1

Electromagnetics Around Us: Some Basic Concepts

1.1 Introduction

Electromagnetics is a brief name for the subject that deals with the theory and applications of electric and magnetic fields. Its implications are of fundamental importance in almost all segments of electrical engineering. Limitations on the speed of modern computers, the range of validity of electrical circuit theory, and the principles of signal transmission by means of optical fibers are just a few examples of topics for which knowledge of electromagnetics is indispensable. Electricity and magnetism also affect practically all aspects of our lives. Probably the most spectacular natural manifestation of electricity is lightning, but without tiny electrical signals buzzing through our nervous system we would not be what we are, and without light (an electromagnetic wave) life on our planet would not be possible.

The purpose of this chapter is to give you a glimpse of what you will learn in this course and how powerful this knowledge is. You will find that you are familiar with some of the information. However, you may also find that some concepts or equations mentioned in this chapter are not easy to understand. Don't let this problem bother you, because we will explain everything in detail later. What is expected at this point is that you refresh some of your knowledge, note some relationships, get a rough understanding of the unity of electricity and magnetism, and above all, understand how important this subject is in most of electrical engineering.

Electromagnetic devices are almost everywhere: in TV receivers, car ignition systems, elevators, and mobile phones, for instance. Although it may sometimes be hard to see the fundamental electromagnetic concepts on which their operation is based, you certainly cannot design these devices and understand how they work if you do not know basic electromagnetic principles.

In this chapter we first look at a few examples that show how the knowledge you will gain through this course can help you understand, analyze, and design different electrical devices. We will start with a typical office, which is likely to have a computer and a printer or a copier. We will list the different components and mechanisms inside the computer, relating them to chapters we will study later in the course. You may not yet understand what all the words mean, but that should not alarm you. During the course we will come back to these examples, each time with more understanding.

Questions and problems: Q1.1 to Q1.3

1.2 Electromagnetics in Your Office

Let us consider a personal desktop computer connected to a printing device and list the different components and mechanisms that involve knowledge of electricity or magnetism (Fig. 1.1).

1. The computer needs energy. It has to be plugged into a wall socket—that is, to an ac voltage generator. An ac voltage generator converts some form of energy into electrical energy. For example, hydroelectric power plants have large

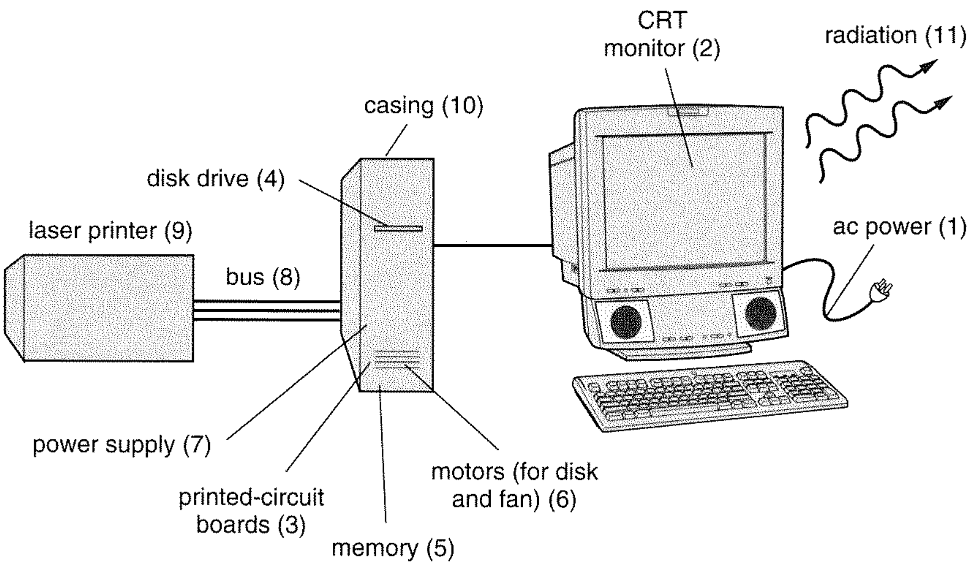


Figure 1.1 A personal desktop computer plugged into the wall socket and connected to a printer

generators in which the turbines, powered by water, produce rotating magnetic fields. We will study in Chapter 14 how such a generator can be built. These generators are made of copper conductors and iron or other magnetic materials, the properties of which we will study in Chapter 13.

2. Most desktop computers use a cathode-ray tube (CRT) monitor. In Chapter 17, we will explain how a CRT works. It involves understanding charge motion in electric and magnetic fields. Basically, a stream of electrons (negatively charged particles) is accelerated by an electric field and then deflected by a magnetic field, to trace a point on the front surface of the monitor and, point by point, a full image. The CRT runs off very high voltages, so the 110-V (or 220-V) socket voltage needs to be transformed into a voltage of a few kilovolts, which accelerates the electron beam. This is done using a magnetic circuit, or transformer, which we will study in Chapters 13 and 17.
3. The computer cabinet, or system unit, contains numerous printed-circuit boards. They contain conductive traces (Chapter 6) on dielectric substrates (Chapter 7); chips with many transistors, which are essentially charge-control devices (Chapter 7); and elements such as capacitors, resistors, and inductors (Chapters 8, 10, and 15). Signals flowing through the board traces couple to each other by electric (capacitive) and magnetic (inductive) coupling, which we will study in Chapters 8 and 15.
4. Many disks are read by magnetic heads from ferromagnetic traces. This is the topic of Chapters 14 and 17.
5. Computer memory used to be magnetic, built of small ferromagnetic toruses (Chapter 17). Now it is made of transistors, which serve as charge storage devices. We describe this mechanism in Chapter 8.
6. Inside the computer a motor operates the cooling fan. A motor converts electric energy to mechanical energy.
7. The semiconductor chips in the computer need typically 5 V or 3 V dc, instead of the 60-Hz 110 V (or 50-Hz 220 V) available from the socket. The power supply inside the computer performs the conversion. It uses components such as inductors, capacitors, and transformers, which we have already listed above.
8. The computer is connected to the printer by a multi-wire bus. The different lines of the bus can couple to each other capacitively (Chapter 8) and inductively (Chapter 15), and the bus can have an electromagnetic wave traveling along it, which we will discuss in Chapters 18, 23, and 25.
9. The printer will probably be a laser printer or an ink-jet printer. The laser printer operates essentially the same way as a copier machine, which is based on recording an electrostatic charge image and then transferring it to paper. The ink-jet printer is also an electrostatic device, and we will describe operations of both types of printers in Chapter 11.
10. The computer parts are shielded from outside interference by their metal casings. We are all bathing constantly in electromagnetic fields of different frequencies and intensities, which have different penetration properties into different materials (Chapter 20). However, some of the computer parts sometimes act

as receiving antennas (Chapter 24), which couple the interference onto signal lines, causing errors. This is called *electromagnetic interference (EMI)*. The regulations that are imposed on frequency band allocations, allowed power levels, and shielding properties are generally referred to as *electromagnetic compatibility (EMC)* regulations.

11. The computer also radiates a small amount of energy—that is, it acts as a transmitting antenna at some frequencies. We will study basic antenna principles in Chapter 24.
12. Finally, when we use the computer we are (we hope) thinking, which makes tiny voltage impulses in our neurons. Since our cells are mostly salty water, which is a liquid conductor, the current in the neurons will roughly have the same properties as the one through wire conductors.

1.3 Electromagnetics in Your Home

Now let us look at some uses of electromagnetics in your home. We know that most household appliances need ac voltage for their operation and that most of them (for example, blenders, washers, dryers, fans) contain some kind of electric motor. Both motors and generators operate according to principles that are covered in the third part of this book. An electric oven, as well as any other electric heating element (such as the one in a hair dryer or curling iron), operates according to Joule's law, which is covered in Chapter 10. Your washer, dryer, and car have been painted using electrostatic coating techniques, which we will briefly describe in Chapter 11.

Your TV receiver contains a cathode-ray tube, which, as we mentioned earlier, is described in Chapter 17. It is connected to the cable distribution box with a coaxial cable, a transmission line we will study throughout this book (Chapter 18). A transmission line supports an electromagnetic wave (Chapters 21 and 22). A similar wave traveling in free space is captured by an antenna, which you might also own. It could be a simple "rabbit ears" wire antenna or a highly directional reflector (dish) antenna. Basic antenna principles are covered in Chapter 24. Your cordless phone also contains an antenna, as well as high-frequency (rf) circuitry. All these applied electromagnetics topics are discussed in higher level courses in this field. Some of these applications are briefly described in Chapter 25 in the context of communications engineering.

