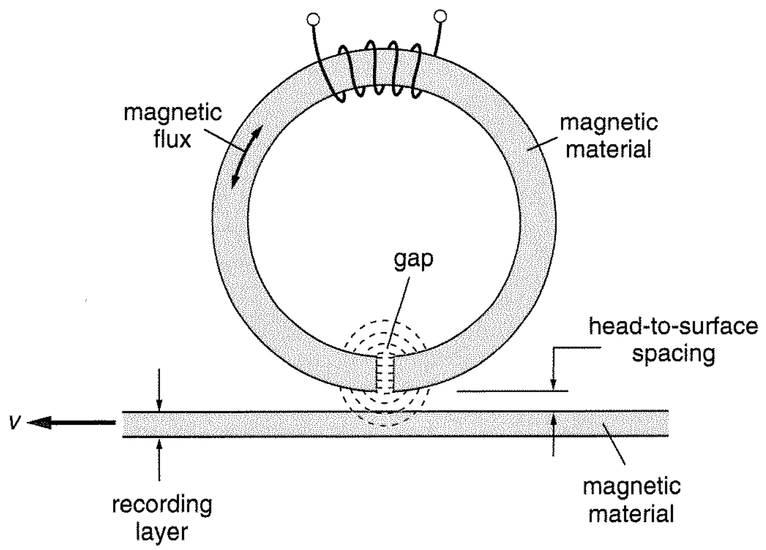


Figure 17.10 Magnetic head

In the "write" process, a current flows through the winding of the magnetic head, thus creating a fringing magnetic field in the gap. The gap is as small as 5 μm. As the head moves along the track (usually the track rotates), the fringing field magnetizes a small part of the track, creating a south and a north pole in the direction of rotation. These small magnets are about 5 μm long by 25 μm wide. A critical design parameter is the height of the head above the track: the head cannot hit the track and get smashed, but it also needs to be as close as possible to maximize the leakage flux that magnetizes the track. Typically, the surface of the track is flat

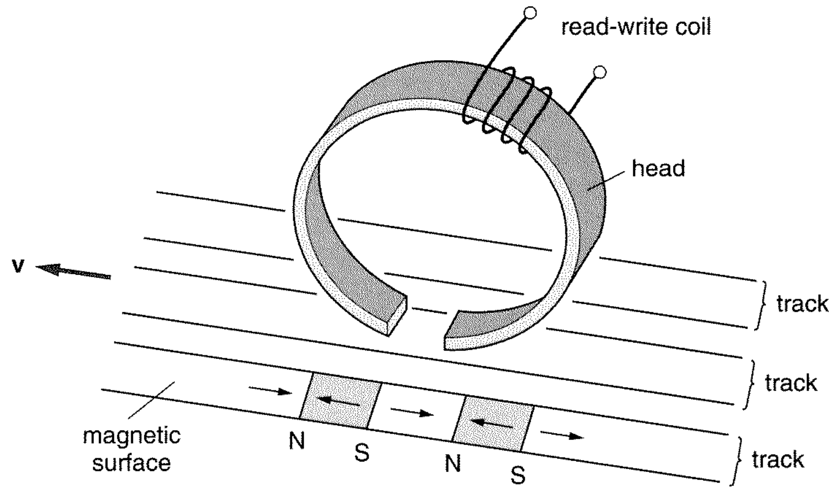


Figure 17.11 The magnetic head aerodynamically flies over the disk surface at a distance of only about 1 micrometer above it, following the surface profile.

to within several micrometers, and the head follows the surface profile at a distance of about 1 micrometer or less above it. This is possible because the head aerodynamically flies above the disk surface, as shown in Fig. 17.11. The current in the head windings should be strong enough to saturate the ferromagnetic track. If the track is saturated, the voltage signal during readout is maximized.

In the "read" process, there is no current in the windings of the magnetic head. The residual magnetization of the magnetic head should be as small as possible, so that the head is demagnetized when the current is turned off in readout. Now the flux from the magnetized track induces a voltage between the winding open ends while the head is moving with respect to the track (according to Faraday's law). Since the largest changes in **B** occur when the magnetic field changes direction, i.e., between two tiny magnets along the track, the output voltage has a waveform consisting of positive and negative pulses, as shown in Fig. 17.12b. The volt-

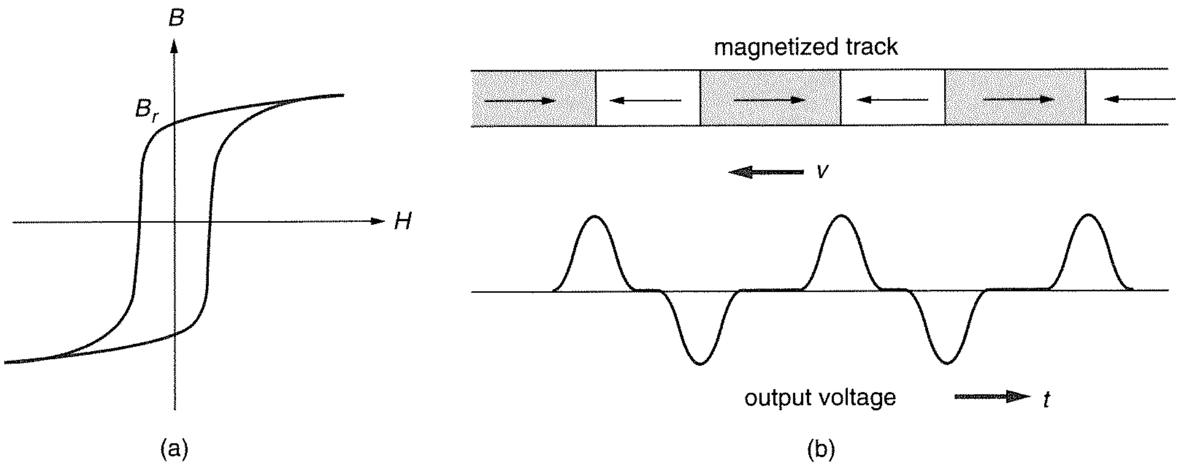


Figure 17.12 (a) Hysteresis curve of the track material. B_r is the remanent magnetic flux density and should be large for good readout. (b) A typical voltage signal read from the disk.

age is proportional to the remanent magnetic flux density, B_r, of the ferromagnetic hysteresis curve in Fig. 17.12a.

The capacity of data storage is given by the information density per unit area of storage surface. The storage density per unit surface area is the product of the storage density per unit track length, times the track density per unit distance normal to the direction of relative motion. An increase in track density reduces the sharpness in magnetic field discontinuity, thus reducing the readout voltage. Note that magnetic disks are inherently binary storage devices and that the frequency of the voltage pulses in readout is doubled compared to the number of actual segments along the track.

Questions and problems: Q17.8 and Q17.9, P17.6 and P17.7

17.5 Transformers

A transformer is a magnetic circuit with (usually) two windings, "primary" and "secondary," on a common ferromagnetic core (Fig. 17.13a). When an ac voltage is applied to the primary coil, the magnetic flux through the core is the same at the secondary and induces a voltage at the open ends of the secondary winding. Ampère's law for this circuit can be written as

$$N_1 i_1 - N_2 i_2 = Hl,$$

where N_1 and N_2 are the numbers of the primary and secondary turns, i_1 and i_2 are the currents in the primary and secondary coils when a generator is connected to the primary and a load to the secondary, H is the magnetic field in the core, and l is the effective length of the core. Since $H = B/\mu$ and, for an *ideal* core, $\mu \to \infty$, both B and H in the ideal core are zero (otherwise the magnetic energy in the core would be infinite). Therefore for an ideal transformer we have

$$\frac{i_1}{i_2} = \frac{N_2}{N_1}. \tag{17.3}$$

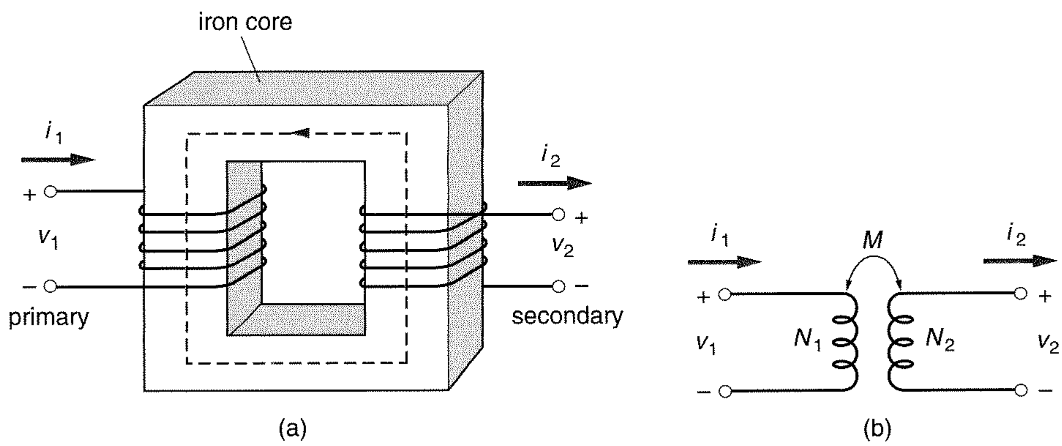


Figure 17.13 (a) A transformer, and (b) the equivalent circuit of an ideal transformer

This is the relationship between the primary and secondary currents in an *ideal transformer*. For good ferromagnetic cores, the permeability is high enough that this is a good approximation.

From the definition of magnetic flux, we know that the flux through the core is proportional to the number of turns in the primary. From Faraday's law, we know that the induced emf in the secondary is proportional to the number of times the magnetic flux in the core passes through the surface of the secondary winding, that is, to N_2. Therefore, we can write the following for the voltages across the primary and secondary windings:

$$\frac{v_1}{v_2} = \frac{N_1}{N_2}. \tag{17.4}$$

From our discussion of mutual inductance in Chapter 15, we see that the equivalent circuit of an ideal transformer is just a mutual inductance, as shown in Fig. 17.13b.

Assume that the secondary winding of an ideal transformer is connected to a resistor of resistance R_2. What is the resistance seen from the primary terminals? From Eqs. (17.3) and (17.4) we obtain

$$R_1 = \frac{v_1}{i_1} = \frac{v_2 N_1/N_2}{i_2 N_2/N_1} = R_2 \left(\frac{N_1}{N_2}\right)^2. \tag{17.5}$$

From this discussion, we see that the transformer's name is appropriate: it transforms the values of the voltage, current, and resistance between the primary and secondary windings. The transformation ratio is dictated by the ratio of the number of turns. In an ideal transformer there are no losses, so all of the power delivered to the primary can be delivered to a load connected to the secondary.

In a realistic transformer there are several loss mechanisms: resistance in the wire of the windings, and eddy current losses and hysteresis losses in the ferromagnetic core. To minimize resistive losses in sometimes very long wires used for a large number of turns, a good metal such as copper is chosen. Eddy current losses are minimized by laminating the core, as discussed in Example 14.6, and hysteresis losses were discussed in Example 16.3. These losses, as well as the inductance of the windings, result in a realistic equivalent circuit for a transformer shown in Fig. 17.14. For

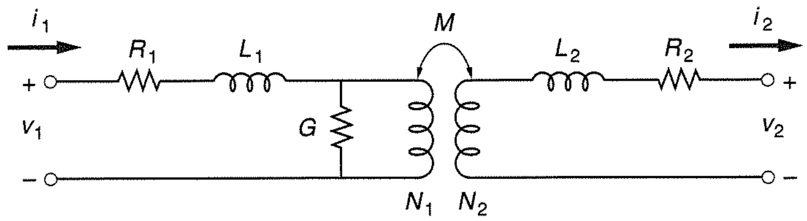


Figure 17.14 Equivalent circuit of a realistic transformer

high-frequency transformers, or in cases where transients are important, the capacitance between the winding turns also needs to be taken into account.

Questions and problems: Q17.10 and Q17.11, P17.8

17.6 Synchronous and Asynchronous (Induction) Electric Motors

Electric motors serve to continuously transform electric energy into mechanical energy. There are several types of electric motors, and we will briefly describe two types that use the concept of rotating magnetic fields.

Imagine we have a U-shaped magnet that rotates with an angular velocity ω, as in Fig. 17.15. The magnetic field will rotate with the magnet; thus it is known as the *rotating magnetic field.* (We will see that a rotating magnetic field can be obtained with appropriate sinusoidal currents in *stationary coils.*) Let a small magnet, e.g., a compass needle, be situated in this field, with the axis of rotation the same as that of the U-shaped magnet. A magnetic torque will act on the small magnet. If the small magnet is stationary, and ω is large, there will be a torque on the small magnet that tends to rotate it in one and then in the other direction, so that it will only oscillate. However, if the small magnet is brought to rotate with the angular velocity ω, the rotating magnetic field will act on it by a continuous torque in one direction, and the small magnet will rotate *in synchronism* with the magnetic field, even if it has to overcome a small friction (or a load). If the rotating magnetic field is obtained with currents in stationary coils, the same will happen, and we will have a simple *synchronous motor*. The name comes from the fact that the motor can rotate only in synchronism with the rotation of the magnetic field.

If instead of the small magnet we have in the rotating magnetic field a short-circuited wire loop, as in Fig. 17.16, a current will be induced in the loop because the magnetic flux through the loop is varying in time. According to Lentz's law, the actual direction of the induced current will be as indicated in the figure. It is seen that there will be a torque on the loop, tending to rotate it with the field. If the rotating field is produced by currents in stationary coils, we obtain a simple *induction motor*.

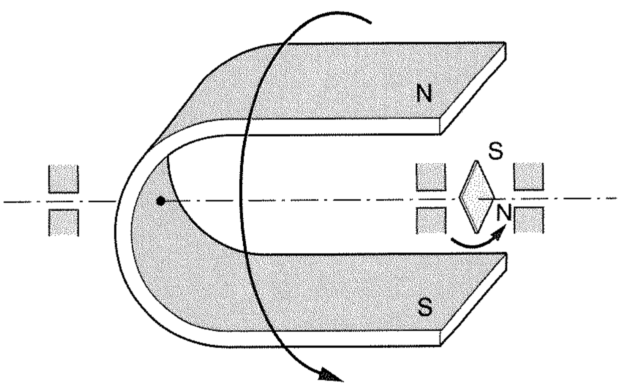


Figure 17.15 A small magnet in the rotating magnetic field of a rotating magnet

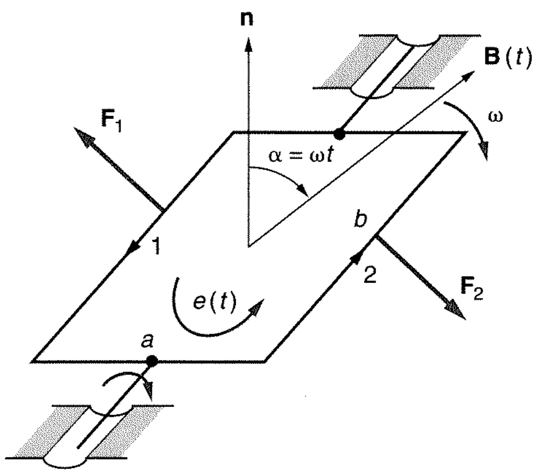


Figure 17.16 A short-circuited wire loop in a rotating magnetic field

Because there will be a time-average torque on the loop for any angular velocity of the loop rotation, whether or not it rotates in synchronism with the field, it is also known as the *asynchronous* (i.e., not synchronous) *motor*. Its rotating part, or *rotor*, is usually made in the form of a number of short-circuited loops at an angle, similar to a cage (the *squirrel-cage rotor*). The short-circuited loops are fixed in grooves in a ferromagnetic rotor core.

For large amounts of power, the rotating magnetic field is obtained directly from a three-phase current system. Let us examine a simpler way of obtaining a rotating magnetic field using two currents of equal amplitude that are 90 degrees out of phase (Fig. 17.17), a method used for low-power synchronous and asynchronous motors.

If the currents in the two coils in Fig. 17.17 are of equal magnitude and shifted in phase by 90 degrees, so are the magnetic flux densities they produce. Therefore (see Fig. 17.17),

$$B_x(t) = B_m \cos \omega t, \qquad \text{and} \qquad B_y(t) = B_m \sin \omega t.$$

The total magnetic flux density has a magnitude of

$$B_{\text{total}}(t) = \sqrt{B_x^2(t) + B_y^2(t)} = B_m,$$

which means that it has a constant magnitude. The vector **B** is rotating, however, since

$$\tan \alpha(t) = \frac{B_y(t)}{B_x(t)} = \tan \omega t,$$

so that

$$\alpha(t) = \omega t,$$

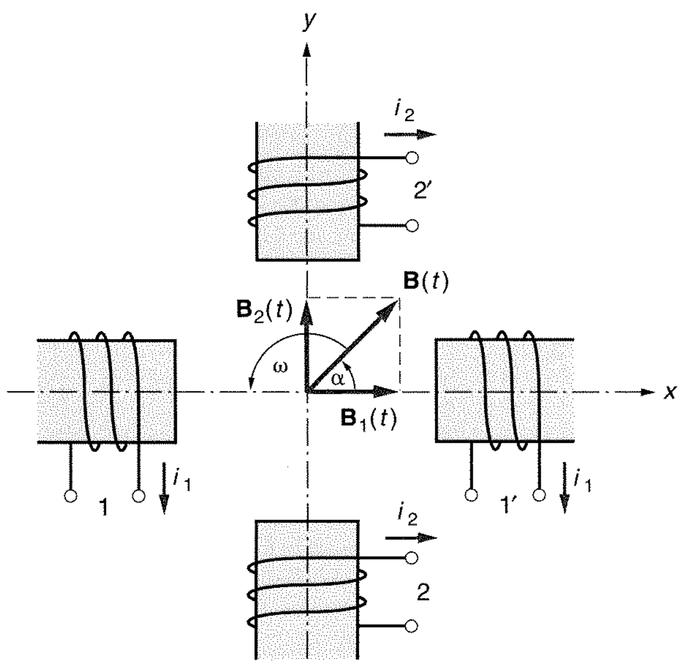


Figure 17.17 A rotating magnetic field produced with two perpendicular coils with sinusoidal currents shifted in phase by 90 degrees

which means that, indeed, the vector **B** rotates with a constant angular velocity ω. We have thus obtained a rotating magnetic field with sinusoidal currents in two stationary coils.

Three-phase motors and generators were invented by Nikola Tesla, a Serbian immigrant who came to America with 4 cents in his pocket. In 1891, he filed about 50 patents related to different kinds of ac generators and motors, but he had to fight Thomas Edison's promotion of dc power. Eventually George Westinghouse, who supported Tesla's inventions, won the battle and the first ac power plant was built on Niagara Falls. In 1891, the same year Tesla filed the patents that provoked strong reactions and resistance in the scientific community, mines in Telluride, Colorado, had already installed polyphase motors and generators based on his patents.

Questions and problems: Q17.12 and Q17.13, P17.9

17.7 Rough Calculation of the Effect of Power Lines on the Human Body

We often hear that the electric and magnetic fields "radiated" from power lines may be harming our bodies. We are now ready to do some rough calculations of the voltages induced in the body. We will do the calculations on the example of a human head (which is the most important and most sensitive part of our body). We will assume that our head is a sphere with a radius of 10 cm, and made mostly of salty water. In this example, let the power lines be as close as 20 m to a human, and let them carry an

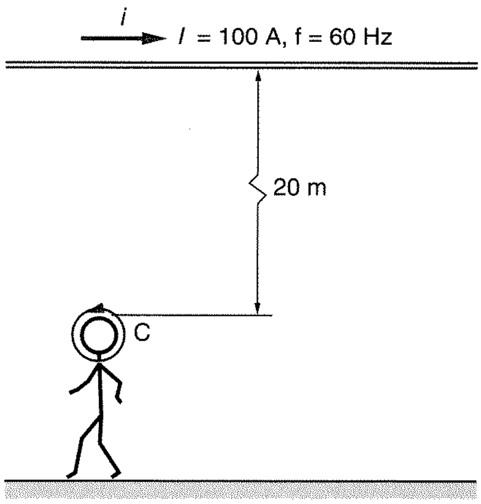


Figure 17.18 The effect of a power line on the human body

unbalanced current of 100 A, as in Fig. 17.18. (The total current in a balanced power line is zero.)

We need to consider two effects: the induced voltage from the magnetic field, since the current in the line is sinusoidally varying in time at 60 Hz, and the voltage due to the electric field of the wire.

First, the magnetic flux density 20 m from a wire carrying 100 A is equal to $B = \mu_0 I/2\pi r = 1\,\mu$T. For comparison, the earth's average dc magnetic field is 50 μT on the earth's surface. (This dc field induces currents in our bodies only if we move, and we are probably adapted to this small effect.) The induced electric field around our head (which is a conductor) can be calculated from Faraday's law:

$$\oint_{\text{head perimeter}} \mathbf{E}_{\text{ind}} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_{\text{head cross section}} \mathbf{B} \cdot d\mathbf{S}.$$

The left-hand side of the equation is approximately $2\pi a E_{\text{ind}}$, and the right-hand side equals $-a^2\pi\ \partial B/\partial t$. In complex notation we thus have

$$2\pi a E_{\text{ind}} = -j\omega B\pi a^2, \qquad \text{hence} \qquad |E|_{\text{ind}} = \omega B a/2. \tag{17.6}$$

The rms value of the voltage due to this induced electric field across a single cell in our head (which is about 10 μm wide) is $V_{\text{cell}} \simeq 33$ pV at 60 Hz. This is very small; for comparison, the normal neural impulses that pass through the cells are spikes on the order of 100 mV in amplitude, and they last about a millisecond, with a frequency between 1 and 100 Hz.

Let us now consider the electric field effect. The approximate value for the electric field around power lines depends on the power line voltage rating and the distance of the point from the line, but a resonable value would be E_0=1 kV/m. Our cells are made of essentially salty water, which has a resistivity of about 1 $\Omega \cdot$ m. To find the voltage across an individual cell we reason as follows.

The sphere approximating the head is conducting. It is situated in an approximately uniform electric field. Therefore, surface charges are induced on its surface as determined in Example 11.3, Eq. (11.9):

$$\sigma(\theta) = 3\epsilon_0 E_0 \cos\theta,$$

where θ is the angle between the radius to the cell considered (a point on the sphere surface) and the direction of vector $\mathbf{E}_0$. However, this charge is not time-constant, as in Example 11.3, but rather time-varying, since E_0 is time-varying. Consequently, there is a time-varying current inside the sphere, which can be determined approximately in the following manner.

The total charge on one hemisphere is given by

$$Q = \int_0^{\pi/2} \sigma(\theta) 2\pi a \sin\theta \, a \, \mathrm{d}\theta = \int_0^{\pi/2} 3\epsilon_0 E \cos\theta \; 2\pi a \; \sin\theta \; a \, \mathrm{d}\theta = 3\pi\epsilon_0 a^2 E_0. \quad (17.7)$$

For $E_0 = 1\,\mathrm{kV/m}$, this amounts to $Q = 835\,\mathrm{pC}$.

If the charge is time-varying, there is a time-varying current in the sphere obtained as $i(t) = \mathrm{d}Q(t)/\mathrm{d}t$. For a sinusoidal field of frequency $f = 60\,\mathrm{Hz}$, the rms value of the current is

$$I = \omega Q = 2\pi f Q = 0.315\,\mu\mathrm{A}.$$

The current density inside the sphere, in the equatorial plane of the sphere and normal to $\mathbf{E}_0$, is hence

$$J = \frac{I}{a^2\pi} \simeq 10\,\mu\mathrm{A/m}^2,$$

so that the electric field inside the sphere is not zero, but has a rms value $E = \rho J = 10\mu\mathrm{V/m}$. So the voltage across a cell equals about $10\,\mu\mathrm{V/m} \times 10\,\mu\mathrm{m} = 100\,\mathrm{pV}$. This is somewhat larger than the voltage due to the time-varying magnetic field, but it is probably still negligible with respect to 100-mV voltage spikes due to normal neural impulses.

In conclusion, voltages induced in our body when we are close to power lines are much smaller than the normal electric impulses flowing through our nerve cells. Nevertheless, it is hard to say with absolute certainty that these orders-of-magnitude lower voltages do not have any effect on us, because biological systems are often at a very unstable equilibrium.

Questions and problems: Q17.14

QUESTIONS

Q17.1. Where is the earth's south magnetic pole?

Q17.2. What is the order of magnitude of the earth's magnetic flux density?

