Appendix 1

A Brief Survey of Vectors and Vector Calculus

A1.1 Introduction

If only one number on an appropriate scale is needed to describe a certain quantity, the quantity is known as a *scalar*. For example, the mass of a body is a scalar described by a single number (e.g., $m = 1.6\,\text{kg}$).

Some quantities need a single number to describe them at any one point in a region of space; thus they need an infinite set of numbers to describe them at all points of the region with respect to a scale. For example, air pressure or temperature inside a room can be specified at all points of the room. These quantities are scalars, but to distinguish them from ordinary scalars that need just a single number for their measure, we say that they represent a *scalar field*. A scalar field is described mathematically by a function of three spatial coordinates (and possibly of time, as for a changing temperature in a room). For example, the function $T(x, y, z) = T_0(1 + x + y^2 + z^3)$ defines a specific temperature field.

Many quantities need more than one number to describe them. An important class of quantities is that which needs *three* numbers to describe them completely. These are known as *vectors*. For example, a straight path traversed by a person from one point to another is a vector. It is described by its length (the *magnitude* of the vector), the line along which the motion takes place, and the direction along that line. Other vectors include velocity, acceleration, force, the electric field vector, and the magnetic field vector. In handwriting, vector quantities are denoted by an arrow above the symbol, for example, $\vec{s}$. In printed text, the boldface type is used much more frequently, for example, **s**. For example, we could write

$$\mathbf{s} = \text{New York} \rightarrow \text{Chicago},$$

which is unconventional but correct. The magnitude of a vector is denoted by the same symbol, but without the arrow, or not boldface. Sometimes the absolute value sign is used instead, so that s, $|\vec{s}|$, and $|\mathbf{s}|$ mean the same: the magnitude of the vector **s**.

Like scalars, many vectors need to be defined by an infinite set of three numbers, e.g., the velocity of all air particles in a region during windy weather. Such vector quantities represent a *vector field*. Vector fields are described mathematically by a *vector function* of coordinates (and possibly of time). For example, in rectangular coordinates, a vector field could take the form $\mathbf{v}(x, y, z, t)$.

Questions and problems: QA1.1 and QA1.2

A1.2 Algebraic Operations with Vectors

Five basic algebraic operations are used for vector quantities: vector addition, vector subtraction, multiplication of a vector with a scalar (resulting in a vector), multiplication of two vectors resulting in a scalar (known as the scalar, or dot, product), and multiplication of two vectors resulting in a new vector (known as the vector, or cross, product).

A1.2.1 ADDITION AND SUBTRACTION OF VECTORS

Let us adopt the straight-line motion from one point to another as the prototype of a vector quantity. The vector **A** that describes this displacement is represented as a straight-line segment from its starting point (point 1) to its end point (point 2) (Fig. A1.1a). The arrowhead is added at the vector endpoint to symbolize its direction. Vector **B**, representing another displacement with a starting point at 2 and the end point at 3, is also shown. The total displacement is as if we moved from point 1 to point 3. Vector **C** in Fig. A1.1a, from point 1 to point 3, is therefore defined as the sum of the vectors **A** and **B**, which is written as

$$\mathbf{C} = \mathbf{A} + \mathbf{B}. \tag{A1.1}$$

The vector in Fig. A1.1a directed from point 2 to point 1 is defined as the *negative* of the vector **A**, that is, $-\mathbf{A}$, and that from point 3 to point 2 as the negative of the vector **B**, that is, $-\mathbf{B}$. From the definition of vector addition in Eq. (A1.1) it follows that the vector **D** in Fig. A1.1a represents the difference between vectors **A** and **B**:

$$\mathbf{D} = \mathbf{A} + (-\mathbf{B}) = \mathbf{A} - \mathbf{B}. \tag{A1.2}$$

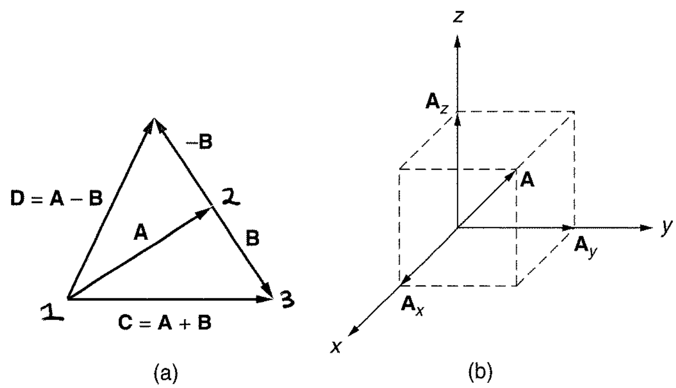


Figure A1.1 (a) Graphical representation of vectors and of vector addition and subtraction. (b) Vector **A** represented as a sum of its three components in the rectangular coordinate system.

Additions and subtractions of more than two vectors are performed two vectors at a time, thus reducing the problem to that of two vectors. A vector in space can always be represented uniquely as a vector sum of three vectors. If these are parallel to the coordinate axes, they are known as the *three vector components* of the vector considered. As an example, vector **A** in Fig. A1.1b is represented by its three vector components in a rectangular coordinate system.

Example A1.1—Sum of vectors forming a polygon. Consider first three vectors forming a triangle in the head-to-tails arrangement. To obtain the sum of the three vectors, we first make the vector sum $\mathbf{A} + \mathbf{B}$ and find that it is equal to $-\mathbf{C}$. Thus the sum of the three vectors is zero.

It is trivial to extend this reasoning to any number of vectors in the head-to-tail arrangement that form a closed polygon. Consider now the limiting case of a polygon with an infinite number of sides formed by differential vector lengths $d\mathbf{l}$ in the head-to-tail arrangement. The sum of these differential length vectors is a line integral around the contour C they form, and it is equal to zero:

$$\oint_C d\mathbf{l} = 0.$$

Note that the integral of the *magnitudes* of the differential vectors is not zero—it equals the contour length.

A1.2.2 MULTIPLICATION OF A VECTOR WITH A SCALAR

Multiplication of a vector **A** with a scalar s results in a vector of the same direction as **A**, but of magnitude sA. To divide a vector with a scalar amounts to multiplying the vector by the reciprocal of the scalar. For example, $\mathbf{A}/s = (1/s)\mathbf{A}$.

A1.2.3 UNIT VECTORS

Of particular importance is the concept of the *unit vector*. This is a vector of magnitude one and a pure number (with no physical dimensions). Because we know its magnitude, to specify it, we need only two pieces of information: its direction and sense along that direction. Thus a unit vector defines an *oriented direction in space.* Several notations are used for unit vectors. For example, the unit vector in the direction x may be denoted $\widehat{\mathbf{x}}$, $\mathbf{a}_x$, $\mathbf{i}_x$, or $\mathbf{u}_x$. In this text, the symbol $\mathbf{u}$ ("unit vector") with appropriate subscripts is adopted, and sometimes the symbol $\mathbf{r}_0$ for a unit vector in the direction of a radius vector (a vector directed from a fixed point to any direction).

The unit-vector concept is extremely important and can be used on many occasions. Note that *any* vector can be represented as a product of its magnitude and a unit vector, and that the unit vector in the direction of a vector $\mathbf{A}$ is obtained by dividing $\mathbf{A}$ by its magnitude:

$$\mathbf{A} = A\mathbf{u}_\mathbf{A}, \qquad \mathbf{u}_A = \frac{\mathbf{A}}{A}. \tag{A1.3}$$

(Relationship between a vector and the unit vector in its direction)

A1.2.4 THE DOT PRODUCT OF TWO VECTORS

The *scalar product*, or *dot product*, of two vectors, such as $\mathbf{A}$ and $\mathbf{B}$, denoted as $\mathbf{A} \cdot \mathbf{B}$, is defined as follows. Let the angle between the two vectors be α, as in Fig. A1.2. Then

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\alpha. \tag{A1.4}$$

[Definition of dot (scalar) product]

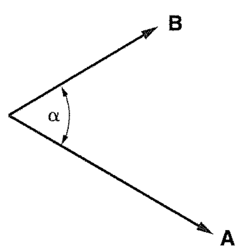


Figure A1.2 The dot (scalar) product of vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as $AB\cos\alpha$.

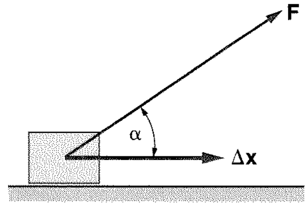


Figure A1.3 The work done by the force **F** in moving the body by $\Delta\mathbf{x}$ is $\mathbf{F} \cdot \Delta\mathbf{x}$.

Note that the dot product is a *commutative operation* (i.e., the two vectors in the product can exchange places without affecting the result):

$$\mathbf{B} \cdot \mathbf{A} = BA\cos\alpha = \mathbf{A} \cdot \mathbf{B} = AB\cos\alpha. \tag{A1.5}$$

Finally, the dot product of a vector with itself results in the square of the vector magnitude:

$$\mathbf{A} \cdot \mathbf{A} = A^2. \tag{A1.6}$$

Example A1.2—Work of a force expressed as the dot product. Consider a force **F** acting on a body (Fig. A1.3). Assume that the body is free to move only along the x axis. If the body is moved by a small distance Δx, this displacement can be due only to the projection F_x of the force **F** on the x axis. Therefore the work done by the force **F** is $\Delta A_{\text{of force }\mathbf{F}} = F_x\,\Delta x\cos\alpha$.

According to the definition of the dot product, this can be written in a compact form, $\Delta A_{\text{of force }\mathbf{F}} = \mathbf{F} \cdot \Delta\mathbf{x}$.

Example A1.3—Dot product of a vector with a unit vector. Let **u** be the unit vector in an arbitrary direction. The dot product of a vector **A** with the unit vector **u** is given by

$$\mathbf{A} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{A} = A\cos(\mathbf{A}, \mathbf{u}) = A_u,$$

where $\cos(\mathbf{A}, \mathbf{u})$ is the cosine of the angle between the two vectors, and A_u is the projection of the vector **A** onto the direction of the unit vector **u**. This is a simple but important result.

A1.2.5 THE CROSS PRODUCT OF TWO VECTORS

The *cross product* (or vector product) of two vectors **A** and **B** is defined to be a new vector, normal to the plane of vectors **A** and **B**, of magnitude $AB\sin\alpha$ (Fig. A1.4). The cross product is written in the form $\mathbf{A} \times \mathbf{B}$. Thus the magnitude of the cross product in Fig. A1.4 is

$$|\mathbf{A} \times \mathbf{B}| = AB\sin\alpha, \tag{A1.7}$$

(Magnitude of cross product of two vectors)

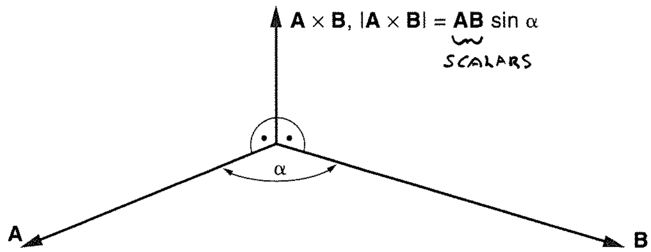


Figure A1.4 Cross product of vectors **A** and **B**, that is, $\mathbf{A} \times \mathbf{B}$. Note that $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$.

and its direction is as indicated in the figure. It is obtained by the *right-hand rule*: if the first vector (**A** in this case) is rotated to coincide with the second vector (**B** in this case) by the smaller of the two angles made by the two vectors, and we imagine rotating our right hand with it, the direction of the cross product is that of the direction of the thumb (or, equivalently, the progression of a right-hand screw). From Fig. A1.4, we have

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}, \tag{A1.8}$$

i.e., *the cross product is not a commutative operation.*

Example A1.4—Moment of force expressed as the cross product. Consider a force **F** acting on a lever of vector length **r** that cannot move but can rotate only about the z axis, to which it is normal (Fig. A1.5). In rotating the lever, only the component normal to the lever and to the z axis is of interest. The strength with which the lever is forced to rotate is proportional to the lever length and to that force component, and is known as the *moment of force* **F** *with respect to the z axis.* Since this component of the force equals $|\mathbf{F}| \sin\alpha$, where α is the angle between vectors **F** and **r**,

$$\text{Moment of force } \mathbf{F} \text{ with respect to } z \text{ axis} = |\mathbf{r}|\ |\mathbf{F}| \sin\alpha = |\mathbf{r} \times \mathbf{F}|$$

The moment of force is defined to be a vector,

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}, \tag{A1.9}$$

so that by definition, in Fig. A1.5 it is in the direction of the z axis.

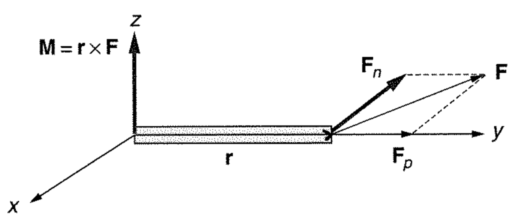


Figure A1.5 The moment of force **F** with respect to the z axis is defined as $\mathbf{r} \times \mathbf{F}$. It is assumed that **F** is in the plane normal to z.

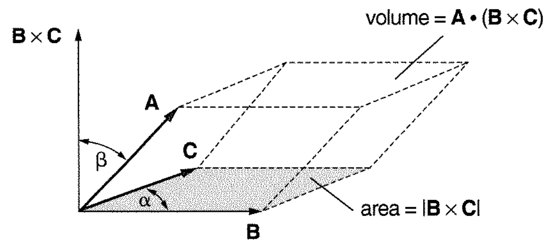


Figure A1.6 Geometrical illustration of the triple scalar product

A1.2.6 THE TRIPLE SCALAR PRODUCT

We will also need to find products of three vectors. There are two such product types. One is of the form $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, and it is called a *triple scalar product*. The triple scalar product has the property that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}), \tag{A1.10}$$

which can be proved as follows. According to the definition of the cross product, the magnitude of the cross product $\mathbf{B} \times \mathbf{C}$ equals the area of the parallelogram formed by vectors $\mathbf{B}$ and $\mathbf{C}$ (Fig. A1.6). Hence $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ equals the volume of the parallelepiped formed by the three vectors in Fig. A1.6, because $A \cos \beta$ is the height of the parallelepiped. It can be demonstrated that the volume of the same parallelepiped is obtained by the other two triple scalar products in Eq. (A1.10).