A microwave oven is essentially a resonant cavity (Chapter 23), in which electromagnetic fields of a very high frequency are contained. The energy of these fields (Chapter 19) is used to heat up water (Chapter 25), whose molecule has a rotational resonance in a broad range around the designated heating frequency of 2.45 GHz. Thus the energy of the electromagnetic wave is transformed into kinetic energy of the water molecules, which on average determines the temperature of water. Because a large percentage of most foods is water, this in turn determines the food temperature.

Many other examples of electromagnetic phenomena occur in everyday life—light, which enables you to read these pages, is an electromagnetic wave. White light covers a relatively narrow range of frequencies, and our eyes are frequency-dependent sensors of electromagnetic radiation (that is, antennas for the visible part of the electromagnetic spectrum).

1.4 A Brief Historical Introduction

A tour through the historical development of the knowledge of electricity and magnetism reveals that this seemingly theoretical subject is entirely based on experimentally discovered laws of nature.

1.4.1 THE BEGINNING

When and where were the phenomena of electricity and magnetism first noticed? Around 600 B.C., the Greek philosopher and mathematician Thales of Miletus found that when amber was rubbed with a woolen cloth, it attracted light objects, such as feathers. He could not explain the result but thought the experiment was worth writing down. Miletus was at the time an important Greek port and cultural center. Ruins of Miletus still exist in today's Turkey, shown on the map in Fig. 1.2. Some 20 km from Miletus is an archaeological site called Magnesia, where the ancient Greeks first found magnetite, a magnetic ore. They noticed that lumps of this ore attracted one another and also attracted small iron objects. The word *magnet* comes from the name of the place where this ore was found.

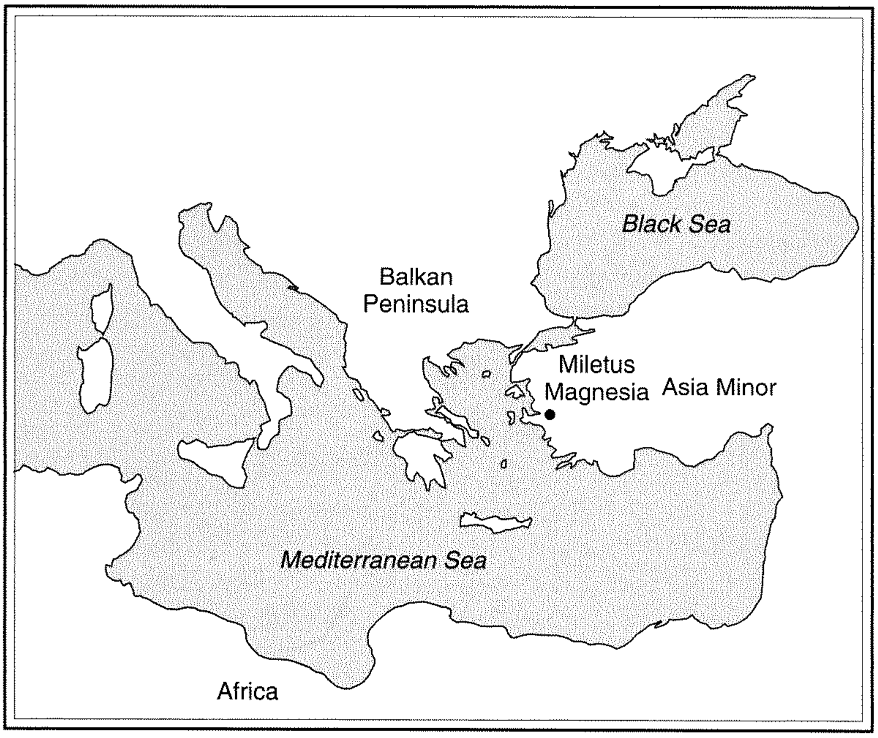


Figure 1.2 Map of the Mediterranean coast. Until Roman times, most coast colonies were Greek. Miletus was an important port and cultural center, connected by a 16-km marble road, lined with statues, to the largest Greek temple ever built (but never finished), at Didime.

Thus the first manifestations of both electricity and magnetism were noticed by the ancient Greeks at about the same time and at almost the same place. This coincidence was in a way an omen: we now know that electricity and magnetism are two facets of the same physical phenomenon.

1.4.2 CHRISTENING OF ELECTRICITY 22 CENTURIES LATER

There is no evidence that people thought about what Thales had observed for the next 2200 years. Around the year 1600 a physician to Queen Elizabeth I, William Gilbert, repeated Thales's experiments in a systematic way. He christened "electricity" from the Greek word for amber, *electron*, in honor of Thales's experiments. He rubbed different materials with woolen or silk cloth and concluded that some repel each other, and others are attracted after they are rubbed. We now know that when a piece of amber is rubbed with wool, some electrons (negative charges) from the wool molecules hop over to the amber molecules and therefore the amber has extra electrons. We say that the amber is *negatively charged*. The wool has fewer electrons, which makes it also different from neutral, and we say it is *positively charged*.

1.4.3 POSITIVE AND NEGATIVE CHARGES

The terms *positive* and *negative electric charges* were introduced by Benjamin Franklin (around 1750) for no particular reason; he could also have called them red and blue. It turned out, however, that for mathematically describing electrical phenomena, associating "+" and "−" signs with the two kinds of electricity was extremely convenient. For example, electrically neutral bodies are known to contain very large but equal amounts of positive and negative electric charges; the "+" and "−" convention allows us to describe them as having zero total charge.

Why were electrical phenomena not noticed earlier? The gravitational force has been known and used ever since the ancient man poured, for example, water in his primitive container. This time lag can be easily understood if we compare the magnitudes of electrical forces and some other forces acting around us.

1.4.4 COULOMB'S LAW

Electrical forces were first investigated systematically by Charles de Coulomb in 1784. By that time it was well established that like charges repel and opposite charges attract each other, but it was not known how this force could be calculated. Using a modified, extremely sensitive torsion balance (with a fine silk thread replacing the torsion spring), Coulomb found experimentally that the intensity of the force between two "point" charges (charged bodies that are small compared to the distance between them) is proportional to the product of their charges (Q_1 and Q_2 in Fig. 1.3), and inversely proportional to the square of the distance r between them:

$$F_e = k_e \frac{Q_1 Q_2}{r^2}. \tag{1.1}$$

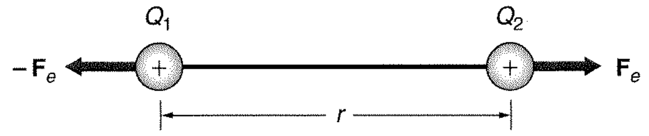


Figure 1.3 Coulomb's electric force between two particles with charges of the same sign, which are small in size compared to the distance r between them

This is *Coulomb's law*. The unit for charge we use is called a *coulomb* (C). With the distance r in meters and force F in newtons (N), the constant k_e is found to be very nearly $9 \times 10^9\ \mathrm{N \cdot m^2/C^2}$. This force is attractive for different charges (one positive and the other negative), and repulsive for like charges (both negative or both positive). The charge of an electron turns out to be approximately -1.6×10^{-19} C.

How large is this force? Let us first look at the formula. If we replace the constant k_e with the gravitational constant $\gamma = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$, and the charges by the masses, m_1 and m_2, of the two particles (in kg), the formula becomes that for the gravitational force between the two particles due to their masses:

$$F_g = \gamma \frac{m_1 m_2}{r^2}. \tag{1.2}$$

Let us calculate how the electric force in a hydrogen atom (which has one electron and one proton) compares to the gravitational force. Using the preceding formulas and the data for the masses of an electron and a proton given in Appendix 3, we find that the ratio of the electric to gravitational forces between the electron and the proton of a hydrogen atom is astonishing:

$$\frac{F_e}{F_g} \simeq 10^{39}.$$

We know that atoms of matter are composed of elemental charges that include protons and electrons. If this is the ratio of electric to gravitational force acting between one proton and one electron, we should also expect enormous electric forces acting around us. Yet we can hardly notice them. They include such minor effects as our hair rising after we pull off a sweater. There are simply no appreciably larger electric forces in everyday life. How is this possible? To understand it, let us do a simple calculation.

Assume two students are sitting 1 m apart and their heads are charged. Let us find the force between the two heads, assuming they are point charges (for most students, of course, this is not at all true, but we are doing only an approximate calculation). Our bodies consist mostly of water, and each water molecule has 10 electrons and the same number of protons in one oxygen atom and two hydrogen atoms. Thus we are nothing but a vast ensemble of electric charges. In normal circumstances, the amount of positive and negative charges in the body is practically balanced, i.e., the net charge of which our body is composed is very nearly zero.