Q17.3. Approximately how fast would you need to spin around your axis in the magnetic field of the earth to induce 1 mV around the contour of your body?

Q17.4. Turn your computer monitor sideways or upside down while it is on (preferably with some brightly colored pattern on it). Do you notice changes in the screen? If yes, what and why?

Q17.5. What do you expect to happen if a magnet is placed close to a monitor? If you have a small magnet, perform the experiment (note that the effect might remain after you remove the magnet, but it is not permanent). Explain.

Q17.6. Explain how the Hall effect can be used to measure the magnetic flux density.

Q17.7. Explain how the Hall effect can be used to determine whether a semiconductor is p- or n-doped.

Q17.8. What magnetic material properties are chosen for the tracks and heads in a hard disk?

Q17.9. Sketch and explain the time-domain waveform of the induced emf (or current) in the magnetic head coil in "read" mode as it passes over a piece of information recorded on a computer disk as "110." (Assume that a "1" is a small magnet along the track with a N-S orientation from left to right, and a "0" is in the opposite direction.)

Q17.10. Write Ampère's law for an ideal transformer, and derive the voltage, current, and impedance (resistance) transformation ratio. The number of turns in the primary and secondary are N_1 and N_2.

Q17.11. What are the loss mechanisms in a real transformer, and how does each of the contributors to loss depend on frequency?

Q17.12. Explain how a synchronous motor works.

Q17.13. How is an asynchronous motor different from the synchronous type?

Q17.14. Describe the two mechanisms by which ac currents can affect our body. Use formulas in your description.

PROBLEMS

P17.1. What is the minimum magnitude of a magnetic flux density vector that will produce the same magnetic force on an electron moving at 100 m/s that a 10-kV/cm electric field produces?

P17.2. Calculate the velocity of an electron in a 10-kV CRT. The electric field is used to accelerate the electrons, and the magnetic field to deflect them.

P17.3. How large is the magnetic flux density vector needed for a 20-cm deflection in the CRT in problem P17.2, if the length of the tube is 25 cm?

P17.4. A thin conductive ribbon is placed perpendicularly to the field lines of a uniform **B** field. When the current is flowing in the direction shown in Fig. 17.6a, there is a measured negative voltage V_{12} between the two edges of the ribbon. Are the free charges in the conductive ribbon positive or negative?

P17.5. What is the voltage V_{12} equal to in problem P17.4 if $B = 0.8\,\mathrm{T}$, the ribbon thickness $t = 0.5\,\mathrm{mm}$, $I = 0.8\,\mathrm{A}$, and the concentration of free carriers in the ribbon is $N = 8 \cdot 10^{28}\,\mathrm{m}^{-3}$?

P17.6. The magnetic head in Figure 17.10 is in write mode. Calculate the magnitude of the current i in the winding that would be needed to produce a $B_0 = 1\,\mu\mathrm{T}$ field in the gap. There are $N = 5$ turns on the core, the core can approximately be considered as linear, of relative permeability of $\mu_r = 1000$, the gap is $L_0 = 20\,\mu\mathrm{m}$ wide, the cross-sectional area of the core is $S = 10^{-9} m^2$, and the mean radius of the core is $r = 0.1$ mm. Assume that the fringing field in the gap makes the gap cross-sectional area effectively 10% larger than that of the core.

P17.7. The head and the tracks in magnetic hard disks are made of different magnetic materials because they perform different functions. Sketch and explain the preferred hysteresis curves for the two materials, indicating the differences. Which has higher loss in the ac regime?

P17.8. A CRT needs 10 kV to produce an electric field for electron acceleration. Design a wall-plug transformer to convert from 110 V in the U.S. and Canada or 220 V in Europe and Asia. Assume you have a core made of a magnetic material that has a very high permeability.

P17.9. Assume that in Fig. 17.17 you have three instead of two coils. The axes of the coils are now at 60 degrees, not 90 degrees, with respect to each other. What is the relative phasing of three sinusoidal currents in the coils that will give a rotating magnetic field, as described in section 17.6 for the case of two currents? Plot the current waveforms as a function of time.

18

Transmission Lines

18.1 Introduction

We have by now learned what capacitors, resistors, and inductors are from the standpoint of electromagnetic field theory. In circuit theory, we usually assume that these elements are lumped (pointlike) and that they are interconnected by means of wire conductors. The current along a wire conductor is assumed to be the same at all points.

Transmission lines consist most frequently of two conductors (some have more, e.g., a three-phase power line). Examples are a coaxial line, a two-wire line, and a stripline. Transmission lines are rare electromagnetic systems that can also be analyzed by circuit-theory tools, although we need electromagnetic theory for determining the transmission-line parameters (i.e., the circuit elements).

Consider a very long section of a transmission line, such as a coaxial line, with perfect conductors and an imperfect dielectric. Let a dc generator of voltage V be connected at one end of the line and a resistor of resistance R at the other. Is there a current along the line? The answer is, of course, yes. However, because there are stray currents through the imperfect dielectric from one line conductor to the other, the current intensity along the two line conductors is not constant. The largest current will be at the generator end, as at that point all the stray currents add up. At the other end of the line, the current intensity through the conductors is the smallest, equal to V/R, as sketched in Fig. 18.1a. Note that if the conductors are perfect, the current intensity in the resistor does not depend on the stray currents.

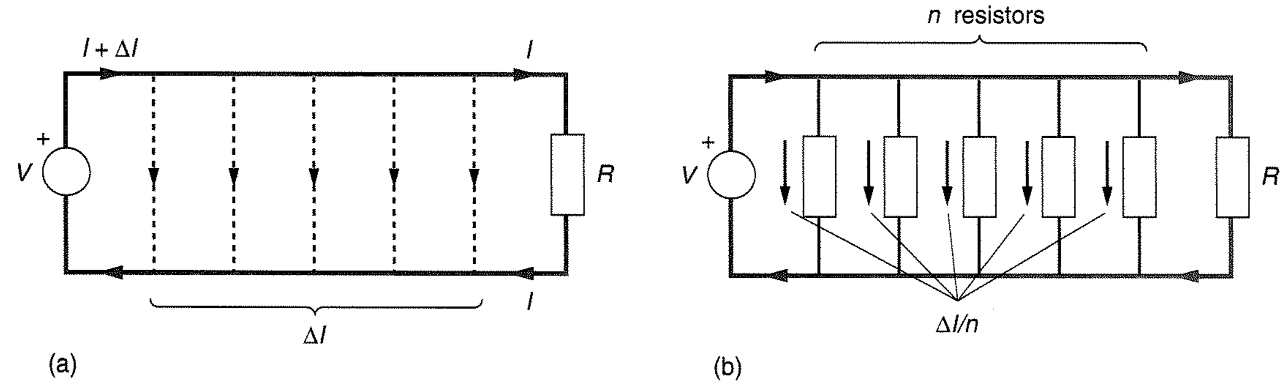


Figure 18.1 (a) A section of a transmission line with perfect conductors and an imperfect dielectric shows stray currents. (b) A ladderlike circuit-theory approximation of the line in (a).

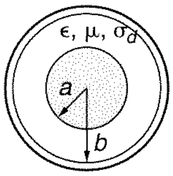
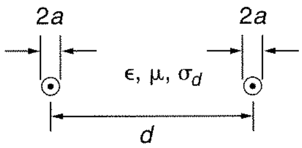
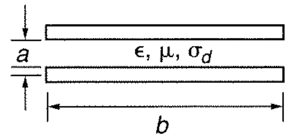
The conclusion that currents at the generator and load ends are different does not fit into the circuit-theory postulate that the current is the same all along a wire that connects circuit elements. Is it possible nevertheless to use circuit theory to analyze this simple circuit? We can subdivide the line section into short segments and represent it as a ladderlike structure with appropriate resistors connected between the conductors of these short line elements, as in Fig. 18.1b. The accuracy of this approximation will increase with the number of segments. For exact representation we need an infinite number of infinitely small segments, but a large number of segments should also give us an accurate result.

If instead of a dc generator we connect an ac generator, the same effect occurs even if the line dielectric is perfect, for now we have *capacitive* stray currents between the two conductors. However, now the voltage across the load will also differ from that at the generator, in spite of the line conductors being perfect. This is due to small inductive voltage drops across short segments of the line; we know that a line segment of length Δz has an inductance $L' \Delta z$ (L' is the line inductance per unit length). Of course, a real transmission line also has a resistance per unit length (due to imperfect conductors), so in addition we will have a resistive voltage drop across segments of the line.

Shown in Table 18.1 are the parameters C', G', L', and R' of the three mentioned transmission-line types. Note that most frequently $\mu = \mu_0$, that the conductivity of the conductors is approximately $\sigma_c \simeq 56 \times 10^6$ S/m (copper), that the relative permittivity of the dielectric is usually 1.0 (air) or 2.1 to 4.0 (most other dielectrics, although dielectrics with considerably higher relative permittivity are also used), and that the conductivity of the dielectrics other than air is on the order of 10^{-12} S/m.

Thus if we wish to analyze any transmission line with ac excitation by circuit-theory concepts, we need to represent it as a series connection of many small cells containing series inductors and resistors, and parallel capacitors and resistors, as in Fig. 18.2. Such circuits are said to have *distributed parameters*. If losses in a line are very small, the line is referred to as a *lossless line*. Although all lines have losses, they can frequently be neglected, so analysis of lossless lines is of considerable practical interest.

TABLE 18.1 Parameters of Some Transmission Lines at High Frequencies

Parameter	Coaxial line	Two-wire line ($d \gg 2a$)	Strip line ($b \gg a$)
			
$C'\left(\frac{\mathrm{F}}{\mathrm{m}}\right)$	$\frac{2\pi\epsilon}{\ln b/a}$	$\frac{\pi\epsilon}{\ln d/a}$	$\epsilon\frac{b}{a}$
$G'\left(\frac{\mathrm{S}}{\mathrm{m}}\right)$ S	$\frac{\sigma_d}{\epsilon}C'$	$\frac{\sigma_d}{\epsilon}C'$	$\frac{\sigma_d}{\epsilon}C'$
$L'_{\mathrm{ext}}{}^{*}\left(\frac{\mathrm{H}}{\mathrm{m}}\right)$	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	$\frac{\mu}{\pi}\ln\frac{d}{a}$	$\mu\frac{a}{b}$
$R'\left(\frac{\Omega}{\mathrm{m}}\right)$	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{R_s}{\pi a}$	$\frac{2R_s}{b}$

*$L'_{\mathrm{int}} = R'/\omega$ in all three cases, and $R_s = \sqrt{\omega\mu/2\sigma_c}$ (see Examples 21.7 and 21.9 for proof); σ_c is the conductivity of the line conductors. PROOFS IN EX 20.4-20.6.

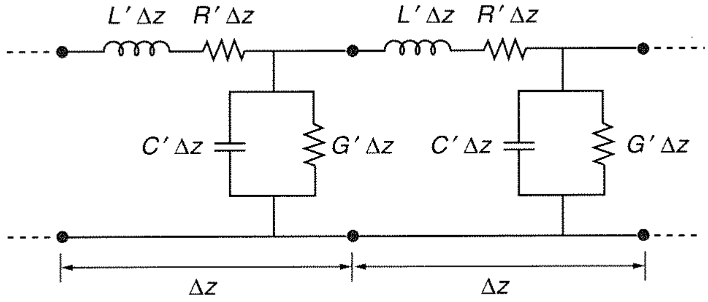


Figure 18.2 Circuit-theory approximation of a transmission line with losses

We will develop first the theory for the analysis of lossless lines and then introduce losses in a simple manner. The analysis will show that the time-varying voltage and current along the line vary continuously and that these variations propagate along the line. These are known as *voltage* and *current waves.* The analysis will also show that the voltages, currents, and the ratio of voltage and current at a point along the line depend on what load is connected at the end of the line. Typically, we wish to efficiently deliver power from a generator, through a line, to the load.

Questions and problems: P18.1 and P18.2

18.2 Analysis of Lossless Transmission Lines

The circuit-theory approximations of short and long sections of a lossless transmission line are sketched in Figs. 18.3a and b. Consider the three short sections in

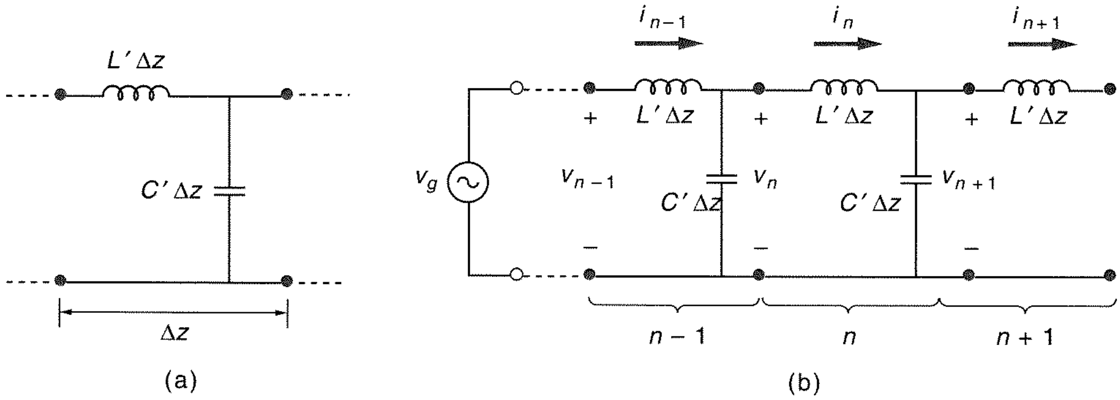


Figure 18.3 (a) A very short piece of a lossless transmission line of length Δz can be represented as a circuit consisting of a series inductor and a shunt (parallel) capacitor. (b) A longer piece of the line can be represented as many cascaded short sections.

Fig. 18.3b, labeled $(n-1)$, n, and $(n+1)$. Let us apply Kirchhoff's voltage and current laws. The voltage across the nth inductor and the current through the nth capacitor are

$$\Delta L \frac{\mathrm{d}i_n}{\mathrm{d}t} = v_n - v_{n+1} \qquad \text{and} \qquad \Delta C \frac{\mathrm{d}v_n}{\mathrm{d}t} = i_{n-1} - i_n. \tag{18.1}$$

Dividing both equations by Δz and noting that

$$\frac{\Delta L}{\Delta z} = L' \qquad \text{and} \qquad \frac{\Delta C}{\Delta z} = C', \tag{18.2}$$

we can rewrite Eqs. (18.1) as follows:

$$L' \frac{\mathrm{d}i_n}{\mathrm{d}t} = -\frac{v_{n+1} - v_n}{\Delta z} \qquad \text{and} \qquad C' \frac{\mathrm{d}v_n}{\mathrm{d}t} = -\frac{i_n - i_{n-1}}{\Delta z}. \tag{18.3}$$

As Δz approaches zero, the right-hand sides become derivatives with respect to the coordinate z (note that the left-hand sides are true derivatives with respect to time), and Eqs. (18.3) become

$$\frac{\partial v(t,z)}{\partial z} = -L' \frac{\partial i(t,z)}{\partial t} \qquad \text{and} \qquad \frac{\partial i(t,z)}{\partial z} = -C' \frac{\partial v(t,z)}{\partial t}. \tag{18.4}$$

(Transmission-line equations, or telegraphers' equations, for lossless lines)

Partial derivatives need to be used because $u(z,t)$ and $i(z,t)$ are functions of time, t, and distance along the line, z. It is clear that if voltage and current vary in *time*, they also vary *along the line*. Equations (18.4) are called the *transmission-line equations* or the *telegraphers' equations*. These two equations are coupled differential equations in two unknowns, i and v.

It is easy to obtain instead equations with only voltage or only current. To that aim, take the derivative with respect to z $(\partial/\partial z)$ of the first equation and the time derivative $(\partial/\partial t)$ of the second equation and eliminate the current (or voltage) by substitution. Following this procedure, we obtain

$$\frac{\partial^2 v(t,z)}{\partial t^2} - \frac{1}{L'C'}\frac{\partial^2 v(t,z)}{\partial z^2} = 0 \qquad \frac{\partial^2 i(t,z)}{\partial t^2} - \frac{1}{L'C'}\frac{\partial^2 i(t,z)}{\partial z^2} = 0. \tag{18.5}$$

(Wave equations for voltage and current along lossless transmission lines)

These equations are known as the *wave equations*. They describe the variation of voltage and current along a line and in time. The same or similar type of equation can be used to describe the electric and magnetic fields in a radio wave or optical ray, sound waves in acoustics, etc. We will later derive the same equation for the electric and magnetic field strength vectors, **E** and **H**, instead of voltages and currents.

18.2.1 FORWARD AND BACKWARD VOLTAGE WAVES IN THE TIME DOMAIN

Consider the voltage wave equation. It is not difficult to show (see Example 18.1) that its solution is

$$v(t,z) = V_+ f(t - z/c) + V_- g(t + z/c) \qquad \text{(V)}, \tag{18.6}$$

(Forward and backward voltage wave on a transmission line)

where V_+ and V_- are constants, f and g are *arbitrary* functions of the indicated arguments, and

$$c = \frac{1}{\sqrt{L'C'}} \qquad \text{(m/s)}. \tag{18.7}$$

(Velocity of a wave propagating along a transmission line)

The physical meaning of the solution in Eq. (18.6) is as follows. Consider the function $f(t,z) = f(t - z/c)$ (the constant V_+ is irrelevant). Let the function at $t = 0$ be as $f(0,z)$ in Fig. 18.4. At a somewhat later instant, say $t = \Delta t$, the difference $t - z/c$ will have the same value as for $t = 0$ if we consider a point $z + c\,\Delta t = z + \Delta z$ instead of point z. This means that the bell-shaped voltage pulse $f(0,z)$ will have exactly the same form, but will be moved by $\Delta z = c\Delta t$, as indicated by the pulse labeled $f(\Delta t, z + c\,\Delta t)$ in Fig. 18.4. Because Δt is arbitrary, this means that the voltage pulse moves from left to right in Fig. 18.4 (i.e., in the direction of the z axis) with a velocity $c = 1/\sqrt{L'C'}$. The wave moving in the $+z$ direction is the *forward traveling (voltage) wave* or the *incident (voltage) wave.*

It is simple to conclude in the same way that the function $g(t + z/c)$ represents a voltage wave propagating in the $-z$ direction. Such a wave is the *backward traveling (voltage) wave* or the *reflected (voltage) wave.*

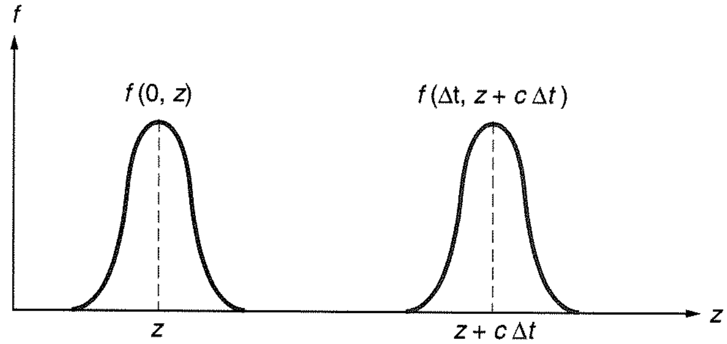


Figure 18.4 A voltage wave moves unchanged in shape, with constant velocity c, along a lossless transmission line.

What is the velocity of propagation of voltage waves along a typical cable? For a 50-Ω coaxial cable with a typical dielectric, $C' \simeq 1$ pF/cm, and $L' \simeq 2.5$ nH/cm. The velocity c in Eq. (18.7) for these C' and L' is about two-thirds of the speed of light in air (i.e., about 2×10^8 m/s).

Example 18.1—Proof that $f(t - z/c)$ and $g(t + z/c)$ are solutions of the wave equation. The proof is simple if we recall the rules for finding the derivatives of a function of several variables. Suppose we have a function $f(x)$, where $x = x(t, z)$ is an arbitrary function of two independent variables, t and z. The partial derivative of $f(x)$ with respect to t, for example, is obtained using the chain rule as follows:

$$\frac{\partial f(x)}{\partial t} = \frac{\partial f(x)}{\partial x}\frac{\partial x(t, z)}{\partial t}.$$

The second partial derivative is obtained in an analogous manner.

Let $x(t, z) = (t \pm z/c)$. Then

$$\frac{\partial f(x)}{\partial t} = \frac{\partial f(x)}{\partial x}\frac{\partial (t \pm z/c)}{\partial t} = \frac{\partial f(x)}{\partial x}, \tag{18.8a}$$

because $\partial(t \pm z/c)/\partial t = 1$. Hence also

$$\frac{\partial^2 f(x)}{\partial t^2} = \frac{\partial^2 f(x)}{\partial x^2}.$$

The derivative with respect to z is somewhat different because z is multiplied by a constant, $\pm 1/c$:

$$\frac{\partial f(x)}{\partial z} = \frac{\partial f(x)}{\partial x}\frac{\partial (t \pm z/c)}{\partial z} = \pm\frac{1}{c}\frac{\partial f(x)}{\partial x}, \tag{18.8b}$$

and

$$\frac{\partial^2 f(x)}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 f(x)}{\partial x^2}.$$

Substituting the second derivatives with respect to t and z into the wave equation (18.5), we see that it is indeed satisfied for *any* function $f(t \pm z/c)$.

Note that according to Eqs. (18.8*a*) and (18.8*b*),

$$\frac{\partial f(x)}{\partial z} = \pm\frac{1}{c}\frac{\partial f(x)}{\partial x} = \pm\frac{1}{c}\frac{\partial f(x)}{\partial t}. \tag{18.9}$$

18.2.2 FORWARD AND BACKWARD VOLTAGE WAVES IN THE COMPLEX (FREQUENCY) DOMAIN

Here we deal mostly with sinusoidally time-varying voltages and currents and linear materials, which means that we can use phasor (complex) notation. Both voltage and current have an assumed exponential time variation,

$$v, i \propto \mathrm{e}^{\mathrm{j}\omega t} \qquad \mathrm{j} = \sqrt{-1} \quad \text{(the imaginary unit)}, \tag{18.10}$$

but these exponentials cancel out when we write equations involving V and I in phasor form. We use capital letters for rms (root mean square) complex quantities. The derivative with respect to time becomes just a multiplication with $\mathrm{j}\omega$. Consequently, we can write the transmission-line equations (18.4) for sinusoidal time variation as

$$\frac{\mathrm{d}V(z)}{\mathrm{d}z} = -\mathrm{j}\omega L' I(z) \qquad \text{and} \qquad \frac{\mathrm{d}I(z)}{\mathrm{d}z} = -\mathrm{j}\omega C' V(z). \tag{18.11}$$

[Lossless-transmission-line equations in phasor (complex) form]

Eliminating the current from these equations results in

$$\frac{\mathrm{d}^2 V(z)}{\mathrm{d}z^2} = -\omega^2 L'C'V(z) = (\mathrm{j}\omega\sqrt{L'C'})^2 V(z) = (\mathrm{j}\beta)^2 V(z). \tag{18.12}$$

[Voltage wave equation along lossless transmission lines in phasor (complex) form]

The solution to this second-order differential equation is of the form

$$V(z) = V_+\mathrm{e}^{-\mathrm{j}\beta z} + V_-\mathrm{e}^{+\mathrm{j}\beta z} \qquad \text{(V)}, \tag{18.13}$$

(Total voltage wave in phasor notation)

where

$$\beta = \omega\sqrt{L'C'} = \frac{\omega}{c} \qquad (1/\mathrm{m}). \tag{18.14}$$

(Definition of phase constant)

The constant β is known as the *phase constant* (or *phase coefficient*) because it determines the phase of the voltage at a distance z from the origin ($z = 0$). Comparing Eq. (18.13) with Eq. (18.6), it can be inferred that $V_+ e^{-j\beta z}$ is the complex representation of a forward traveling wave and $V_- e^{+j\beta z}$ that of a backward traveling wave.

18.2.3 WAVELENGTH ALONG TRANSMISSION LINES

Consider the expression for the forward traveling cosine voltage wave in the time domain,

$$v_+(t, z) = V_+\sqrt{2}\cos(\omega t - \beta z).$$

The argument of the cosine function remains the same if any multiple of 2π is added to it, or by moving by $\beta\,\Delta z = n \cdot 2\pi,\ n = \pm 1, \pm 2, \dots$ along the line. The smallest distance $\Delta z = \lambda$ for which this happens is obtained from the equation $\beta\lambda = 2\pi$, from which we derive

$$\boxed{\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{(\omega/c)} = \frac{c}{f} \quad \text{(m)}. \qquad (18.15)}$$

(Definition of wavelength of sinusoidal waves)

This distance, λ, is known as the *wavelength* of the sinusoidal wave.