A1.2.7 THE TRIPLE VECTOR PRODUCT

The other product of three vectors has the form $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and is known as a *triple vector product*. This is a vector normal to both vectors $\mathbf{A}$ and $\mathbf{B} \times \mathbf{C}$. Since the vector $\mathbf{B} \times \mathbf{C}$ is normal to the plane of vectors $\mathbf{B}$ and $\mathbf{C}$, it follows that the vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ lies in the plane of vectors $\mathbf{B}$ and $\mathbf{C}$. Therefore, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = a\mathbf{B} + b\mathbf{C}$, where a and b are scalars to be expressed in terms of vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. It can be proved (e.g., by expanding the left-hand side and the right-hand side in the rectangular coordinate system) that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\,\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}. \tag{A1.11}$$

Questions and problems: QA1.3 to QA1.13, PA1.1 to PA1.6

A1.3 Orthogonal Coordinate Systems

When vectors are used in actual calculations, they are usually represented in a convenient coordinate system. Generally speaking, a *coordinate system* is a structure that enables unique definition of a point in space by three variables, known as the coordinates of the point. The lines along which two of the three variables are constant are known as coordinate lines. If coordinate lines are mutually orthogonal at all points of space, the coordinate system is termed an *orthogonal coordinate system*. We describe three orthogonal coordinate systems most often needed in electromagnetics: the rectangular, the cylindrical, and the spherical. The rectangular system is useful for a rectangular waveguide cavity, for example. The cylindrical system is needed, for example, for a coaxial cable. The spherical system is a natural one to use when dealing with antennas, which radiate in all radial directions in space.

A1.3.1 RECTANGULAR COORDINATE SYSTEM

The rectangular, or cartesian, coordinate system (Fig. A1.7a) is the most commonly used. A point in the rectangular system is defined by coordinates (x, y, z), which represent distances of the point from the planes $x = 0$, $y = 0$, and $z = 0$. A point can also be defined as an intersection of three *coordinate surfaces*—in this case planes (Fig. A1.7b).

The directions of the three axes in a rectangular coordinate system are characterized by three mutually orthogonal unit vectors, $\mathbf{u}_x$, $\mathbf{u}_y$, $\mathbf{u}_z$, known as the *base unit vectors*, shown in Fig. A1.8a. Note that they have the same direction at all points, which is the case only for the rectangular system. A vector represented by its three rectangular components is shown in Fig. A1.8b. The magnitudes of these components, A_x, A_y, and A_z, are known as the *scalar components* of the vector.

Note the sequence of the base unit vectors. It is such that if the first rotates toward the second following the smaller angle between them, the direction of the third is obtained by the right-hand rule. For this reason the rectangular coordinate system is known as a *right-handed coordinate system*. The cylindrical and spherical coordinate systems are also defined so as to be right-handed systems.

We will need the differential lengths dl_1, dl_2, and dl_3 corresponding to a small increase in *one* coordinate, with the other two kept constant. In a rectangular coordinate system these differential lengths (Fig. A1.9) evidently are

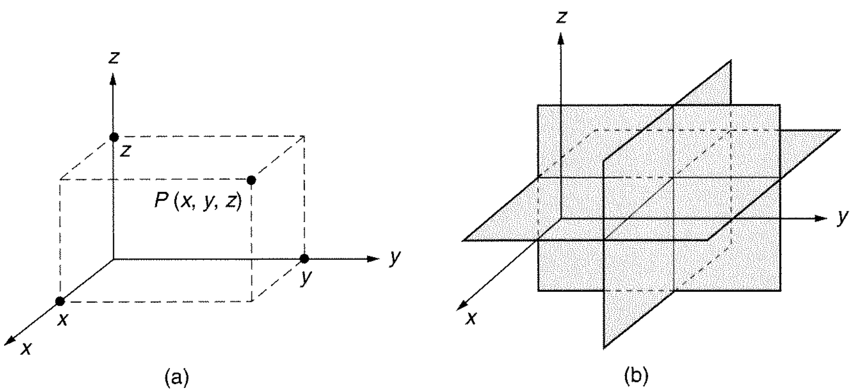


Figure A1.7 (a) Rectangular coordinates of a point. (b) The definition of a point in a rectangular coordinate system can also be an intersection of three coordinate surfaces.

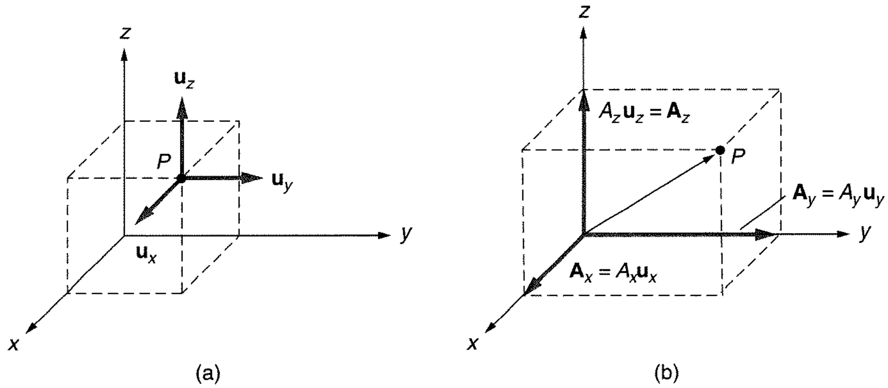


Figure A1.8 (a) The base unit vectors in a rectangular coordinate system. (b) A vector with ~~a startin~~g point P decomposed into its three rectangular components.
THE END

$$dl_1 = dx, \qquad dl_2 = dy, \qquad dl_3 = dz. \tag{A1.12}$$

(Differential lengths along coordinate lines in rectangular coordinate system)

Example A1.5—Dot product in the rectangular coordinate system. In the rectangular coordinate system, vectors **A** and **B** are represented in terms of their scalar components as

$$\mathbf{A} = A_x\mathbf{u}_x + A_y\mathbf{u}_y + A_z\mathbf{u}_z$$

and

$$\mathbf{B} = B_x\mathbf{u}_x + B_y\mathbf{u}_y + B_z\mathbf{u}_z.$$

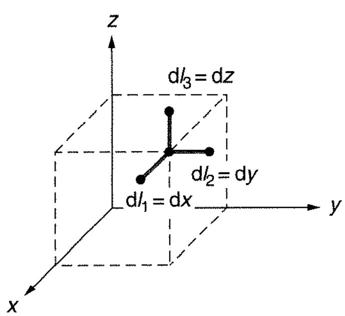


Figure A1.9 Illustration of the differential lengths along coordinate lines in a rectangular coordinate system

Since the base unit vectors are mutually orthogonal, the dot product of any two is zero, and the dot product of any of them by itself is unity. So we obtain

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \tag{A1.13}$$

(Dot product in the rectangular coordinate system)

Example A1.6—Cross product in the rectangular coordinate system. With the vectors **A** and **B** represented as in Example A1.5, we have

$$\mathbf{A} \times \mathbf{B} = (A_x\mathbf{u}_x + A_y\mathbf{u}_y + A_z\mathbf{u}_z) \times (B_x\mathbf{u}_x + B_y\mathbf{u}_y + B_z\mathbf{u}_z).$$

Note that the cross product of any unit vector with itself is zero (because the angle is zero), and that

$$\mathbf{u}_x \times \mathbf{u}_y = -\mathbf{u}_y \times \mathbf{u}_x = \mathbf{u}_z \qquad \mathbf{u}_x \times \mathbf{u}_z = -\mathbf{u}_z \times \mathbf{u}_x = -\mathbf{u}_y \qquad \mathbf{u}_y \times \mathbf{u}_z = -\mathbf{u}_z \times \mathbf{u}_y = \mathbf{u}_x,$$

so that

$$\mathbf{A} \times \mathbf{B} = \mathbf{u}_x(A_y B_z - A_z B_y) + \mathbf{u}_y(A_z B_x - A_x B_z) + \mathbf{u}_z(A_x B_y - A_y B_x). \tag{A1.14}$$

(Cross product in the rectangular coordinate system)

A1.3.2 CYLINDRICAL COORDINATE SYSTEM

In the cylindrical coordinate system (Fig. A1.10a), the coordinates of a point are the distance r from the coordinate origin of the projection of the point onto the plane $z = 0$, the angle ϕ made by this projection with the x axis, and the z coordinate as in

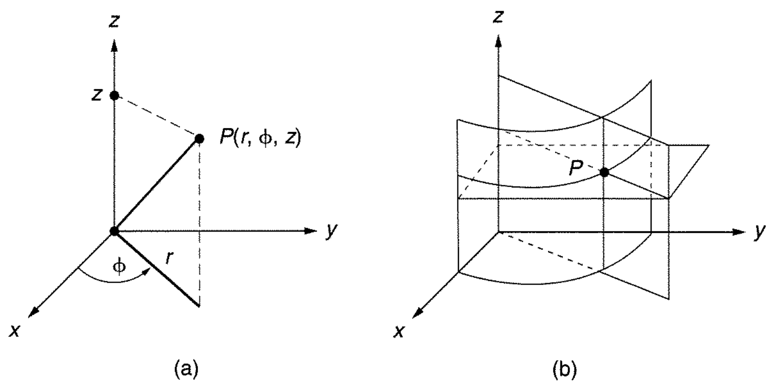


Figure A1.10 (a) Cylindrical coordinates of a point. (b) The definition of a point in the cylindrical coordinate system as the intersection of three coordinate surfaces.

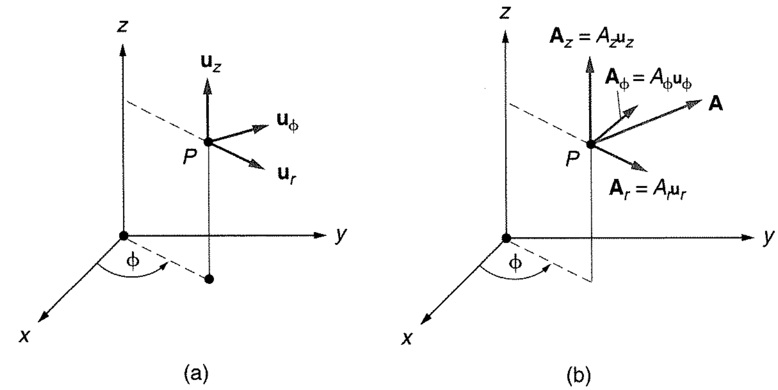


Figure A1.11 (a) The base unit vectors in a cylindrical coordinate system. (b) A vector with a starting point P decomposed into its three cylindrical components.

the rectangular system. The coordinate surfaces (whose intersection defines a point) are circular cylinders of radius r, half-planes ϕ = constant, and planes z = constant (Fig. A1.10b).

The base unit vectors are in the directions of motion *if* the coordinate of interest is given a small positive increment and the other two are kept fixed. Thus the base unit vectors $\mathbf{u}_r$, $\mathbf{u}_\phi$, and $\mathbf{u}_z$ are as in Fig. A1.11a. Note that although the base unit vector $\mathbf{u}_z$ has the same direction at all points, the base unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\phi$ generally differ from one point to another. Note that the three unit vectors are mutually orthogonal. A vector represented by its three cylindrical components is shown in Fig. A1.11b.

Note that the order of the base unit vectors is again such that if the first rotates toward the second following the smaller angle between them, the direction of the third is obtained by the right-hand rule. Therefore the cylindrical coordinate system is also an orthogonal right-handed system.

The differential lengths dl_1, dl_2, and dl_3 along coordinate lines corresponding to a small increase in *one* coordinate with the other two kept constant (Fig. A1.12) have the form

$$dl_1 = dr, \qquad dl_2 = r\,d\phi, \qquad dl_3 = dz. \tag{A1.15}$$

(Differential lengths along coordinate lines in the cylindrical coordinate system)

A1.3.3 SPHERICAL COORDINATE SYSTEM

The last coordinate system we need is the spherical one. In terms of rectangular coordinates, a point in the spherical system is defined by its distance r from the origin, the angle θ that r makes with the z axis, and the angle ϕ that the projection of r onto

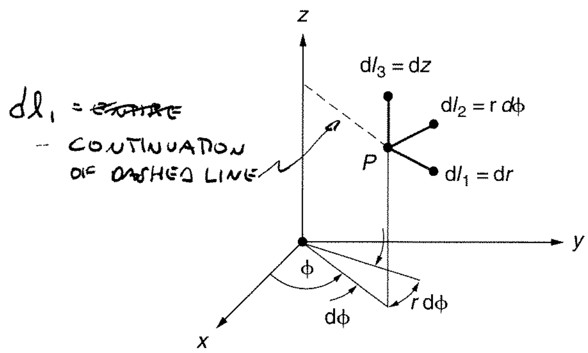


Figure A1.12 Differential lengths in a cylindrical coordinate system

the plane $z = 0$ makes with the x axis (Fig. A1.13a). A point can also be considered as an intersection of three coordinate surfaces, a sphere of radius r, a circular cone of half-angle θ, and a half plane containing the z axis and making an angle ϕ with the x axis, as in Fig. A1.13b.

The directions of the base unit vectors at a point in a spherical coordinate system are obtained if one coordinate at a time is given as a small positive increment, the other two being kept constant (Fig. A1.14a). Note that they are mutually orthogonal and that the direction of *all three* varies from point to point. A vector decomposed into its three spherical components is shown in Fig. A1.14b. The sequence of the base unit vectors ($\boldsymbol{u}_r, \boldsymbol{u}_\theta, \boldsymbol{u}_\phi$) is again such that the defined spherical coordinate system is a right-handed system.