1.4.5 PERCENTAGE OF EXCESS CHARGE ON CHARGED BODIES

Let us assume, however, that a small percentage of the total electron charge, say 0.1%, exists in excess of the total positive charge. If each head has a volume of roughly $10^{-3}\,\mathrm{m}^3$, and solids and liquids have about 10^{28} atoms/m^3, each head has on the order of 10^{25} atoms. Assume an average of 10 electrons per atom (human tissue consists of various atoms). One tenth of a percent of this is roughly 10^{23} electrons/head. Since every electron has a charge of -1.6×10^{-19} C, this is an extra charge of about -1.6×10^{4} C. When we substitute this value into Coulomb's law, we find that the force between the two students' heads 1 m apart is on the order of 2×10^{20} newtons (N).

How large is this force? The "weight" of the earth, if such a thing could be defined, would be on the order of 10^{20} N, that is, of same order of magnitude as the previously estimated force between the two students. How is it then possible that we do not notice the electric force? Where did our calculation go wrong? The answer is obvious: we assumed too high a percentage (0.1%) of excess electrons. Since we do not notice electric forces in common life, this tells us that the charges in our world are *extremely well balanced*, i.e., that only a very small percentage of protons or electrons in a body is in excess over the other.

1.4.6 CAPACITORS AND ELECTRIC CURRENT

We know that extra charge can be produced by rubbing one material against another. This charge can stay on the material for some time, but it is very difficult to collect from there and put somewhere else. It is of extreme practical importance to have a device analogous to a water container in which it is possible to store charge. Devices that are able to act as charge containers are called *capacitors*. They consist of two conducting pieces known as *capacitor electrodes* that are charged with *charges of equal magnitude but opposite signs*. An example is in Fig. 1.4a.

If the medium between the two electrodes is air, and if many small charged particles are placed there, the electric forces due to *both* electrodes will move the charges systematically toward the electrode of the opposite sign. Such an ordered motion of a large number of electric charges is called the *electric current* because it resembles

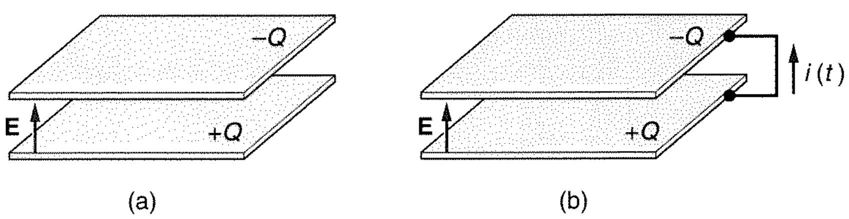


Figure 1.4 (a) A simple capacitor consists of two oppositely charged bodies. (b) If the two capacitor electrodes are connected by a wire, a short flow of charges occurs until the capacitor is discharged.

the current of a fluid. We can get the same effect more easily if we connect the two electrodes by a metallic (conducting) wire. A short flow of electrons in the metal wire will result, until the capacitor is discharged (Fig. 1.4b), i.e., until all of the negative charges neutralize the positive ones. Thus a charged capacitor cannot sustain a permanent electric current.

1.4.7 ELECTRIC GENERATORS

This flow of charges, more precisely an effect of this flow, was first noticed around 1790 by Luigi Galvani when he placed metal tweezers on a frog's leg and noticed that the leg twitched. Soon after that, between 1800 and 1810, Alessandro Volta made the first battery—a device that was able to maintain a continuous charge flow for a reasonable time.

A sketch of Volta's battery is shown in Fig. 1.5. The battery consisted of zinc and copper disks separated by leather soaked in vinegar. The chemical reactions between the vinegar and the two types of metal result in opposite charges on copper and zinc disks. These charges exert a force on freely movable electrons in a wire connecting them, resulting in electric current in the wire. Obviously, the larger these charges, the stronger the force on electrons in the wire. A quantity that is directly proportional to the charge on one of the disks is known as *voltage.* The unit of voltage is the *volt* (V), in honor of Volta. Volta "measured" the voltage by placing two pieces of wire on his tongue (the voltage is about 1 V per cell).

The chemical reaction that governs the process in a zinc-copper battery that uses a solution of sulfuric acid (H_2SO_4) is given by the following equation, assuming

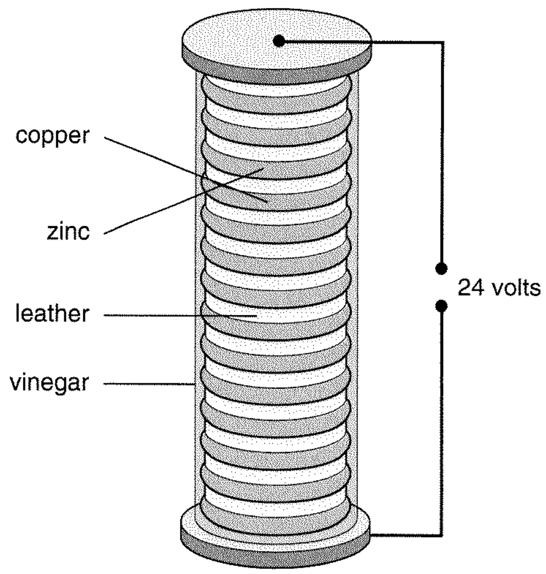


Figure 1.5 Volta's first battery consisted of 24 pairs of copper and zinc disks separated by leather soaked in vinegar.

the end copper (Cu) and zinc (Zn) plates to be connected with a conducting wire:

$$Cu + Zn + 3H_2SO_4 = Zn^{++} + 2SO_4^{--} + Cu^{++} + SO_2 + H_2 + 2H_2O. \quad (1.3)$$

Hydrogen gas molecules (H_2) are given off at the copper plate, which loses electrons to the solution and becomes positively charged. Zinc dissolves from the zinc plate, leaving electrons behind. The electrons move through the wire from the zinc to the copper plate, making an electric current. The process stops when the zinc plate is eaten away, or when no more acid is left.

Volta's battery is just one type of *electric generator*. Other chemical generators operate like Volta's battery but with different substances. However, generators can separate positive and negative electric charges, that is, can produce a voltage between their terminals, in many different ways: by a wire moving in a magnetic field; by light charging two electrodes of a specific semiconductor device; by heating one connection of two wires made of different materials; and even by moving charges mechanically (which, however, is extremely inefficient). All electric generators have one common property: they use some other kind of energy (chemical, mechanical, thermal, solar) to separate electric charges and to obtain two charged electrodes.

1.4.8 JOULE'S LOSSES

When there is an electric current in a substance, the electric force accelerates charged particles that can move inside the substance (e.g., electrons in metals). After a very short trip, however, these particles collide with atoms within the substance and lose some energy they acquired by acceleration. This lost energy is transformed into heat—more vigorous vibrations of atoms inside the substance. This heat is known as *Joule's heat* or *Joule's losses*.

1.4.9 MAGNETISM

As mentioned, the phenomenon of magnetism was first noticed at about the same time as that of electricity. The magnetic needle (a small magnet suspended to rotate freely about a vertical axis) was observed by the Chinese about 120 B.C. The magnetic force was even more mysterious than the electric force. Every magnet *always has two "poles"* that cannot be separated by cutting a magnet in half. In addition, one pole of the magnetic needle, known as its *north pole*, always turns itself toward the north.

People could not understand why this happened. An "explanation" that lasted for many centuries (until about A.D. 1600) was that the north pole of the needle was attracted by the North Star. This does not show, of course, that our ancestors were illogical, for without the knowledge we have today we would probably accept the same explanation. Instead it shows at least two things typical of the development of human knowledge: we like simple explanations, and we tend to take explanations for granted. Whereas the desire to find a simpler explanation presents a great positive challenge, the tendency to take explanations for granted presents a great danger.

The magnetic forces were also studied experimentally by Coulomb. Using long magnets and his torsion balance, he concluded that the magnetic poles exert forces on each other and that these forces are of the same form as those between two point charges. This is known as the Coulomb force for magnetic poles, and it represents another approach we frequently use in trying to understand things: the use of analogies. We will see shortly that magnetic poles actually do not exist. This example, therefore, demonstrates that we should be careful about analogies and be critical of them.

1.4.10 ELECTROMAGNETISM AND ORIGIN OF MAGNETISM IN PERMANENT MAGNETS

Because of Coulomb's law for magnetic poles, magnetism was for some time considered to be separate from electricity but to have very similar laws. Around 1820, however, the Danish physicist Hans Christian Oersted noticed that a magnetic needle is deflected from its normal orientation (north-south) if placed close to a wire with electric current. Knowing that two magnets act on each other, he concluded that a wire with electric current is a kind of magnet, i.e., that *magnetism is due to moving electric charges*. This "magnet" is, of course, different from a piece of magnetic ore (a permanent magnet) because it can be turned on and off and its value can be controlled. It is called an *electromagnet* and has many uses, for example cranes and starter motors.