Note again that in complex (phasor) notation, the forward traveling wave has a minus sign in the exponential. This means that *for a fixed moment in time* the phase of the wave lags along the z direction. (Because the wave is propagating in that direction, this must be the case.)

18.2.4 CURRENT WAVES IN THE COMPLEX (PHASOR) DOMAIN, AND THE CHARACTERISTIC IMPEDANCE

Expressions analogous to those for the voltage wave can be obtained for the current wave along the line. From Eqs. (18.4), (18.6), and (18.9), we find

$$\frac{\partial i(t,z)}{\partial t} = -\frac{1}{L'}\frac{\partial v(t,z)}{\partial z} = \frac{1}{cL'}\left[V_+\frac{\partial f(t - z/c)}{\partial t} - V_-\frac{\partial g(t + z/c)}{\partial t}\right].$$

This equation can be integrated directly with respect to time. Assuming zero dc components of voltages and currents and having in mind Eq. (18.7), the integration results in

$$i(t,z) = \frac{V_+}{\sqrt{L'/C'}} f(t - z/c) - \frac{V_-}{\sqrt{L'/C'}} g(t + z/c) \quad \text{(A)}. \qquad (18.16)$$

(Forward and backward current waves along transmission lines)

In phasor notation this equation becomes

$$I(z) = \frac{V_+}{Z_0} e^{-j\beta z} - \frac{V_-}{Z_0} e^{+j\beta z} \quad \text{(A)}, \tag{18.17}$$

(Forward and backward current waves in phasor notation)

where

$$Z_0 = \sqrt{\frac{L'}{C'}} \quad (\Omega). \tag{18.18}$$

(Characteristic impedance of lossless line)

Z_0 is called the *characteristic impedance* of the lossless transmission line. Like L' and C', it depends only on how the line is built (its dimensions and the materials used in it).

Example 18.2—Numerical values of c and Z_0 for some lossless transmission lines. Because for lossless lines $L'_{\text{int}} = 0$, $R' = 0$, and $G' = 0$, for the three lines given in Table 18.1 the velocity of propagation, c, becomes

$$c = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\epsilon\mu}} \quad \text{(for the three lines in Table 18.1)}.$$

It can be shown that this simple relation is valid not only for the three lines considered but for all lossless transmission lines with a homogeneous dielectric.

In particular, if the dielectric in the line is air we have

$$c_0 = \frac{1}{\sqrt{\epsilon_0\mu_0}} = \frac{1}{\sqrt{8.854 \cdot 10^{-12} \times 4\pi \cdot 10^{-7}}} \simeq 3 \times 10^8 \ \text{m/s},$$

i.e., the velocity of propagation of voltage and current waves along air lines equals the velocity of light in a vacuum. This is a conclusion to be remembered. Note that it is valid only for lines with air dielectric. Because for any dielectric $\epsilon > \epsilon_0$, the propagation velocity along lines with dielectric other than air is always less than the velocity of light in a vacuum.

The characteristic impedance, Z_0, is different for the three lines. Let us write the explicit expression for the coaxial line:

$$Z_0 = \sqrt{\frac{L'}{C'}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \ln\frac{b}{a} \quad \text{(lossless coaxial line)}.$$

For example, if $b/a = \mathrm{e} = 2.71828$ and the dielectric is air, we obtain that $Z_0 \simeq 60\ \Omega$. Characteristic impedance of commercial coaxial lines (for which $\epsilon > \epsilon_0$) ranges from about 50 Ω to about 90 Ω.

Generally, we have both a forward and a backward wave on the line. To calculate the ratio $v(t, z)/i(t, z)$ at any point along the line and at any instant, the complete expressions for $v(t, z)$ and $i(t, z)$ in Eqs. (18.6) and (18.16) must be used. If only a forward wave exists along a line, the ratio of the forward voltage and current waves is found from Eqs. (18.6) and (18.16) to be

$$\frac{v(t, z)}{i(t, z)} = Z_0. \tag{18.19}$$

(Only a forward wave along the line)

This means that if only a forward wave exists along the line, the ratio of voltage and current at *any* point along the line and at *any* instant of time is the same, equal to Z_0.

If only the backward wave propagates along the line, Eqs. (18.6) and (18.16) yield

$$\frac{v(t, z)}{i(t, z)} = -Z_0. \tag{18.20}$$

(Only a backward wave along the line)

These two equations have a simple physical meaning. Consider Fig. 18.5a and assume reference directions of voltage v_+ and current i_+ as indicated. If only a forward wave exists, the generator is at the left in Fig. 18.5a, and the line to the right is such that no backward (reflected) wave is created. This will be the case if the line is infinitely long to the right. This means that a generator connected to the input terminals of an infinitely long lossless line will see the line as a resistor of resistance equal to Z_0.

The last conclusion enables us to understand an extremely important fact: because an infinite section of any lossless line with respect to its input terminals behaves as a resistor of resistance Z_0, we can eliminate the reflected wave on a line of *any* length by terminating the line in its characteristic impedance. If this is done, we say that the line is *matched*.

Why do we have the minus sign in Eq. (18.20)? Note that the reference directions of voltage and current have been adopted as indicated in Figs. 18.5a and b (the reference conductor for voltage is designated by a "+" sign and the reference direc-

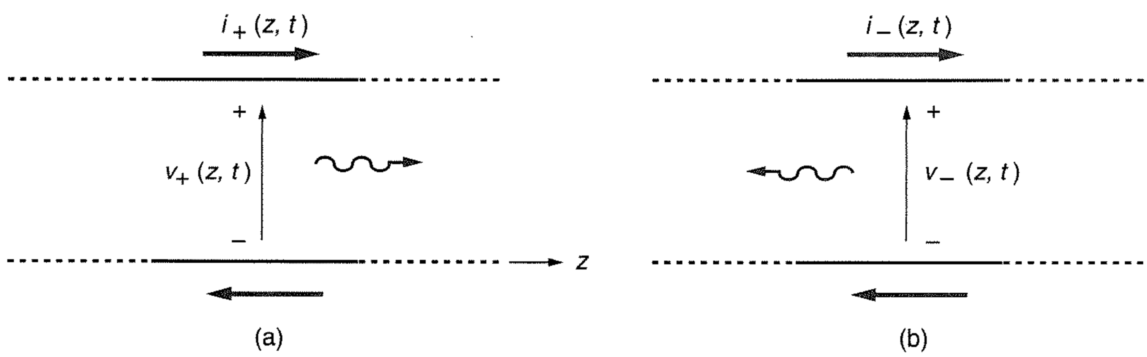


Figure 18.5 (a) Forward and (b) backward voltage and current waves in a transmission line. Note that the adopted reference directions of voltage and current are the same in both cases.

tion for current by an arrow). If the backward wave alone propagates along the line, the generator must be at the right end of the line, feeding an infinite line extending to the left. In Fig. 18.5b the same reference directions are adopted for voltage and current as in Fig. 18.5a; therefore one of these quantities for a reflected wave must be negative so that the power flow is from right to left. Because we retained the same sign for the backward voltage wave, the current wave must change sign with respect to the forward wave. The meaning of the negative sign is exactly the same as in circuit theory: the current is in the direction opposite to the reference direction.

Assume that there is a backward (reflected) wave along the line. Let the generator be connected at the line end toward which the backward wave is propagating (to the left in Fig. 18.5b). When the backward wave reaches the generator, will it produce a backward-backward (i.e., a new forward) wave? The answer is evident: such a wave *will* be produced unless the internal resistance of the generator equals the line characteristic impedance, Z_0. For this reason generators (or equivalent Thévenin's generators), if possible, are made to have this internal resistance, usually 50 Ω. In what follows, we assume that generators driving transmission lines satisfy this condition, i.e., that they are matched to the line.

Questions and problems: Q18.1 to Q18.11, P18.3 to P18.7

18.3 Analysis of Terminated Lossless Transmission Lines in Frequency Domain

The excitation of transmission lines can have any time variation. Frequently the excitation is sinusoidal or nearly sinusoidal. In this and the next section we restrict our attention to sinusoidal voltages and currents along transmission lines and use the phasor (complex) notation. This section is devoted to lossless lines, and the next to lines with losses.

In reality, a line may be terminated in *any* load, which is not necessarily the same as its characteristic impedance, as shown in Fig. 18.6. We now know that a forward (or incident) wave travels from the generator to the right in Fig. 18.6. When

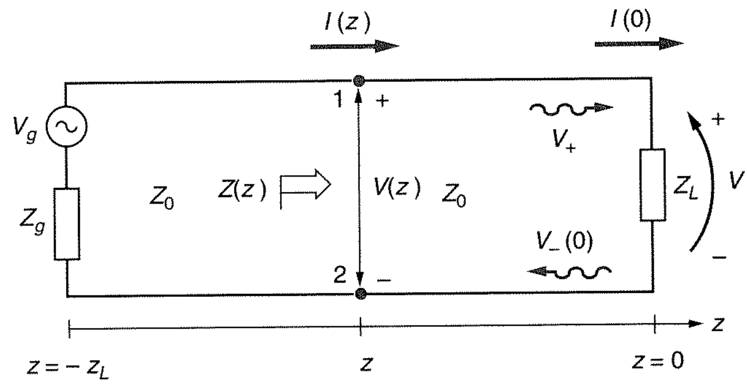


Figure 18.6 A transmission line of characteristic impedance Z_0 terminated in a load Z_L at a distance z_L from the generator of internal impedance Z_0. The coordinate origin, $z = 0$, is adopted at the position of the load.

it reaches the load, some of the power is absorbed, and some is reflected, giving rise to a backward wave. The line being linear (the fundamental transmission-line equations are linear), the amplitude of the reflected wave is proportional to that of the incident wave. Because we normally assume that the generator is matched to the line, when the reflected wave reaches the generator it is absorbed in its internal impedance.

The coordinate origin, $z = 0$, can be anywhere along the line. In the analysis of transmission lines, we are mostly concerned with the load because this is where we wish the generator power to be delivered. Therefore, it is convenient to shift the origin from the generator to the load, as in Fig. 18.6.

18.3.1 THE REFLECTION AND TRANSMISSION COEFFICIENTS

The *(voltage) reflection coefficient* is defined as the ratio of the complex amplitudes (or rms values) of the reflected and incident voltage waves at the load. If $z = 0$ is at the load, as in Fig. 18.6, the reflection coefficient is given by

$$\rho = \frac{V_-}{V_+} \qquad \text{(dimensionless)}. \tag{18.21}$$

(Definition of the reflection coefficient)

With this definition, the phasor voltage and current along the line in Eqs. (18.13) and (18.17) can be written in the form

$$V(z) = V_+ \mathrm{e}^{-\mathrm{j}\beta z}(1 + \rho \mathrm{e}^{2\mathrm{j}\beta z}) \qquad \text{(V)}, \tag{18.22a}$$

$$I(z) = \frac{V_+}{Z_0} \mathrm{e}^{-\mathrm{j}\beta z}(1 - \rho \mathrm{e}^{2\mathrm{j}\beta z}) \qquad \text{(A)}. \tag{18.22b}$$

(Total voltage and current along a transmission line in terms of the reflection coefficient)

When a generator is connected at one end of a line and a load at the other end, part of the power is reflected from the load (if the load is not perfectly matched) and part of the power is delivered to the load. Generally the goal is to deliver as much power to the load as possible. A quantity that describes the voltage across the load as a function of the incident voltage is called the *transmission coefficient*, defined by

$$\tau = \frac{V_{\text{load}}}{V_+} = \frac{V_+ + V_-}{V_+} = 1 + \rho \qquad \text{(dimensionless)}. \tag{18.23}$$

(Definition of transmission coefficient)

The magnitude of the voltage reflection coefficient for passive loads is smaller than or equal to unity, whereas the magnitude of the transmission coefficient is smaller than or equal to 2. The following examples illustrate this range of values.

Example 18.3—Reflection and transmission coefficients for shorted, open, and matched transmission lines. Let us look at a few simple and extreme examples of terminations (loads) in Fig. 18.6: (1) an open circuit ($Z_L = \infty$), (2) a short circuit ($Z_L = 0$), and (3) a matched load ($Z_L = Z_0$).

At an open end of a line, no current flows between the two conductors. As the adopted reference directions of forward and backward current waves are the same (Figs. 18.5a and b) the reflected current at that point has to be of the same magnitude as the incident current wave, but of opposite sign. According to Eqs. (18.13) and (18.17), the reflected voltage wave at that point is then equal to the incident wave (note reference directions for the two voltages in Fig. 18.5). Consequently, the voltage reflection coefficient at the load for an open end is $\rho = 1$. From Eq. (18.22*a*) it follows that at the open-circuited line end the total voltage is twice the incident voltage, corresponding to a voltage transmission coefficient $\tau = 2$.

At a short-circuited line end, there is no voltage between the two line conductors, corresponding to a transmission coefficient $\tau = 0$. Referring to reference directions for voltage in Fig. 18.5, as the total voltage at the end of the line has to be zero, the reflected voltage is the negative of the incident voltage. The (voltage) reflection coefficient for a zero load is therefore $\rho = -1$. According to Eq. (18.22*b*), the current at the short-circuited end is twice the current of the incident current wave.

For a matched case (load impedance equal to the line characteristic impedance), if we divide Eq. (18.22*a*) by Eq. (18.22*b*) and set $z = 0$ this ratio is equal to the load impedance, in this case Z_0. So we obtain the following equation:

$$Z_0 = Z_0 \frac{1+\rho}{1-\rho}.$$

This equation can be satisfied only if $\rho = 0$. This was to be expected because we know that a matched load absorbs the incident wave completely, corresponding to a transmission coefficient of $\tau = 1$.

Example 18.4—Time-average power delivered to the load. From circuit theory we know that the time-average power delivered to a load is obtained from the phasor voltage across the load, V, and the phasor current in the load, I, as $P_{\text{av}} = \text{Re}\{V \cdot I^*\}$, where the asterisk denotes a complex conjugate. The voltage and current across the load are obtained from Eqs. (18.22*a*) and (18.22*b*) if we set $z = 0$. Thus the average power delivered to the load terminating a transmission line is

$$P_{\text{load av}} = \text{Re}\{V(0)I^*(0)\} = \text{Re}\left\{ \frac{|V_+|^2}{Z_0}[1 - |\rho|^2 + (\rho - \rho^*)] \right\}.$$

Recall that for a complex number $a = b + \text{j}c$, $a - a^* = (b + \text{j}c) - (b - \text{j}c) = \text{j}2c$. Therefore $(\rho - \rho^*)$ is purely imaginary, so that

$$P_{\text{load av}} = \frac{|V_+|^2}{Z_0}(1 - |\rho|^2). \tag{18.24}$$

(Average power delivered to a transmission-line load)

Note that this is precisely the difference of average power of the incident wave, $|V_+|^2/Z_0$, and of the reflected wave, $|V_-|^2/Z_0 = |\rho|^2|V_+|^2/Z_0$.

Usually we wish to express the power delivered to the load in terms of the voltage V_g of the generator connected at the input transmission-line terminals. As mentioned, we always assume that the generator is matched to the line, i.e., that its internal impedance is equal to the line characteristic impedance. Because for the incident voltage wave we assume an infinite line, the impedance of the line seen by the *incident wave* at the generator terminals is simply Z_0. Therefore, $V_+ = V_g/2$ for a matched generator.

18.3.2 IMPEDANCE OF A TERMINATED TRANSMISSION LINE

Consider again the cross section of the line at z in Fig. 18.6. Looking to the right from points 1 and 2, we have a passive network (containing no generators) with two terminals (points 1 and 2). Considering it as a black box, we define its impedance in the usual manner as the ratio of the voltage between the terminals (which is the *total* voltage) and the corresponding current (which is the *total* current). (Note that the adopted reference directions of voltage and current in Fig. 18.6 are precisely as needed for an impedance element to the right of points 1 and 2.) This impedance is a function of z. According to Eqs. (18.22*a*) and (18.22*b*), we have

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{1 + \rho e^{j2\beta z}}{1 - \rho e^{j2\beta z}}.$$

This is the impedance of the line looking toward the load at a distance z from the coordinate origin (assumed at the load). Due to the adopted coordinate origin, the z coordinate of any point is negative (Fig. 18.7). To avoid the minus sign in the expressions to follow, it is convenient to introduce a new coordinate, $\zeta = -z$ (Fig. 18.7), representing the distance from the load. With this change in variable along the line, the expression for the impedance in the last equation becomes

$$Z(\zeta) = Z_0 \frac{1 + \rho\, e^{-j2\beta\zeta}}{1 - \rho\, e^{-j2\beta\zeta}}. \tag{18.25}$$

In particular, for $\zeta = 0$ we have that $Z(0) = Z_L$. Thus

$$Z_L = Z_0 \frac{1 + \rho}{1 - \rho}, \tag{18.26}$$

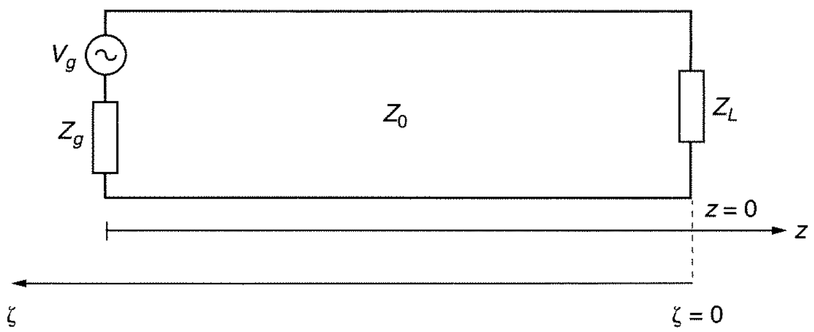


Figure 18.7 Coordinate origin, $z = 0$, at the load, and the coordinate $\zeta = -z$

which is used for determining Z_L if ρ has been determined experimentally. Solving the last equation, we can obtain expressions for ρ and τ as a function of the load impedance:

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0} \quad \text{(dimensionless)}, \tag{18.27a}$$

(Alternative expression for reflection coefficient)

$$\tau = \frac{2Z_L}{Z_L + Z_0} \quad \text{(dimensionless)}, \tag{18.27b}$$

(Alternative expression for transmission coefficient)

so that the reflection and transmission coefficients can be determined knowing only the load impedance and the line characteristic impedance.

Finally, if we substitute ρ from Eq. (18.27*a*) into Eq. (18.25) and recall that $e^{j\alpha} = \cos\alpha + j\sin\alpha$ (Euler's formula), the input impedance of a section of line of length ζ terminated in Z_L, Eq. (18.25), after simple manipulations becomes

$$Z(\zeta) = Z_0\frac{Z_L\cos\beta\zeta + jZ_0\sin\beta\zeta}{Z_0\cos\beta\zeta + jZ_L\sin\beta\zeta} = Z_0\frac{Z_L + jZ_0\tan\beta\zeta}{Z_0 + jZ_L\tan\beta\zeta} \quad (\Omega). \tag{18.28}$$

(Input impedance of line of length ζ terminated in impedance Z_L)

The important thing to remember is that the characteristic impedance Z_0 depends only on the way the line is made. The impedance along the line (looking toward the terminating impedance) is quite different: it depends on both Z_0 and the terminating impedance but also on the coordinate along the line.

Example 18.5—Input impedance of an open line. Assume that the line is open. This corresponds to $Z_L = \infty$ in Eq. (18.28), so that the input impedance of a section of the line of length ζ becomes

$$Z(\zeta) = -j\frac{Z_0}{\tan\beta\zeta} = -jZ_0\overset{\text{cotan}}{\cancel{\tan}}\,\beta\zeta.$$

If $\beta\zeta < \pi/2$, that is, if $\zeta < \pi/2 \cdot \lambda/(2\pi) = \lambda/4$, $Z(\zeta)$ is a negative imaginary number. This means that the line behaves as a capacitor. (Note, however, that this line behavior is valid only for a line length less than $\lambda/4$!)

You might recall from Chapter 2 that the parasitic inductance of rf chip capacitors makes these elements look predominantly inductive above a certain frequency (the lead inductance is on the order of 1 nH). At microwave frequencies, short sections of open-ended lines are frequently used to obtain in a simple manner a capacitive reactance of a desired value at a given frequency. Note that this reactance depends on frequency in a different way than in the case of a capacitor (for which $X_C = -1/\omega C$).

As an example, consider a short segment of length 1 cm of a 50-Ω line. Assume that the velocity of wave propagation along the line is $0.67c_0$. The reactance of this line segment at $f = 1000$ MHz is

$$Z(1\ \text{cm})_{1000\text{MHz}} = -jZ_0 \overset{\text{COTAN}}{\cancel{\tan^{-1}}} \left(\frac{2\pi}{c/f} \times \zeta\right)$$
$$= -j50 \overset{\text{COTAN}}{\cancel{\tan^{-1}}} \left(\frac{2\pi}{0.67 \cdot 3 \cdot 10^8/10^9} \times 0.01\right) \simeq -j320\Omega,$$

which corresponds to a capacitance of $1/(2\pi \cdot 10^9 \cdot 320) \simeq 0.5$ pF (only at 1000 MHz!). It is suggested as an exercise for the reader to calculate the capacitance of the line between 900 MHz and 1100 MHz.

If $\lambda/4 < \zeta < \lambda/2$, the line behaves as an inductive element; for a still greater length it behaves again as a capacitive element, and so on. It is left as an exercise for the reader to plot $Z(\zeta)$ as (1) a function of frequency and (2) a function of the length of the line in wavelengths.

Example 18.6—Input impedance of a shorted line. Assume now that the line is shorted, i.e., that in Eq. (18.28) $Z_L = 0$. The input impedance of a section of the line of length ζ in this case is

$$Z(\zeta) = +jZ_0 \tan \beta\zeta.$$

If $\beta\zeta < \pi/2$, that is, if $\zeta < \pi/2 \cdot \lambda/(2\pi) = \lambda/4$, $Z(\zeta)$ is a *positive* imaginary number, i.e., the line behaves as an inductor.

You might also recall from Chapter 2 that an inductor has parasitic capacitance between the windings, and as the frequency increases the element looks more and more like a capacitor. Therefore it is hard to make inductors at microwave frequencies. Shorted sections of transmission lines are used frequently at microwave frequencies to obtain in a simple manner an inductive reactance of desired value. Note, however, that this reactance depends on frequency in a different way from that of an inductor ($X_L = \omega L$).

If $\lambda/4 < \zeta < \lambda/2$, the line behaves as a capacitive element; for a still greater length it behaves again as an inductive element, and so on. It is left as an exercise for the reader to plot $Z(\zeta)$ as (1) a function of frequency and (2) as a function of the length of the line in wavelengths.

Example 18.7—Quarter-wave transformers. An interesting and important case is when the length of the transmission line is a quarter of a wavelength. Because then $\beta\zeta = (2\pi/\lambda) \times (\lambda/4) = \pi/2$, from Eq. (18.28) we obtain

$$Z\left(\frac{\lambda}{4}\right) = \frac{Z_0^2}{Z_L}. \tag{18.29}$$

The load impedance is transformed from a value Z_L to a value Z_0^2/Z_L. For example, if Z_L is a high impedance, Z will be low and vice versa. Quarter-wavelength transmission-line sections often play the same role at rf and microwave frequencies as impedance transformers at lower frequencies. (At high frequencies, it is difficult to build good transformers due to parasitic capacitances in inductors and also losses in the conductors and cores.)

Quarter-wave transformers are especially used for matching resistive loads. For example, if we want to match a 100-Ω load to a 50-Ω transmission line, we could use a quarter-wavelength section of a line with a characteristic impedance of $Z_0 = \sqrt{100 \cdot 50} = 70.7\ \Omega$.

However, unlike in a low-frequency transformer there is a phase lag in the section of the transmission line. This type of impedance transformer does not work for voltage and current transformation. Note also that the quarter-wavelength transformer effect works only in a narrow range of frequencies (it exactly works only at the frequency for which the length of the line is a quarter of a wavelength).

Analogous ideas are used in optics to make antireflection coatings for lenses. We explain this in more detail in later chapters.