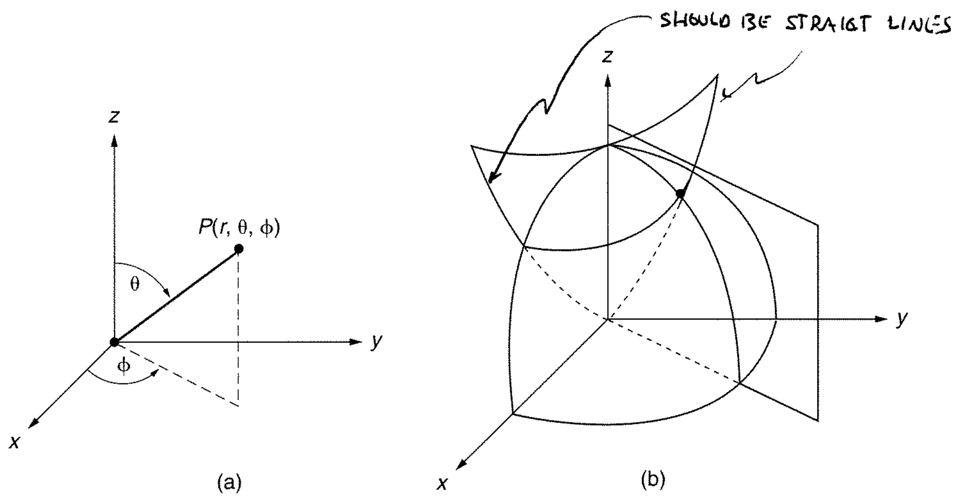


Figure A1.13 (a) Spherical coordinates of a point. (b) The definition of a point in spherical coordinate system as the intersection of three coordinate surfaces.

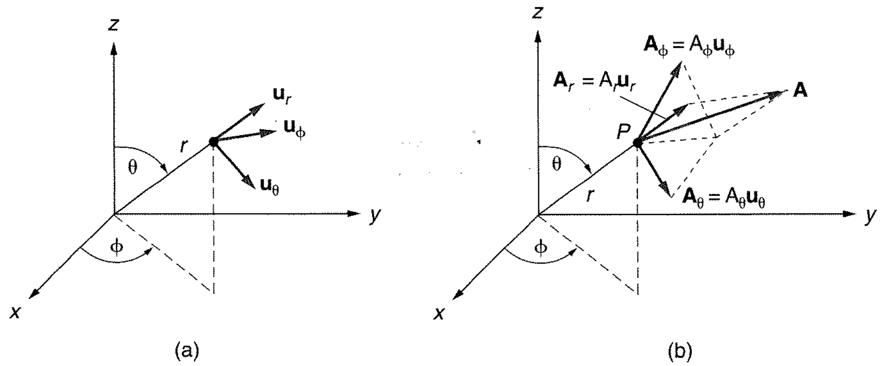


Figure A1.14 (a) The base unit vectors in a spherical coordinate system. (b) A vector with a starting point P decomposed into its three spherical components.

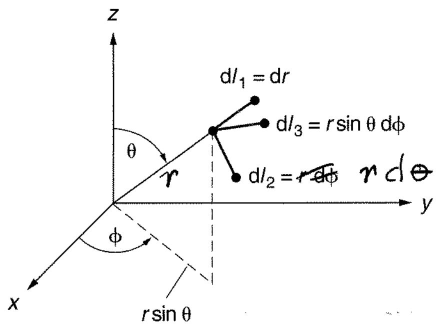


Figure A1.15 Differential lengths along coordinate lines in a spherical coordinate system

The differential lengths dl_1, dl_2, and dl_3 along the three coordinate lines in a spherical coordinate system (Fig. A1.15) are

$$dl_1 = d, \qquad dl_2 = r\, d\theta, \qquad dl_3 = r \sin\theta\, d\phi. \tag{A1.16}$$

(Differential lengths along coordinate lines in the spherical coordinate system)

Questions and problems: QA1.14 to QA1.27, PA1.7 to PA1.24

A1.4 Elements of Vector Calculus

Vector calculus deals with various forms of differentiations and integrations of vector functions. The bases for most of the conclusions are the definitions of three types of "derivatives" (actually, a combination of derivatives with respect to the three coordinates) that can be associated with scalar and vector fields. The first of these is the gradient. It operates on a scalar field and yields a vector field. The second is the divergence. It operates on a vector field, resulting in a scalar field. The third is the curl. It also operates on a vector field, but the result is a new vector field.

These basic operations can be combined to give second-order operations such as divergence of a gradient, divergence of a curl, curl of a gradient, and gradient of a scalar product of two vector fields.

A1.4.1 GRADIENT

The values of the function describing a scalar field in the neighborhood of a point are generally not the same. If we move in various directions from the point considered, in some directions the values of the function gradually decrease, and in some they gradually increase. The *gradient* is a *vector* function of coordinates pointing in the direction of the *most rapid increase* of these gradual variations (hence the name) of the scalar field in the neighborhood of a point. For example, a temperature inside a room generally differs from one point to another. The gradient of temperature at a point will be directed toward the warmest neighboring points.

Let a scalar field be described by the function $f(c_1, c_2, c_3)$, where (c_1, c_2, and c_3) are the coordinates in any coordinate system (rectangular, cylindrical, spherical, or any other). Consider a point A in the field and assume that an axis ζ passes through that point, as in Fig. A1.16. Let the value of the function at A be f, and at a very close point B on the ζ axis let a distance $\mathrm{d}\zeta$ from the point A in the reference axis direction be $f + \mathrm{d}f_\zeta$. The expression

$$\frac{(f + \mathrm{d}f_\zeta) - f}{\mathrm{d}\zeta} = \frac{\mathrm{d}f_\zeta}{\mathrm{d}\zeta}$$

is evaluated as the ordinary derivative of a function of a single unknown, ζ. Becuase it is associated with a specific direction (that of the ζ axis), it is known as the *directional derivative*. If we multiply this expression by the unit vector $\mathbf{u}_\zeta$ along the ζ axis, we obtain a vector with the following properties: (1) the magnitude of the vector equals the directional derivative of f in the ζ axis direction, and (2) the direction of the vector coincides with that of the ζ axis. Thus, instead of giving the value of the directional derivative and of explaining in which direction it was obtained, we specify

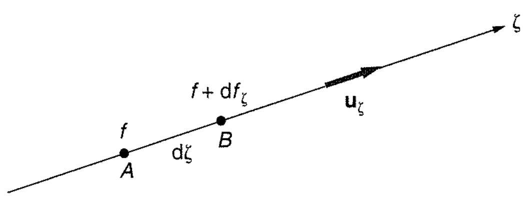


Figure A1.16 A ζ axis in a scalar field described by a function f

just one vector, which contains all the information we need. This vector is known as the *component of the gradient of the function f in the direction of the ζ axis* and is denoted as $\mathrm{grad}_\zeta f = \nabla_\zeta f$:

$$\nabla_\zeta f = \frac{\mathrm{d}f_\zeta}{\mathrm{d}\zeta}\mathbf{u}_\zeta. \tag{A1.17}$$

(Component of the gradient in the direction of a ζ axis)

This procedure can be used to obtain the component of the gradient in any direction, and thus also along the three local coordinate axes defined by the base unit vectors. Of course, the derivatives will be partial derivatives, because in determining the derivative in Eq. (A1.17) in such a case we change only one coordinate at a time. The total vector ∇f, as any vector, is obtained as a sum of these three components. Note that in Eq. (A1.17) the ζ coordinate is assumed to be a *length coordinate*, so that the increment $\mathrm{d}\zeta$ is also a differential length.

That the vector ∇f is in the direction of the maximal increase of the function f at a point is almost obvious: if we make a vector sum of the rate of increase of f in three orthogonal directions, the result will be the direction of the maximal increase of f in space.

We know differential length elements in rectangular, cylindrical, and spherical coordinate systems. In a rectangular coordinate system, from Eqs. (A1.17) and (A1.12) we obtain the following expression for the gradient of the function $f(x, y, z)$:

$$\nabla f(x, y, z) = \frac{\partial f(x, y, z)}{\partial x}\mathbf{u}_x + \frac{\partial f(x, y, z)}{\partial y}\mathbf{u}_y + \frac{\partial f(x, y, z)}{\partial z}\mathbf{u}_z. \tag{A1.18}$$

(Gradient in the rectangular coordinate system)

Note that the *del operator*, or *nabla operator*, is defined in the rectangular system as

$$\nabla = \frac{\partial}{\partial x}\mathbf{u}_x + \frac{\partial}{\partial y}\mathbf{u}_y + \frac{\partial}{\partial z}\mathbf{u}_z. \tag{A1.19}$$

(Definition of del operator)

Although defined in a rectangular coordinate system, the symbol ∇f is used frequently to denote the gradient in *any* coordinate system.

The differential lengths along coordinate lines in the cylindrical and spherical coordinate systems are given in Eqs. (A1.15) and (A1.16). From Eq. (A1.17) the expressions for the gradient in the two systems are

$$\nabla f(r, \phi, z) = \frac{\partial f(r, \phi, z)}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial f(r, \phi, z)}{\partial \phi}\mathbf{u}_\phi + \frac{\partial f(r, \phi, z)}{\partial z}\mathbf{u}_z. \tag{A1.20}$$

(Gradient in the cylindrical coordinate system)

and

$$\nabla f(r, \theta, \phi) = \frac{\partial f(r, \theta, \phi)}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial f(r, \theta, \phi)}{\partial \theta}\mathbf{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial f(r, \theta, \phi)}{\partial \phi}\mathbf{u}_\phi. \tag{A1.21}$$

(Gradient in the spherical coordinate system)

A1.4.2 DIVERGENCE

The *divergence* of a vector field is a *scalar* function indicating integrally how much the field "diverges," i.e., originates, from a small closed surface enclosing a point in the field. For example, if we imagine a point in air from which there is a permanent outflow of air, the divergence at that point would be a positive number. If there is an inflow of air at the point, the divergence would be a negative number. At points with no outflow or inflow, the divergence is zero. The divergence of a vector function **F** is denoted as $\nabla \cdot \mathbf{F}$.

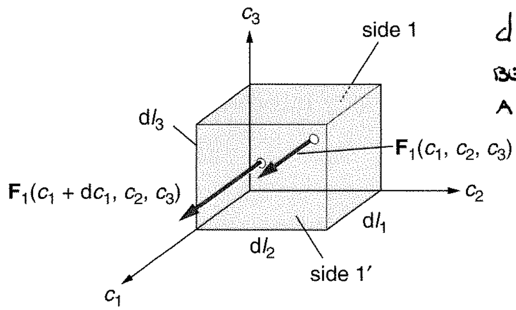
To represent this explanation mathematically, consider a point in the vector field described by a vector function $\mathbf{F}(c_1, c_2, c_3)$ of three arbitrary orthogonal coordinates c_1, c_2, and c_3. Let ΔS be a small surface enclosing the point, and let Δv be the volume enclosed by ΔS. The divergence of $\mathbf{F}(c_1, c_2, c_3)$ at that point is then defined as

$$\nabla \cdot \mathbf{F}(c_1, c_2, c_3) = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oint_{\Delta S} \mathbf{F} \cdot d\mathbf{S}, \tag{A1.22}$$

(Definition of divergence of a function **F***)*

The product $\mathbf{F} \cdot d\mathbf{S}$ is known as the *flux* of the vector **F** through the differential surface $d\mathbf{S}$. Note that only the component of **F** *normal* to $d\mathbf{S}$ contributes to the flux.

If we have a differential volume in the form of a parallelepiped (box) defined by three differential length elements in a coordinate system, we have to integrate over its six sides. Pairs of opposite parallelepiped surfaces are normal to one of the three base unit vectors. Therefore the flux through such a pair will be due to only one component of the vector. For example, the flux through the sides 1 and 1′ in Fig. A1.17 is due only to the component $\mathbf{F}_1$ of the vector function **F**.



$dl_1 \cdots dl_3$ SHOULD BE ON THE $c_1 \ldots c_3$ AXES.

Figure A1.17 A pair of differential surfaces corresponding to fixed values of the second and third coordinate, c_2 and c_3, for two values, c_1 and $c_1 + dc_1$, of the first coordinate

There is only one problem that remains to be solved. How can we determine the flux of the vector function $\mathbf{F}_1(c_1, c_2, c_3)$ when it is given as a *general* function of the coordinates? Let this be the value of the function on side 1 in Fig. A1.17. The flux through side 1 (note that it is an inward flux, i.e., negative) is then

$$\text{Flux through side 1} = -F_1(c_1, c_2, c_3)\, dl_2\, dl_3.$$

To obtain the outward, i.e., positive flux through side 1′, we need to consider the *flux through side 1*, and not only the function $F_1(c_1, c_2, c_3)$, as a function of the coordinate c_1. This is necessary because the size of the side 1′ is also a function of c_1. So

$$\text{Flux through side 1}' = F_1(c_1, c_2, c_3)\, dl_2\, dl_3 + \frac{\partial[F_1(c_1, c_2, c_3)\, dl_2\, dl_3]}{\partial c_1}\, dc_1.$$

Adding the two fluxes, we obtain that the flux through the pair of surfaces 1 and 1′ in Fig. A1.17 is

$$\text{Flux through sides 1 and 1}' = \frac{\partial[F_1(c_1, c_2, c_3)\, dl_2\, dl_3]}{\partial c_1}\, dc_1$$

The flux through the other two pairs is obtained in exactly the same way. When we sum the three fluxes and divide by the differential volume, $dl_1\, dl_2\, dl_3$, we obtain a general expression for the divergence:

$$\nabla \cdot \mathbf{F} = \frac{1}{dl_1\, dl_2\, dl_3}\left[\frac{\partial(F_1\, dl_2\, dl_3)}{\partial c_1}\, dc_1 + \frac{\partial(F_2\, dl_1\, dl_3)}{\partial c_2}\, dc_2 + \frac{\partial(F_3\, dl_1\, dl_2)}{\partial c_3}\, dc_3\right], \tag{A1.23}$$

(Divergence in orthogonal coordinates c_1, c_2, and c_3)

where

$$\mathbf{F} = \mathbf{F}(c_1, c_2, c_3) = F_1(c_1, c_2, c_3)\mathbf{u}_1 + F_2(c_1, c_2, c_3)\mathbf{u}_2 + F_3(c_1, c_2, c_3)\mathbf{u}_3.$$

Using this formula and the expressions for differential lengths in rectangular, cylindrical, and spherical coordinate systems and noting that the coordinates are independent variables, we obtain directly the expressions for the divergence in rectangular, cylindrical, and spherical coordinate systems. In a rectangular system $c_1 = x$, $c_2 = y$, $c_3 = z$, so that

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_x}{\partial y} + \frac{\partial F_x}{\partial z}. \tag{A1.24}$$

(Divergence in a rectangular coordinate system)

In a cylindrical system $c_1 = r$, $c_2 = \phi$, $c_3 = z$, and the expression in Eq. (A1.23) becomes

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}. \tag{A1.25}$$

(Divergence in a cylindrical coordinate system)

In a spherical system $c_1 = r$, $c_2 = \theta$, $c_3 = \phi$, so that Eq. (A1.23) takes the form

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta\, F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}. \tag{A1.26}$$

(Divergence in a spherical coordinate system)

A1.4.3 CURL

In the velocity vector field of air, for example, you will notice that a light feather is not only being translated but usually also rotates. This is due to different air velocities normal to the main motion of air, resulting in miniature whirlwinds all over the vector field. This property of any vector field (not necessarily describing a motion), of having different magnitude at close points transverse to the field lines, is described by a vector quantity known as the *curl* of the vector field. It is a vector normal to the plane in which this curling action of the field at a point is maximal, and is denoted as $\nabla \times \mathbf{F}$.