Soon after Oersted's discovery, the French physicist André Marie Ampère offered an explanation of the origin of magnetism in permanent magnets. He argued that inside a permanent magnet there must be a large number of tiny loops of electric current. He also proposed a mathematical expression describing the force between two short segments of wire with current in them. We will see in a later chapter that this expression is more complicated than Coulomb's law. However, for the particular case of two parallel short wire segments l_1 and l_2 with currents I_1 and I_2, shown in Fig. 1.6, *and only in that case*, this expression is simple:

$$F_m = k_m \frac{(I_1 l_1)(I_2 l_2)}{r^2}, \tag{1.4}$$

where k_m is a constant. The direction of the force in the case in Fig. 1.6 (parallel elements with current in the same direction) is *attractive*. It is repulsive if the currents in the elements are in opposite directions. Note that an analogy with electric forces might tempt us to anticipate (erroneously) different force directions than the actual ones.

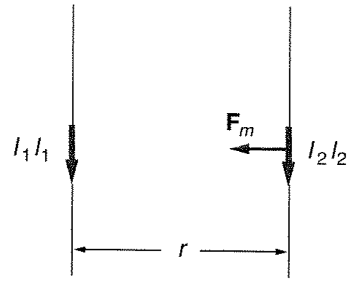


Figure 1.6 Magnetic force between two parallel current elements

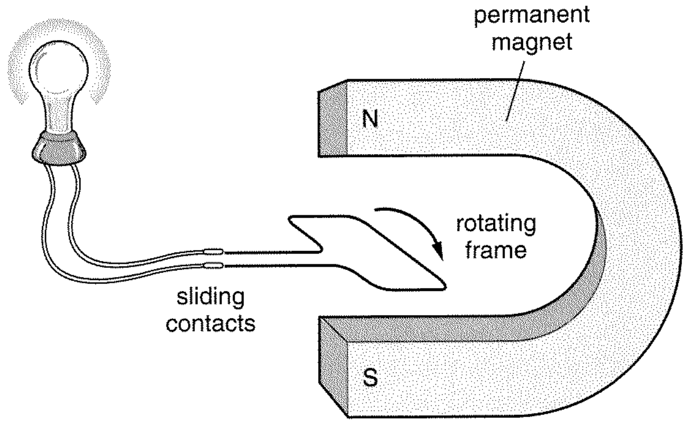


Figure 1.7 A simple generator can be made by turning a wire frame in a dc magnetic field.

1.4.11 ELECTROMAGNETIC INDUCTION

The final important physical fact of electricity and magnetism we mention was discovered in 1831 by the British physicist Michael Faraday. He performed experiments to check whether Oersted's experiment was reciprocal, i.e., whether current will be produced in a wire loop placed near a magnet. He did not find that, but he realized that *a current in the loop was obtained while the magnet was being moved toward or away from it.* The law that enables this current to be calculated is known as *Faraday's law of electromagnetic induction.*

As an example, consider a simple generator based on electromagnetic induction. It consists of a wire frame rotating in a time-constant magnetic field, as in Fig. 1.7, with the ends of the frame connected to the "outer world" by means of sliding contacts. Let the sliding contacts be connected by a separate and stationary wire, so that a closed conducting loop is obtained. When the wire frame turns, its position with respect to the magnet varies periodically in time, which induces a varying current in the frame and the wire that completes the closed conducting loop.

Questions and problems: Q1.4 to Q1.16, P1.1 to P1.4, P1.10

1.5 The Concept of Electric and Magnetic Field

Let us now assume that we know the position of the charge Q_1 in Coulomb's law, but that there are several charges close to charge Q_1, of unknown magnitudes and signs and at unknown locations (Fig. 1.8). We cannot then calculate the force on Q_1 using Coulomb's law, but from Coulomb's law, and knowing that mechanical forces add as vectors, we anticipate that there will be a force on Q_1 proportional to Q_1 itself:

$$\mathbf{F}_e = Q_1\mathbf{E}. \tag{1.5}$$

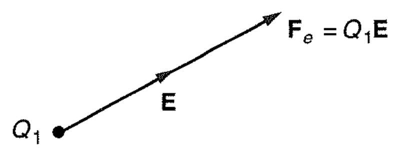


Figure 1.8 The electric field vector, **E**, is defined by the force acting on a charged particle.

(It is customary in printed text to use boldface fonts for vectors, e.g., **r**. In handwriting, vectors are denoted by an arrow above the letter, e.g., $\vec{r}$. A brief survey of vectors is given in Appendix 1.)

This is the definition of the *electric field strength*, **E**. It is a vector, equal to the force on a small charged body at a point in space, divided by the charge of the body. Note that **E** generally differs from one point to another, and that it frequently varies in time (for example, if we move the charges producing **E**). The domain of space where there is a force on a charged body is called the *electric field*. Thus, we can describe the electric field by **E**, a *vector* function of space coordinates (and possibly of time). For example, in a Cartesian coordinate system we would write: $\mathbf{E}(x, y, z, t) = \mathbf{E}_x(x, y, z, t) + \mathbf{E}_y(x, y, z, t) + \mathbf{E}_z(x, y, z, t)$. Obviously, sources of the electric field are electric charges and currents. If sources producing the field are not moving, the field can be calculated from Coulomb's law. This kind of field is termed the *electrostatic field*, meaning "the field produced by electric charges that are not moving."

Consider now Eq. (1.4) for the magnetic force between two current elements and assume that several current elements of unknown intensities, directions, and positions are close to current element $I_1 l_1$. The resulting magnetic force will be proportional to $I_1 l_1$. We know that current elements are nothing but small domains with moving charges. Let the velocity of charges in the current element $I_1 l_1$ be **v**, and the charge of individual charge carriers in the current element be Q. The force on the current element is the result of forces on individual moving charge carriers, so that the force on a single charge carrier should be expected to be proportional to $Q\mathbf{v}$. Experimentally, the expression for this force is found to be of the form

$$\mathbf{F}_m = Q\mathbf{v} \times \mathbf{B}, \tag{1.6}$$

where the sign "$\times$" implies the vector, or cross, product of two vectors (Appendix 1). The vector **B** is known as the *magnetic induction vector* or the *magnetic flux density vector*. If in a region of space a force of the form in Eq. (1.6) exists on a moving charge, we say that in that region there is a *magnetic field*.

Questions and problems: Q1.17 to Q1.20, P1.5 to P1.9

1.6 The Electromagnetic Field

Faraday's law shows that a time-varying magnetic field produces a time-varying electric field. Is the converse also true? About 1860 the British physicist James Clerk Maxwell stated that this must be so, and he formulated general differential equations of the electric and magnetic fields that take this assumption into account. Because the electric and magnetic fields in these equations are interrelated in such a manner that if they are variable in time one cannot exist without the other, this resulting field is known as the *electromagnetic field*. These famous equations, known as *Maxwell's equations*, have proven to be exact in all cases of electromagnetic fields considered since his time. In particular, Maxwell theoretically showed from his equations that an electromagnetic field can detach itself from its sources and propagate through space as a field package, known as an *electromagnetic wave*. He also found theoretically that the speed of this wave in air is the same as the speed of light measured earlier by several scientists (for example, Roemer in 1675 estimated it to be about 2.2×10^8 m/s, and Fizeau in 1849 and Foucault in 1850 determined it to be about 3×10^8 m/s). This led him to the conclusion that light must be an electromagnetic wave and he formulated his famous electromagnetic theory of light. Maxwell's equations break down, however, at the atomic level because the field quantities used in the equations are averaged over many atoms. Such quantities are called *macroscopic*. (The science that deals with electromagnetic phenomena at the atomic and subatomic levels is called *quantum physics*.)

The first person who experimentally verified Maxwell's theory was the German physicist Heinrich Hertz. Between 1887 and 1891 he performed a large number of ingenious experiments at frequencies between 50 MHz and 5 GHz. At that time, these were incredibly high frequencies. One of his experiments proved the existence of electromagnetic waves. A device that launches or captures electromagnetic waves is called an *antenna*. Hertz used a high voltage spark (intense current in air of short duration, and therefore rich in high frequencies) to excite an antenna at about 60 MHz (Fig. 1.9). This was his transmitter. The receiver was an adjustable loop of wire with another spark gap. When he adjusted the resonance of the receiving antenna to that of the transmitting one, he was able to notice a weak spark in the gap of the receiving antenna. Hertz thus demonstrated for the first time that Maxwell's predictions about

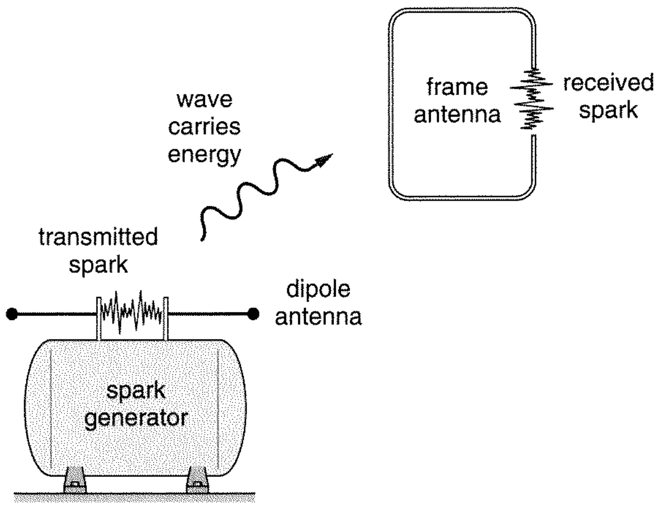


Figure 1.9 Hertz's first demonstration of an electromagnetic wave

the existence of electromagnetic waves were correct. Hertz also introduced the first reflector antennas, predicted the finite velocity of waves in coaxial transmission lines and the existence of standing electromagnetic waves, as well as a number of radio techniques used today. He was, in fact, the first radio engineer.