Example 18.8—Thévenin equivalent of an open-ended section of transmission line fed by a generator. Line terminated in an infinite line of different characteristic impedance. Assume that a line of characteristic impedance Z_1 (line 1) is terminated in an infinite (or matched) line of characteristic impedance Z_2 (line 2). We know that line 2 from its input terminals represents a load of resistance Z_2. So line 1 can be regarded as terminated in a load $Z_L = Z_2$. Consequently we know the voltage and current distribution along line 1.

Along line 2 we have only a forward voltage and a forward current wave, propagating along it with a velocity determined by its capacitance and inductance per unit length. For determining these waves we need only determine the voltage at the input terminals of line 2. This can be done very simply by applying Thévenin's theorem.

Line 1 as seen from the input terminals of line 2 represents an equivalent real voltage generator. The voltage of the generator equals the open-circuit voltage at the end of line 1. From Example 18.3 we know that this voltage is twice the incident voltage along line 1, $V_{\text{Th}} = V_+(1 + \rho) = 2V_+$. The internal impedance of the Thévenin generator is the impedance of the infinite (or matched) line 1, so $Z_{\text{Th}} = Z_1$.

Let us summarize this very useful result. The equivalent Thévenin voltage source and impedance for an open-ended section of line of characteristic impedance Z_1 fed by a generator that gives an incident voltage V_+ are

$$Z_{\text{Th}} = Z_1 \qquad \text{and} \qquad V_{\text{Th}} = 2V_+.$$

After we replace line 1 with its Thévenin equivalent, the input voltage of line 2 can be found as in a voltage divider:

$$V_{2\ \text{input}} = V_{\text{Th}} \frac{Z_2}{Z_1 + Z_2} = 2V_+ \frac{Z_2}{Z_1 + Z_2}.$$

We know the reflection coefficient for line 1, given in Eq. (18.27*a*), which in this case becomes

$$\rho = \frac{Z_2 - Z_1}{Z_2 + Z_1}.$$

The transmission coefficient, from line 1 to line 2, is given by Eq. (18.27*b*):

$$\tau = \frac{V_{2\ \text{input}}}{V_+} = \frac{2Z_2}{Z_1 + Z_2}.$$

18.3.3 THE VOLTAGE STANDING-WAVE RATIO (VSWR)

A useful and frequently used concept related to the reflection coefficient is the *voltage standing-wave ratio*, or VSWR. The VSWR is the ratio of the maximal to minimal voltage along the line. Because $|e^{2j\beta z}| = 1$, according to Eq. (18.22*a*) the VSWR is given by

$$\text{VSWR} = \frac{V(z)_{\text{max}}}{V(z)_{\text{min}}} = \frac{1+|\rho|}{1-|\rho|}. \tag{18.30}$$

(Definition of voltage standing-wave ratio, VSWR)

Note that for a matched load, VSWR $= 1$, and for open and for short circuits, VSWR $= \infty$.

Example 18.9—Standing waves on transmission lines. When a line is matched at its end, we know that there is only a forward wave propagating along the line. To visualize such a voltage wave for sinusoidal excitation, imagine a sine function that moves along the line with a velocity c.

When the line is not matched there is another sinusoid, usually of smaller amplitude, moving from the load toward the generator (where we assume a matched load, i.e., no more reflected waves) with the same velocity, c. So in the general case we have two sine waves of unequal amplitudes moving in opposite directions with the same velocity. The total voltage at any point along the line (and at any moment) is obtained as their sum. Due to their equal velocities, however, there will be fixed minima and maxima of the total wave, as the following example shows.

We know that for a shorted line the voltage reflection coefficient $\rho = -1$ (see Example 18.3). Consequently, according to Eq. (18.22*a*), the total voltage along the line is of the form

$$V(z) = V_+ e^{-j\beta z}(1 - e^{2j\beta z}) = V_+(e^{-j\beta z} - e^{j\beta z}) = -j2V_+ \sin(\beta z),$$

because $e^{-j\alpha} - e^{j\alpha} = (\cos\alpha - j\sin\alpha) - (\cos\alpha + j\sin\alpha) = -j2\sin\alpha$.

The instantaneous value of the voltage along the line, $v(t, z)$, is hence obtained as

$$v(t, z) = \text{Re}\{-j2V_+\sqrt{2}\sin(\beta z)e^{j\omega t}\} = 2V_+\sqrt{2}\sin(\beta z)\sin\omega t.$$

This voltage does *not* have any argument of the form $(t \mp z/c)$! Consequently this is not a forward or a backward traveling voltage wave. Instead, it has zero values at all points where the sine has a zero value, and it oscillates *between* these fixed, stationary zeros of the total voltage. For this reason, this kind of wave is termed the *standing wave*. A sketch of the voltage standing wave for a sequence of time instants is shown in Fig. 18.8a. Note that a standing wave in phasor (complex) notation is easily recognized by the absence of the "traveling-wave" factor $e^{\mp j\beta z}$ (or any other coordinate instead of z).

The distance along two fixed, zero-voltage points along the line is given by

$$\Delta z_{\text{between two voltage zeros}} = \frac{\pi}{\beta} = \pi\frac{\lambda}{2\pi} = \frac{\lambda}{2}.$$

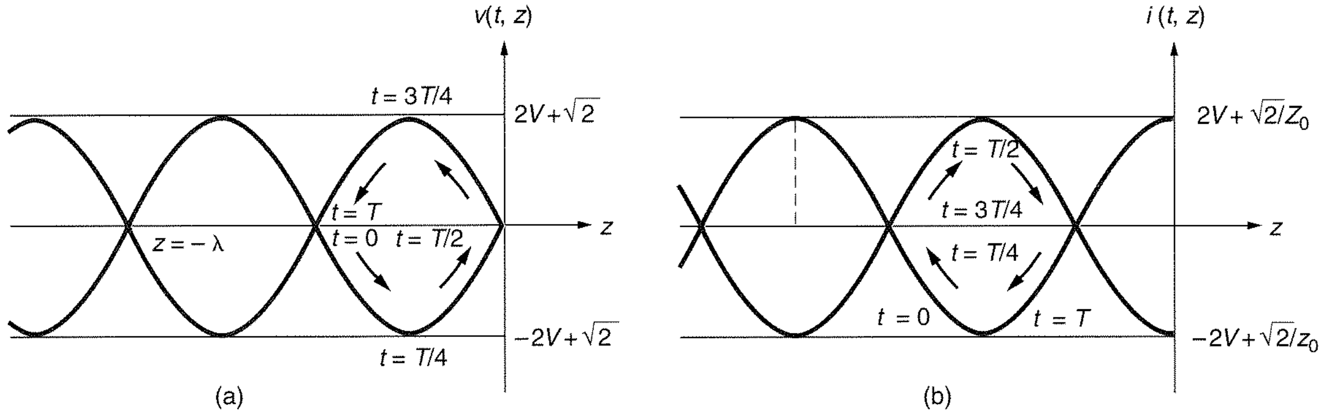


Figure 18.8 (a) Voltage and (b) current standing wave along a shorted transmission line at indicated time instants (corresponding to the expressions derived in Example 18.9)

The total current along a shorted line has the same property, i.e., it is also a standing wave. From Eq. (18.22*b*), it is easily found that

$$i(t, z) = -2\frac{V_+}{Z_0}\sqrt{2}\cos(\beta z)\cos\omega t.$$

A sketch of the current standing wave for a sequence of time instants is shown in Fig. 18.8b.

It is left as an exercise for the reader to derive the expressions for standing voltage and current waves for an open transmission line.

Thus if we are able to measure the voltage along a transmission line we can easily conclude whether the line is shorted or open. The next example will show how we can measure the impedance of any load terminating a line by measuring, essentially, the VSWR.

Example 18.10—Measurement of load impedance using a slotted line. Figure 18.9a shows what is called the slotted coaxial line. Slotted lines may be used for measuring impedances at very high frequencies. To understand how this can be done, let the generator have a fixed but unknown frequency, and let the dielectric in the slotted line of characteristic impedance Z_0 be air. The coax is rigid and the outer conductor tube has a narrow slot along its length. The slot is made along the current-flow lines in the outer line conductor, so it only slightly affects the distribution of current and voltage in the line. A movable fixture is attached to the tube and contains a pinlike probe that protrudes through the slot and samples the electric field (voltage) inside the cable. Recall that the electric field vector in the cable is radial, so a voltage $v = \int_{\text{pin}} \mathbf{E} \cdot \mathbf{dl}$ is induced in the probe. This voltage is converted to a dc voltage by means of a diode detector and gives a measure of the relative electric field along the line. The position of the probe is measured along an arbitrary scale, e.g., like the one in Fig. 18.9b.

Usually the probe cannot slide all the way to the end on the line where the load is attached, and also the connector at the end of the line adds an unknown line length. So the first step in measuring an unknown load is to determine exactly where the line ends. We know that everything along a line is repeated every half wavelength. This means that we can determine the position of the end of the line displaced by an integer number of half wavelengths, so that it falls along our scale.

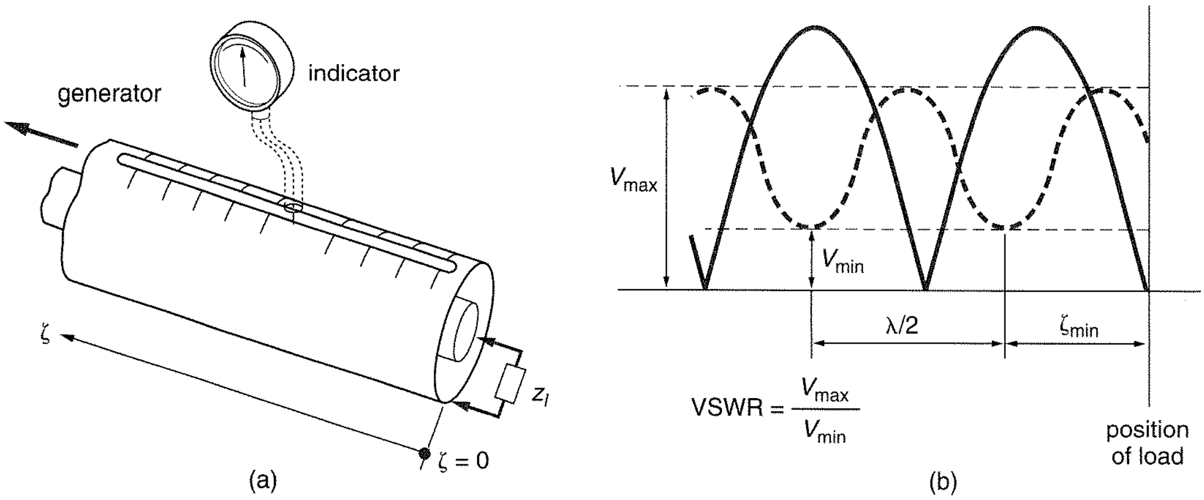


Figure 18.9 (a) Sketch of a slotted coaxial line and (b) sketch of measured voltage along the line for a short circuit (solid line) and arbitrary complex load (dashed line)

How is the position of the end of the line determined? The easiest way is to connect a known load to the end, so that a standing wave is set up. If a short (or open) is connected, we know that the minima (maxima) of the standing wave occur at the load and every half wavelength away toward the generator. The wavelength measured along the line is practically equal to the free-space wavelength, $\lambda_0 = c_0/f$. Therefore we can measure the frequency of the generator simply by moving the probe back and forth and determining the two successive minima. Usually a short is used because minima are sharper and therefore more precise than maxima. (Sketch the derivative, or slope, of a standing wave to convince yourself.) The standing wave pattern due to a short is sketched in Fig. 18.9b in solid line, and the real and displaced positions of the end of the line are indicated.

After this calibration is performed, the unknown load is connected to the end of the line and the standing wave sampled once more. Again, two successive minima will be at a distance $\lambda_0/2$ apart, but they are displaced in position from the minima obtained with a short. This is because the phase of the load is different from that of the short. By moving the probe back and forth, we determine the maximum and minimum readings of the indicator, which gives us the voltage standing-wave ratio, VSWR, defined in Eq. (18.30), as sketched in Fig. 18.9b in dashed line. Then we locate as precisely as possible the distance ζ_{min} of the first minimum from the minimum obtained with a short, in terms of wavelength. From Eq. (18.22*a*) we find that the voltage minimum occurs when $\rho e^{-j2\beta\zeta}$ is real and negative, that is, equal to $-|\rho|$. So the impedance $Z(\zeta)$ given by Eq. (18.28) for $\zeta = \zeta_{min}$ is real, and equal to

$$Z(\zeta_{min}) = Z_0 \frac{1 - |\rho|}{1 + |\rho|} = \frac{Z_0}{\text{VSWR}}.$$

As we now know $Z(\zeta_{min})$ and ζ_{min} (in terms of wavelength), we can evaluate the unknown load impedance Z_L from Eq. (18.28).

Questions and problems: Q18.12 to Q18.16, P18.8 to P18.27

18.4 Lossy Transmission Lines

We know that a real transmission line has losses in the conductors (due to the finite conductivity of the metal) as well as in the dielectric between the conductors (due principally to the polarization losses in the dielectric). If the line is represented as a series connection of many short cells, these losses can be accounted for by a series and a shunt resistor in every cell, as in Fig. 18.10. The total series impedance per unit length is thus $R' + j\omega L'$ (instead of $j\omega L'$ for lossless lines), and the total shunt admittance per unit length is $G' + j\omega C'$ (instead of $j\omega C'$). The phasor equations (18.11) therefore take the form

$$\frac{dV(z)}{dz} = -(R' + j\omega L')I(z), \qquad \text{and} \qquad \frac{dI(z)}{dz} = -(G' + j\omega C')V(z). \tag{18.31}$$

Noting that the lossless-line characteristic impedance in phasor form, $\sqrt{L'/C'}$, originally was $\sqrt{j\omega L'/j\omega C'}$, the characteristic impedance of a lossy line is given by

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}. \tag{18.32}$$

(Characteristic impedance of a lossy line)

Similarly, the expression $j\beta = j\omega\sqrt{L'C'} = \sqrt{(j\omega L')(j\omega C')}$ in the exponential terms in Eq. (18.13) now becomes

$$\gamma = \alpha + j\beta = \sqrt{(R' + j\omega L')(G' + j\omega C')}. \tag{18.33}$$

(Propagation constant of a lossy line)

The constant γ is known as the *propagation constant* (or *propagation coefficient*) of the line, α as the *attenuation constant (coefficient)*, and β, as earlier, the *phase constant (coefficient)*.

Thus, for lossy lines and a forward wave, instead of $e^{-j\beta z}$ in the expressions for voltages and currents we now have $e^{-(\alpha+j\beta)z} = e^{-\alpha z}e^{-j\beta z}$. The factor $e^{-\alpha z}$ means that in addition to traveling in the z direction, the amplitudes of the forward voltage and current waves also fall off in the direction of propagation. This is called *attenuation* and is a characteristic of every real transmission line. The phase of the wave is determined by β (phase constant), and its attenuation by α.

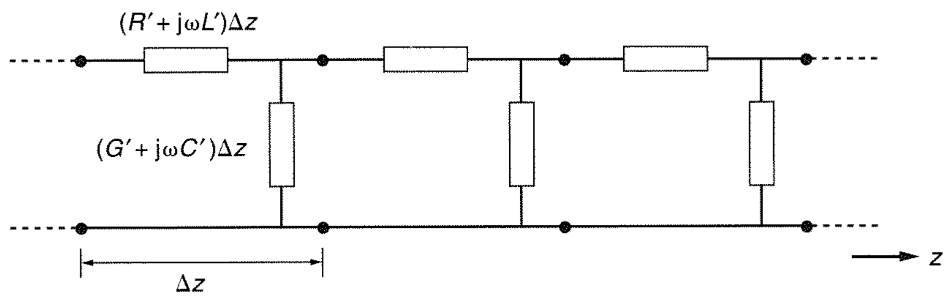


Figure 18.10 Schematic of a transmission line with distributed losses included.

Rearranging Eq. (18.33) we obtain

$$\gamma = \sqrt{j\omega L' j\omega C' \left(1 + \frac{R'}{j\omega L'}\right)\left(1 + \frac{G'}{j\omega C'}\right)} =$$
$$= j\omega\sqrt{L'C'}\sqrt{1 - j\left(\frac{R'}{\omega L'} + \frac{G'}{\omega C'}\right) - \frac{R'G'}{\omega^2 L'C'}}. \tag{18.34}$$

For transmission lines with small losses ($R' \ll \omega L'$ and $G' \ll \omega C'$), this can be written in approximate form

$$\gamma \simeq j\omega\sqrt{L'C'}\sqrt{1 - j\left(\frac{R'}{\omega L'} + \frac{G'}{\omega C'}\right)} \simeq j\omega\sqrt{L'C'}\left[1 - \frac{j}{2}\left(\frac{R'}{\omega L'} + \frac{G'}{\omega C'}\right)\right]. \tag{18.35}$$

So we find that for transmission lines with small losses

$$\alpha \simeq \frac{1}{2}\left(R'\sqrt{\frac{C'}{L'}} + G'\sqrt{\frac{L'}{C'}}\right) \qquad \beta \simeq \omega\sqrt{L'C'}. \tag{18.36}$$

(Attenuation and phase constant for lines with small losses)

If along a transmission line only the forward wave is propagating, both current and voltage along the line have the attenuation factor $e^{-\alpha z}$. Therefore the average power transmitted at a point z in the direction of the wave, being the product of phasor rms voltage and conjugate current, is of the form $P(z) = P(0)e^{-2\alpha z}$.

If a quantity (e.g., voltage) at z is of amplitude $V_+e^{-\alpha z}$, at $z + d$ it is of amplitude $V_+e^{-\alpha(z+d)}$. The attenuation of the voltage along this line section is frequently expressed as the natural logarithm of the ratio of the voltage amplitude at z and that at $z + d$. The unit of this measure of attenuation is termed the *neper* (Np) [after the Scottish mathematician John Neper (Napier), who at the turn of the 16th century invented the logarithm]:

Attenuation of forward voltage wave in nepers

$$= \ln\frac{V_+e^{-\alpha z}}{V_+e^{-\alpha(z+d)}} = \ln e^{\alpha d} = \alpha d \qquad \text{(Np)} \tag{18.37}$$

The unit of the attenuation constant, α, is thus *neper per meter* (Np/m).

The attenuation of voltage or current along a line section is more often expressed in terms of decimal logarithm in *decibels* (dB) (after Alexander Graham Bell, 1847–1922, inventor of the telephone), as

Attenuation of forward voltage wave in decibels

$$= 20\log\frac{V_+e^{-\alpha z}}{V_+e^{-\alpha(z+d)}} = 20\log e^{\alpha d} = (20\log e)\,\alpha d \qquad \text{(dB)}. \tag{18.38}$$

Since $20\log e = 8.686$, the attenuation in decibels is 8.686 times the attenuation in nepers, or 1 Np = 8.686 dB.

Questions and problems: Q18.17, P18.28

18.5 Basics of Analysis of Transmission Lines in the Time Domain

For various reasons, cables might have, or develop in use, faults along their length. It is useful to know where, so that they can be quickly repaired without pulling the whole cable out. The instrument used today to find faults in cables is called the *time domain reflectometer (TDR)*. Its operating principle is very simple: the instrument sends a voltage step and waits for the reflected signal. If there is a fault in the cable it will be equivalent to some rapid change in cable properties, and part of the voltage step wave will reflect off the discontinuity. As both the transmitted and reflected waves travel at the same velocity, the distance of the fault from the place where the TDR was connected can be calculated exactly. Not only can we learn where the fault is, but the TDR can also tell us something about the nature of the fault.

So far, we have looked at transmission lines only in the frequency domain (we assumed sinusoidal voltages and currents). Now we will look at what happens when a step function (in time) is launched down a transmission line terminated in a load. To analyze the time-domain response, we first replace the entire line with its Thévenin equivalent with respect to the load, as derived in Example 18.8. We thus obtain a simple circuit with a Thévenin generator connected to a load. Transients in such a circuit can next be analyzed by solving a differential equation, or by the Laplace (or Fourier) transform (the two procedures are basically the same). We will use the latter method, where we multiply the Laplace (or Fourier) transform of the reflection coefficient (i.e., the reflection coefficient in complex form) with the transform of a step function and then transform back to the time domain with the inverse Laplace (or Fourier) transform.

Example 18.11—Reflection from an inductive load. Let us consider reflection from an inductive load (Fig. 18.11a). The transmission line has a characteristic impedance Z_0 and the incident voltage wave is $v_+(t)$. The Thévenin equivalent generator and impedance for this line are $Z_{\mathrm{Th}} = Z_0$ and $v_{\mathrm{Th}}(t) = 2v_+(t)$ (Fig. 18.11b).

If we now assume that the incident voltage wave is a unit step function starting at $t = 0$, $v_+(t) = 1,\ t > 0$, the Laplace transform is

$$v_+(s) = \frac{1}{s},$$

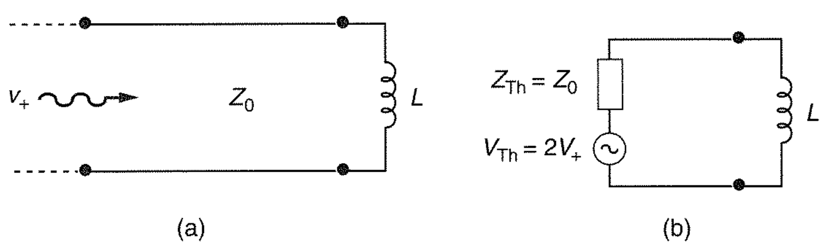


Figure 18.11 (a) A transmission line with an incident wave v_+, terminating in an inductive load. (b) The lossless transmission line is replaced by its Thévenin equivalent circuit.

and because the impedance of the inductor is

$$Z = sL,$$

we find that the load voltage is equal to

$$v(s) = \frac{2L}{sL + Z_0} = \frac{2}{s + Z_0/L}. \tag{18.39}$$

We can recognize this as the Laplace transform of a decaying exponential with a time constant $t_L = L/Z_0$:

$$v(t) = 2e^{-t/t_L} \qquad t > 0. \tag{18.40}$$

Because the voltage of an inductor is $v(t) = L\,di/dt$, we can find the current through the inductive load by integrating the voltage:

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt = \frac{2}{Z_0}(1 - e^{-t/t_L}) \qquad t > 0. \tag{18.41}$$

This describes the buildup of current in an inductor through a resistor, which we already understand from circuit theory. The reflected wave is the difference between the transmitted wave and the incident wave:

$$v_-(t) = v(t) - v_+(t) = 2e^{-t/t_L} - 1 \qquad t > 0. \tag{18.42}$$

We can see that initially the inductor has no current, and the voltage is just v_+, so it looks like an open circuit and the reflection coefficient is +1. The current then builds up to the short circuit current (the Norton equivalent current) and the voltage drops to zero, so the inductor appears as a short circuit. The reflected and transmitted waves are shown in Fig. 18.12.

Example 18.12—Reflection from a short circuit. As another example, let us look at a transmission line that is shorted at one end. If a voltage source is turned on at the other end, what will the reflected wave look like back at the source? We know that the reflection coef-

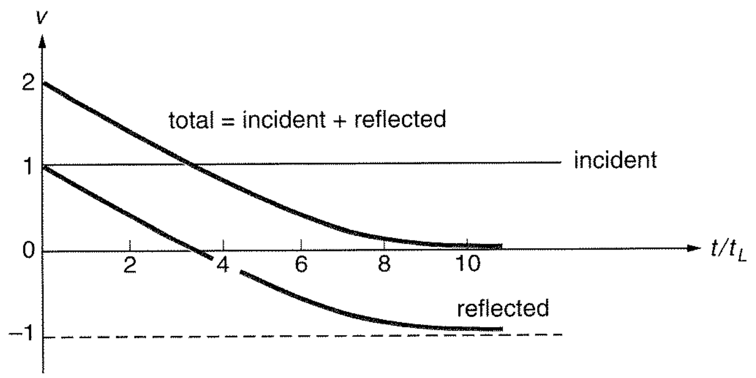


Figure 18.12 The incident, reflected, and total voltages for an inductive load

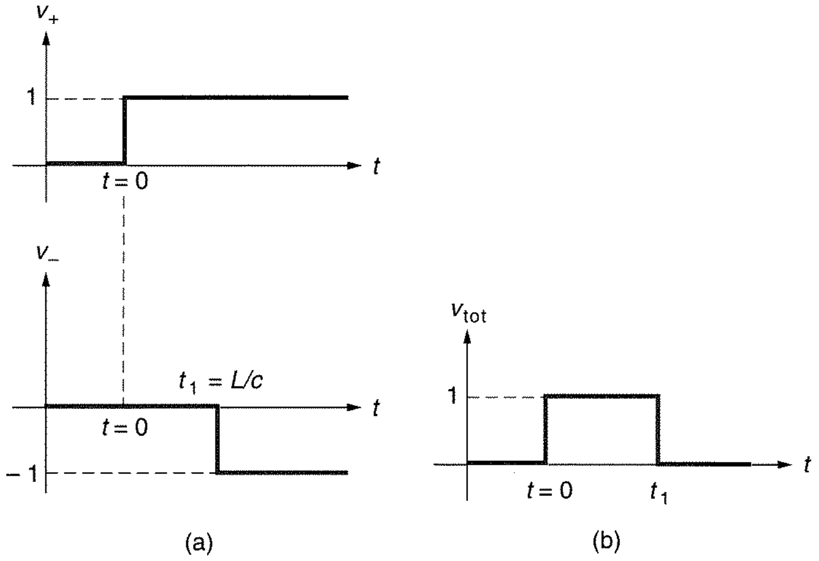


Figure 18.13 (a) Reflected voltage wave off a short-circuited transmission line with an incident unity step function. (b) A standard TDR display shows the reflected wave added on to the incident step function.

ficient of a short circuit is -1, so the reflected wave is as shown in Fig. 18.13a. In TDRs, the reflected wave is added on to the incident step (which goes on forever in time), so in this case the instrument display would appear as shown in Fig. 18.13b. The duration of the "pulse" tells us how long the line is (it corresponds to the round-trip time of the leading edge of the step).