As a vector quantity, the curl can be obtained as a sum of its three components. So we define first the component of the curl of a vector function $\mathbf{F}$ in the direction of a unit vector $\mathbf{n}$, as in Fig. A1.18. Assume a small surface of area ΔS bounded by a contour ΔC, *normal* to $\mathbf{n}$. Let the direction around the contour be connected with the direction of the unit vector $\mathbf{n}$ by the right-hand ~~screw~~ rule. The component of $\nabla \times \mathbf{F}$ in the direction of the unit vector $\mathbf{n}$, that is, $\mathbf{n} \cdot \nabla \times \mathbf{F}$, is then defined as

$$\mathbf{n} \cdot \nabla \times \mathbf{F} = \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_{\Delta C} \mathbf{F} \cdot d\mathbf{l}. \tag{A1.27}$$

[Component of curl of vector **F** *in the direction of unit vector* **n** *(Fig. A1.18)]*

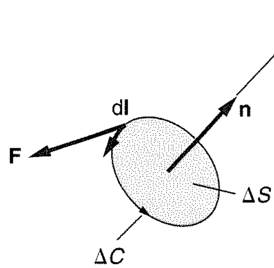


Figure A1.18 A surface ΔS bounded by the contour ΔC normal to unit vector $\mathbf{n}$

Using this definition, it is not difficult to obtain a general expression for the curl in orthogonal coordinate systems. Again let the vector field be described by a vector function $\mathbf{F}(c_1, c_2, c_3)$, where (c_1, c_2, c_3) are any orthogonal coordinates. Let us find the expression for the c_1 component of $\nabla \times \mathbf{F}$, Fig. A1.19.

In calculating the integral around the rectangular contour ΔC in Fig. A1.19, we have four terms. Since the dot product is integrated, we need only components of vector $\mathbf{F}$ along the rectangle sides. Omitting for simplicity the dependence on coordinates (c_1, c_2, c_3), the contribution to the integral around ΔC along sides 2 and 2′ is

$$\int_{\text{side } 2} \mathbf{F} \cdot \mathrm{d}\mathbf{l} + \int_{\text{side } 2'} \mathbf{F} \cdot \mathrm{d}\mathbf{l} = F_2\, \mathrm{d}l_2 - \left[F_2\, \mathrm{d}l_2 + \frac{\partial(F_2\, \mathrm{d}l_2)}{\partial c_3}\, \mathrm{d}c_3\right] = -\frac{\partial(F_2\, \mathrm{d}l_2)}{\partial c_3}\, \mathrm{d}c_3.$$

Similarly, the contribution to the integral along sides 3 and 3′ is

$$\int_{\text{side } 3} \mathbf{F} \cdot \mathrm{d}\mathbf{l} + \int_{\text{side } 3'} \mathbf{F} \cdot \mathrm{d}\mathbf{l} = -F_3\, \mathrm{d}l_3 + \left[F_3\, \mathrm{d}l_3 + \frac{\partial(F_3\, \mathrm{d}l_3)}{\partial c_2}\, \mathrm{d}c_2\right] = \frac{\partial(F_3\, \mathrm{d}l_3)}{\partial c_2}\, \mathrm{d}c_3.$$

Thus, the area of the rectangle being $\mathrm{d}l_2\, \mathrm{d}l_3$, according to the definition in Eq. (A1.27) the c_1 component of $\nabla \times \mathbf{F}$ is

$$\mathbf{u}_1 \cdot \nabla \times \mathbf{F} = \frac{1}{\mathrm{d}l_2\, \mathrm{d}l_3}\left[\frac{\partial(F_3\, \mathrm{d}l_3)}{\partial c_2}\, \mathrm{d}c_2 - \frac{\partial(F_2\, \mathrm{d}l_2)}{\partial c_3}\, \mathrm{d}c_3\right].$$

The c_2 and c_3 components of the curl are obtained in the same manner, so that the final general expression for the curl of vector $\mathbf{F}$ in any orthogonal coordinate

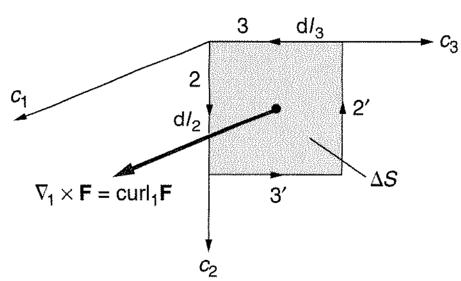


Figure A1.19 Determination of the c_1 component of $\nabla \times \mathbf{F}$

system is

$$\nabla \times \mathbf{F} = \mathbf{u}_1 \frac{1}{dl_2\, dl_3}\left[\frac{\partial(F_3\, dl_3)}{\partial c_2} dc_2 - \frac{\partial(F_2\, dl_2)}{\partial c_3} dc_3\right]$$
$$+ \mathbf{u}_2 \frac{1}{dl_1\, dl_3}\left[\frac{\partial(F_1\, dl_1)}{\partial c_3} dc_3 - \frac{\partial(F_3\, dl_3)}{\partial c_1} dc_1\right]$$
$$+ \mathbf{u}_3 \frac{1}{dl_1\, dl_2}\left[\frac{\partial(F_2\, dl_2)}{\partial c_1} dc_1 - \frac{\partial(F_1\, dl_1)}{\partial c_2} dc_2\right]. \tag{A1.28}$$

(Curl in orthogonal coordinates c_1, c_2, and c_3)

Because we know the length elements in rectangular, cylindrical, and spherical coordinate systems, using this general formula we easily find the expression for the curl in the three systems. In a rectangular system, where $c_1 = x$, $c_2 = y$, and $c_3 = z$, we obtain

$$\nabla \times \mathbf{F} = \mathbf{u}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{u}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{u}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right). \tag{A1.29}$$

(Curl in a rectangular coordinate system)

In a cylindrical system, $c_1 = r$, $c_2 = \phi$, $c_3 = z$, and the expression in Eq. (A1.28) results in

$$\nabla \times F = \mathbf{u}_r \left(\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right) + \mathbf{u}_\phi \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right) + \mathbf{u}_z \frac{1}{r}\left[\frac{\partial(rF_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi}\right]. \tag{A1.30}$$

(Curl in a cylindrical coordinate system)

In a spherical system, $c_1 = r$, $c_2 = \theta$, $c_3 = \phi$, so that according to Eq. (A1.28) the curl becomes

$$\nabla \times F = \mathbf{u}_r \frac{1}{r \sin\theta}\left[\frac{\partial(\sin\theta\, F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi}\right]$$
$$+ \mathbf{u}_\theta \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial(rF_\phi)}{\partial r}\right] + \mathbf{u}_\phi \frac{1}{r}\left[\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right]. \tag{A1.31}$$

(Curl in a spherical coordinate system)

Example A1.7—Divergence of the cross product of two vector functions. Consider the expression $\nabla \cdot (\mathbf{A} \times \mathbf{B})$, where $\mathbf{A}$ and $\mathbf{B}$ are vector functions. This is a legitimate operation, since the cross product of two vectors is again a vector. Because the del operator is a differential operator, we have to treat this expression as a derivative of a product. So we have

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \nabla \cdot (\mathbf{A} \times [\mathbf{B}]) + \nabla \cdot ([\mathbf{A}] \times \mathbf{B}),$$

where the brackets indicate that the vector function in the particular operation should be considered as a constant.

The del operator is a vector operator, so formally we can make use of the property of the triple scalar product in Eq. (A1.10), to obtain

$$\nabla \cdot (\mathbf{A} \times [\mathbf{B}]) = [\mathbf{B}] \cdot (\nabla \times \mathbf{A}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}), \tag{A1.32}$$

and

$$\nabla \cdot ([\mathbf{A}] \times \mathbf{B}) = [\mathbf{A}] \cdot (\mathbf{B} \times \nabla) = -\mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

So we finally have

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \tag{A1.33}$$

A1.4.4 LAPLACIAN OF A SCALAR FIELD

Because the gradient is a vector, a legitimate operation is to find the divergence of that vector, i.e., to find $\nabla \cdot (\nabla V)$. This is known as the *Laplacian of* V. Since the gradient implies differentiation and so does the divergence, a combination of second derivatives of the scalar function V results from this operation.

The symbol for the Laplacian of V is denoted by $\nabla^2 V$ (symbolizing the operation $\nabla \cdot (\nabla V)$, or sometimes ΔV. The symbol $\nabla^2 = \nabla \cdot \nabla$ is known as the *Laplacian operator*. In the rectangular coordinate system it has the form

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{A1.34}}$$

(Laplacian operator in rectangular coordinates)

Because we know the gradient and the divergence in rectangular, cylindrical, and spherical coordinate systems, it is not difficult to determine the Laplacian of a scalar function in these systems. For example, in the rectangular system, where we know the Laplacian operator,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \tag{A1.35}$$

(Laplacian of a scalar function in a rectangular coordinate system)

A1.4.5 LAPLACIAN OF A VECTOR FIELD

Curl of a curl of a vector function is a legitimate operation because curl of a vector is also a vector. It is again a second-order operation, resulting in a combination of second derivatives of the components of the vector function. It is written as $\nabla \times (\nabla \times \mathbf{F})$. Although it is an operator, ∇ can be considered as a vector, so that the last expression can be expanded using the identity in Eq. (A1.11), to obtain

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - (\nabla \cdot \nabla)\mathbf{F}, \tag{A1.36}$$

from which, recalling that $\nabla \cdot \nabla = \nabla^2$,

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}). \tag{A1.37}$$

(Definition of the Laplacian of a vector function)

The expression on the left-hand side of this equation is known as the *Laplacian of the vector function* $\mathbf{F}$, and the right-hand side tells us how it can be evaluated in any coordinate system. In the rectangular system it can be obtained directly from its definition:

$$\nabla^2 \mathbf{F} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_x \mathbf{u}_x + F_y \mathbf{u}_y + F_z \mathbf{u}_z).$$

After performing the indicated formal multiplications, this becomes

$$\nabla^2 \mathbf{F} = \left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2}\right)\mathbf{u}_x + \left(\frac{\partial^2 F_y}{\partial x^2} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_y}{\partial z^2}\right)\mathbf{u}_x + \left(\frac{\partial^2 F_z}{\partial x^2} + \frac{\partial^2 F_z}{\partial y^2} + \frac{\partial^2 F_z}{\partial z^2}\right)\mathbf{u}_x. \tag{A1.38}$$

Note that F_x, F_y, and F_z are *scalar functions* (these are scalar components of the vector $\mathbf{F}$). Therefore the expressions in the parentheses in the last equation are the Laplacians of the three scalar components of the vector $\mathbf{F}$. Thus we finally obtain

$$\nabla^2\mathbf{F} = (\nabla^2 F_x)\mathbf{u}_x + (\nabla^2 F_y)\mathbf{u}_y + (\nabla^2 F_z)\mathbf{u}_z. \tag{A1.39}$$

(Laplacian of vector **F** *in rectangular coordinate system)*

A1.4.6 DIVERGENCE THEOREM

The divergence theorem is a direct consequence of the definition of the divergence in Eq. (A1.22). It states that the volume integral of the divergence of a vector function **F** over any volume equals the flux of **F** through the surface S enclosing v.

For a small surface ΔS, without going to the limit and multiplying Eq. (A1.22) by Δv, we obtain

$$\nabla \cdot \mathbf{F}\,\Delta v = \oint_{\Delta S} \mathbf{F} \cdot d\mathbf{S}.$$

Imagine now a large domain of volume v, enclosed by a large surface S, subdivided into small volumes Δv enclosed by small surfaces ΔS. For all these small volumes we can write the preceding equation. Adding these equations, the left-hand side becomes the integral over volume v of $\nabla \cdot F\,dv$. The right-hand side represents the sum of the fluxes through all the small closed surfaces pressed one onto another and enclosed by the large surface S.