Electric, magnetic, or electromagnetic fields are present in any device we use in electrical engineering. Therefore, Maxwell's equations should strictly be used for the analysis and design of all such devices. This would be quite a complicated process, however. Fortunately, in many cases approximations that simplify the analysis process are possible. For example, circuit theory is essentially a very powerful and simple approximation of the exact field theory. In the next chapter we look at the interconnection between fields and circuits, and explore briefly the electromagnetic foundations of circuit theory and its limitations.

Questions and problems: Q1.21, Q1.22

1.7 Chapter Summary

1. The principal developments in the history of the science of electricity and magnetism began with the ancient Greeks. Key concepts, however, have been described only in the past 400 years.
2. The objects in the world around us are composed of very nearly equal numbers of elemental positive and negative electric charges. The excess charge of one kind over the other can be only an extremely small fraction of the total charge of that kind.
3. Between stationary bodies with excess charges, which we term *charged bodies*, there is a force known as the *electric force*.
4. If there is a force on a charge Q in a region of space of the form $\mathbf{F}_e = Q\mathbf{E}$, we say that an *electric field* exists in that region. The vector $\mathbf{E}$ is known as the *electric field vector*.
5. If charges are moving, there is an additional force acting between them. It is called the *magnetic force*.
6. If there is a force on an electric charge Q moving with a velocity $\mathbf{v}$ in a region of space, of the form $\mathbf{F}_m = Q\mathbf{v} \times \mathbf{B}$, we say that a *magnetic field* exists in that region. The vector $\mathbf{B}$ is known as the *magnetic induction vector* or *magnetic flux density vector*.
7. An electric field that varies in time is always accompanied by a magnetic field that varies in time, and vice versa. This combined field is known as the *electromagnetic field*.
8. The equations that mathematically describe any electric, magnetic, and electromagnetic field are known as *Maxwell's equations*. They are mostly based on experimentally obtained physical laws.

Questions

Q1.1. What is electromagnetics?

Q1.2. Think of a few examples of animals that use electricity or electromagnetic waves. What about a bat?

Q1.3. The basis of plant life is photosynthesis, i.e., synthesis (production) of life-sustaining substances by means of light. Is an electromagnetic phenomenon included?

Q1.4. What is the origin of the word *electricity*?

Q1.5. What is the origin of the word *magnetism*?

Q1.6. When did Thales of Miletus and William Gilbert make their discoveries?

Q1.7. Why is it convenient to associate plus and minus signs with the two kinds of electric charges?

Q1.8. When did Coulomb perform his experiments with electric forces?

Q1.9. What is the definition of a capacitor?

Q1.10. What is electric current?

Q1.11. What are electric generators?

Q1.12. What common property do all electric generators have?

Q1.13. Describe in your own words the origin of Joule's losses.

Q1.14. What is the fundamental cause of magnetism?

Q1.15. What is an electromagnet?

Q1.16. What did Faraday notice in 1831 when he moved a magnet around a closed wire loop? What did he expect to see?

Q1.17. Explain the concept of the electric field.

Q1.18. Define the electric field strength vector.

Q1.19. Explain the concept of the magnetic field.

Q1.20. Define the magnetic induction (magnetic flux density) vector.

Q1.21. What is an electromagnetic wave?

Q1.22. What are macroscopic quantities?

Problems

P1.1. How many electrons are needed to obtain one coulomb (1 C) of negative charge? Compare this number with the number of people on earth (about $5 \cdot 10^9$).

P1.2. Calculate approximately the gravitational force between two glasses of water a distance $d = 1$ m apart, containing 2 dl (0.2 liter) of water each.

P1.3. Estimate the amount of equal negative electric charge (in coulombs) in the two glasses of water in problem P1.2 that would cancel the gravitational force.

P1.4. Two small equally charged bodies of masses $m = 1$ g are placed one above the other at a distance $d = 10$ cm. How much negative charge would the bodies need to have so that the electric force on the upper body is equal to the gravitational force on it (i.e., so the upper body levitates)? Do you think this charge can be realized?

P1.5. Calculate the electric field strength necessary to make a droplet of water of radius $a = 10\,\mu\text{m}$, with an excess charge of 1000 electrons, levitate in the gravitational field of the earth.

P1.6. How large does the electric field intensity need to be in order to levitate a body 1 kg in mass and charged with -10^{-8} C? Is the answer of practical value, and why?

P1.7. A drop of oil, $r = 2.25\,\mu\text{m}$ in radius, is negatively charged and is floating above a very large, also negatively charged body. The electric field intensity of the large body happens to be $E = 7.83 \cdot 10^4\,\text{V/m}$ at the point where the oil drop is situated. The density of oil is $\rho_m = 0.851\,\text{g/cm}^3$. (1) What is the charge of the drop equal to? (2) How large is this charge compared to the charge of an electron? *Note: the values given in this problem can realistically be achieved in the lab. Millikan used such an experiment at the beginning of the 20th century to show that charge is quantized.*

P1.8. Find the force between the two parallel wire segments in Fig. P1.8 if they are 1 mm long and 10 cm apart, and if they are parts of current loops that carry 1 A of current each. The constant k_m is equal to 10^{-7} in SI units (N/A^2).

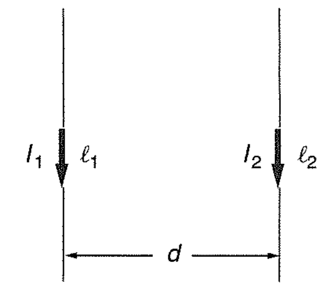


Figure P1.8 Two parallel wire segments

P1.9. A small body charged with $Q = -10^{-10}$ C finds itself in a uniform electric and magnetic field as shown in Fig. P1.9. The electric field vector and the magnetic flux density vector are **E** and **B**, respectively, everywhere around the body. If the magnitude of the electric field is $E = 100\,\text{N/C}$, and the magnetic flux density magnitude is $B = 10^{-4}\,\text{N}\cdot\text{s/C}\cdot\text{m}$, find the force on the body if it is moving with a velocity **v** as shown in the figure, where $v = 10\,\text{m/s}$ (the speed of a slow car on a mountain road). How fast would the body need to move to maintain its direction of motion?

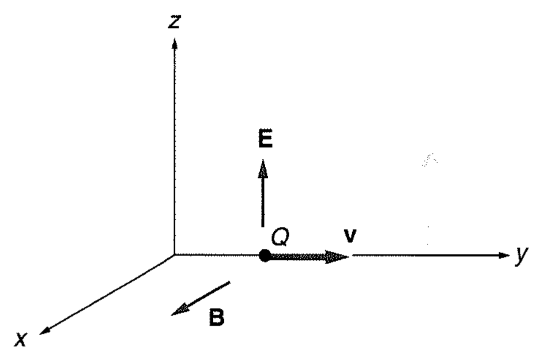


Figure P1.9 Point charge in an electric and magnetic field

P1.10. Volta used a chemical reaction to make the first battery that could produce continuous electric current. Use the library, or any other means, to find out if electric current can be used to make chemical reactions possible. Write one page on the history and implications of these processes.

2

Circuit Theory and Electromagnetics

2.1 Introduction

One of the most important tools of electrical engineers is circuit theory. Circuits have charges and currents, which we know produce electric and magnetic fields. Thus circuits are actually electromagnetic systems and strictly speaking, they should be analyzed starting from the general electromagnetic-field equations, i.e., Maxwell's equations. We start, however, from the two Kirchhoff's laws instead, well aware that circuit theory can be used to accurately predict circuit behavior.

Circuit theory is an approximate theory that can be obtained from Maxwell's equations with a set of approximations. We will return to this point throughout this book. In this chapter we review some simple circuit examples and look at where these approximations are made, arriving at two important conclusions. First, circuit theory is an approximation (but fortunately a very good and useful one in most applications). Second, the limitations of circuit theory can be understood only if we understand electromagnetic-field theory. In the next section we consider the effects of some simple electromagnetic properties of electric circuits, which will help you understand these conclusions. You can perform the examples shown here in the lab, using just a function generator and oscilloscope.