Example 18.13—Reflection from a series RL circuit. A third, slightly more complicated, example is that of a series combination of an inductor L and a resistor R. The incident voltage is a step of unit amplitude. The voltage across the inductor is, as before,

$$v_L(t) = 2e^{-t/t_L}, \quad t > 0, \tag{18.43}$$

where the time constant is now $t_L = L/(R + Z_0)$, because the inductor sees a series connection of the characteristic impedance and the resistive load. The inductor current is

$$i_L(t) = \frac{2}{Z_0 + R}(1 - e^{-t/t_L}), \quad t > 0, \tag{18.44}$$

and the load voltage becomes

$$v(t) = v_L(t) + Ri_L(t) = 2\left[\frac{R}{R + Z_0} + \frac{Z_0}{R + Z_0}e^{-t/t_L}\right], \quad t > 0. \tag{18.45}$$

The reflected voltage wave, shown in Fig. 18.14a, is now

$$v_-(t) = v(t) - v_+(t) = \left[\frac{R - Z_0}{R + Z_0} + 2\frac{Z_0}{R + Z_0}e^{-t/t_L}\right], \quad t > 0. \tag{18.46}$$

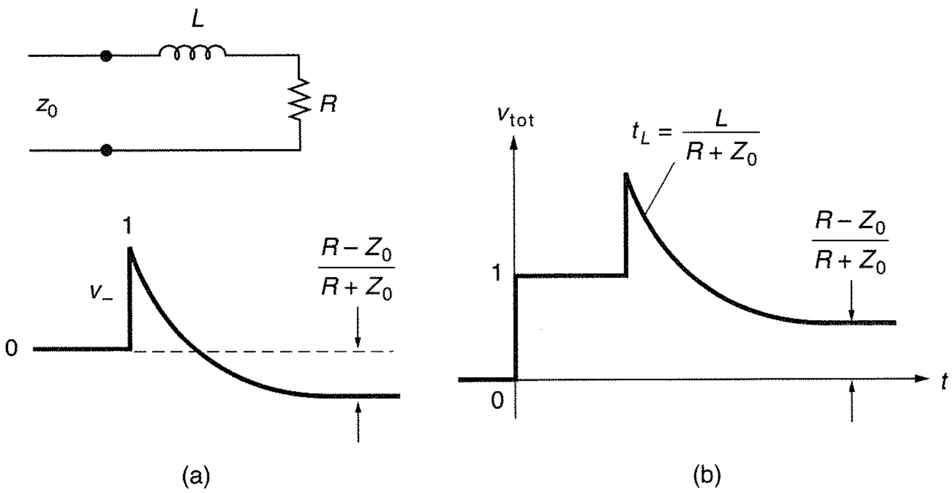


Figure 18.14 (a) Reflected voltage wave off a series RL combination with an incident unity step function. (b) A standard TDR instrument display shows the reflected wave added on to the incident step function.

A simpler qualitative analysis can be done by just evaluating the reflected voltage at $t = 0$ (the time when the reflected wave gets back to the launching port, for example) and $t = \infty$, and assuming any transition between these two values to be exponential. In the previously analyzed case of a series RL circuit, at $t = 0$ the reflected voltage is $v_-(0) = +1$, since the inductor looks like an open circuit initially. On the other hand, as time goes by the current through the inductor builds up, and at $t = \infty$ the inductor looks like a short, so $v_-(\infty) = (R - Z_0)/(R + Z_0)$ and is determined by the resistive part of the load. The resulting plot out of a TDR (incident step plus reflected wave) is shown in Fig. 18.14b.

Example 18.14—Measuring the time constant of the reflected wave from complex loads. The most straightforward way to measure the time constant (such as t_L in the inductor examples) is to measure the time t_1 needed to complete half of the exponential transition from $v_-(0)$ to $v_-(\infty)$. This corresponds to $t_L = t_1/0.69$, where t_L is the time constant we used for an inductive load, but it also holds for a capacitive load. This procedure is shown qualitatively in Fig. 18.15.

Questions and problems: Q18.18 to Q18.21, P18.29 and P18.30

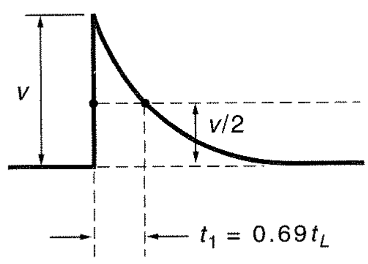


Figure 18.15 Determining the time constant of an exponential TDR response

18.6 The Graphical Solution of Lossless-Line Problems Using the Smith Chart

Until the advent of digital computers, the solution of transmission-line problems was done most often with the aid of a graphical tool known as the Smith chart (P. H. Smith, "Transmission-line Calculator," *Electronics*, **12,** January 1939, p. 29; "An Improved Transmission-line Calculator," *Electronics*, **17,** January 1944, p. 130). The Smith chart is a polar plot of the reflection coefficient with some additional details. We restrict our attention to the Smith chart used for solving problems with lossless lines, that is, for Z_0 real.

The Smith chart enables us to make a direct determination of the complex reflection coefficient $\rho(0)$ at the load, corresponding to a given load impedance Z_L and the characteristic impedance of the line. Conversely, if $\rho(0)$ is determined experimentally, we can read directly from the chart the load impedance Z_L if Z_0 is known. However, the usefulness of the Smith chart far surpasses these two relatively simple tasks, and its use does not seem to decline. Even in the most advanced measurement instruments, such as network analyzers, a Smith chart can be generated on the screen to represent the measurement results, because of the very compact form of such representation. Therefore, we will illustrate the use of a Smith chart with a number of examples. The theoretical basis of the Smith chart is given in most higher-level books on electromagnetics and microwave engineering.

A chart in its usual form, with some additional details whose use will be explained later, is shown in Fig. 18.16. As we have already mentioned, the Smith chart is used for plotting impedances and reflection coefficients. An impedance is plotted on the chart as a *normalized impedance,* defined as

$$z = \frac{Z}{Z_0} = \frac{R + \mathrm{j}X}{Z_0} = r + jx \qquad \text{(dimensionless)}. \tag{18.47}$$

(Definition of normalized load impedance)

The real part of the impedance, r, is defined by the complete circles on the chart. The imaginary part x is defined by the circular arcs. A normalized complex impedance, $z = r + \mathrm{j}x$, is defined by the intersection of a circle and an arc. For example, the circle labeled $r = 1$ in Fig. 18.16 intersects the arc labeled $\mathrm{j}x = \mathrm{j}1$ at the point labeled z, which corresponds to an impedance of $Z = z \cdot Z_0 = Z_0(1 + \mathrm{j}1)$. If $Z_0 = 50\ \Omega$, this corresponds to $Z = 50 + j50\ \Omega$.

The complex reflection coefficient corresponding to z is plotted in polar form, $\rho = |\rho|\angle\phi$, by drawing a straight line from the center of the chart to point z. The distance of the point z from the chart center gives $|\rho|$, which can be read off the scale on the horizontal line going through the center of the chart. The angle $\angle\phi$ is read off the (innermost) angular scale on the outer circle of the chart.

The basic properties of the Smith chart are the following:

- All points on the horizontal axis (labeled r) correspond to purely real impedances.

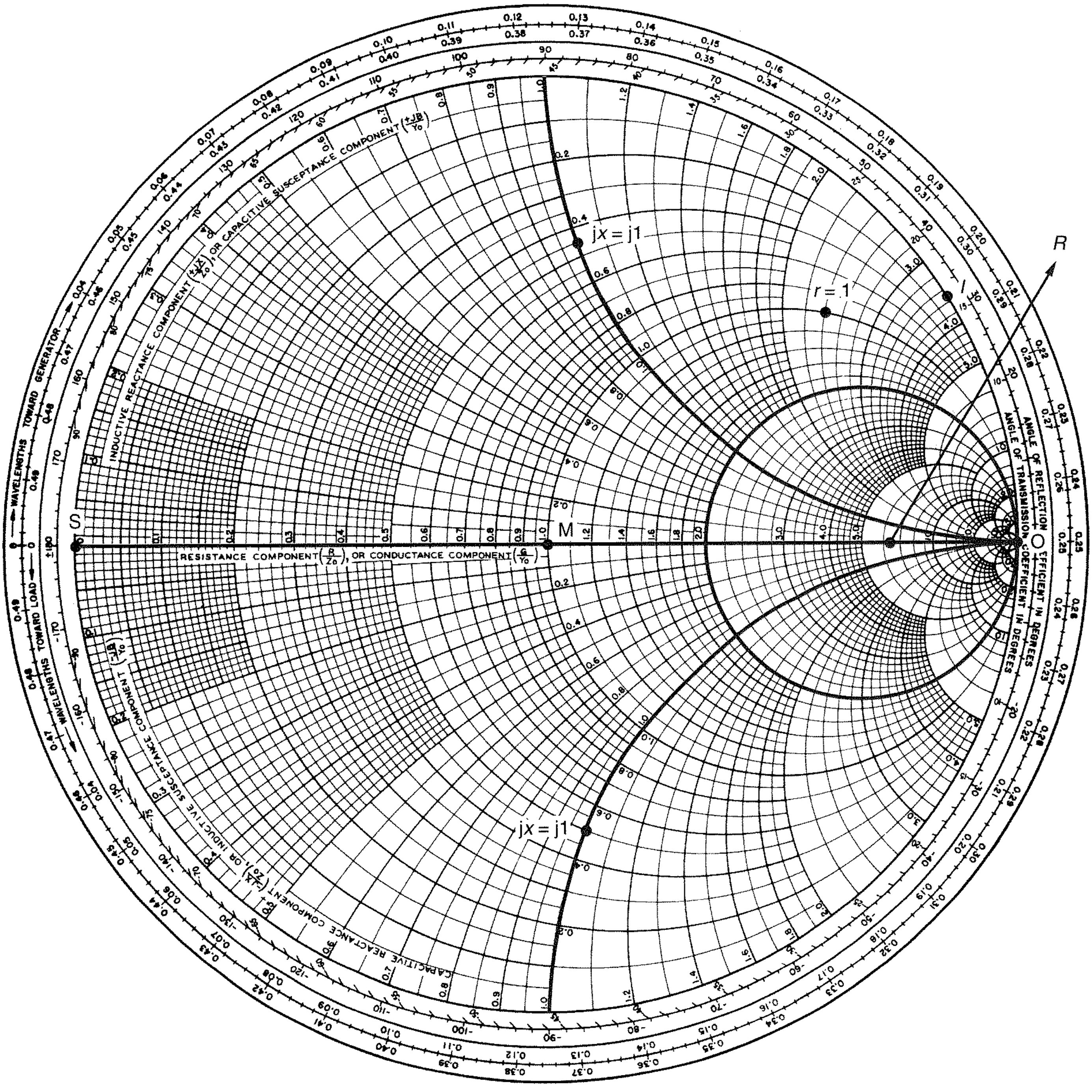


Figure 18.16 The Smith chart

- All points on the circle bounding the chart (labeled x) correspond to purely imaginary impedances.
- The rightmost point on the chart corresponds to an open circuit (labeled O).
- The leftmost point on the chart corresponds to a short circuit (labeled S).
- The center of the chart has a reflection coefficient equal to zero and corresponds to a matched load (labeled M).
- All points in the lower chart half correspond to loads with a capacitive (negative) reactance.
- All points in the upper chart half correspond to loads with an inductive (positive) reactance.
- The points on the circle labeled $r = 1$ correspond to loads with a real part equal to the adopted normalizing characteristic impedance, Z_0 (usually 50 Ω).
- The points on the arc labeled $\mathrm{j}x = \mathrm{j}1$ correspond to loads with a positive imaginary part equal to the adopted normalizing impedance, Z_0 (usually 50 Ω).
- The points on the arc labeled $\mathrm{j}x = -\mathrm{j}1$ correspond to loads with a negative imaginary part equal to the adopted normalizing impedance, Z_0 (usually 50 Ω).
- The length of the straight-line segment between the chart center and a point in the chart represents the magnitude of the reflection coefficient, the points on the boundary circle being of magnitude one. The angle scale on the chart boundary gives the reflection coefficient angle.
- The inside of the Smith chart corresponds to passive impedances (no generators). The magnitude of the reflection coefficient is smaller than or equal to unity inside the chart.
- The outside of the Smith chart (not plotted in Fig. 18.16) corresponds to impedances that give reflection coefficients of magnitudes larger than unity. This means that we can use the chart for active circuits, such as amplifiers and oscillators (i.e., generators). This external part of the chart is sometimes also plotted, and such a chart is referred to as an "extended Smith chart."

Example 18.15—Determination of the reflection coefficient at the load. The information that can be obtained directly from a Smith chart is the complex reflection coefficient at the load, $\rho = \rho(0)$, corresponding to a certain normalized load impedance $z = r + \mathrm{j}x$. For example, for $Z_0 = 50\ \Omega$ and $Z_L = (40 + \mathrm{j}60)\ \Omega$ we have $z = Z_L/Z_0 = 0.8 + \mathrm{j}1.2$, so $r = 0.8$ and $x = +1.2$. From Fig. 18.16, we find that $|\rho| \simeq 0.57$, and $\theta_\rho \simeq 66°$.

The magnitude of ρ is obtained by first measuring the distance of the point M from the chart center (point $\rho' = \rho'' = 0$), using a compass. Below the chart a linear scale is provided, which we can use to obtain the distance measured by the compass in terms of the chart radius (unit circle in the complex ρ plane). For easy reading of the angle θ_ρ, an angle scale marked "angle of reflection coefficient in degrees" is provided around the main chart. So we only need to draw a straight line from the origin through M to determine its intersection with the angle scale.

Example 18.16—Determination of the load impedance from the reflection coefficient. The converse problem of determining the normalized load impedance for a given (say, experimentally determined) reflection coefficient at the load is equally simple. For example, according to Fig. 18.16, for $\rho = 0.8e^{-j45°}$, that is, $|\rho| = 0.8$ and $\theta_\rho = -45°$, we obtain $z \simeq 0.75 - j2.20$. So if, say, $Z_0 = 60\ \Omega$, the load impedance is $Z_L = z \cdot Z_0 \simeq (45.0 - j132)\ \Omega$.

In addition to these two simple applications of the Smith chart, there are several more sophisticated ones. Perhaps the most important is that of determining the input impedance of a line of a given length and characteristic impedance, terminated by a given load impedance. This problem can be solved by means of Eq. (18.28), but an approximate solution using the Smith chart is quite simple. Let us consider the position along the line at a distance ζ from the load, as in Fig. 18.7. We first normalize $Z(\zeta)$ given in Eq. (18.25) with respect to Z_0:

$$z(\zeta) = \frac{Z(\zeta)}{Z_0} = \frac{1 + \rho(0)\,e^{-j2\beta\zeta}}{1 - \rho(0)\,e^{-j2\beta\zeta}}. \tag{18.48}$$

For $\zeta = 0$, $z(\zeta)$ is identical to $z = Z_L/Z_0$. But $z(\zeta)$ is of *exactly* the same form as z, except that $\rho(0)$ in z is replaced by $\rho(\zeta) = \rho(0)e^{-j2\beta\zeta}$:

$$z(\zeta) = \frac{1 + \rho(\zeta)}{1 - \rho(\zeta)} \qquad \rho(\zeta) = \rho(0)\,e^{-j2\beta\zeta}. \tag{18.49}$$

Because $|\rho(0)| \le 1$ and $|e^{-2j\beta\zeta}| = 1$, $|\rho(\zeta)| \le 1$. So the chart for $z(\zeta)$ and $\rho(\zeta)$ is exactly the same as for z and $\rho = \rho(0)$. Now, if we know z (for example, Z_L and Z_0), we can locate the point on the Smith chart that determines ρ directly. To obtain $z(\zeta)$, however, $\rho(\zeta) = \rho e^{-j2\beta\zeta}$ is needed rather than ρ. But multiplying a complex number by $e^{-j2\beta\zeta}$ implies changing its angle by $-2\beta\zeta$, leaving its magnitude constant, which means that we simply have to rotate ρ (corresponding to z) by $2\beta\zeta$, in the clockwise (negative) direction. Thus we obtain $\rho(\zeta)$ and can read $z(\zeta)$ directly from the chart.

Noting that

$$2\beta\zeta = 2\frac{2\pi}{\lambda}\zeta = \frac{2\zeta}{\lambda}2\pi, \tag{18.50}$$

it follows that an angle of rotation 2π corresponds to $\zeta = \lambda/2$. This must be so because we know from Eq. (18.28) that $z(\zeta) = z(\zeta + \lambda/2)$. To facilitate the rotation of ρ by the proper angle, an additional scale around the Smith chart is provided, with 0.5 (wavelengths) corresponding to one complete revolution around the unit circle $|\rho| = 1$. In the chart shown in Fig. 18.16 this wavelength scale is designated by "wavelengths toward generator." For some applications the same scale in the opposite (counterclockwise) direction is also useful and is designated by "wavelengths toward load" in Fig. 18.16.

So we have the following additional properties of the Smith chart related to the reflection coefficient $\rho(\zeta)$:

- Moving around the chart in the clockwise direction corresponds to moving down the line from the load toward the generator (the phase increases).

- Moving around the chart in the counterclockwise direction corresponds to moving down the line from the generator toward the load (the phase decreases).
- One full circle around the chart corresponds to 180 degrees of phase (or half a wavelength). This is because the phase of the reflection coefficient changes as $e^{2j\beta z}$, so everything repeats every half wavelength down a line.

Example 18.17—Input impedance of a line terminated in an arbitrary impedance. As an example, let us consider a line of characteristic impedance $Z_0 = 60\ \Omega$ and of length $\zeta = 0.40\,\lambda$ at the frequency used. Let us suppose that $Z_L = (90 - j60)\ \Omega$, and that we wish to determine the input impedance of the line thus terminated, using the Smith chart.

First, $z = Z_L/Z_0 = 1.5 - j1$, and we start with this value in the chart. This point has to be rotated in a clockwise direction by 0.4 units on the wavelength scale. Therefore we draw a straight line from the center of the chart through the point $z = 1.5 - j1$. The intersection of this line and the "wavelengths toward generator" scale is at 0.308 on the scale. We add 0.40 to this and get 0.708. This is 0.208 farther than point 0.000 on the scale. We draw a straight line from the 0.208 point of the wavelength scale toward the chart center and measure along this line the distance of the point $z = 1.5 - j1$ from the center. The point found in this way determines $z(\zeta) = z(0.4\lambda)$. From the chart we find that $z(0.4\lambda) \simeq (1.83 + j0.95)$. So the input impedance of a 0.4λ long 60-Ω line terminated with $Z_L = (90 - j60)\ \Omega$ is $Z(0.4\lambda) = Z_0 z(0.4\lambda) \simeq (110 + j57.0)\ \Omega$.

Example 18.18—Examples of matching by transmission-line segments. As already mentioned, at high frequencies (above about 100 MHz) it is not simple to make passive elements like resistors, capacitors, inductors, and transformers. For example, shunt (parallel) susceptance of interwinding capacitances of coils at these frequencies becomes pronounced and may completely distort the frequency behavior of the inductor. Shorted or open sections of transmission lines do not have this problem, so they are frequently used to replace reactive circuit elements in such cases. Such transmission-line reactive elements are often used for matching a high-frequency load to a desired impedance.

Another possible use of transmission-line segments for matching is as components for matching a load to a transmission line of a given characteristic impedance. Three principal ways of using transmission-line matching sections are sketched in Fig. 18.17.

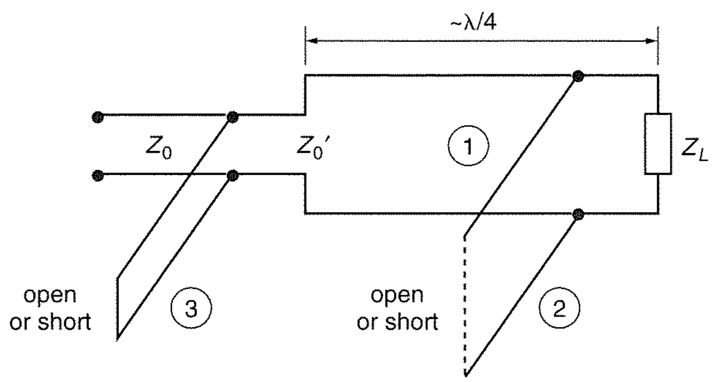


Figure 18.17 Three principal ways of using transmission-line sections for matching purposes: (1) a matching line section; (2) stub matching at the load; (3) stub matching along the line

Suppose the load impedance is $Z_L = R_L + jX_L$ and the transmission-line characteristic impedance is $Z_0 \neq R_L$. We can attempt to match the load to the line by following two steps:

1. Add a shorted line section in parallel to the load (a "stub" labeled 2 in Fig. 18.17), such that the admittance (and impedance) of the combination becomes real. Let the impedance of the combination be Z_L'.
2. Add a quarter-wavelength *matching line section* like that labeled 1 in Fig. 18.17 to match the load Z_L' to Z_0.

In some instances, a matching line section of length different than quarter wavelength may do the entire job when it transforms the load impedance to approximately Z_0 without the stub at the load.

Finally, it is possible to add another stub, labeled 3 in Fig. 18.17, at a convenient location along the line to improve matching. Although all such problems can be solved with ease by programmable calculators or computers, they can also be solved simply using the Smith chart. Several specific examples of matching are given in the problems at the end of the chapter.

These examples illustrate only some of the simplest applications of the Smith chart. Several others will be found in the problems at the end of the chapter. Applications of the Smith chart are much more diverse than these examples suggest. The chart can also be used for analyzing plane waves perpendicularly incident on a plane boundary surface, for analyzing lossy lines, and for many other problems. Even though we can use a computer to perform such tasks, the Smith chart is useful for presenting the results and getting an intuitive feel for what the analysis tells us.

Questions and problems: Q18.22 and Q18.23, P18.31 to P18.40

18.7 Chapter Summary

1. A transmission line is an electromagnetic structure, so strictly speaking it should be analyzed by means of the field equations. However, a very short segment of a line can be approximated by a simple circuit, and the complete line by a chain of such circuits. Consequently, transmission lines can also be analyzed using circuit theory.
2. The voltage and current along transmission lines have a property not encountered in "normal" circuits: they move along the line with a certain velocity. These moving voltages and currents are known as voltage and current waves.
3. If the line is infinitely long, the ratio of the voltage and current waves propagating in one direction along the line, at any point and at any instant, is constant. This constant is known as the line characteristic impedance, and it depends only on how the line is made (dimensions and materials).
4. For lossless air lines, the velocity of propagation of voltage and current waves along them equals the velocity of light in a vacuum, whereas in all other cases this velocity is smaller.

5. The input impedance of open- or short-circuited transmission-line segments is purely reactive. Therefore such segments are used at high frequencies as capacitors and inductors.
6. Transmission-line segments act as specific, frequency-dependent transformers of impedances of loads connected at their end. This, combined with adding appropriate segments (stubs) of shorted (or open) lines in parallel, can be used for matching a transmission line to the load it is terminated in.