The flux is evaluated always with respect to the *outward* normal, as indicated in Fig. A1.20. The figure shows the cross section of a few small surfaces in the vicinity of the large surface. The fluxes through shared sides of two small closed surfaces is the same, but of opposite sign. Consequently, the flux through those sides of any small closed surface shared by another small closed surface cancels out. Only the flux through that side of a small surface that is a part of the large surface, S, is not canceled out. Therefore the sum of fluxes through all the small surfaces equals the

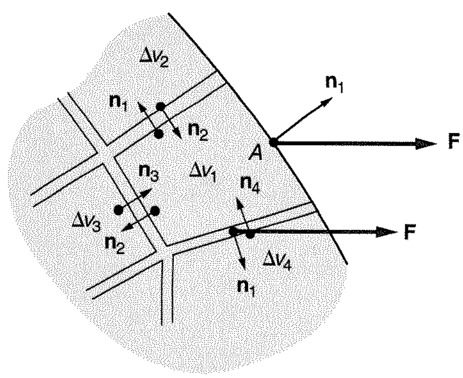


Figure A1.20 Cross section of a volume v consisting of many small subvolumes near the surface S enclosing v

flux through the large surface. We obtain an identity, often used in electromagnetics, called the *divergence theorem*:

$$\int_v \nabla \cdot \mathbf{F}\, dv = \oint_S \mathbf{F} \cdot d\mathbf{S}. \tag{A1.40}$$

(*Divergence theorem*)

A1.4.7 STOKES'S THEOREM

Stokes's theorem follows from the definition of the vector component of the curl of a vector function $\mathbf{F}$ in an arbitrary direction, Eq. (A1.29). It states that the flux of $\nabla \times \mathbf{F}$ through any open surface S equals the line integral of $\mathbf{F}$ around the contour bounding S.

To prove the theorem, consider a large open surface S and imagine the surface subdivided into many small surface elements ΔS. Figure A1.21 shows one part of S so subdivided, with a few small surface elements near the contour C bounding S.

Consider Eq. (A1.27) for one small surface ΔS, but without going to the limit. We multiply this equation by ΔS to obtain

$$\nabla \times \mathbf{F} \cdot \Delta \mathbf{S} = \oint_{\Delta C} \mathbf{F} \cdot d\mathbf{l};$$

because the product $\Delta S\mathbf{n}$ is the vector surface element, $\Delta \mathbf{S}$.

By adding such equations for all the surface elements, the left-hand side becomes simply the flux of the vector * $\mathbf{F}$ through the surface S. The line integral along any side of a small contour ΔC shared by an adjacent small contour cancels out (Fig. A1.21). We are left with the line integral along segments of the small contours belonging to the large contour C bounding S. Thus, writing $d\mathbf{S}$ instead of $\Delta \mathbf{S}$,

$* \nabla \times F$

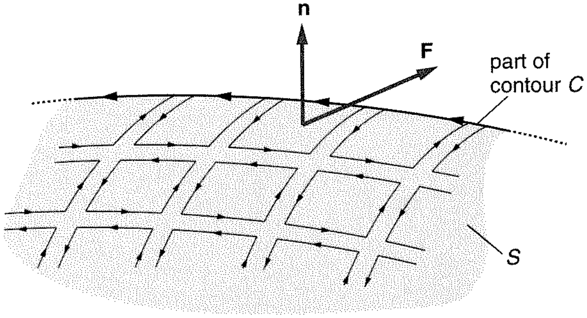


Figure A1.21 Part of the surface S consisting of many small subsurfaces, near the contour C bounding S

$$\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{l}. \tag{A1.41}$$

(Stokes's theorem)

THE DIVERGENCE + Stokes's theoremS ~~has~~ HAVE many applications in electromagnetic field theory. We use the two theorems now to derive two important identities of vector calculus.

Example A1.8—Divergence of the curl is identically zero. The operation $\nabla \cdot (\nabla \times \mathbf{A})$ is a legitimate operation, since $\nabla \times \mathbf{A}$ is a vector. In Eq. (A1.~~41~~ 40) let $\mathbf{F} = \nabla \times \mathbf{A}$. We have

$$\int_v \nabla \cdot (\nabla \times \mathbf{A})\, \mathrm{d}v = \oint_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S}. \tag{A1.42}$$

The surface integral on the right side will not be changed if we assume the closed surface to have a miniature hole. However, in such a case we can consider the surface to be an *open surface*, with the boundary being the hole boundary. So we can apply Stokes's theorem to the right-hand side. The line integral of $\mathbf{A}$ around the miniature hole tends to zero as the hole shrinks. Therefore we first find that the following integral identity holds:

$$\oint_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} \equiv 0. \tag{A1.43}$$

So the left-hand side in Eq. (A1.4~~3~~2) is zero for *any volume*. This is possible only if the integrand is zero, i.e., if

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0. \tag{A1.44}$$

(Valid for any vector function **A***)*

Example A1.9—Curl of the gradient is identically zero. The operation $\nabla \times (\nabla V)$ is a legitimate operation, since ∇V is a vector. In Eq. (A1.41) let $F = \mathbf{A} = \nabla V$, to obtain

$$\int_S \nabla \times (\nabla V) \cdot \mathrm{d}\mathbf{S} = \oint_C \nabla V \cdot \mathrm{d}\mathbf{l}. \tag{A1.45}$$

The integral on the right-hand side can be written in the form

$$\oint_C \nabla V \cdot \mathrm{d}\mathbf{l} = \oint_C \frac{\partial V}{\partial l} \mathrm{d}l = \oint_C \mathrm{d}_l V,$$

where $\mathrm{d}_l V$ is the increment of the function V when the point moves from the beginning to the end of the line element $\mathrm{d}\mathbf{l}$ of the contour C. Since in calculating the integral we start at a point of C, describe the full contour, and return to the starting point, the integration amounts to integrating a function from a point to that same point. We know that the result is zero. So the right-hand side in Eq. (A1.45) is zero,

$$\oint_C \nabla V \cdot \mathrm{d}\mathbf{l} \equiv 0. \tag{A1.46}$$

For the left-hand side in Eq. (A1.~~46~~45) to be zero for *any surface S*, the integrand must be identically zero. So we have another identity,

$$\nabla \times (\nabla V) = \cancel{\nabla \times (\nabla V)} = 0. \qquad \text{(A1.47)}$$

(Valid for any scalar function V)

Questions and problems: QA1.28 to QA1.41, PA1.25 to PA1.35

QUESTIONS

QA1.1. Classify the following quantities as either scalar or vector quantities: mass, time, weight, course of a ship, position of a point with respect to another point, acceleration, power of an engine, current intensity, voltage.

QA1.2. Classify the following quantities as either scalar or vector fields: temperature in a room, mass density of an inhomogeneous body, weight per unit volume of an inhomogeneous body, velocity of air particles in a room, velocity of water particles in a river.

QA1.3. Discuss whether the definition of subtraction of vectors follows from the definition of vector addition, or whether an additional definition is indispensable.

QA1.4. Sketch the dependence of the dot product of two unit vectors on the angle between them.

QA1.5. Why is the cross product not commutative?

QA1.6. Which of the following expressions does not make sense, and why? (1) $\mathbf{A} \times (\mathbf{B} \cdot \mathbf{C})$, (2) $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$, (3) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\cdot\mathbf{D})$, (4) $(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\times\mathbf{D})$, (5) $[(\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}]\times\mathbf{D}$, (6) $[(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}]\times\mathbf{D}$, (7) $[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})] \cdot \mathbf{D}$

QA1.7. What is the necessary and sufficient condition for three vectors, **A**, **B**, and **C**, to be in the same plane?

QA1.8. Prove that $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A}$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{B}$ are zero.

QA1.9. If $\mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C}$, does it mean that $\mathbf{A} = \mathbf{B}$? Explain.

QA1.10. If $\mathbf{A} \times \mathbf{C} = \mathbf{B} \times \mathbf{C}$, does it mean that $\mathbf{A} = \mathbf{B}$? Explain.

QA1.11. Can the dot product of two vectors be negative? Can the magnitude of the cross product of two vectors be negative?

QA1.12. In which cases is (1) $\mathbf{A} \cdot \mathbf{B} = 0$, (2) $\mathbf{A} \times \mathbf{B} = 0$, and (3) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = 0$?

QA1.13. Explain the meaning of $(\mathbf{A} \cdot \mathbf{B})\mathbf{C}$. Is this the same as $\mathbf{C}(\mathbf{A} \cdot \mathbf{B})$?

QA1.14. Which of the following sets of coordinates define a point? (1) $x = 2$, $y = -4$, $z = 0$, (2) $r = -4$, $\phi = 0$, $z = -1$, (3) $r = 3$, $\theta = -90°$, $\phi = 0$

QA1.15. How do you obtain the components of a vector **A** in the direction of the three base unit vectors in any coordinate system?

QA1.16. Define coordinate lines.

QA1.17. Define differential lengths along coordinate lines.

QA1.18. Define orthogonal coordinate systems.

QA1.19. What is a "right-handed coordinate system"?

QA1.20. A vector is defined by its three orthogonal components. Determine the vector itself and the unit vector in its direction.

QA1.21. A vector is defined by its starting and end points in the rectangular system. Determine its components, the vector itself, and the unit vector in its direction.

QA1.22. A point is defined (a) in a cylindrical coordinate system by its coordinates (r, ϕ, z), and (b) in a spherical coordinate system by its coordinates (r, θ, ϕ). Find the rectangular coordinates of the point in both cases.

QA1.23. A point is defined in a rectangular coordinate system by its coordinates (x, y, z). Find the coordinates of the point in (a) cylindrical and (b) spherical coordinate systems.

QA1.24. If $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$ are base unit vectors in the rectangular coordinate system, and $\mathbf{u}_r$ that in the cylindrical coordinate system, what are the values of the following products? (1) $\mathbf{u}_r \cdot \mathbf{u}_x$, (2) $\mathbf{u}_r \cdot \mathbf{u}_y$, (3) $\mathbf{u}_r \cdot \mathbf{u}_z$, (4) $\mathbf{u}_r \times \mathbf{u}_x$, (5) $\mathbf{u}_r \times \mathbf{u}_y$, (6) $\mathbf{u}_r \times \mathbf{u}_z$

QA1.25. If $\mathbf{u}_\phi$ is the base unit vector in the cylindrical coordinate system, what are the following products equal to? (1) $\mathbf{u}_\phi \cdot \mathbf{u}_x$, (2) $\mathbf{u}_\phi \cdot \mathbf{u}_y$, (3) $\mathbf{u}_\phi \cdot \mathbf{u}_z$, (4) $\mathbf{u}_\phi \times \mathbf{u}_x$, (5) $\mathbf{u}_\phi \times \mathbf{u}_y$, (6) $\mathbf{u}_\phi \times \mathbf{u}_z$

QA1.26. If $\mathbf{u}_\zeta$ is the base unit vector in the cylindrical coordinate system in the direction of the z axis, what are the values of the following products? (1) $\mathbf{u}_\zeta \cdot \mathbf{u}_x$, (2) $\mathbf{u}_\zeta \cdot \mathbf{u}_y$, (3) $\mathbf{u}_\zeta \cdot \mathbf{u}_z$, (4) $\mathbf{u}_\zeta \times \mathbf{u}_x$, (5) $\mathbf{u}_\zeta \times \mathbf{u}_y$, (6) $\mathbf{u}_\zeta \times \mathbf{u}_z$

QA1.27. If $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$ are base unit vectors in the rectangular coordinate system and $\mathbf{u}_r$ that in the spherical coordinate system, what are the values of the following products? (1) $\mathbf{u}_r \cdot \mathbf{u}_x$, (2) $\mathbf{u}_r \cdot \mathbf{u}_y$, (3) $\mathbf{u}_r \cdot \mathbf{u}_z$, (4) $\mathbf{u}_r \times \mathbf{u}_x$, (5) $\mathbf{u}_r \times \mathbf{u}_y$, (6) $\mathbf{u}_r \times \mathbf{u}_z$

QA1.28. What is the physical meaning of the vector $-\nabla f$, where f is a scalar function?

QA1.29. Vector $\mathbf{r}$ is the position vector in a scalar field described by a function f. What is the directional derivative of the function f in the direction defined by the vector $\mathbf{r} \times \nabla f$?

QA1.30. What is the unit of the del operator in the cartesian (rectangular) coordinate system?

QA1.31. What is the divergence of the velocity field of the water flow in a pipe?

QA1.32. A pipe with a liquid flowing through it has a very small hole through which the liquid leaks out of the pipe. If the surface in the definition of the divergence is assumed to be finite, and if it encloses part of the pipe and the hole, is the divergence of the liquid velocity field nonzero at that point? Explain.

QA1.33. Propose a model in a time-variable flow of a compressible gas for which the divergence of the velocity field might be nonzero. Explain.

QA1.34. A fluid flows through a pipe with a rough wall. Would you expect the curl of the fluid velocity field to be nonzero? Explain.

QA1.35. A small spherical pressured cloud of gas is suddenly freed to disperse. Assuming completely symmetrical gas dispersion, which of the three functions of the gas velocity (a function of coordinates and time), the gradient, the divergence, and the curl, would be zero, and which nonzero? Explain.

QA1.36. Does the divergence theorem apply to time-dependent fields? Explain.

QA1.37. A volume v is limited by a surface S_0 from outside, but has holes limited by surfaces S_1, S_2, and S_3 from inside. Can the divergence theorem be applied to such a domain? If it can, explain in detail the formulation of the theorem in such a case.

QA1.38. Is it possible to apply the divergence theorem to a vector function of the form $\mathbf{F} \times \mathbf{G}$? Explain.

QA1.39. Does the Stokes's theorem apply to time-dependent fields?

QA1.40. An open surface S is limited by a large contour C_0, but has holes limited by small contours C_1 and C_2. Is it possible to apply Stokes's theorem to such a surface? If so, explain in detail the expression for the theorem in such a case.

QA1.41. Is it possible to apply Stokes's theorem to the vector function of the form $(\mathbf{F} \cdot \mathbf{G})\mathbf{F}$? Explain.

PROBLEMS

PA1.1. Prove that the distributive law is valid for vector addition and for the three types of vector products (product of a vector with a scalar, dot product, and cross product).

PA1.2. Prove that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$.

PA1.3. Prove that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$.

PA1.4. Let $\mathbf{n}$ be a unit vector in an arbitrary direction, and $\mathbf{r}_0$ be the position vector of a point in a plane normal to $\mathbf{n}$. What is the equation of the plane (i.e., which equation must be satisfied by the position vector of any point belonging to the plane)?