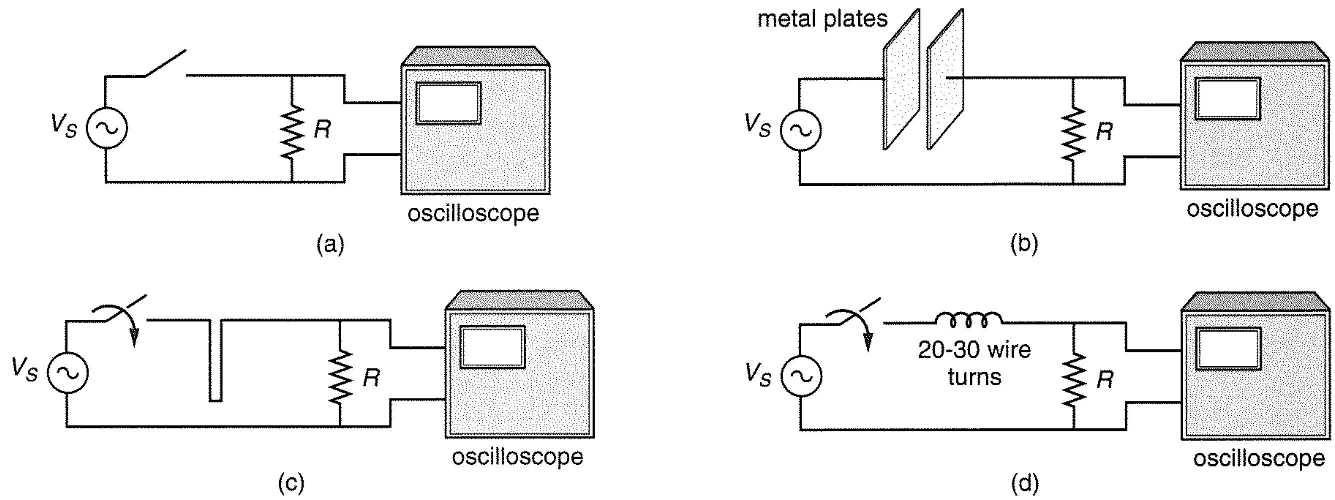


Figure 2.1 (a) A resistor connected to a function generator. The voltage across the resistor is observed on an oscilloscope. (b) The switch in (a) is replaced by two metal plates. (c) An interconnecting conductor is shaped to form a short-circuited two-wire line. (d) The interconnecting conductor is wound around a pen.

2.2 Circuit Elements as Electromagnetic Structures

Let us first consider three basic circuit elements: a switch, a conducting wire, and a resistor (Fig. 2.1). We shall find out later how Ohm's law is derived from Maxwell's equations, but right now let us start from what you learned in circuits: the voltage across the resistor is $v_R(t) = Ri(t)$. (Actually, this relation should be considered as the definition of an *ideal* resistor.)

We connect the resistor to a function generator and look at the voltage across it on an oscilloscope, Fig. 2.1a, when the switch is open. The classical expectation is that the voltage is zero. However, note that the switch consists of two contacts that are separated by an insulator (e.g., air) when the switch is open. These two contacts, therefore, form a capacitor. Only if we can neglect the capacitance of this capacitor, i.e., if we can consider it to be zero, is the voltage across the resistor also zero.

To understand this, imagine changing the *shape* and *size* of the switch. For example, let us replace the actual switch by two parallel rectangular metal plates, say 10 by 10 cm. Let the plates be separated by 1 cm when the switch is open, as in Fig. 2.1b, and pressed tightly together when the switch is closed. The resistance of the large plates is certainly less than that of the switch contacts, and close to 0 Ω, so we should see no change in $v_R(t)$ on the oscilloscope screen. However, in the open position this new switch may influence the current in the circuit considerably because it has a sizable capacitance. Indeed, if we bring the two plates closer together, we will notice on the oscilloscope that this open switch influences the voltage between the resistor terminals more than when the plates are farther apart.

This capacitance is present in any switch, but if—as mentioned—the capacitance is small enough, the switch will *behave* as if the capacitance is zero. However, in strict electromagnetic theory even an element as simple as a switch does not exist. Only if we can neglect its capacitance does a switch behave according to the definition in circuit theory, i.e., that it is either an open switch or a short circuit. To analyze the open switch more accurately, we must consider its capacitance and use electromagnetic-field theory. Recall that the reactance of a capacitor is inversely proportional to the product of its capacitance and frequency. So we may infer that, at extremely high frequencies, it may not be easy to make a switch that, if open, acts indeed as an open circuit.

Let us now concentrate on the influence of the size and shape of a conducting wire. In circuit theory, the wire form and size are assumed to have no effect on circuit behavior and they are assumed to be short-circuit interconnections. We now analyze this assumption in more detail.

Assume that the switch is closed. According to circuit theory, we may vary the length and shape of the wire connecting the resistor to the generator as much as we wish without changing either the voltage across the resistor or current in it. What will happen, however, if we substantially extend one of the conductors and bend it as in Fig. 2.1c, so that we get two relatively long parallel close wires? Experiments show that this changes the voltage across the resistor to a large degree. How can we explain this?

The bent conductor represents a section of short-circuited transmission line. If the frequency is low enough, this is just a wire loop having a certain inductance. This inductance is connected in series with the resistor and changes the current, and hence the voltage across the resistor.

So the circuit-theory assumption that the interconnecting conductors have no effect on the circuit behavior is only an approximation. Electromagnetic-field theory tells us that in the case of varying currents, the same circuit will behave differently if we twist it, extending, shortening, or deforming the interconnecting conductors and generally changing the circuit's shape. This is indeed a strange conclusion if one adheres to circuit-theory explanations, but it is true. At high frequencies, even as low as about 10 MHz, and for circuit dimensions exceeding about 10 cm, circuit theory frequently cannot predict circuit properties with sufficient accuracy, but electromagnetic theory can.

As a more specific example, consider the circuit in Fig. 2.2. It consists of one resistor and one capacitor of very small dimensions (known as "chip" or "surface mount" resistors and capacitors). If we compute the input impedance of the circuit as a function of frequency, we get the solid line in Fig. 2.3. Experimentally obtained results, indicated by the square symbols, are quite different, however. Above a certain

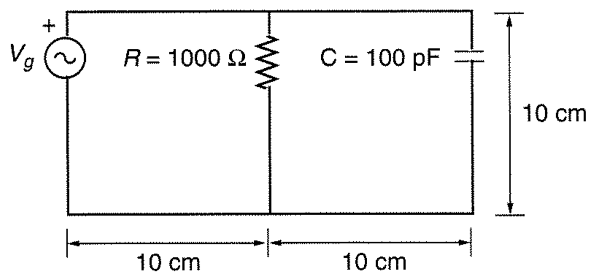


Figure 2.2 A simple circuit with a small resistor and a small capacitor

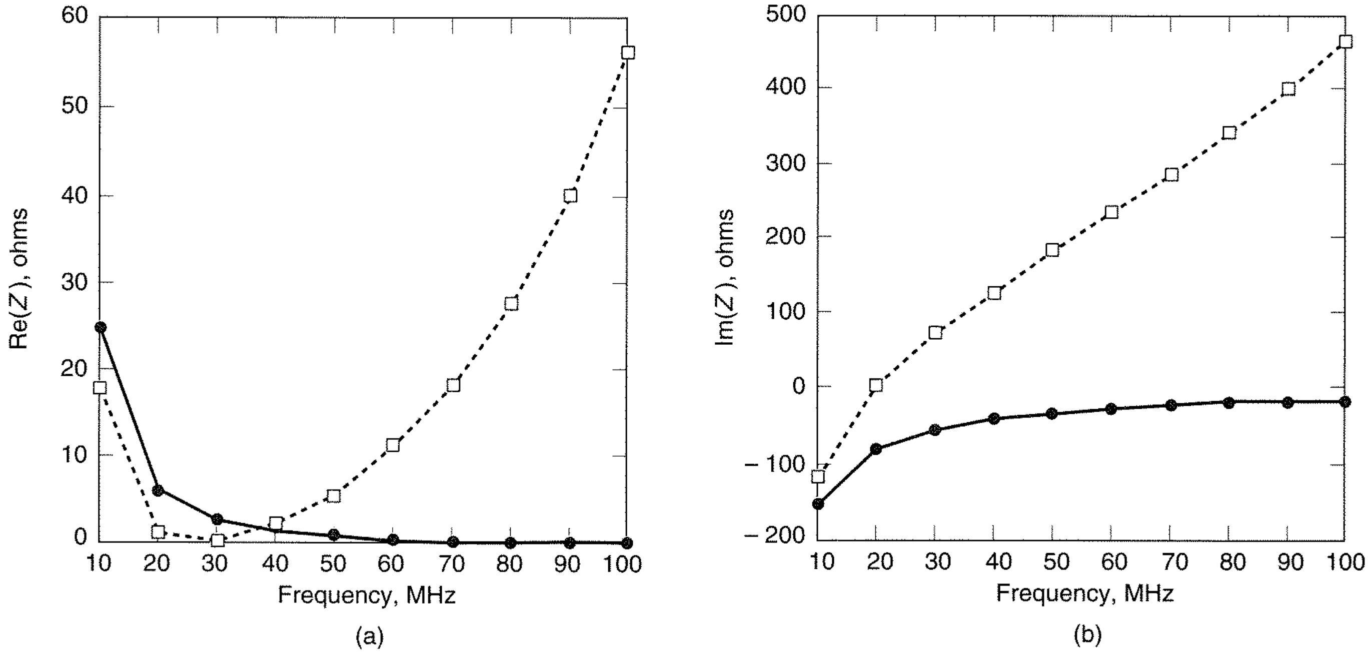


Figure 2.3 Real (a) and imaginary (b) parts of the input impedance of the circuit shown in Fig. 2.2 versus frequency, obtained by circuit theory (solid line with dots), by electromagnetic analysis of the circuit (dashed line), and by experiment (small squares).