QUESTIONS

Q18.1. Why is it not practically possible to obtain a coaxial cable of characteristic impedance $Z_0 = 500\ \Omega$? Can you have a two-wire line of this characteristic impedance?

Q18.2. Assume that a transmission line is made of two parallel, highly resistive wires. Can this line be analyzed using fundamental transmission-line equations? Explain.

Q18.3. A coaxial cable is filled with water. Does it represent a transmission line? Explain.

Q18.4. Two wires several wavelengths long serve as a connection between a generator and a receiver. The distance between the wires is small but not constant, varying as a smooth function of the coordinate along the line. Can you use the transmission-line equations for the analysis of this line? Explain.

Q18.5. Explain how you can obtain (1) a forward wave only; (2) a backward wave only along a transmission line.

Q18.6. Describe at least three ways of obtaining simultaneously a forward and a backward sinusoidal wave of the same amplitude along a transmission line.

Q18.7. Why can you replace an infinitely long end of a transmission line with a resistor of resistance equal to the line characteristic impedance?

Q18.8. Can a voltage (or a current) wave along a transmission line be described by the expression of the form $u(xy)$, where $u(xy)$ is a function of the product of the arguments $x = (t - z/c)$ and $y = (t + z/c)$? Explain.

Q18.9. Can we adopt the negative instead of positive value of the square root in Eq. (18.7) for the velocity of wave propagation along transmission lines? Explain.

Q18.10. Why must the exponent of the forward voltage and current waves in Eqs. (18.13) and (18.17) be negative? Why must those of the backward waves be positive?

Q18.11. Is the wavelength along an air line greater or less than that in the same line filled with a dielectric? What is the answer if the dielectric has relative permeability greater than one? Explain.

Q18.12. What are the SI units for the following quantities: (1) the attenuation constant (α), (2) the phase constant (β), (3) the reflection coefficient (ρ), and (4) the voltage standing-wave ratio (VSWR)?

Q18.13. What is the magnitude of the reflection coefficient, $|\rho|$, and of the VSWR, for which one half of the power of the incident wave is transferred to the load?

Q18.14. Why is the voltage at the termination Z of a transmission line with characteristic impedance Z_0 equal to $V = 2V_+Z/(Z + Z_0)$?

Q18.15. What are the input impedances to lossless lines of lengths $\lambda/4$ and $\lambda/2$, if they are (1) open-circuited or (2) short-circuited?

Q18.16. Can a resistive load of *any* resistance R be matched in practice to a transmission line of characteristic impedance Z_0? Explain.

Q18.17. The characteristic impedance of a lossy line in Eq. (18.32) is real if $R' = 0$ and $G' = 0$. Can it be real for some other relation between R', L', G', and C'? Explain.

Q18.18. Why could we not use simple transmission-line analysis when calculating the step response of an inductor, as in Fig. Q18.18?

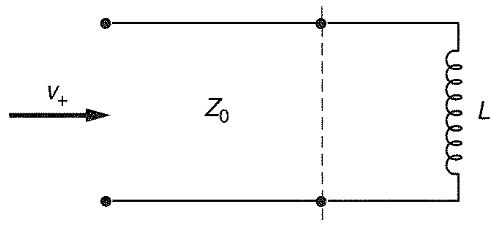


Figure Q18.18 Calculating the step response of an inductor

Q18.19. If you had a break in the dielectric of a cable causing a large shunt conductance, what do you expect to see reflected if you excite the cable with a short pulse (practical delta function)?

Q18.20. If you had a break in the outer conductor of a cable, causing a large series resistance, what do you expect to see reflected if you excite the cable with a short pulse (imperfect delta function)?

Q18.21. What do the reflected waves off a series inductor and shunt capacitor in the middle of a transmission line look like for a short pulse excitation, assuming that $\omega L \gg Z_0$ and $\omega C \gg 1/Z_0$?

Q18.22. Using the Smith chart, determine the complex reflection coefficient on a 60-Ω line if it is terminated by (1) 80 Ω, (2) $(30 - j40)$ Ω, or (3) $(40 + j90)$ Ω.

Q18.23. Using the Smith chart, determine the terminating impedance of a 70-Ω line if it was found experimentally that the complex voltage reflection coefficient is (1) 0.8, (2) $0.2e^{-j\pi/4}$, or (3) $0.5e^{j\pi/3}$.

PROBLEMS

P18.1. Given a high-frequency RG-55/U coaxial cable with $a = 0.5$ mm, $b = 2.95$ mm, $\epsilon_r = 2.25$ (polyethylene), and $\mu_r = 1$, find the values for the capacitance and inductance per unit length of the cable.

P18.2. Assume that the coaxial cable from problem P18.1 is not lossless but that the losses are small, resulting in an attenuation constant in decibels per meter at 10 GHz of $\alpha = 0.5$ dB/m. Assuming the dielectric in the cable to be perfect, find the resistance per unit length that causes the losses in the conductors.

P18.3. The distance d between wires of a lossless two-wire line is a smooth, slowly varying function of the coordinate z along the line so that the line capacitance and inductance per unit length, L' and C', are also smooth functions of z, $L' = L'(z)$, and $C' = C'(z)$. Derive the transmission-line equations for such a nonuniform transmission line. Check if these equations become the transmission-line equations (18.4) for $L'(z)$ and $C'(z)$ constant.

P18.4. Using circuit theory, analyze approximately a matched, lossless, air-filled coaxial transmission line of length $l = \lambda$ and conductor radii $a = 1$ mm and $b = 3$ mm as a connection of n cells of the type in Fig. 18.3b, for $n = 1, 2, \ldots, 20$. Such a circuit-theory approximation to transmission lines is known as an *artificial transmission line.* Note that an artificial transmission line can be analyzed as a simple ladder network. Assume the artificial line to be terminated in the actual characteristic impedance, and compare current in series-concentrated inductive elements and voltage across parallel concentrated capacitive elements with exact results. Solve the problem so that you can vary L', C', and n.

P18.5. Noting that $c = 1/\sqrt{\epsilon\mu}$ for all transmission lines in Table 18.1, prove that for these lines the inductance per unit length and the characteristic impedance of a lossless transmission line can be expressed in terms of c and C'.

P18.6. Express $V(z)$ in Eq. (18.13) and $I(z)$ in Eq. (18.17) for lossless lines in terms of the sending-end voltage $V(0)$ and sending-end current $I(0)$.

P18.7. Prove that it is possible to obtain the characteristic impedance of any lossless line by measuring the input impedance of a section of the line when it is open-circuited, and when it is short-circuited.

P18.8. A lossless line of characteristic impedance Z_{01} and length l_1 is terminated in an impedance Z_L. The line serves as a load for another lossless line of characteristic impedance Z_{02} and length l_2. The dielectric in both lines is air and the angular frequency of the current is ω. Determine general expressions for the input impedance of the second line, the reflection coefficient in both lines, and the voltage standing-wave ratio in both lines.

P18.9. A short and then an open load are connected to a 50-Ω transmission line at $z = 0$. Make a plot of the impedance, normalized voltage ("normalized" means that you divide the voltage by its maximal value to get a maximum normalized voltage of 1), and normalized current along the line up to $z = -3\lambda/2$ for the two cases.

P18.10. A *lumped* capacitor is inserted into a transmission-line section, as shown in Fig. P18.10. Find the reflection coefficient for a wave incident from the left. Assume the line is terminated to the right so that there is no reflection off the end of the line. Find a simplified expression that applies when C is small. The characteristic impedance of the line is Z_0.

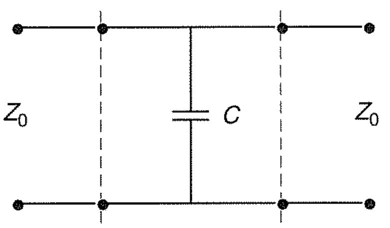


Figure P18.10 A shunt capacitor in a line

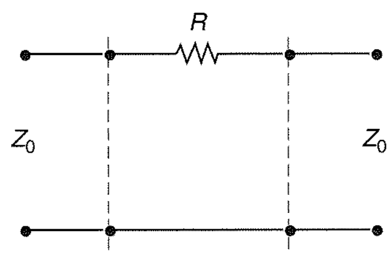


Figure P18.11 A series resistor in a line

P18.11. Repeat problem P18.10 assuming that a lumped resistor is inserted into a transmission-line section as shown in Fig. P18.11. Find a simplified expression that applies when R is small.

P18.12. Repeat problem P18.10 assuming that a lumped inductor is inserted into a transmission-line section as shown in Fig. P18.12. Find a simplified expression that applies when L is small.

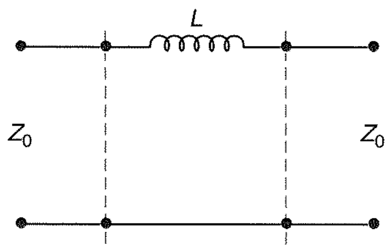


Figure P18.12 A series coil in a line

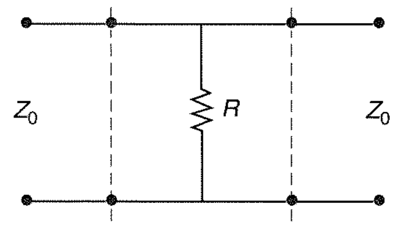


Figure P18.13 A shunt resistor in a line

P18.13. Repeat problem P18.10 assuming that a lumped resistor is inserted into a transmission-line section as shown in Fig. P18.13. Find a simplified expression that applies when R is large.

P18.14. A 50-Ω transmission line needs to be connected to a 100-Ω load. The setup is used at 1 GHz. What* would you connect between the line and the load to have no reflected voltage on the line? * LOSSLESS ELEMENTS

P18.15. In problem P18.14, the load is a 100-Ω resistor but the leads are long and represent a 2 nH inductor in series with the resistor. How would you get rid of the reflected voltage on the line in this case?

P18.16. Find the transmission coefficient for the transmission line in Fig. P18.10.

P18.17. Find the reflection and transmission coefficients for the transmission line in Fig. P18.17. Because the reflection coefficient is defined by voltage, the power is given by its square. What are the reflected and transmitted power equal to? Does the power balance make sense?

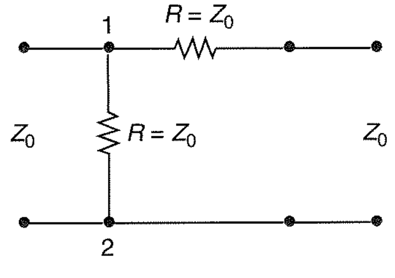


Figure P18.17 Two resistors in a line

P18.18. Derive the normalized input impedance (i.e., the impedance divided by Z_0) for a section of line that is $n\lambda/8$ long and shorted at the other end, for $n = 1, 2, 3, 4$, and 5. Plot the impedance as a function of electrical line length from the load (length measured in wavelengths along the line).

P18.19. Repeat problem P18.18 for an open-ended line.

P18.20. Find the total current and voltage at the beginning of a $\lambda/4$ shorted transmission line of characteristic impedance Z_0. What circuit element does this line look like? Plot the total current and voltage as a function of electrical line length from the load (length measured in wavelengths along the line).

P18.21. Repeat the previous problem for an open-ended line.

P18.22. Find the reflection and transmission coefficients for an ideal $n : 1$ transformer, as in Fig. P18.22, where n is the voltage transformation ratio.

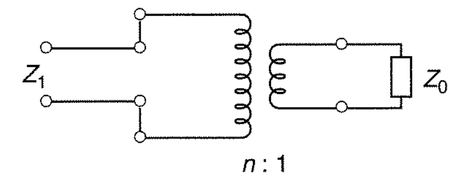


Figure P18.22 An ideal transformer

P18.23. Find the input impedance for the circuit in Fig. P18.23.

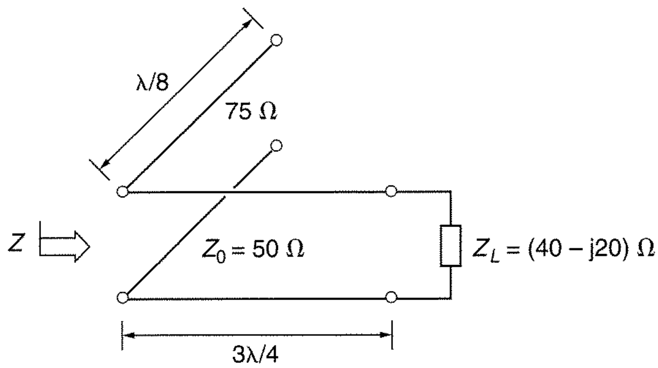


Figure P18.23 Impedance of a line with a shunt stub

P18.24. A coaxial transmission line with a characteristic impedance of 150 Ω is 2 cm long and is terminated in a load impedance of $Z = 75 + j150\ \Omega$. The dielectric in the line has a relative permittivity of $\epsilon_r = 2.56$. Find the input impedance and VSWR on the line at $f = 3$ GHz.

P18.25. Match a 25-Ω load to a 50-Ω line using (1) a single quarter-wave section of line, or (2) two quarter-wave line sections.

P18.26. Match a purely capacitive load, $C = 10$ pF, to a 50-Ω line at 1 GHz. How many different ways can you think of doing this?

P18.27. Calculate and plot magnitude and phase of $\rho(f)$ between 1 and 3 GHz for a 50-Ω open transmission line that is $\lambda/4$ long at 2 GHz.

P18.28. If you had a cable like the one in problem P18.2 spanning the Atlantic and you sent a continuous signal of 1-MW power from the United States to England, how much power approximately would you get in England? (Look up the approximate distance across the Atlantic in an atlas if you need to.)

P18.29. A printed-circuit board trace in a digital circuit is excited by a voltage $v(t)$, as in Fig. P18.29. Derive an equation for the coupled (cross-talk) signal on an adjacent line, $v_c(t)$, assuming the adjacent line is connected to a load at one end and a scope (infinite impedance) at the other end so that no current flows through it. (*Hint:* the coupling is capacitive and you can approximate it by a capacitor between the two traces and use circuit theory.)

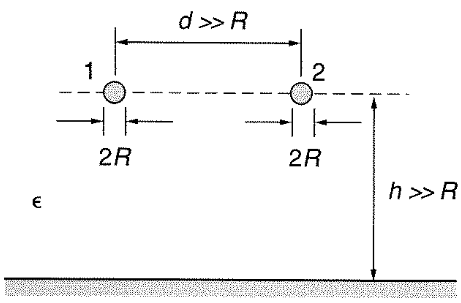


Figure P18.29 An example of two coupled lines

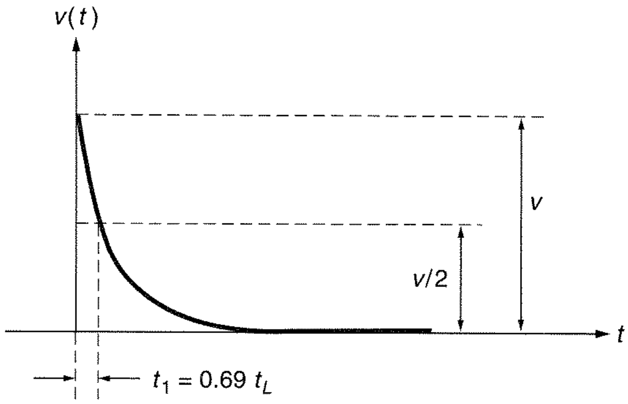


Figure P18.30 Measuring the reflected wave from a complex load

P18.30. Derive the expression $t_1 = 0.69\, t_L$ discussed in Example 18.14. This expression shows a practical way to measure the time constant of the reflected wave for the case of complex loads, as in Fig. P18.30.

P18.31. Trace the procedure for solving problem P18.8 by means of the Smith chart.

P18.32. The reciprocals of complex numbers can be determined easily from the Smith chart. Starting with Eq. (18.49), deduce how this can be done.

P18.33. A fixed, known complex impedance Z_L is to be connected to a lossless line having a characteristic impedance Z_0. Show that it is possible to eliminate the reflected wave along the line if an appropriate length of the same line, assumed to be short-circuited, is connected at an appropriate place on the line near Z_L (see Fig. P18.33).

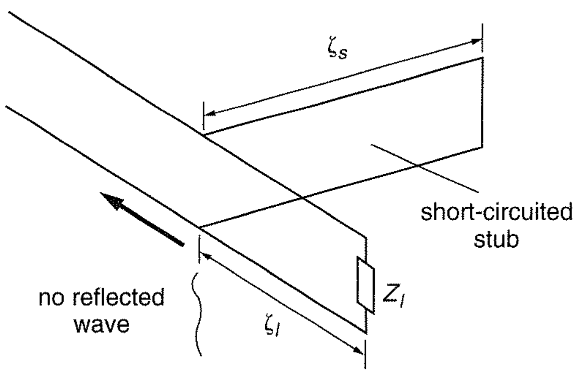


Figure P18.33 A configuration for matching a load to a transmission line

P18.34. What circuit element corresponds to the point on the Smith chart that is defined by the intersection of the circle $r = 1$ and the arc $jx = j1.2$ at 1 GHz, if the normalizing impedance is 50 Ω?

P18.35. What circuit element corresponds to the point on the Smith chart that is defined by the intersection of the circle $r = 1$ and the arc $jx = -j0.4$ at 500 MHz, if the normalizing impedance is 50 Ω?

P18.36. At the load of a terminated transmission line of characteristic impedance $Z_0 = 100\ \Omega$, the reflection coefficient is $\rho = 0.56 + j0.215$. What is the load impedance?

P18.37. A 50-Ω line is terminated in a load impedance of $Z = 80 - j40\ \Omega$. Find the reflection coefficient of the load and the VSWR.

P18.38. A 50-Ω slotted line measurement (see Example 18.10) was done by first placing a short at the place of the unknown load. This results in a large VSWR on the line with sharply defined voltage minima. On an arbitrarily positioned scale along the air-filled coaxial line, the voltage minima are observed at $z_s = 0.1, 1.1,$ and 2.1 cm. The short is then replaced by the unknown load, the VSWR is measured to be 2, and the voltage minima (not as sharp as with the short termination) are found at $z = 0.61, 1.61,$ and 2.61 cm. Use the Smith chart to find the complex impedance of the load. Explain all your steps.

P18.39. Use a shorted parallel stub to match a 200-Ω load to a 50-Ω transmission line. Include a Smith chart plot with step-by-step explanations.

P18.40. A load consists of a 100-Ω resistor in series with a 10-nH inductor at 1 GHz. Use an open single stub to match it to a 50-Ω line.

19

Maxwell's Equations

19.1 Introduction

This chapter is devoted to the extension of the equations we have derived so far to the most general equations for the electromagnetic field. These general equations are known as *Maxwell's equations*. Any engineering problem that includes electromagnetic fields is solved starting from these equations, although in some instances the application may not be obvious. For example, ac circuits are in fact described by an approximation of Maxwell's equations valid for specific fields existing in such circuits.

We will see that Maxwell's equations can be written in two forms: integral and differential. We will also see that numerous general conclusions follow from these equations. For example, the problem of energy transfer by means of an electromagnetic field can be understood and solved only if we start from Maxwell's equations and derive what is known as *Poynting's theorem*. General boundary conditions will also be derived. Finally, we will show that in many important instances electromagnetic field vectors can be derived from auxiliary functions, known as *potentials*.

This is probably the most important chapter in the entire book. It unifies all the concepts we have studied so far. It also adds the concept of displacement current that couples Gauss', Ampère's, and Faraday's laws with the current continuity and conservation of magnetic flux equations. Maxwell's equations enable us to solve many practical engineering problems that deal with electromagnetic fields.

19.2 Displacement Current

We know from Faraday's law that a time-varying magnetic field is always accompanied by a time-varying electric (induced) field. This also means that a time-varying electric field is accompanied by a time-varying magnetic field. We have learned so far that sources of a magnetic field are electric currents. From the preceding inverse statement, we can say that a *time-varying magnetic field* is not caused solely by time-varying electric currents but also by a *time-varying electric field.*

This conclusion is the essence of Maxwell's contribution to the theory of electricity and magnetism. To stress that this time-varying electric field is the source of a magnetic field, as is a current, a quantity tightly connected with time variation of the electric field is termed the *displacement current,* even though it is not a current in the usual sense.

Consider a circuit containing an air-filled parallel-plate capacitor and with time-varying current flowing through it, as sketched in Fig. 19.1. Imagine two surfaces, S_1 and S_2, shown in the figure. The surface S_1 intersects a part of the wire. The surface S_2 intersects *one electrode of the capacitor only.*

If we apply the current continuity equation [Eq. (10.14)],

$$\int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_v \rho \, \mathrm{d}v, \tag{10.14}$$

to S_1, we find that it is satisfied because a current $i(t)$ enters the surface, the same current leaves the surface, and there is no charge accumulation along the enclosed wire segment. If, however, we apply Eq. (10.14) to S_2, we are working with a current entering S_2, but no current leaving this surface. Instead, we have an increase of charge in S_2 such that Eq. (10.14) is satisfied.

Suppose we wish to express the general equation for current continuity in Eq. (10.14) as a surface integral on the left-hand side, and a zero on the right-hand side. This can be done easily if we recall Gauss' law in Eq. (7.20). The volume integral

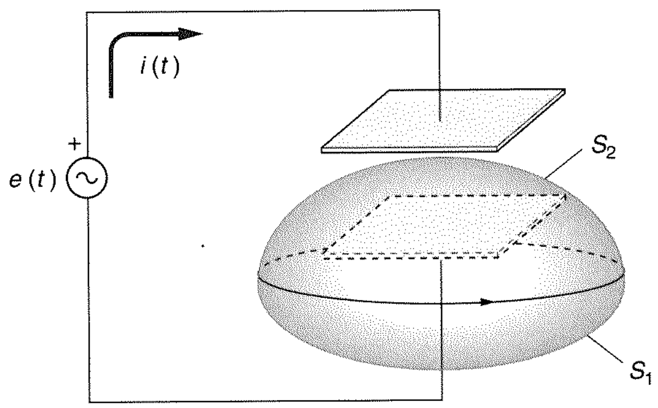


Figure 19.1 Circuit containing an air-filled capacitor and with time-varying current flowing through it

in Eq. (10.14) is precisely $Q_{\text{free in } S}$ in Eq. (7.20), except that this charge now varies in time. So instead of Eq. (10.14) we can write an equivalent equation

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S}. \tag{19.1}$$

If we assume that the surface S is not varying in time, the time derivative can be introduced under the integral sign to act on the vector $\mathbf{D}$ only. Noting that the surface integrals on the two sides of the equation refer to the same surface, we can write Eq. (19.1) in the form

$$\oint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathrm{d}\mathbf{S} = 0. \tag{19.2}$$

We have arrived at an interesting conclusion: the flux through a closed surface of the vector sum $(\mathbf{J} + \partial\mathbf{D}/\partial t)$ is always zero. The expression $\partial\mathbf{D}/\partial t$ has the dimension of current density. It is therefore termed the *displacement current density.*

We know that if the flux of a vector function through any closed surface is zero, then the flux of that vector through all *open* surfaces bounded by the same contour is the same. Consider the contour C indicated in Fig. 19.1, and two surfaces bounded by the contour, S_1 and S_2. The surface S_1 cuts the wire, so the flux of $(\mathbf{J} + \partial\mathbf{D}/\partial t)$ through it is simply $i(t)$. The surface S_2 passes between the capacitor electrodes and does not cut the wire. Therefore, there is no current through that surface, and the flux of $(\mathbf{J} + \partial\mathbf{D}/\partial t)$ equals that of vector $\partial\mathbf{D}/\partial t$ through it. We will now show that these two integrals are equal.

Open surfaces S_1 and S_2 make the *closed* surface S. The flux of $(\mathbf{J} + \partial\mathbf{D}/\partial t)$ through S is calculated with respect to the *outward* unit vector normal to S. Recall the right-hand rule of defining a unit vector normal to a surface defined by a contour. The flux through the part S_1 of S is calculated with respect to the outward normal, but the flux through the part S_2 of S should be calculated with respect to the opposite normal. Consequently Eq. (19.2) yields

$$\int_{S_1} \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \int_{S_2} \frac{\partial \mathbf{D}}{\partial t} \cdot \mathrm{d}\mathbf{S}.$$

This could be interpreted as if the conductive current in the metallic wire continues between the capacitor plates in the form of the displacement current. In other words, if we consider a time-varying conductive current only, it has sources and sinks. The *total current* (the sum of conductive and displacement currents), however, does not have sources and sinks, but rather closes onto itself, as a dc current. With this in mind, Maxwell postulated that in time-varying fields the source of the magnetic field is not solely the conductive current, but rather the total current, the density of which is $(\mathbf{J} + \partial\mathbf{D}/\partial t)$.