PA1.5. Let $\mathbf{n}$ be the unit vector along a ζ axis, and $\mathbf{r}_0$ be the position vector of the point $\zeta = 0$ on the axis. What is the expression for the position vector, $\mathbf{r}$, of a point with a coordinate ζ on the ζ axis?

PA1.6. Two unit vectors, $\mathbf{u}_1$ and $\mathbf{u}_2$, are in the half-plane $x > 0$ of the $z = 0$ plane. The unit vector $\mathbf{u}_1$ makes an angle $\alpha_1 > 0$ with the x axis, and $\mathbf{u}_2$ an angle α_2. Find the dot product of the two unit vectors (1) if $\alpha_1 > \alpha_2 > 0$, and (2) if $\alpha_2 < 0$. Do the results remind you of some trigonometric formulas?

PA1.7. Determine the unit vector in the direction of a vector $\mathbf{R}$, with the origin at a point $M'(x', y', z')$, and the tip at a point $M(x, y, z)$.

PA1.8. Express the base unit vectors of the cylindrical coordinate system in terms of those of the rectangular system, and conversely.

PA1.9. A point in the cylindrical coordinate system is defined by the coordinates (r, ϕ, z). Determine the rectangular coordinates of the point.

PA1.10. A point in the spherical coordinate system is defined by the coordinates (r, θ, ϕ). Determine the rectangular coordinates of the point.

PA1.11. A point in the rectangular system is defined by the coordinates (x, y, z). Determine the cylindrical and spherical coordinates of the point.

PA1.12. A vector is described at a point $M(r, \phi, z)$ in the cylindrical coordinate system by its rectangular components A_x, A_y, and A_z. Determine the cylindrical components of the vector.

PA1.13. A vector is described at a point $M(r, \theta, \phi)$ in the spherical coordinate system by its rectangular components A_x, A_y, and A_z. Determine the spherical components of the vector.

PA1.14. Given that $\mathbf{A} = A_x\mathbf{u}_x + A_y\mathbf{u}_y + A_z\mathbf{u}_z$ and $\mathbf{B} = B_x\mathbf{u}_x + B_y\mathbf{u}_y + B_z\mathbf{u}_z$, determine the smaller angle between the two vectors.

PA1.15. The *direction cosines* of a vector are cosines of angles between the vector and the base unit vectors. Determine the sum of the squares of the direction cosines.

PA1.16. If l_A, m_A, and n_A are direction cosines of a vector $\mathbf{A}$, and l_B, m_B, and n_B are those of a vector $\mathbf{B}$, find the cosine of the smaller angle between the two vectors.

PA1.17. A vector $\mathbf{A}$ is given by its components A_x, A_y, and A_z. Find the angles the vector makes with the three rectangular coordinate axes.

PA1.18. Express the vector product of two vectors given by their rectangular components in the form of a determinant.

PA1.19. Express the triple scalar product of three vectors given by their rectangular components in the form of a determinant.

PA1.20. Given that $\mathbf{A} = 3\mathbf{u}_x - 5\mathbf{u}_y + 8\mathbf{u}_z$, $\mathbf{B} = 4\mathbf{u}_x + \mathbf{u}_y - 3\mathbf{u}_z$, and $\mathbf{C} = -2\mathbf{u}_x + 3\mathbf{u}_y - 4\mathbf{u}_z$, determine: (1) $\mathbf{A} + \mathbf{B} - \mathbf{C}$, (2) $\mathbf{A} \cdot \mathbf{B}$, (3) $\mathbf{B} \times \mathbf{C}$, (4) $\mathbf{A} \times \mathbf{C}$, (5) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, (6) $(\mathbf{A} \cdot \mathbf{B})\mathbf{C}$, (7) $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$, and (8) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$. Find the smaller angle between $\mathbf{A}$ and $\mathbf{B}$, and between $\mathbf{A}$ and $\mathbf{C}$, the magnitudes of the three vectors, and the unit vectors in their direction.

PA1.21. If $\mathbf{A} = 2\mathbf{u}_r - \mathbf{u}_\phi + \mathbf{u}_z$, with the origin at the point $M(3, 0, 0)$ of the cylindrical coordinate system, and $\mathbf{B} = 2\mathbf{u}_r - \mathbf{u}_\phi + \mathbf{u}_z$, with the origin at the point $N(5, \pi/2, 5)$, determine: (1) $\mathbf{A} + \mathbf{B}$, (2) $\mathbf{A} \cdot \mathbf{B}$, (3) $\mathbf{A} \times \mathbf{B}$, (4) the smaller angle between the two vectors, and (5) their magnitudes.

PA1.22. If $\mathbf{A} = 3\mathbf{u}_r + 2\mathbf{u}_\theta - \mathbf{u}_\phi$, with the origin at the point $M(1, \pi/2, 0)$ of the spherical coordinate system, and $\mathbf{B} = -2\mathbf{u}_r - 4\mathbf{u}_\theta + 2\mathbf{u}_\phi$, with the origin at the point $N(3, \pi/2, \pi)$, determine: (1) $\mathbf{A} + \mathbf{B}$, (2) $\mathbf{A} - \mathbf{B}$, (3) $\mathbf{A} \cdot \mathbf{B}$, (4) $\mathbf{A} \times \mathbf{B}$, (5) the smaller angle between the two vectors, and (6) their magnitudes.

PA1.23. If $\mathbf{A} = \mathbf{u}_x + 4\mathbf{u}_y + 3\mathbf{u}_z$, and $\mathbf{B} = 2\mathbf{u}_x + \mathbf{u}_y - \mathbf{u}_z$, determine the smaller angle between the vectors, the unit vectors along the two vectors, and the ratio of their magnitudes.

PA1.24. Determine the differential volume and three differential areas normal to the three base unit vectors in rectangular, cylindrical, and spherical coordinate systems.

PA1.25. From questions QA1.9 and QA1.10, we know that the relations $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$ and $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$, taken separately, do not mean that $\mathbf{B} = \mathbf{C}$. Is this still true if *both* these relations are satisfied?

PA1.26. Determine the gradient of the scalar function $f(x, y, z) = x \cos 3y \exp(-4z)$, and the divergence and curl of the vector function $\mathbf{F}(x, y, z) = (2x^2yz)\mathbf{u}_x + x \sin y \cos z\mathbf{u}_y + (x + y + z)\mathbf{u}_z$.

PA1.27. Prove that the curl of the gradient of the function $f(x, y, z)$ from problem PA1.26 is zero, and that the divergence of the curl of $\mathbf{F}(x, y, z)$ is zero.

PA1.28. Prove that the identities $\nabla \times [\nabla V(x, y, z)] = 0$ and $\nabla \cdot [\nabla \times \mathbf{A}(x, y, z)] = 0$ are satisfied for any twice differentiable functions $V(x, y, z)$ and $\mathbf{A}(x, y, z)$.

PA1.29. Let $\mathbf{A}(x, y, z) = xyz\mathbf{u}_x - \sin x \cos y e^z \mathbf{u}_y + xy^2z^3\mathbf{u}_z$. (1) Evaluate the line integral of $\mathbf{A}(x, y, z)$ around a rectangular contour in the plane $z = 0$, with a vertex at the origin, a side a along the x axis and a side b along the y axis; start from the origin along side a. (2) Evaluate the flux of $\nabla \times \mathbf{A}(x, y, a)$ through the contour in the direction of the base unit vector $\mathbf{u}_z$. Can you conclude that the results you obtained are correct?

PA1.30. Let $\mathbf{A}(x, y, z) = xyz\mathbf{u}_x + x^2y^2z^2\mathbf{u}_y + x^3y^3z^3\mathbf{u}_z$. (1) Evaluate the flux of $\mathbf{A}(x, y, z)$ through a cube with a vertex at the origin, and with sides of length 1 along coordinate lines for $x \geq 0$, $y \geq 0$, and $z \geq 0$. (2) Evaluate the volume integral of $\nabla \cdot \mathbf{A}(x, y, z)$ over the cube. Do you have a simple check for the accuracy of the results?

PA1.31. The identity in Eq. (A1.44), $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, was proved by considering the closed surface S having a small hole that shrinks to zero. Prove the identity by considering the closed surfaces to consist of two arbitrary open surfaces with a common boundary C.

PA1.32. The gradient of a scalar function f can alternatively be defined in the form very similar to that of the divergence in Eq. (A1.22),

$$\nabla f(c_1, c_2, c_3) = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oint_{\Delta S} f \, d\mathbf{S}.$$

Using arguments similar to those for deriving the divergence in orthogonal coordinate systems, derive the analogous general expression for the gradient of f.

PA1.33. From the general formula for the gradient obtained in problem PA1.32, prove that the same expressions for the gradient in rectangular, cylindrical, and spherical coordinate systems are obtained as in section A1.4.1.

PA1.34. The curl of a vector function $\mathbf{F}$ can alternatively be defined in the form very similar to that of the divergence in Eq. (A1.22),

$$\nabla \times \mathbf{F}(c_1, c_2, c_3) = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oint_{\Delta S} d\mathbf{S} \times \mathbf{F}.$$

Using arguments similar to those for deriving the divergence in orthogonal coordinate systems, derive the analogous general expression for the curl of $\mathbf{F}$.

PA1.35. From the general formula for the curl obtained in problem PA1.34, prove that the same expressions for the curl in rectangular, cylindrical, and spherical coordinate systems are obtained as in section A1.4.3.

Appendix 2

Summary of Vector Identities

In the relationships described in this Appendix, **A**, **B**, **C**, **D**, and **F** are vector functions, and V, W, and f are scalar functions of coordinates. It is assumed that they have all necessary derivatives.

ALGEBRAIC IDENTITIES

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
3. $\mathbf{A} \cdot \mathbf{B} = AB\cos(\mathbf{A}, \mathbf{B})$
4. $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2$
5. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
6. $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$
7. $V(\mathbf{A} + \mathbf{B}) = V\mathbf{A} + V\mathbf{B}$
8. $\mathbf{A} \times \mathbf{B} = AB\sin(\mathbf{A}, \mathbf{B})\mathbf{n}$, where $\mathbf{n}$ is the unit vector normal to the plane of vectors **A** and **B**, and its direction is determined by the right-hand rule when vector **A** is rotated to coincide with vector **B** in the shortest way.

9. $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$
10. $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
11. $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$
12. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$
13. $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{D} \cdot [(\mathbf{A} \times \mathbf{B}) \times \mathbf{C})] = \mathbf{C} \cdot [\mathbf{D} \times (\mathbf{A} \times \mathbf{B})] = \ldots$ (using nos. 11 and 12, several other forms can be obtained)
14. $\mathbf{u}_A = \mathbf{A}/A$ (unit vector in the direction of vector $\mathbf{A}$)

DIFFERENTIAL IDENTITIES

15. $\mathbf{u}_x \cdot \nabla V = \partial V/\partial x$ (x: arbitrary axis)
16. $\nabla(V + W) = \nabla V + \nabla W$
17. $\nabla(VW) = V\nabla W + W\nabla V$
18. $\nabla f(V) = f'(V)\nabla V$
19. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$
20. $\nabla \cdot (V\mathbf{A}) = V\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla V$
21. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$
22. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
23. $\nabla \times (V\mathbf{A}) = (\nabla V) \times \mathbf{A} + V\nabla \times \mathbf{A}$
24. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
25. $\nabla \cdot (\nabla V) = \nabla^2 V = \Delta V =$ Laplacian of V
26. $\nabla \cdot [\nabla(VW)] = V\nabla \cdot (\nabla W) + 2\nabla V \cdot \nabla W + W\nabla \cdot (\nabla V)$
27. $\nabla \times (\nabla V) = 0$
28. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

INTEGRAL IDENTITIES

Basic integral identities

29. $\displaystyle\int_v \nabla f \, \mathrm{d}v = \oint_S f \, \mathrm{d}\mathbf{S}$
30. $\displaystyle\int_v \nabla \cdot \mathbf{F} \, \mathrm{d}v = \oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ (the divergence theorem)
31. $\displaystyle\int_v \nabla \times \mathbf{F} \, \mathrm{d}v = \oint_S \mathrm{d}\mathbf{S} \times \mathbf{F}$
32. $\displaystyle\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{l}$ (Stokes's theorem)

Some integral identities derived from basic integral identities

33. $\displaystyle\oint_S \nabla \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} = 0$
(set $\mathbf{F} = \nabla \times \mathbf{A}$ in no. 30, and take into account no. 24)

34. $\int_v (V\nabla^2 W + \nabla V \cdot \nabla W)\,dv = \oint_S V\frac{\partial W}{\partial n}\,dS$
(set $\mathbf{F} = V\nabla W$ in no. 30, and use nos. 20 and 15)

35. $\int_v (V\nabla^2 W - W\nabla^2 V)\,dv = \oint_S \left(V\frac{\partial W}{\partial n} - W\frac{\partial V}{\partial n}\right)\,dS$
(set $\mathbf{F} = V\nabla W - W\nabla V$ in no. 30, and use nos. 20 and 15)

36. $\int_v [\nabla \times \mathbf{A} \cdot \nabla \times \mathbf{B} - \mathbf{A} \cdot \nabla \times (\nabla \times \mathbf{B})]\,dv = \oint_S (\mathbf{A} \times \nabla \times \mathbf{B}) \cdot d\mathbf{S}$
(set $\mathbf{F} = \mathbf{A} \times \nabla \times \mathbf{B}$ in no. 30, and use nos. 20 and 15)

37. $\int_v \nabla V \cdot \nabla \times \mathbf{A}\,dv = \int_S V\nabla \times \mathbf{A} \cdot d\mathbf{S}$
(set $\mathbf{F} = V\nabla \times \mathbf{A}$ in no. 30, and use nos. 20 and 24)

38. $\int_S d\mathbf{S} \times \nabla V = \oint_C V\,d\mathbf{l}$
(set $\mathbf{F} = \mathbf{C}V$ in no. 29, where $\mathbf{C}$ is a constant vector, use nos. 23 and 11, and take $\mathbf{C}$ in front of both integrals)