frequency, they differ greatly from those predicted by circuit theory. We now know why: in addition to the circuit elements themselves (the resistor and the capacitor), the shape of interconnecting wires in Fig. 2.2 also influences the circuit behavior. This simple circuit can also be analyzed using electromagnetic theory and computer programs that take the shape of the interconnecting conductors into account. The result for the circuit impedance using such a program is shown in Fig. 2.3 in dashed line. You can observe excellent agreement between measurement and theory at frequencies considerably above those where circuit theory loses accuracy. For the moment, you may trust (or not trust) the dashed line results.

Another effect observed in the circuit from Fig. 2.2 is associated with the assumption that the chip (surface mount) components are very small, or *lumped* (which is always assumed in circuit theory). The chip capacitor and resistor in Fig. 2.2 will in reality not have the exact impedance values given in their specification sheets. It turns out that most chip capacitors and resistors have an associated series lead inductance of about 1 nH. That means that above a certain frequency, the chip capacitor will start behaving like an inductor. It is left as an exercise for the reader to calculate this resonant frequency for 1, 10, and 100-pF chip capacitors.

As the next example, let us again close the switch in Fig. 2.1a. Then we take the wire connecting the resistor to the generator and wind it tightly around a pen 20 to 30 times, as shown in Fig. 2.1d. The result is a more conventional inductor than the bent conductor of Fig. 2.1c. Again the shape of the wire has a huge effect on the voltage across the resistor. It has an even larger effect if an iron rod is used instead of the pen. We can find the value of inductance only by using electromagnetic theory, or by measurements. What we will learn in this book is how to take into account the actual

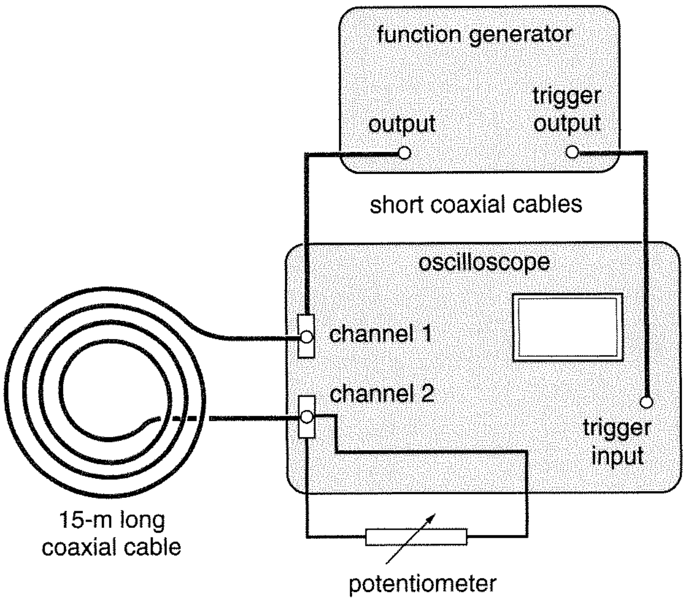


Figure 2.4 Observing electromagnetic effects in a coaxial cable

shape of conductive and nonconductive bodies, and the properties of the materials they are made of, in order to predict the behavior not only of simple circuits but also of different devices used in electrical engineering.

The last example is related to electromagnetic waves and transmission-line theory. Figure 2.4 shows a 15-m coaxial cable connected at one end to a function generator. At the other end it is connected to channel 2 of a two-channel oscilloscope and in parallel with a potentiometer (variable resistor). Channel 1 of the oscilloscope monitors the output of the function generator (which is the input to the coaxial cable). If you used only basic circuit theory, you would expect to see the same voltage for all values of the resistor at the end of the cable, and the voltage should be the same as that coming out of the signal generator. However, due to electromagnetic wave effects, the waveforms at the two channels (the voltages at the beginning and the end of the coaxial cable) can be very different. For example, a 1-V pulse from the signal generator could result in a negative, zero, or greater-than-1-V pulse at channel 2 of the oscilloscope. To explain this result we need so-called transmission-line theory, which turns out to be a special case of electromagnetic wave theory. We shall consider transmission lines in Chapter 18.

Questions and problems Q2.1 to Q2.7, P2.1 to P2.4

2.3 Oscillations in Circuits from the Electromagnetic Point of View

Let us review a more complex (but still very simple) circuit shown in Fig. 2.5, which is a combination of the previous cases. This is a series resonant circuit. For a voltage

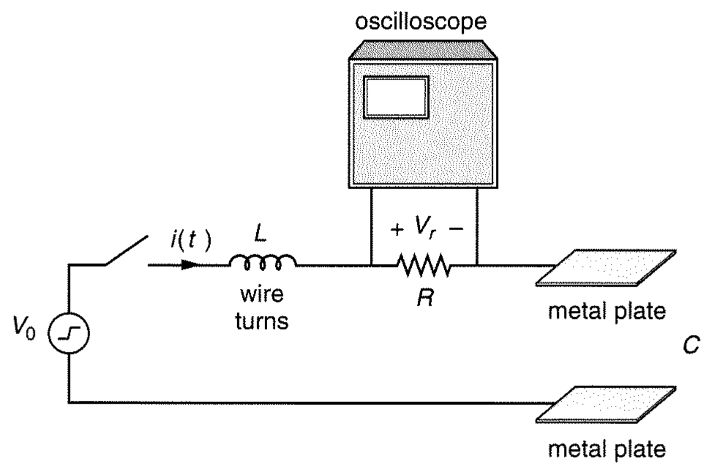


Figure 2.5 A possible physical realization of a series resonant circuit

step V_0 turned on at $t = 0$, Kirchhoff's voltage law gives

$$Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int_0^t i\,\mathrm{d}t + V_0 = 0. \tag{2.1}$$

By differentiating with respect to t and rearranging the terms, we get

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{i}{LC} = 0, \tag{2.2}$$

which is a second-order ordinary differential equation with exponential solutions of the form

$$i(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t). \tag{2.3}$$

A_1 and A_2 are constants determined from the initial conditions, and s_1 and s_2 are complex roots of the characteristic equation of (2.2), given by

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}. \tag{2.4}$$

Let $\omega_0 = 1/\sqrt{LC}$. For $\omega_0^2 > (R/2L)^2$, $s_{1,2} = -\alpha \pm \mathrm{j}\omega$ are complex numbers with real and imaginary parts, as seen in Eq. (2.4), and the solution for the current is

$$i(t) = B_1 e^{-\alpha t}\cos\omega t + B_2 e^{-\alpha t}\sin\omega t. \tag{2.5}$$

The constants $B_{1,2}$ are given by the initial conditions. At $t = 0$ there is no current through the inductor before the voltage is turned on at the input, $i(0) = 0$ and $B_1 = 0$, so $i(t) = B_2 e^{-\alpha t}\sin\omega t$, and B_2 can be found from knowing what $\mathrm{d}i(0)/\mathrm{d}t$ is. Since $i(0) = 0$ immediately after the switch is closed, there is no voltage drop across the resistor and the initial voltage V_0 on the capacitor shows up across the inductor,

$L(\mathrm{d}i/\mathrm{d}t)_{t=0} = V_0$. The final expression for the voltage across the resistor after the switch is closed is

$$v_R(t) = Ri(t) = R\frac{V_0}{L}e^{-\alpha t}\sin\omega t. \tag{2.6}$$

This last expression shows that the voltage is a sinusoid with an exponential amplitude decay. This is called a *damped oscillation*. In electromagnetic terms, the energy in an undamped case is stored in the inductor for one half of the cycle, and in the capacitor in the other half. In a damped case, some of the electromagnetic energy goes into heat in the resistive parts of the circuit.