From Ampère's law we know that the line integral of the magnetic field intensity vector, $\mathbf{H}$, along a closed contour equals the current through any surface defined by the contour. From the reasoning we just did, we see that it is also equal to the flux of the *displacement current* through the contour (i.e., through a surface bounded by the contour):

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = \int_{S_2} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}. \tag{19.3}$$

This equation tells us that if we wish Ampère's law to be valid for time-varying currents, we must replace $\mathbf{J}$ in it by $(\mathbf{J} + \partial\mathbf{D}/\partial t)$. So the generalized Ampère's law has the form

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{S} \qquad \text{(generalized Ampère's law).} \tag{19.4}$$

This is the fundamental contribution of Maxwell, which can be interpreted as follows: *the displacement current produces a magnetic field according to the same law as "normal" current.* We will see that the addition of the displacement current density in Ampère's law has far-reaching consequences. For example, without it we cannot explain the existence of electromagnetic waves. An electromagnetic wave is a moving electromagnetic field that, once created by charges and currents, continues to exist with no connection whatsoever to the charges and currents that created it.

Example 19.1—Displacement current density in dielectrics and in a vacuum. Since $\mathbf{D} = (\epsilon_0 \mathbf{E} + \mathbf{P})$, the displacement current density, $\partial\mathbf{D}/\partial t$, can be written in the form

$$\frac{\partial \mathbf{D}}{\partial t} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t}.$$

We know that the polarization vector, $\mathbf{P}$, represents the transfer of *real* charge per unit area normal to vector $\mathbf{P}$. It is measured in $\mathrm{C/m^2}$. Therefore the expression $\partial\mathbf{P}/\partial t$ is in $\mathrm{A/m^2}$, and represents a *real* current density, resulting from the motion of the polarization charges. This part of the displacement current density is termed the *displacement current density in the dielectric*, or frequently, the *density of polarization current.*

The other part of the displacement current density, $\epsilon_0\, \partial\mathbf{E}/\partial t$, is measured in the same units, $\mathrm{A/m^2}$, but it does not represent any motion of real charges. This is the *displacement current density in a vacuum.* This part of the displacement current can be very misleading, however, if one does not keep in mind its physical meaning: the time-varying electric field is the source of the time-varying magnetic field. In other words, as far as the source of the magnetic field is concerned, $\epsilon_0\, \partial\mathbf{E}/\partial t$ is completely equivalent to an electric current of the same density, although it does *not* represent any real motion of electric charges.

Questions and problems: Q19.1, P19.1 to P19.3

19.3 Maxwell's Equations in Integral Form

We are now ready to formulate the general equations of the electromagnetic field in integral form. In fact, what we need to do is to review all the equations we have postulated or derived, and make sure they are not contradictory. If they do not contradict each other, we can, following Maxwell, *postulate* them to be true for all electromagnetic fields. The sole criterion for the validity of these equations is, of course, experiment. Ever since Maxwell postulated in the 1860s the equations that bear his name, no experimental evidence has indicated even the slightest disagreement with these equations.

We now write the integral form of the four most general equations we have derived. With no particular reason except that it is customary, we start with Faraday's law in Eq. (14.6) for a fixed contour, so that the time derivative can be introduced under the integral sign. This equation is usually followed by the generalized Ampère's law. Gauss' law in Eq. (7.20), in which the total free charge enclosed by a closed surface is replaced by a volume integral of the charge density, is the third equation. The last equation is the law of conservation of magnetic flux.

Thus Maxwell's equations in integral form are

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}, \tag{19.5}$$

[Faraday's law for a fixed contour, Eq. (14.6) = Maxwell's first equation]

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathrm{d}\mathbf{S}, \tag{19.6}$$

[Generalized Ampère's law, Eq. (19.4) = Maxwell's second equation]

$$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \int_v \rho \, \mathrm{d}v, \tag{19.7}$$

[Gauss' law, Eq. (7.20) = Maxwell's third equation]

$$\oint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0. \tag{19.8}$$

[Law of conservation of magnetic flux, Eq. (12.11) = Maxwell's fourth equation]

Finally, we add to these equations the current continuity equation,

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\int_v \frac{\partial \rho}{\partial t} \mathrm{d}v. \tag{19.9}$$

[Current continuity equation, Eq. (10.15) = law of conservation of electric charge]

These equations can be paraphrased as follows. Equation (19.5) tells us that a time-varying magnetic field is a source of an (evidently time-varying) electric field. Equation (19.6) states that the sources of a magnetic field are electric currents *and a time-varying electric field*. According to Eq. (19.7), the only source that produces a nonzero flux through a closed surface of the electric displacement vector are free electric charges. Finally, Eq. (19.8) can be interpreted as stating that no analogue of free electric charges exists for a magnetic field. Equation (19.9) is not a field equation, but the law of conservation of electric charge must be satisfied by all real sources of the electromagnetic field.

If we try to find any logical deficiencies in these equations, we will see that there are none, in spite of the fact that they have been derived separately, for specific types of fields. For this reason we *postulate* that these equations are always valid and represent the equations of the general electromagnetic field.

There are numerous applications of Maxwell's equations in integral form. One group of applications relates to the derivation of some general conclusions about electromagnetic fields. One of the most important applications of this type is the derivation of general boundary conditions.

Example 19.2—General boundary conditions. We know that boundary conditions are relations between values of any field quantity at two close points on the two sides of a surface between two different media. For the four basic field vectors, **E**, **H**, **D**, and **B**, they are but special forms of the integral Maxwell's equations (19.5) to (19.8).

In order to be able to derive them in the most usual form, we need to consider also the possibility of surface currents. We shall see in the next chapter that at high frequencies, currents in good conductors are distributed essentially over conductor surfaces, and are practically surface currents. This is why we need to include surface currents in boundary conditions, and to specialize the boundary conditions at the surface of a "perfect" conductor.

If one of the two media on two sides of a boundary surface is a perfect conductor, let it be medium 2. Inside a perfect conductor *there can be no electric field* (it would result in infinite current density). We know that a time-varying electric field is accompanied by a time-varying magnetic field. Therefore, inside a perfect conductor, *there can also be no time-varying magnetic field*.

We derived boundary conditions in the electrostatic field starting, in fact, from Eq. (19.5) (with zero right-hand side), and from Eq. (19.7). Does the nonzero right-hand side in Eq. (19.5) change anything? Recall that in the derivation of the boundary condition for the tangential components of vector **E** we assumed that the contour was infinitely narrow. Therefore, the flux of vector $\partial\mathbf{B}/\partial t$ is zero also if we start from Eq. (19.5). On the other hand, Eq. (19.7) is the same as Gauss' law in electrostatics. So we conclude that in *any* electromagnetic field, on the two sides of *any* boundary surface, both electrostatic conditions, Eqs. (7.26) and (7.27), remain valid. If one of the media is a perfect conductor, these equations take the forms that are also valid in electrostatics (but for any, not necessarily perfect, conductor):

$$\mathbf{E}_{1\,\mathrm{tang}} = \mathbf{E}_{2\,\mathrm{tang}}, \qquad \text{or} \qquad \mathbf{E}_{\mathrm{tang}} = 0 \quad \text{on surface of perfect conductor,} \tag{19.10}$$

(General boundary condition for tangential components of **E***)*

and

$$\mathbf{D}_{1\,\mathrm{norm}} - \mathbf{D}_{2\,\mathrm{norm}} = \sigma, \quad \text{or} \quad \mathbf{D}_{\mathrm{norm}} = \sigma \quad \text{on surface of perfect conductor.} \tag{19.11}$$

(General boundary condition for normal components of **D***)*

The condition for the tangential components of the magnetic field intensity vector was derived from Ampère's law, applied to an infinitely narrow contour. Displacement current through such a contour is zero, but conduction current may be nonzero if there is a surface current on the boundary.

Consider Fig. 19.2 and assume the surface-current density vector $\mathbf{J}_s$ to be locally in the y direction. The magnetic field of these currents is then in the x direction, as indicated. The current through the narrow rectangular contour in the figure, which is in the x-z plane, i.e., normal to $\mathbf{J}_s$, is $J_s\,\Delta l$. The integral of vector $\mathbf{H}$ around the contour is $(H_{1x} - H_{2x})\Delta l$. Noting that the unit vector normal to the boundary is directed into medium 1, from the integral form of Ampère's law we obtain

$$\mathbf{H}_{1\,\mathrm{tang}} - \mathbf{H}_{2\,\mathrm{tang}} = \mathbf{J}_s \times \mathbf{n}, \quad \text{or} \quad \mathbf{H}_{\mathrm{tang}} = \mathbf{J}_s \times \mathbf{n} \quad \text{on surface of perfect conductor.} \tag{19.12}$$

(General boundary condition for tangential components of **H***—see Fig. 19.2)*

The condition for the normal components of vector **B**, Eq. (13.8), also remains the same, since it was derived from the law of conservation of magnetic flux, Eq. (19.8). If medium 2 is a

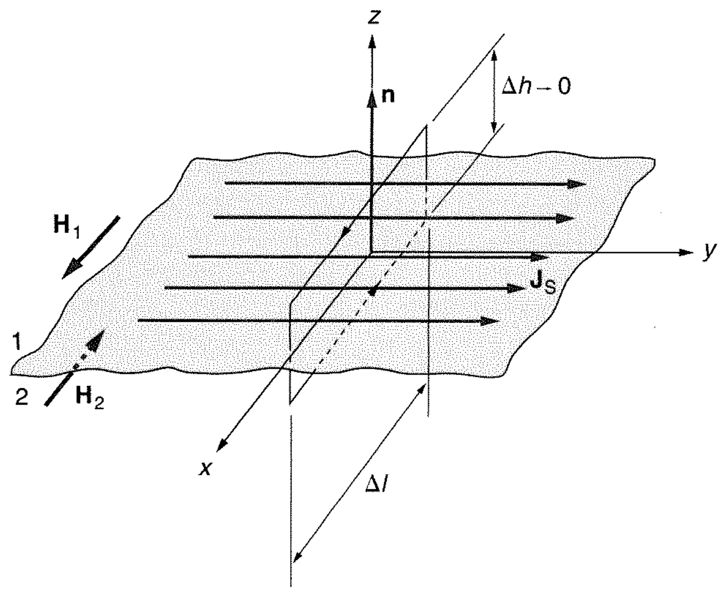


Figure 19.2 Surface current on boundary between two media. The magnetic field due to this current is locally normal to the surface-current density vector.

perfect conductor, no field is there, and we have

$$\mathbf{B}_{1\text{norm}} = \mathbf{B}_{2\text{norm}}, \qquad \text{or} \qquad \mathbf{B}_{\text{norm}} = 0 \qquad \text{on surface of perfect conductor.} \tag{19.13}$$

(General boundary condition for normal components of **B***)*

It is worthwhile repeating what we need boundary conditions for. These equations are, in fact, Maxwell's equations specialized to boundary surfaces. Therefore in a medium consisting of several bodies of different properties, the field transition from one body to the adjacent body, through a boundary surface, *must* be as required by the boundary conditions. If this were not so, such an electromagnetic field could not be a real field, because it would not satisfy the field equations *everywhere*.

Questions and problems: Q19.2 to Q19.4

19.4 Maxwell's Equations in Differential Form

Maxwell's equations in integral form, Eqs. (19.5) to (19.8), can be transformed into a set of differential equations, known as Maxwell's equations in differential form. They can easily be obtained from the integral forms by applying the Stokes's and the divergence theorems of vector analysis. (If necessary, consult Appendix 1, Sections A1.4.6 and A1.4.7, to refresh your knowledge of these two theorems before proceeding further.)

Consider the first and second Maxwell's equations. By Stokes's theorem, the line integral of **E** in the first equation can be transformed into the flux of the vector curl $\mathbf{E} = \nabla \times \mathbf{E}$ through *any surface bounded by the contour* C. Therefore, instead of Eq. (19.5) we can write the equivalent equation

$$\int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \tag{19.14}$$

The two surfaces have the same boundary contour, but they may or may not be the same. If they are the same, *any* surface bounded by C can be chosen. Such an equation can be satisfied, however, only if the integrands in the two integrals are equal at all points, that is, if $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$. Maxwell's second equation can be transformed in exactly the same manner.

The third and fourth Maxwell's equations can also be written in an equivalent form from which we can obtain their differential counterparts. For example, apply the divergence theorem to the left side of Eq. (19.7), to obtain

$$\int_v \nabla \cdot \mathbf{D}\, dv = \int_v \rho\, dv. \tag{19.15}$$

Note that the domains v on the two sides of the equation are the same. This equation can be satisfied for any domain v only if the integrands are equal at all points, that is, if $\nabla \cdot \mathbf{D} = \rho$. In the same manner, we can transform the fourth Maxwell's equation.

From these derivations, Maxwell's equations in differential form read

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{19.16}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{19.17}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{19.18}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{19.19}$$

(Maxwell's equations in differential form)

Let us add here the current continuity equation in differential form, obtained in the same manner as the last two equations:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{19.20}$$

(Current continuity equation in differential form)

To these equations (as well as to their integral counterparts) it is necessary to add the relationships between vectors (1) **D**, **E**, and **P**; (2) **B**, **H**, and **M**; and (3) **J** and **E**:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \mathbf{P} = \mathbf{P}(\mathbf{E}) \tag{19.21}$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \qquad \mathbf{M} = \mathbf{M}(\mathbf{B}) \tag{19.22}$$

$$\mathbf{J} = \mathbf{J}(\mathbf{E}). \tag{19.23}$$

For linear media, which are practically the only media we consider in this text, we have

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H} \qquad \mathbf{J} = \sigma \mathbf{E}. \tag{19.24}$$

(Constitutive relations for linear media)

Equations (19.21) to (19.23) and (19.24) are often referred to as the *constitutive relations*.

The differential Maxwell's equations are used for solving many electromagnetic problems. There are modern, extremely powerful numerical methods for solving these equations directly. As computers evolve, increasingly complex electromagnetic problems can be solved numerically in a reasonable amount of time.

It is interesting that the fourth equation follows from the first. Indeed, if we take the divergence of the left-hand and right-hand sides of Eq. (19.16), the left-hand side is equal to zero, because $\nabla \cdot (\nabla \times \mathbf{F})$ (divergence of the curl) of any vector functions $\mathbf{F}$ is identically zero. Therefore, $\partial(\nabla \cdot \mathbf{B})/\partial t = 0$, which means that $\mathbf{B}$ does not depend on time. So if at *any time in the past* $\mathbf{B} = 0$ (and therefore also $\nabla \cdot \mathbf{B}$), which certainly was the case, then $\nabla \cdot \mathbf{B} = 0$ generally. In a similar manner one can prove that with the aid of the current continuity equation, the third Maxwell's equation follows from the second.

Questions and problems: Q19.5 to Q19.20

19.5 Maxwell's Equations in Complex (Phasor) Form

Maxwell's equations in differential form, Eqs. (19.16) to (19.19), are partial differential equations with three space coordinates and time as independent variables. Very often, the time variation of the sources is sinusoidal. *If the medium is also linear*, we know that all quantities vary in time sinusoidally. It is then possible to eliminate time from the equations, and thus simplify them. The procedure is very similar to that in circuit theory. The difference is that here we have *vector quantities* in addition to scalar quantities, and that these quantities are functions of space coordinates.

Quantities varying sinusoidally in time are often called *time-harmonic*. Their time dependence can be written in the form $\cos(\omega t + \varphi)$, where $\omega = 2\pi f$ is the angular frequency (in radians per second), f is the frequency (in Hz), and φ is the initial phase. In general, φ is a function of coordinates. In the case of vector quantities, the initial phases of the three vector components at a point can be different.

Example 19.3—Complex field quantities. To understand the logic of complex representation of time-harmonic vectors, consider the x component of a time-harmonic electric field of angular frequency ω:

$$E_x(x, y, z, t) = E_{x\,\max}(x, y, z)\cos[\omega t + \varphi(x, y, z)]. \tag{19.25}$$

Euler's identity allows us to express the cosine as a sum of complex exponentials:

$$\cos(\omega t + \varphi) = \frac{e^{j\omega t}e^{j\varphi} + e^{-j\omega t}e^{-j\varphi}}{2}, \tag{19.26}$$

where $j = \sqrt{-1}$ is the imaginary unit.

The time derivative of $E_x(x, y, z, t)$ can be written as

$$\frac{\partial}{\partial t}E_x(x, y, z, t) = E_{x\,\max}(x, y, z)\,\frac{1}{2}\left(j\omega e^{j\omega t}e^{j\varphi(x,y,z)} - j\omega e^{-j\omega t}e^{-j\varphi(x,y,z)}\right). \tag{19.27}$$

All the quantities from Maxwell's equations can be expressed in this form. The equations written in such a way will contain some parts with a factor $e^{j\omega t}$, and the same parts with a factor $e^{-j\omega t}$. Because the two functions, $e^{j\omega t}$ and $e^{-j\omega t}$, are independent, the factors they multiply must be zero in order that the equations be satisfied at any t. In other words, instead of each equation, we get *two equivalent complex equations*. In these equations, time does not appear explicitly, and the time derivatives are replaced by $j\omega$, or $-j\omega$.

Formally, one of these complex equations can be obtained from the initial equation by replacing all the cosines with $e^{j\omega t}$, and the other by replacing the cosines with $e^{-j\omega t}$. Then, after differentiating with respect to time, all factors with $e^{j\omega t}$ and $e^{-j\omega t}$ cancel out. Although both $e^{j\omega t}$ and $e^{-j\omega t}$ can be used, it is customary in electrical engineering to replace the cosine with $e^{j\omega t}$, so that the first time derivative is replaced by the factor $j\omega$, the second time derivative by the factor $-\omega^2$, etc.

A phasor quantity in electrical engineering is written as a complex root-mean square (rms) value. To stress that a quantity is a phasor or complex, the International Electronics Commission (IEC) recommends that it be underlined, as follows:

$$\underline{A} = A(x, y, z)\, e^{j\varphi(x,y,z)} = \frac{A_{\max}(x, y, z)}{\sqrt{2}}\, e^{j\varphi(x,y,z)}. \tag{19.28}$$

The magnitude of the complex quantity is represented with the rms value instead of the maximum value because the expressions for average power and energy are conveniently expressed with rms values. Most instruments show rms values.

When dealing with complex vectors, it is important to keep in mind the following. A *real vector* has three components and, at any given moment, can be drawn as an arrow in space. The arrow describes the direction and magnitude of the vector. A *complex vector* is a set of *six* numbers, three real and three imaginary parts of its components. This is why, in general, a complex vector cannot be represented with an arrow.

After all these explanations, we can finally write down the simplest and most often quoted (but least general) form of Maxwell's equations—their complex form:

$$\nabla \times \underline{\mathbf{E}} = -j\omega\underline{\mathbf{B}}, \tag{19.29}$$

$$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + j\omega\underline{\mathbf{D}}, \tag{19.30}$$

$$\nabla \cdot \underline{\mathbf{D}} = \underline{\rho}, \tag{19.31}$$

$$\nabla \cdot \underline{\mathbf{B}} = 0. \tag{19.32}$$

(Maxwell's equations in complex form)

It is important to keep in mind that these equations are valid *only for linear media*. Otherwise, as explained, all quantities cannot simultaneously be time-harmonic.

In addition, we have the current continuity equation in complex form,

$$\nabla \cdot \underline{\mathbf{J}} = -j\omega\underline{\rho}, \tag{19.33}$$

(Current continuity equation in complex form)

as well as the constitutive relations with complex vectors (phasors), and with complex permittivity, permeability, and conductivity,

$$\underline{\mathbf{D}} = \underline{\epsilon}\,\underline{\mathbf{E}}, \qquad \underline{\mathbf{B}} = \underline{\mu}\,\underline{\mathbf{H}}, \qquad \underline{\mathbf{J}} = \underline{\sigma}\,\underline{\mathbf{E}}. \tag{19.34}$$

(Constitutive relations in complex form)

Questions and problems: Q19.21 to Q19.24, P19.4

19.6 Poynting's Theorem

Poynting's theorem is the mathematical expression of the law of conservation of energy as applied to electromagnetic fields.

To obtain an energy expression from Maxwell's equations, we have to combine them in an appropriate way. We know that the expression $\mathbf{J} \cdot \mathbf{E}$ is dissipated (Joule's) power per unit volume. Note that $\mathbf{E}$ stands for the electric field due to charges and time-varying currents. In Section 10.5 we introduced the concept of the impressed electric field, $\mathbf{E}_i$. It was defined as a field equivalent to nonelectric forces acting on electric charges. Therefore the expression $\mathbf{J} \cdot \mathbf{E}_i$ is the power of impressed (external) distributed sources per unit volume.

With this in mind, consider Maxwell's differential equations (19.16) and (19.17), which we repeat for convenience:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{19.35 = 19.16}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{19.36 = 19.17}$$

To obtain a power expression that must be satisfied by an electromagnetic field, we must combine *both* of these equations because both must simultaneously be satisfied for a real field. Let us therefore multiply (find the dot product of) the first of these equations by $\mathbf{H}$, the second by $-\mathbf{E}$, and then add the two equations thus obtained. The result is

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \mathbf{J} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}. \tag{19.37}$$

Now, from vector analysis (see Appendix 2, No. 21)

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = \nabla \cdot (\mathbf{E} \times \mathbf{H}). \tag{19.38}$$

If we assume the medium to be linear, we can write

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu \mathbf{H} \cdot \mathbf{H}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right), \tag{19.39}$$

and

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2\right). \tag{19.40}$$

For linear media, $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i)$, so that $\mathbf{E} = (\mathbf{J}/\sigma - \mathbf{E}_i)$. If we substitute this expression of $\mathbf{E}$ into the term $\mathbf{E} \cdot \mathbf{J}$ in Eq. (19.37), taking into account Eqs. (19.38) to (19.40), after simple manipulations Eq. (19.37) becomes

$$\mathbf{E}_i \cdot \mathbf{J} = \frac{J^2}{\sigma} + \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) + \nabla \cdot (\mathbf{E} \times \mathbf{H}). \tag{19.41}$$

Let us multiply this equation by a volume element $\mathrm{d}v$ and integrate over an arbitrary volume v of the field. The last term of the equation thus obtained is a volume integral of the divergence of the vector $(\mathbf{E} \times \mathbf{H})$. By the use of the divergence theorem, this volume integral can be transformed into a surface integral over the surface S bounding the volume v. So we finally obtain

$$\boxed{\int_v \mathbf{E}_i \cdot \mathbf{J}\, \mathrm{d}v = \int_v \frac{J^2}{\sigma} \mathrm{d}v + \frac{\partial}{\partial t}\int_v \left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) \mathrm{d}v + \oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathrm{d}\mathbf{S}. \qquad (19.42)}$$

(Poynting's theorem)

This is *Poynting's theorem*. It tells us about power balance inside a volume v of the electromagnetic field.

The term on the left represents the power of all the sources inside v. The terms on the right show how this power is used. One part (represented by the first term) is transformed inside v into heat. The other part (represented by the second term) is used to change (increase if positive, decrease if negative) the energy localized in the electric and magnetic field inside v. Because we consider a finite volume of the field, we need a term representing possible exchange of energy with the rest of the field, through the boundary of v, that is, surface S. According to Poynting, the last term on the right has precisely that meaning:

$$\boxed{\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathrm{d}\mathbf{S} = \text{power transferred through } S \text{ to a region outside } S. \qquad (19.43)}$$

This statement is also frequently considered as Poynting's theorem.

According to Poynting's theorem, the cross product $(\mathbf{E} \times \mathbf{H})$ can be interpreted as the power transferred by the electromagnetic field per unit area. The direction of the vector $(\mathbf{E} \times \mathbf{H})$ then shows the direction of transfer of energy through a surface perpendicular to it. The vector $(\mathbf{E} \times \mathbf{H})$ is referred to as the *Poynting vector*. We will designate it by $\mathcal{P}$ (calligraphic P):

$$\mathcal{P} = \mathbf{E} \times \mathbf{H} \qquad (\mathrm{W/m^2}). \tag{19.44}$$

(Definition of the Poynting vector)

The unit of Poynting's vector is $\mathrm{W/m^2}$ (watts per square meter).