39. $\int_S (\nabla V \times \nabla W) \cdot d\mathbf{S} = \oint_C V\nabla W \cdot d\mathbf{l}$
(set $\mathbf{F} = V\nabla W$ in no. 29, and use nos. 23 and 27)

GRADIENT, DIVERGENCE, CURL, AND LAPLACIAN IN ORTHOGONAL COORDINATE SYSTEMS

Rectangular coordinate system

Notation: $f = f(x, y, z)$, $\mathbf{F} = \mathbf{F}(x, y, z)$, $F_x = F_x(x, y, z)$, $F_y = F_y(x, y, z)$, $F_z = F_z(x, y, z)$

40. $\nabla f = \frac{\partial f}{\partial x}\mathbf{u}_x + \frac{\partial f}{\partial y}\mathbf{u}_y + \frac{\partial f}{\partial z}\mathbf{u}_z$

41. $\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

42. $\nabla \times \mathbf{F} = \mathbf{u}_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{u}_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{u}_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$

43. $\nabla^2 f \equiv \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

44. $\nabla^2 \mathbf{F} = (\nabla^2 F_x)\mathbf{u}_x + (\nabla^2 F_y)\mathbf{u}_y + (\nabla^2 F_z)\mathbf{u}_z$

Cylindrical coordinate system

Notation: $f = f(r, \phi, z)$, $\mathbf{F} = \mathbf{F}(r, \phi, z)$, $F_r = F_r(r, \phi, z)$, $F_\phi = F_\phi(r, \phi, z)$, $F_z = F_z(r, \phi, z)$

45. $\nabla f = \frac{\partial f}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{u}_\phi + \frac{\partial f}{\partial z}\mathbf{u}_z$

46. $\nabla \cdot \mathbf{F} = \dfrac{1}{r}\dfrac{\partial(rF_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial F_\phi}{\partial \phi} + \dfrac{\partial F_z}{\partial z}$

47. $\nabla \times \mathbf{F} = \mathbf{u}_r\left(\dfrac{1}{r}\dfrac{\partial F_z}{\partial \phi} - \dfrac{\partial F_\phi}{\partial z}\right) + \mathbf{u}_\phi\left(\dfrac{\partial F_r}{\partial z} - \dfrac{\partial F_z}{\partial r}\right) + \mathbf{u}_z\dfrac{1}{r}\left[\dfrac{\partial(rF_\phi)}{\partial r} - \dfrac{\partial F_r}{\partial \phi}\right]$

48. $\nabla^2 f \equiv \nabla \cdot (\nabla f) = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial \phi^2} + \dfrac{\partial^2 f}{\partial z^2}$

49. $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$

Spherical coordinate system

Notation: $f = f(r,\theta,\phi)$, $\mathbf{F} = \mathbf{F}(r,\theta,\phi)$, $F_r = F_r(r,\theta,\phi)$, $F_\theta = F_\theta(r,\theta,\phi)$, $F_\phi = F_\phi(r,\theta,\phi)$

50. $\nabla f = \dfrac{\partial f}{\partial r}\mathbf{u}_r + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\mathbf{u}_\theta + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi}\mathbf{u}_\phi$

51. $\nabla \cdot \mathbf{F} = \dfrac{1}{r^2}\dfrac{\partial(r^2F_r)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial(\sin\theta\, F_\theta)}{\partial \theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial F_\phi}{\partial \phi}$

52. $\nabla \times \mathbf{F} = \mathbf{u}_r\dfrac{1}{r\sin\theta}\left[\dfrac{\partial(\sin\theta\, F_\phi)}{\partial \theta} - \dfrac{\partial F_\theta}{\partial \phi}\right] + \mathbf{u}_\theta\dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{\partial F_r}{\partial \phi} - \dfrac{\partial(rF_\phi)}{\partial r}\right]$
$+ \mathbf{u}_\phi\dfrac{1}{r}\left[\dfrac{\partial(rF_\theta)}{\partial r} - \dfrac{\partial F_r}{\partial \theta}\right]$

53. $\nabla^2 f \equiv \nabla \cdot (\nabla f) = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial f}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 f}{\partial \phi^2}$

54. $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$

Appendix 3

Values of Some Important Physical Constants

Name	Symbol	Value
Velocity of light in a vacuum	c_0	$2.99792458 \cdot 10^8$ m/s
Absolute value of electron charge	e	$(1.60210 \pm 0.00007) \cdot 10^{-19}$C
Magnetic moment of electron	—	$(9.2837 \pm 0.0002) \cdot 10^{-32}$A $\cdot$ m^2
Mass of electron at rest	m_e	$(9.1083 \pm 0.0003) \cdot 10^{-31}$ kg
Mass of neutron at rest	m_n	$(1.67470 \pm 0.00004) \cdot 10^{-27}$ kg
Mass of proton at rest	m_p	$(1.67239 \pm 0.00004) \cdot 10^{-27}$ kg
Permeability of a vacuum	μ_0	$4\pi \cdot 10^{-7}$H/m$(\pi = 3.14159265\ldots)$
Permittivity of a vacuum	ϵ_0	$1/(\mu_0 c_0^2) = 8.85419 \cdot 10^{-12}$F/m
Standard acceleration of free fall	g_g	9.80665 m/s^2
Gravitational constant	γ	$(6.673 \pm 0.003) \cdot 10^{-11}$N $\cdot$ m^2/kg^2

Appendix 4

Electrical Properties of Some Materials at Room Temperature and Low Frequencies

Material	ϵ_r	σ(S/m)	Comment
Silver		$6.14 \cdot 10^7$	oxidizes
Copper		$5.65 - 5.8 \cdot 10^7$	oxidizes
Gold		$4.1 \cdot 10^7$	inert
Aluminum		$3.8 \cdot 10^7$	oxidizes
Tungsten		$1.8 \cdot 10^7$	very hard
Zinc		$1.74 \cdot 10^7$	
Brass			
(30% zinc)		$1.5 \cdot 10^7$	
Nickel		$1.28 \cdot 10^7$	
Bronze		$1 \cdot 10^7$	

(continued)

Material	ϵ_r	σ(S/m)	Comment
Iron		$1 \cdot 10^7$	
Steel		$0.5 - 1 \cdot 10^7$	
Tin		$0.87 \cdot 10^7$	
Nichrome		$0.1 \cdot 10^7$	
Graphite		$7 \cdot 10^4$	
Seawater	70	$3 - 5$	
Wet earth	5 to 15	10^{-2} to 10^{-3}	
Dry earth	2 to 6	10^{-4} to 10^{-5}	
Fresh water (lake)	80	10^{-3}	
Distilled water	80	$2 \cdot 10^{-4}$	
Alcohol	25		
Air	1.006		breakdown 3 kV/m
Styrofoam	1.03		
Teflon	2.1		
Polystyrene	2.56		
Rubber	2.5 to 3	10^{-15}	
Paper	2 to 4		
Quartz	3.8		
Glass	4 to 10	10^{-12}	
Mica	5.4		
Porcelain	6	10^{-10}	
Diamond	5 to 6	$2 \cdot 10^{-13}$	good heat conductor
Silicon	11	semiconductor, σ depends on doping level	
Gallium arsenide	13	semiconductor, σ depends on doping level	
Barium titanate	60 to 3600	anisotropic in crystalline form, dependent on mechanical conditions	

Appendix 5

Magnetic Properties of Some Materials

Material	μ_r	Comment
Silver	0.9999976	diamagnetic
Bismuth	0.999982	diamagnetic
Copper	0.99999	diamagnetic
Gold	0.99996	diamagnetic
Water	0.9999901	diamagnetic
Aluminum	1.000021	paramagnetic
Platinum	1.0003	paramagnetic
Tungsten	1.00008	paramagnetic
Ferrite	1000	e.g. $NiO \cdot Fe_2O_3$, insulator
Ferroxcube 3	1500	Mn-Zn ferrite powder
Cobalt	250	ferromagnetic
Nickel	600	ferromagnetic
Steel	2000	ferromagnetic
Iron (0.2 impurity)	5000	ferromagnetic
Silicon iron	7000	ferromagnetic
Purified iron (0.05 impurity)	$2 \cdot 10^5$	ferromagnetic
Supermalloy	as high as 10^6	ferromagnetic

Appendix 6

Standard (IEC) Multipliers of Fundamental Units

Multiple	Prefix	Symbol	Example
10^{12}	tera	T	$1.2\,\text{Tm} = 1.2 \cdot 10^{12}\,\text{m}$
10^{9}	giga	G	$12\,\text{GW} = 12 \cdot 10^{9}\,\text{W}$
10^{6}	mega	M	$5\,\text{MHz} = 5 \cdot 10^{6}\,\text{Hz}$
10^{3}	kilo	k	$22\,\text{kV} = 22 \cdot 10^{3}\,\text{V}$
10^{2}	hecto	h	$100\,\text{hN} = 100 \cdot 10^{2}\,\text{N}$
10	deca	da	$32\,\text{dag} = 320\,\text{g}$
10^{-1}	deci	d	$2\,\text{dm} = 2 \cdot 10^{-1}\,\text{m}$
10^{-2}	centi	c	$75\,\text{cm} = 70 \cdot 10^{-2}\,\text{m}$
10^{-3}	milli	m	$56\,\text{m}\Omega = 56 \cdot 10^{-3}\,\Omega$
10^{-6}	micro	μ	$25\,\mu\text{H} = 25 \cdot 10^{-6}\,\text{H}$
10^{-9}	nano	n	$56\,\text{nA} = 56 \cdot 10^{-9}\,\text{A}$
10^{-12}	pico	p	$40\,\text{pF} = 40 \cdot 10^{-12}\,\text{F}$
10^{-15}	femto	f	$1.2\,\text{fm} = 1.2 \cdot 10^{-15}\,\text{m}$
10^{-18}	atto	a	$0.16\,\text{aC} = 0.16 \cdot 10^{-18}\,\text{C}$

Appendix 7

The Greek Alphabet

Letter		Name	Letter		Name
A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omikron
Δ	δ	delta	Π	π	pi
E	ϵ, ε	epsilon	R	ρ, ϱ	rho
Z	ζ	zeta	Σ	σ	sigma
E	η	eta	T	τ	tau
Θ	θ, ϑ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ, φ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega

Appendix 8

Theory of Lossless Metallic Waveguides

A8.1 General Theory of Metallic Waveguides

Although waveguides are never lossless, assuming that they have no losses greatly simplifies the analysis. We limit our analysis to cylindrical waveguides, i.e., no bends of waveguides are allowed. Consider a general lossless waveguide with a cross section as shown in Fig. 23.1 (Chapter 23), guiding a time-harmonic wave. Assume that the dielectric in the waveguide has parameters ϵ and μ. Let us write Maxwell's equations in phasor (complex) form for electromagnetic waves propagating in the $+z$ direction along the waveguide.

We know that waves propagating along the waveguide, if represented in phasor (complex) form, generally have a factor $e^{-(\alpha+j\beta)z} = e^{-\gamma z}$. Because the material of the waveguide is lossless, we might be tempted to omit the attenuation coefficient, α. However, as mentioned in the introduction to Chapter 23, below a certain frequency waveguides do not allow wave propagation, i.e., they behave as wave attenuators, so it is necessary to use the complex propagation coefficient, γ.

Because we wish to investigate the types of waves that can propagate along the z axis, the sole dependence of the field vectors on z must be contained in the propagation factor $e^{-\gamma z}$. So the phasor field components are of the form

$$\mathbf{E}_{\text{tot}}(x, y, z) = \mathbf{E}(x, y)e^{-\gamma z} \quad \text{and} \quad \mathbf{H}_{\text{tot}}(x, y, z) = \mathbf{H}(x, y)e^{-\gamma z}, \tag{A8.1}$$

where $\mathbf{E}(x, y)$ and $\mathbf{H}(x, y)$ are complex values of the two vectors *in the plane* $z = 0$.

Complex Maxwell's equations in differential form contain derivatives with respect to coordinates only. The derivatives with respect to coordinates x and y will act on $\mathbf{E}(x, y)$ and $\mathbf{H}(x, y)$ only, and the derivative with respect to z on $e^{-\gamma z}$ only. For example, for the electric field vector the derivatives with respect to x and z are

$$\frac{\partial \mathbf{E}_{\text{tot}}(x, y, z)}{\partial x} = \frac{\partial \mathbf{E}(x, y)}{\partial x}e^{-\gamma z} \quad \text{and} \quad \frac{\partial \mathbf{E}_{\text{tot}}(x, y, z)}{\partial z} = -\gamma \mathbf{E}(x, y)e^{-\gamma z}. \tag{A8.2}$$

Thus the factor $e^{-\gamma z}$ is common to *all* terms in Maxwell's equations, and can be canceled out, just like we canceled out the factor $e^{j\omega t}$ to obtain the phasor form of the equations. Differentiation with respect to z should be replaced by $-\gamma$. Of course, once we have determined the field vectors in the plane $z = 0$, that is, $\mathbf{E}(x, y)$ and $\mathbf{H}(x, y)$, the total phasor field vectors are obtained from Eq. (A8.1).

With this in mind, let us write Maxwell's equation in differential form for $\mathbf{E} = \mathbf{E}(x, y)$ and $\mathbf{H} = \mathbf{H}(x, y)$ [we omit dependence on (x, y) for brevity]. Using the expressions for curl and divergence in the rectangular coordinate system, the first Maxwell's equation in scalar form becomes

$$\frac{\partial E_z}{\partial y} - \gamma E_y = -j\omega\mu H_x, \tag{Ia}$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y, \tag{Ib}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z. \tag{Ic}$$

Similarly, the second equation takes the form

$$\frac{\partial H_z}{\partial y} - \gamma H_y = j\omega\epsilon E_x, \tag{IIa}$$

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y, \tag{IIb}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z. \tag{IIc}$$

Finally, the third and fourth equations become

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \gamma E_z, \tag{III}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = \gamma H_z. \tag{IV}$$

If we substitute H_y from Eq. (Ib) into Eq. (IIa), we obtain an equation in which E_x is expressed in terms of the derivatives of z components of E_z and H_z. Similarly, from Eqs. (I*a*) and (II*b*) we can express E_y in terms of these derivatives. If next the expression for E_y is substituted in Eq. (I*a*), and that for E_x in Eq. (I*b*), we can also express H_x and H_y in terms of the derivatives of E_z and H_z. After simple manipulations we obtain the following expressions:

$$E_x = -\frac{1}{K^2}\left(\gamma\frac{\partial E_z}{\partial x} + \mathrm{j}\omega\mu\frac{\partial H_z}{\partial y}\right), \tag{A8.3}$$

$$E_y = -\frac{1}{K^2}\left(\gamma\frac{\partial E_z}{\partial y} - \mathrm{j}\omega\mu\frac{\partial H_z}{\partial x}\right), \tag{A8.4}$$

$$H_x = -\frac{1}{K^2}\left(-\mathrm{j}\omega\epsilon\frac{\partial E_z}{\partial y} + \gamma\frac{\partial H_z}{\partial x}\right), \tag{A8.5}$$

$$H_y = -\frac{1}{K^2}\left(\mathrm{j}\omega\epsilon\frac{\partial E_z}{\partial x} + \gamma\frac{\partial H_z}{\partial y}\right), \tag{A8.6}$$

where

$$K^2 = \gamma^2 + \beta^2, \qquad \beta^2 = \omega^2\epsilon\mu. \tag{A8.7}$$

Note that the propagation coefficient γ (and therefore also the coefficient K) *is not known*. So there are seven scalar unknowns (the six field components and γ).

A8.2 Quasi-Static Nature of TEM Waves

There is an interesting general conclusion concerning TEM waves. For $H_z = 0$, Eq. (A8.3) simply states that the curl of (transversal) vector $\mathbf{E}(x, y)$ is zero. We know that this means not that $\mathbf{E}(x, y) = 0$, but that $\mathbf{E}(x, y) = -\Delta V(x, y)$, or

$$E_x(x, y) = -\frac{\partial V(x, y)}{\partial x} \qquad \text{and} \qquad E_y(x, y) = -\frac{\partial V(x, y)}{\partial y}. \tag{A8.8}$$

Substituting these expressions for E_x and E_y into Eq. (III), we obtain (FOR $E_z = 0$)

$$\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = 0. \tag{A8.9}$$

So the electric field is derivable from a potential function which at $z = 0$ satisfies Laplace's equation in x and y. Because boundary conditions require that the tangential $\mathbf{E}$ on conductor surfaces be zero, we reach the following conclusion: for TEM waves, *the electric field in planes where z is constant is the same as the electrostatic field corresponding to the potentials of waveguide conductors at that cross section.*

A8.3 Derivation of General Properties of TE Wave Types

For TE waves, $E_z = 0$, so that the expressions for the transversal field components in Eqs. (A8.3) to (A8.6) reduce to

$$E_x = -\frac{\mathrm{j}\omega\mu}{K^2}\frac{\partial H_z}{\partial y}, \tag{A8.10}$$

$$E_y = \frac{j\omega\mu}{K^2}\frac{\partial H_z}{\partial x}, \tag{A8.11}$$

$$H_x = -\frac{\gamma}{K^2}\frac{\partial H_z}{\partial x}, \tag{A8.12}$$

$$H_y = -\frac{\gamma}{K^2}\frac{\partial H_z}{\partial y}. \tag{A8.13}$$

So, we can find all the field components of a TE wave if we determine H_z. How can we do that? We need an equation in which H_z is a single unknown. This is the Helmholtz equation (21.8). [It could, of course, also be derived from Eqs. (I–IV).] It is a vector equation representing three scalar equations in three components of the vector $\mathbf{H}$. For $\sigma = 0$ and in a rectangular coordinate system, having in mind that

$$\nabla^2[\mathbf{H}(x, y)e^{-\gamma z}] = \nabla^2[H_x(x, y)e^{-\gamma z}]\mathbf{u}_x + \nabla^2[H_y(x, y)e^{-\gamma z}]\mathbf{u}_y + \nabla^2[H_z(x, y)e^{-\gamma z}]\mathbf{u}_z,$$

the Helmholtz equation for the H_z component becomes

$$\nabla^2[H_z(x, y)e^{-\gamma z}] + \omega^2\epsilon\mu H_z(x, y)e^{-\gamma z} = 0.$$

Now, the first term of the last equation is just the Laplacian of the *scalar* function $H_z e^{-\gamma z}$, and we know the Laplacian in rectangular coordinates. Omitting the dependence of H_z on (x, y), and having in mind Eq. (A8.7), we obtain

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z + \omega^2\epsilon\mu H_z = \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + K^2 H_z = 0. \tag{A8.14}$$

A8.3.1 EXPRESSIONS FOR TE WAVES IN RECTANGULAR WAVEGUIDES

The first step in determining the TE waves is to find solutions of the Helmholtz equation (A8.14), that is, $H_z(x, y)$. Once this has been determined, the field components are obtained from Eqs. (A8.10) to (A8.13).

Let us attempt to find the solution of Eq. (A8.14) in the form

$$H_z(x, y) = X(x)Y(y). \tag{A8.15}$$

Substituting this H_z into the Helmholtz equation (A8.14), and assuming nonzero $X(x)$ and $Y(y)$, we obtain

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + K^2 = 0. \tag{A8.16}$$

Partial derivatives are replaced by ordinary derivatives because $X(x)$ is a function of x only, and $Y(y)$ a function of y only. Assume any functions $X(x)$ and $Y(y)$. After performing the indicated differentiations, the first term becomes a function of x alone, for example, $f(x)$, and the second of y alone, for example, $g(y)$. $f(x)$ and $g(y)$ must be such that $f(x) + g(y) + K^2 = 0$ *for all x and all y*. How can this be achieved, having in mind that x and y are independent and can have *any* values? There is only one answer: both $f(x)$ and $g(y)$ *must be equal to a constant*, for example, $f(x) = k_x^2$ and $g(y) = k_y^2$, and the sum of these constants plus K^2 must be zero. Thus Eq. (A8.16) can be satisfied for all x and all y only if

$$\frac{1}{X(x)}\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2} = k_x^2, \qquad \frac{1}{Y(y)}\frac{\mathrm{d}^2Y(y)}{\mathrm{d}y^2} = k_y^2,$$

or

$$\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2} = k_x^2X(x), \qquad \frac{\mathrm{d}^2Y(y)}{\mathrm{d}y^2} = k_y^2Y(y), \tag{A8.17}$$

and

$$k_x^2 + k_y^2 = K^2 = \gamma^2 + \omega^2\epsilon\mu. \tag{A8.18}$$

Thus the propagation coefficient of TE waves is

$$\gamma = \sqrt{k_x^2 + k_y^2 - \omega^2\epsilon\mu} = \mathrm{j}\omega\sqrt{\epsilon\mu}\sqrt{1 - \frac{k_x^2 + k_y^2}{\omega^2\epsilon\mu}}. \tag{A8.19}$$

The solutions of Eqs. (A8.17) are well known. They are of the form

$$X(x) = A_x\sin(k_xx) + B_x\cos(k_xx), \tag{A8.20}$$

$$Y(y) = A_y\sin(k_yy) + B_y\cos(k_yy). \tag{A8.21}$$

So the expression for $H_z(x, y)$ becomes

$$H_z(x, y) = [A_x\sin(k_xx) + B_x\cos(k_xx)][A_y\sin(k_yy) + B_y\cos(k_yy)], \tag{A8.22}$$

where $A_x, \ldots, B_y, k_x$, and k_y are constants to be determined from boundary conditions.

We know that on perfectly conducting waveguide walls in Fig. 23.2, boundary conditions require that $\mathbf{E}_{\text{tang}} = 0$ and $\mathbf{H}_{\text{norm}} = 0$. From Eqs. (A8.10) to (A8.13) we see that if one of these conditions is satisfied, the other is satisfied automatically. So we request that $E_y = 0$ for $x = 0$ (the left wall in Fig. 23.2) and for $x = a$ (the right wall), and that $E_x = 0$ for $y = 0$ (the bottom wall) and for $y = b$ (the top wall). According to Eqs. (A8.10), (A8.11), and (A8.22) this means that

$$\frac{\partial H_z}{\partial x} = [A_xk_x\cos(k_xx) - B_xk_x\sin(k_xx)][A_y\sin(k_yy) + B_y\cos(k_yy)] = 0$$
$$\text{for } x = 0 \text{ and } x = a, \tag{A8.23}$$

and

$$\frac{\partial H_z}{\partial y} = [A_x\sin(k_xx) + B_x\cos(k_xx)][A_yk_y\cos(k_yy) - B_yk_y\sin(k_yy)] = 0$$
$$\text{for } y = 0 \text{ and } y = b. \tag{A8.24}$$

Eq. (A8.23) can be satisfied if $A_x = 0$ (then the cosine term in the first brackets, which is nonzero for $x = 0$, vanishes), and if

$$k_xa = m\pi, \qquad \text{or} \qquad k_x = \frac{m\pi}{a}, \qquad m = {}^{0,}1, 2, \ldots, \tag{A8.25}$$

since for $x = a$ and these values of k_x the sine in the first brackets on the right of Eq. (A8.23) is zero.

Similarly, Eq. (A8.24) can be satisfied if $A_y = 0$, and if

$$k_y b = n\pi, \qquad \text{or} \qquad k_y = \frac{n\pi}{b}, \qquad n = 0, 1, 2, \ldots. \tag{A8.26}$$

Noting that $\omega = 2\pi f$, the final expression for the propagation coefficient of a TE wave along a rectangular waveguide becomes

$$\gamma = \mathrm{j}\beta, \qquad \beta = \omega\sqrt{\epsilon\mu}\sqrt{1 - \frac{f_c^2}{f^2}}, \tag{A8.27}$$

where, noting that $1/\sqrt{\epsilon\mu} = c$,

$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}, \qquad \cancel{c = \frac{1}{\sqrt{\epsilon\mu}}}. \tag{A8.28}$$

where f_c is the *cutoff frequency*, as explained in Chapter 23. It depends on both the numbers m and n, and on the waveguide dimensions.

Finally, the H_z component in the cross section $z = 0$ of the waveguide is

$$H_z(x, y) = H_0 \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \qquad (\text{at } z = 0), \tag{A8.29}$$

where H_0 is a constant depending on the level of excitation of the wave in the waveguide. The other components at $z = 0$ are obtained from Eqs. (A8.10) to (A8.13) (note that $E_z = 0$):

$$E_x(x, y) = \frac{\mathrm{j}\omega\mu}{K^2}\frac{n\pi}{b}H_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \qquad (\text{at } z = 0), \tag{A8.30}$$

$$E_y(x, y) = -\frac{\mathrm{j}\omega\mu}{K^2}\frac{m\pi}{a}H_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \qquad (\text{at } z = 0), \tag{A8.31}$$

$$H_x(x, y) = \frac{\gamma}{K^2}\frac{m\pi}{a}H_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \qquad (\text{at } z = 0), \tag{A8.32}$$

$$H_y(x, y) = \frac{\gamma}{K^2}\frac{n\pi}{b}H_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \qquad (\text{at } z = 0). \tag{A8.33}$$

The values of the wave components for any z are obtained by simply multiplying the preceding expressions by $\mathrm{e}^{-\gamma z} = \mathrm{e}^{-\mathrm{j}\beta z}$, where γ and β are given in Eqs. (A8.27). ~~and (23.21).~~

So we have obtained the expressions for the components of the TE wave propagating in a rectangular waveguide as functions of two integer parameters, $m = 0, 1, 2, \ldots$, and $n = 0, 1, 2, \ldots$. Evidently, not only to the field distributions for different pairs of m and n values differ greatly, *but also the propagation coefficient, β, is different for different pairs of numbers m and n.* Note that m represents the number of half-waves along the x axis, and n the number of half-waves along the y axis.

We see that there is a double infinite number of TE wave types, corresponding to any possible pair of m and n. The wave determined by a pair of numbers m and n is known as a *TE$_{mn}$ mode.*

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Index

Units

Quantity	*Symbol*	*Unit*	*Approximate practical range*
Charge	$Q,\ q^*$	coulomb (C)	10^{-19}–10
Surface charge density	$\sigma,\ \rho_s$	coulomb per square meter (C/m^2)	10^{-15}–10^{-5}
Voltage, potential	$V,\ v^*$	volt (V)	10^{-9}–10^{8}
Electric field strength	$\mathbf{E}$	volt per meter (V/m)	10^{-6}–10^{7}
Displacement vector	$\mathbf{D}$	coulomb per square meter (C/m^2)	10^{-15}–10^{-5}
Current intensity	$I,\ i^*,$	ampere (A)	10^{-13}–10^{5}
Current density	$\mathbf{J}$	ampere per square meter (A/m^2)	10^{0}–10^{7}
Capacitance	C	farad (F)	10^{-15}–10^{0}
Resistance	R	ohm (Ω)	0–∞
Inductance	L	henry (H)	10^{-10}–10^{1}
Conductivity	σ	siemens per meter (S/m)	see Appendix 4
Permittivity	ϵ	farad per meter (F/m)	see Appendix 4
Resistivity	ρ	ohm per meter (Ω/m) (Ωm)	see Appendix 4
Permeability	μ	henry per meter (H/m)	see Appendix 5
Magnetic flux density	$\mathbf{B}$	tesla (T)	10^{-15}–10^{0}
Magnetic field strength	$\mathbf{H}$	ampere per meter (A/m)	10^{-8}–10^{7}
Magnetic flux	Φ	weber (Wb)	10^{-15}–10^{0}
Frequency	f	hertz (Hz)	0–10^{22}
Force	$\mathbf{F}$	newton (N)	10^{-9}–10^{5}
Energy, work	$W,\ A$	joule (J)	10^{-15}–10^{12}
Power (work per unit time)	$P,\ p*$	watt (W)	10^{-12}–10^{10}

*In some cases it is customary to use lower-case letters for time-varying quantities.