Poynting's theorem, as a mathematical expression of the law of conservation of energy in the electromagnetic field, is an extremely useful theorem. Note, however, that it is valid only for electromagnetic fields that are described *simultaneously* by the first and second of Maxwell's equations. The following example shows that in other cases Poynting's theorem does not make sense.

Example 19.4—Formal application of Poynting's theorem to crossed electrostatic and magnetostatic fields. Consider the system shown in Fig. 19.3. A charged parallel-plate capacitor and a permanent magnet are positioned so that their fields (electrostatic and magnetostatic) overlap. Consequently, considered formally, Poynting's vector in the figure is directed into the page. This could be interpreted as if energy is perpetually circulating through this region, and the only problem is how to capture it. This reasoning, however, is not correct. These electric and magnetic fields *are not coupled* (we can move the magnet, for example, without affecting the electric field of the capacitor). Combining the two fields in this case is like combining potatoes and oranges.

Example 19.5—Energy transfer through a coaxial cable. The cross section of a coaxial cable is sketched in Fig. 19.4. Assume that the voltage between the cable conductors is V, and that there is a dc current in the cable of intensity I, as indicated. It is a simple matter to conclude that the generator is connected in the direction toward the reader, and the load away from the reader. The Poynting vector is directed away from the reader. According to the interpretation of the Poynting vector, this means that energy is flowing through the cable away from the generator, as it should.

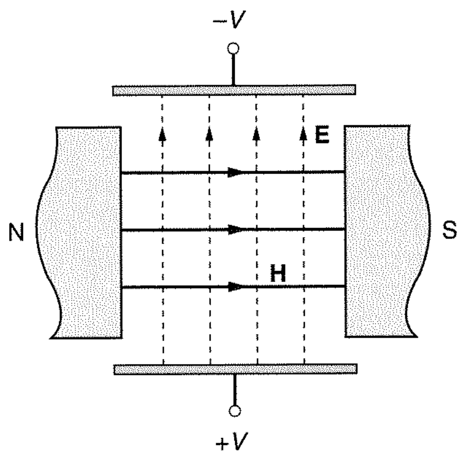


Figure 19.3 Crossed electrostatic and magnetostatic fields

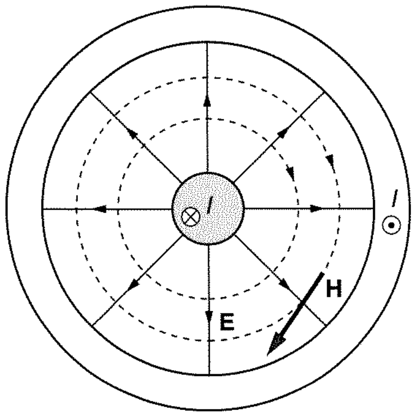


Figure 19.4 Cross section of a coaxial cable with lines of vectors **E** and **H**

It is left as an exercise for the reader to prove that the flux of the Poynting vector through the cross section of the cable equals exactly VI. Note that the intensity of the Poynting vector is the largest at the inner conductor surface, which means that most of the power flows near that surface.

Example 19.6—Poynting's theorem in complex form. Starting from the complex form of Maxwell's equations, it is not difficult to obtain Poynting's theorem in complex form. The principal difference of the derivation is that we start from the complex form of the first equation, from the *complex conjugate form* of the second equation, and multiply the first equation (find the dot product) with the *complex conjugate*, $\underline{\mathbf{H}}^*$, of the vector $\underline{\mathbf{H}}$. The rest of the derivation is quite similar to that given for Poynting's theorem for arbitrary time dependence, and it is left as an exercise for the reader. The result is

$$\int_v \underline{\mathbf{E}}_i \cdot \underline{\mathbf{J}}^* \mathrm{d}v = \int_v \frac{J^2}{\sigma} \mathrm{d}v + 2\mathrm{j}\omega \int_v \left(\frac{1}{2}\mu H^2 - \frac{1}{2}\epsilon E^2 \right) \mathrm{d}v + \oint_S (\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*) \cdot \mathrm{d}\mathbf{S}. \tag{19.45}$$

(Poynting's theorem in complex form)

This is Poynting's theorem in complex form. The vector

$$\underline{\mathcal{P}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}}^* \qquad (\mathrm{W/m^2}) \tag{19.46}$$

(The complex Poynting vector)

is known as the *complex Poynting vector*.

The equation expressing the Poynting theorem in complex form has a real and an imaginary part. It is left to the reader as an exercise to write these two parts of the equation and to discuss their meaning.

Questions and problems: Q19.25 to Q19.30, P19.5 to P19.9

19.7 The Generalized Definition of Conductors and Insulators

For linear media and time-harmonic variation of the fields, it is possible to clearly distinguish what a good conductor and a good insulator are. Let a time-harmonic electromagnetic field of angular frequency ω exist in a medium of permittivity ϵ and conductivity σ. The second Maxwell's equation in complex form becomes

$$\nabla \times \mathbf{H} = (\sigma + \mathrm{j}\omega\epsilon)\mathbf{E}. \tag{19.47}$$

For a perfect dielectric, σ in this equation does not exist. For a very good conductor, displacement current is negligible, so the term $\mathrm{j}\omega\epsilon$ is missing. Thus, at a frequency $f = \omega/(2\pi)$, we can define a good conductor by the inequality

$$\sigma \gg \omega\epsilon \qquad \text{(definition of a good conductor)}, \tag{19.48}$$

and a good insulator by the inequality

$$\sigma \ll \omega\epsilon \qquad \text{(definition of a good insulator)}. \tag{19.49}$$

19.8 The Lorentz Potentials

In Chapter 4 we introduced the concept of the electric scalar potential. This is just one in a family of potentials used in the analysis of electromagnetic fields. A potential is an auxiliary scalar or vector function, which is usually easier to calculate than the field vectors themselves, and from which the field vectors are obtained in some simple manner, usually by differentiation.

We will introduce here a pair of potentials that seem to be used most often in electromagnetic field analysis. One of these is the generalized scalar potential we already know. The other is a vector function, known as the *magnetic vector potential*. The specific pair of potentials we will now derive are known as the *Lorentz potentials*. For reasons to become apparent later, they are also known as the *retarded potentials*.

Note first that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for any vector function $\mathbf{A}$ (see Appendix 2, No. 24). Since $\nabla \cdot \mathbf{B} = 0$, it follows that it is always possible to express the magnetic flux density vector $\mathbf{B}$ as

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{19.50}$$

(Definition of magnetic vector potential)

The vector function $\mathbf{A}$ is known as the *magnetic vector potential*.

If the expression for $\mathbf{B}$ in Eq. (19.50) is introduced into the first Maxwell's equation, we obtain

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}). \tag{19.51}$$

A1.47

This means that $\nabla \times (\mathbf{E} + \partial \mathbf{A}/\partial t) = 0$. Now, we know that $\nabla \times (\nabla V) = 0$ always [see Appendix 1, Eq. (~~A1.50~~)]. Therefore Eq. (19.51) implies that $(\mathbf{E} + \partial \mathbf{A}/\partial t) = -\nabla V$, *and not zero.* (The negative gradient is used for convenience.) Thus the electric field strength can be expressed as

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \tag{19.52}$$

(Electric field strength in terms of retarded potentials)

Evidently, for time-invariant fields V becomes the electric scalar potential we know. Therefore we retain the same name for V in this case, where V is an arbitrary function of time.

So we have two equations, (19.50) and (19.52), from which we can easily calculate vectors $\mathbf{E}$ and $\mathbf{B}$, provided we know the two potentials, V and $\mathbf{A}$. For obtaining Eqs. (19.50) and (19.52) we used the first and the fourth Maxwell's equations. For determining these potentials in terms of the field sources, ρ and $\mathbf{J}$, we therefore make use of the other two Maxwell's equations.

Let us assume that the medium is linear and homogeneous, of permittivity ϵ and permeability μ. Then, substituting Eqs. (19.50) and (19.52) into the second and third Maxwell's equation, we obtain, respectively,

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} - \epsilon\mu \frac{\partial}{\partial t}(\nabla V) - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2}, \tag{19.53}$$

and

$$\nabla \cdot (\nabla V) = \nabla^2 V = -\frac{\rho}{\epsilon} - \nabla \cdot \frac{\partial \mathbf{A}}{\partial t}. \tag{19.54}$$

Since $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ [see Appendix 1, Eq. (A1.37)], Eq. (19.53) becomes

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} + \epsilon\mu \nabla \frac{\partial V}{\partial t} + \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla(\nabla \cdot \mathbf{A}). \tag{19.55}$$

There is a theorem in vector analysis called the *Helmholtz theorem*. It says that a vector function is uniquely defined if its curl and divergence are known at every point in space. We already know what the curl of $\mathbf{A}$ is ($\nabla \times \mathbf{A} = \mathbf{B}$), so we need to define its divergence in order that it be unique. Because only $\nabla \times \mathbf{A}$ matters ($\mathbf{B} = \nabla \times \mathbf{A}$), we can define $\nabla \cdot \mathbf{A}$ in an infinite number of ways, resulting in an infinite number of pairs of potentials $\mathbf{A}$ and V. Having this freedom of choice, it is wise to adopt $\nabla \cdot \mathbf{A}$ so that we can solve Eqs. (19.54) and (19.55) most easily.

It is a simple matter to conclude that if we adopt the *Lorentz condition* for $\nabla \cdot \mathbf{A}$,

$$\nabla \cdot \mathbf{A} = -\epsilon\mu \frac{\partial V}{\partial t}, \tag{19.56}$$

(The Lorentz condition)

Eqs. (19.54) and (19.55) take the simplest possible forms, each becoming a partial differential equation in a single unknown, V in the first case and $\mathbf{A}$ in the second case:

$$\nabla^2 V - \epsilon\mu \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}, \tag{19.57}$$

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}. \tag{19.58}$$

Because the x component of the last equation in a rectangular coordinate system is given by [see Appendix 1, Eq. (A1.39)]

$$\nabla^2 A_x - \epsilon\mu \frac{\partial^2 A_x}{\partial t^2} = -\mu J_x, \tag{19.59}$$

and similarly for the y and z components, we need to solve only Eq. (19.57). The solution of Eq. (19.58) will then be obtained as a vector sum of analogous solutions for the vector components of $\mathbf{A}$.

Solving Eq. (19.57) is not simple and does not add anything to the understanding of the final result. We therefore give only the final result:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon} \int_{v'} \frac{\rho(\mathbf{r}', t - R/c)}{R} \mathrm{d}v' \qquad c = \frac{1}{\sqrt{\epsilon\mu}}. \tag{19.60}$$

So the solution of Eq. (19.58) is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu}{4\pi} \int_{v'} \frac{\mathbf{J}(\mathbf{r}', t - R/c)}{R} \mathrm{d}v' \qquad c = \frac{1}{\sqrt{\epsilon\mu}}. \tag{19.61}$$

These are the *Lorentz potentials*. The meaning of $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{R}$ is illustrated in Fig. 19.5.

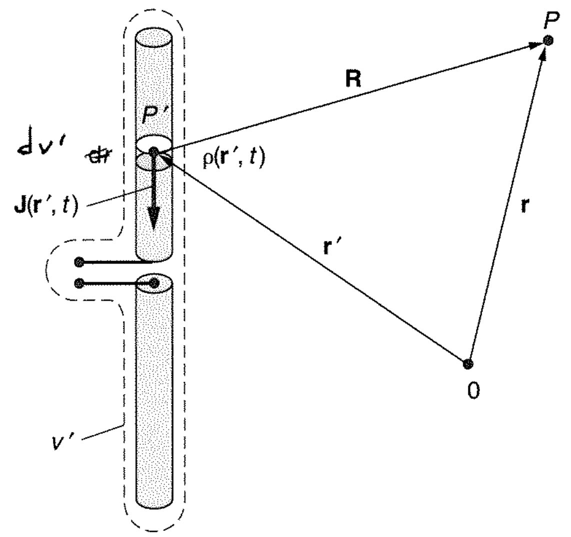


Figure 19.5 Explanation of symbols in Eqs. (19.60) and (19.61)

We stress again that the entire derivation, and therefore also Eqs. (19.60) and (19.61), is valid only for *homogeneous and linear media.*

What is the physical meaning of the expressions for the potentials in Eqs. (19.60) and (19.61)? Say there is an elemental source at a point P' whose position vector is $\mathbf{r}'$, as in Fig. 19.5. We are observing the fields due to this source at a point P defined by the position vector $\mathbf{r}$. The magnitude of the field at point P at a time t is not the one that the source produces at time t, but at an earlier time, $t - R/c$. In other words, in a homogeneous dielectric of permittivity ϵ and permeability μ, the fields propagate with a finite velocity, $c = 1/\sqrt{\epsilon\mu}$, that is, they are *retarded* in reaching the field point. For this reason, the Lorentz potentials are often termed the *retarded potentials.*

In the case of a vacuum ($\epsilon = \epsilon_0$, $\mu = \mu_0$), the velocity c of propagation of the potentials *becomes exactly the speed of light in a vacuum*, c_0, a calculation left as an exercise for the reader.

Example 19.7—Retarded potentials in complex (phasor) form. Very often, sources of an electromagnetic field are time-harmonic. In that case the retarded potentials can be written without explicit time dependence. The procedure for obtaining the complex potentials is simple—we just assume the sources, ρ and $\mathbf{J}$, and the potentials to vary following the law $e^{j\omega t}$. So we obtain

$$\underline{V}(\mathbf{r}) = \frac{1}{4\pi\epsilon}\int_{v'} \frac{\underline{\rho}(\mathbf{r}')e^{-j\omega R/c}}{R}\,dv', \qquad c = \frac{1}{\sqrt{\epsilon\mu}}, \tag{19.62}$$

(Complex retarded scalar potential)

and

$$\underline{\mathbf{A}}(\mathbf{r}) = \frac{\mu}{4\pi}\int_{v'} \frac{\underline{\mathbf{J}}(\mathbf{r}')e^{-j\omega R/c}}{R}\,dv' \qquad c = \frac{1}{\sqrt{\epsilon\mu}}. \tag{19.63}$$

(Complex retarded vector potential)

Example 19.8—Definition of quasi-static fields. It is interesting that for time-harmonic fields it is possible to inspect whether a field in a system can be considered practically as a static field (or a *quasi-static field*), or not.

From Eqs. (19.62) and (19.63) we see that the retardation can be neglected provided that the largest dimension of the field domain we consider, $d_{\max}$, is determined by the inequality $\omega d_{\max}/c \ll 1$, or

$$d_{\max} \ll \frac{c}{\omega} = \frac{1}{\omega\sqrt{\epsilon\mu}} \qquad \text{(the condition for quasi-static fields).}$$

Questions and problems: Q19.31 to Q19.36, P19.10 to P19.13

19.9 Chapter Summary

1. The general equations of the electromagnetic field, known as Maxwell's equations, are as fundamental in electromagnetic field theory as are Newton's laws in mechanics.

2. Some general consequences of Maxwell's equations include general boundary conditions, the Poynting theorem (the theorem on the transfer of energy by the electromagnetic field), and the possibility of making a clear distinction between conductors and insulators for time-harmonic fields.
3. Field vectors can be expressed in terms of auxiliary functions, called potentials.
4. The expressions for potentials indicate that the speed of electromagnetic disturbances in a vacuum is that of light.
5. For sinusoidal field variation, the expressions for the potentials in complex form enables us to define the dimensions of systems in which fields can be considered approximately as static (quasi-static fields).

QUESTIONS

Q19.1. Why (and when) is it allowed to move the time derivative in Eq. (19.1) to act on **D** only, and thus obtain Eq. (19.2)?

Q19.2. Does Eq. (19.5) tell us that a time-varying magnetic field is the source of a time-varying electric field? Explain.

Q19.3. Why would an electric field inside a perfect conductor produce a current of infinite density? Would such a current be physically possible? Explain.

Q19.4. Why are surface currents possible on surfaces of perfect conductors, when a nonzero tangential electric field there is not possible? Is this a current of finite volume density?

Q19.5. Write the full set of Maxwell's equations in differential form for the special case of a static electric field, assuming that the dielectric is linear, but inhomogeneous.

Q19.6. Write the full set of Maxwell's equations in differential form for the special case of a static electric field produced by the charges on a set of conducting bodies situated in a vacuum.

Q19.7. Write the full set of Maxwell's equations in differential form for the special case of a steady current flow in a homogeneous conductor of conductivity σ, with no impressed electric field.

Q19.8. Write the full set of Maxwell's equations in differential form for the special case of a steady current flow in an inhomogeneous poor dielectric, with impressed electric field $\mathbf{E}_i$ present.

Q19.9. Write the full set of Maxwell's equations in differential form for the special case of a time-constant magnetic field in a linear medium of permeability μ, produced by a steady current flow.

Q19.10. Write the full set of Maxwell's equations in differential form for the special case of a time-constant magnetic field, produced by a permanent magnet of magnetization **M** (a function of position).

Q19.11. Write the full set of Maxwell's equations in differential form for the special case of a time-constant magnetic field produced by both steady currents and magnetized matter, if the medium is not linear.

Q19.12. Write the full set of Maxwell's equations in differential form for the special case of a quasi-static electromagnetic field, produced by quasi-static currents in nonferromagnetic conductors.

Q19.13. Write Maxwell's equations in differential form for an arbitrary electromagnetic field in a vacuum, no free charges being present.

Q19.14. Write Maxwell's equations for an arbitrary electromagnetic field in a homogeneous perfect dielectric of permittivity ϵ and permeability μ.

Q19.15. Write the full set of differential Maxwell's equations *in scalar form* in the rectangular coordinate system. Note that *eight* simultaneous, partial differential equations result. Write these equations neatly and save them for future reference.

Q19.16. Repeat question Q19.15 for the cylindrical coordinate system.

Q19.17. Repeat question Q19.15 for the spherical coordinate system.

Q19.18. Write differential Maxwell's equations in scalar form for the particular case of an electromagnetic field in a vacuum ($\mathbf{J} = 0$, $\rho = 0$), if the field vectors are only functions of the cartesian coordinate z and of time t.

Q19.19. Write differential Maxwell's equations in scalar form for a good conductor, for the particular case of an axially symmetrical system with dependence of the field vectors only on the cylindrical coordinate r and time t. Assume that $\mathbf{J} = J_z\mathbf{u}_z$ and $\rho = 0$.

Q19.20. Repeat question Q19.19 for $\mathbf{B} = B_z\mathbf{u}_z$.

Q19.21. Write differential Maxwell's equations in complex form for an arbitrary electromagnetic field in a very good conductor, of conductivity σ and permeability μ.

Q19.22. Write differential Maxwell's equations in complex form for a quasi-static electromagnetic field.

Q19.23. Write differential Maxwell's equations in complex form for an arbitrary electromagnetic field in a perfect dielectric of permittivity ϵ and permeability μ, no free charges being present.

Q19.24. Write the most general integral Maxwell's equations in complex form.

Q19.25. The current intensity through a resistor of resistance R is I. What is the flux of the Poynting vector through any closed surface enclosing the resistor?

Q19.26. A capacitor, of capacitance C, is charged with a charge Q. What is the flux of the Poynting vector through any surface enclosing the capacitor, if the charge Q (1) is constant in time, or (2) varies in time as $Q = Q_m \cos \omega t$?

Q19.27. A coil, of inductance L, carries a current $i(t)$. What is the flux of the Poynting vector through any surface enclosing the coil?

Q19.28. A dc generator of emf $\mathcal{E}$ is open-circuited. What is the flux of the Poynting vector through any surface enclosing the generator?

Q19.29. Repeat question Q19.28 assuming that a current $i(t)$ flows through the generator, and its internal resistance is R.

Q19.30. What is the time-average value of the Poynting vector, if complex rms values are known for the electric and magnetic field strength, $\mathbf{E}$ and $\mathbf{H}$?

Q19.31. The largest dimension of a coil at a very high frequency is on the order of $(\omega\sqrt{\epsilon\mu})^{-1}$. Is it possible at such high frequencies to define the inductance of the coil in the same way as in a quasi-static case? Explain.

Q19.32. The length of a long 60-Hz power transmission line is equal to $0.5(\omega\sqrt{\epsilon_0\mu_0})^{-1}$. Is this a quasi-static system? What is the length of the line?

Q19.33. A parallel-plate capacitor has plates of linear dimensions comparable with $(\omega\sqrt{\epsilon\mu})^{-1}$, where ω is the operating angular frequency. Is it possible to determine the capacitance of such a capacitor in the same way as in the static and quasi-static case? Explain.

Q19.34. An electric circuit operates at a high frequency f. The largest linear dimension of the circuit is $2(\omega\sqrt{\epsilon\mu})^{-1}$. Are Kirchhoff's laws applicable in this case for analyzing the circuit? Explain.

Q19.35. Write the Lorentz condition in complex form.

Q19.36. A current pulse of duration $\Delta t = 10^{-9}$ s was excited in a small wire loop. After how many Δt's is the magnetic and induced electric field of this pulse going to be detected at a point $r = 10$ m from the loop?

PROBLEMS

P19.1. A current $i(t) = I_m \cos \omega t$ flows through the leads of a parallel-plate capacitor of plate area S and distance between them d. If the permittivity of the dielectric of the capacitor is ϵ, prove that the displacement current through the capacitor dielectric is exactly $i(t)$. Ignore fringing effects.

P19.2. A spherically symmetrical charge distribution disperses under the influence of mutually repulsive forces. Suppose that the charge density $\rho(r, t)$, as a function of the distance r from the center of symmetry and of time, is known. Prove that the total current density at any point is zero.

P19.3. Determine the magnetic field as a function of time for the dispersing charge distribution in problem P19.2.

***P19.4.** Small-scale models are used often in engineering practice, including electrical engineering. Starting from differential Maxwell's equations for a linear medium, derive the necessary conditions for the electromagnetic field in a small-scale model to be similar to the field in a real, n times larger model. (These conditions are usually referred to as the conditions of the *electrodynamic similitude*.) (*Hints:* (1) Write the first two differential Maxwell's equations for the full-scale system, and for the model. (2) Note that the coordinates in the latter are n times smaller, and find the conditions under which, in spite of that, the two sets of equations will be the same.)

P19.5. A lossless coaxial cable, of conductor radii a and b, carries a steady current of intensity I. The potential difference between the cable conductors is V. Prove that the flux of the Poynting vector through a cross section of the cable is VI, using the known expressions for vectors $\mathbf{E}$ and $\mathbf{H}$ in the cable. Sketch the dependence of the magnitude of the Poynting vector on the distance r from the cable axis, where $a < r < b$.

P19.6. Repeat problem P19.5 for an air stripline with strips of width a that are a distance d apart, if the current in the strips is I and voltage between them V. Neglect the edge effects.

P19.7. The stripline from the preceding problem is connected to a sinusoidal generator of emf $\mathcal{E}$ and angular frequency ω. The other end of the line is connected to a capacitor of capacitance C. Apply Poynting's theorem in complex form to a closed surface enclosing (1) the generator, or (2) the capacitor.

P19.8. Repeat problem P19.7 assuming that the load is an inductor of inductance L, instead of a capacitor.

P19.9. Repeat problem P19.7, assuming that the line is a lossless coaxial line of conductor radii a and b.

P19.10. Derive Eqs. (19.57) and (19.58) from Eqs. (19.54), (19.55), and (19.56).

P19.11. Derive the retarded potentials in Eqs. (19.62) and (19.63) from Eqs. (19.60) and (19.61).

P19.12. Suppose a system is regarded as approximately quasi-static if its largest dimension d satisfies the inequality $d\omega\sqrt{\epsilon\mu} \leq 0.1$. Determine the largest value of d thus defined for the electrodynamic systems in a vacuum if the frequency of the generators is (1) 60 Hz, (2) 10 MHz, or (3) 10 GHz.

P19.13. Compare the rms values of vectors **J** and $\partial \mathbf{D}/\partial t$ in copper, seawater, and wet ground, for frequencies f of (1) 60 Hz, (2) 10 kHz, (3) 100 MHz, or (4) 10 GHz. For copper, assume $\epsilon = \epsilon_0$, $\sigma = 56 \cdot 10^6$ S/m. For seawater, adopt $\epsilon = 10\epsilon_0$, $\sigma = 4$ S/m, and for the ground $\epsilon = 10\epsilon_0$ and $\sigma = 10^{-2}$ S/m.