This effect can also be explained in a similar way by circuit theory. What *cannot* be answered by circuit theory, however, are the following questions:

- Does the resonant frequency depend on the circuit shape and size?
- Does the damping depend on the circuit shape and size?

We already know the answer to the first question: the resonant frequency *does* depend (at least to some extent) on the shape of the circuit. The reasoning is exactly as in the previous examples.

The second question itself seems a bit strange: how can we have more damping than that resulting from losses in the resistor? We mentioned in the first chapter that Maxwell predicted the existence of electromagnetic waves. We will learn that theoretically these waves are produced by *all* systems with time-varying currents, and that the efficiency in producing these waves depends on the system size and shape. We will also learn that an electromagnetic wave is, in fact, an energy package. Thus, "radiation" of electromagnetic waves actually implies leakage, or loss, of energy from the system producing them. Therefore resonant circuits *do* have damping that depends on their size and shape. Fortunately, in most applications this effect is negligible, but it always exists. It can be predicted only by electromagnetic-field theory—circuit theory is unable to do that. We will learn how large a circuit must be to radiate substantially.

Questions and problems Q2.8 and Q2.9, P2.5 to P2.8

2.4 Chapter Summary

1. Circuit theory is not exact; it is an approximation of electromagnetic-field theory.
2. To understand the limitations of circuit theory, we have to begin from electromagnetic-field theory.
3. To determine theoretically the capacitance of a capacitor or the inductance of a coil, it is necessary to use electromagnetic-field theory. The calculation of the resistance of a resistor also requires some knowledge of electromagnetic-field theory.

4. Along transmission lines, such as two-wire or coaxial lines, exist specific electromagnetic waves with specific effects that cannot be explained in terms of circuit-theory concepts.
5. Resonance effects and damping in circuits depend on the circuit shape and size, a strange phenomenon from the circuit-theory viewpoint.

QUESTIONS

Q2.1. Why does every switch have capacitance?

Q2.2. Try to imagine a "perfect" but real switch (a switch with the smallest possible capacitance). How would you design a good switch? What would be its likely limitations?

Q2.3. Why does it become progressively more difficult to have an "ideal" switch as frequency increases?

Q2.4. Why is the circuit-theory assumption that interconnecting conductors (wires) have no effect on the circuit behavior incorrect?

Q2.5. Imagine a resistor connected to a car battery by wires of fixed length. Does the shape of the wires influence the current in the resistor? Explain.

Q2.6. Answer question Q2.5 if the source is a (1) 60-Hz and (2) 1-GHz generator.

Q2.7. Give at least two reasons for the failure of circuit theory when analyzing the simple circuit in Fig. 2.2.

Q2.8. Explain why the resonant frequency of a circuit is at least to some extent dependent on the circuit shape and size.

Q2.9. Why does the damping in resonant circuits depend at least to some extent on the circuits' shape and size? Can circuit theory explain this?

PROBLEMS

P2.1. The capacitance of a switch ranges from a fraction of a picofarad to a few picofarads. Assume that a generator of variable angular frequency ω is connected to a resistor of resistance of 1 MΩ, but that the switch is open. Assuming a switch capacitance of 1 pF, at what frequency is the open switch reactance equal to the resistor resistance?

P2.2. A surface-mount capacitor has a 1-nH parasitic series lead inductance. Calculate and plot the frequency at which such a capacitor starts looking like an inductor, as a function of the capacitance value.

P2.3. A surface-mount resistor of resistance $R = 100\,\Omega$ has a 1-nH series lead inductance. Plot the real and imaginary part of the impedance as a function of frequency. In which frequency range can this chip be used as a resistor?

P2.4. The windings of a coil have a parasitic capacitance of 0.1 pF, which can be viewed as an equivalent ~~series~~ capacitance. Plot the reactance of such a 1 μH coil as a function of frequency. Parallel

P2.5. A capacitor of capacitance C receives a charge Q. It is then connected to an uncharged capacitor of the same capacitance C by means of conductors with practically no resistance. Find the energy contained in the capacitor before connecting it to the other capacitor, and the energy contained in the two capacitors. [The energy of a capacitor is given by $W_e = Q^2/(2C)$]. Can you explain the results using circuit-theory arguments? Can you explain the results at all?

P2.6. The inductance of a thin circular loop of radius R, made of wire of radius a, where $R \gg a$, is given by the approximate formula

$$L_0 \simeq \mu_0 R \left(\ln \frac{8R}{a} - 2 \right) \text{ (henries)},$$

where $\mu_0 = 4\pi\, 10^{-7}$ H/m, and R and a are in meters. A capacitor of capacitance $C = 100\,\text{pF}$ and a coil of inductance $L = 100\,\text{nH}$ are connected in series by wires of radius a and the shape of a circular loop of radius R ($R \gg a$). Find: (1) the radius of the loop that results in $L_0 = L$ if $a = 0.1\,\text{mm}$; (2) the radius of the wire that results in $L_0 = L$ if $R = 2.5\,\text{cm}$; and (3) the resonant frequency of the circuit versus the loop radius, R, if $a = 0.1\,\text{mm}$.

P2.7. The capacitance between the terminals of a resistor is $C = 0.5\,\text{pF}$, and its resistance is $R = 10^6\,\Omega$. Plot the real and imaginary part of the impedance of this dominantly resistive element versus frequency from 0 MHz to 10 MHz.

P2.8. A coil is made in the form of $N = 10$ tightly packed turns of wire. Predict qualitatively the high-frequency behavior of the coil. Explain your reasoning.

3

Coulomb's Law in Vector Form and Electric Field Strength

3.1 Introduction

We have seen that sources of an electrostatic field are stationary and time-constant electric charges. This is the simplest form of the general electromagnetic field: since there are no moving or time-varying charges, there is no magnetic field. Although the electrostatic field is only a special case of the electromagnetic field, it occurs frequently. It is essentially the field that drives the electric current through wires and resistors in electric circuits; the field driving tiny signals in our nerves and brain; the field in depletion layers of transistors in computer chips; or the field that ionizes air (makes it conducting) just before a lightning bolt.

In this introductory course, the electrostatic field is of specific importance. The simplicity of the physical concepts in the electrostatic field allows us to develop mathematical models in a straightforward way. Later on, in more complex fields, we will be able to solve difficult practical problems using these concepts and tools as they are or with minor modifications.

3.2 Coulomb's Law in Vector Form

We have seen that Coulomb's law is an experimentally established law which describes the force between two charged bodies that are small compared to the distance between them. Such charged bodies are referred to as point charges. Coulomb's law in Eq. (1.1) is an algebraic expression that needs an additional explanation in words. The force directed along the line joining the two bodies is either repulsive or attractive. It is repulsive if the two charges are of the same kind, or sign, and attractive if they are of different kind.

Using vector notation, it is not difficult to write Coulomb's law in a form that does not need such additional explanations. Let $\mathbf{r}_{12}$ be the vector directed from charge Q_1 to charge Q_2 (Fig. 3.1), and $\mathbf{u}_{r12} = \mathbf{r}_{12}/|\mathbf{r}_{12}|$ be the unit vector of $\mathbf{r}_{12}$. (A number of notations have been used for unit vectors, e.g., $\mathbf{a}_r$, $\mathbf{u}_r$, or $\hat{\mathbf{r}}$ for the unit vector of vector $\mathbf{r}$. We will adopt $\mathbf{u}_r$ to remind us that it is a *unit* vector.) We can then express Coulomb's law in Eq. (1.1) in the following form:

$$\mathbf{F}_{e12} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2}\mathbf{u}_{r12} \qquad \text{newtons (N).} \tag{3.1}$$

(Coulomb's law in vector form)

The unit vector $\mathbf{u}_{r12}$, and thus also the force $\mathbf{F}_{e12}$, is directed from Q_1 toward Q_2, so that the additional explanation in words is not necessary anymore. Moreover, we have adopted the convention that a positive charge implies a *positive sign*, and a negative charge a *negative sign*. Complete information about the direction of the force is therefore also contained in this vector expression (recall that $-\mathbf{r}$ means the same vector, but in the opposite direction). If the two charges are of the same sign, the vector $\mathbf{u}_{r12}$ is as in Fig. 3.1, and if they are of different signs, we have instead $-\mathbf{u}_{r12}$, which means that the force is attractive. (If necessary, before proceeding further read Sections A1.1–A1.3 of Appendix 1, "Brief survey of vectors and vector calculus.")

The constant ϵ_0 is known as the *permittivity of free space* or *of a vacuum*. According to Eq. (3.1) its unit is $C^2/(m^2\,N)$. Usually a simpler unit, F/m (farads per meter), is used, as will be explained in Chapter 8. The value of ϵ_0 is

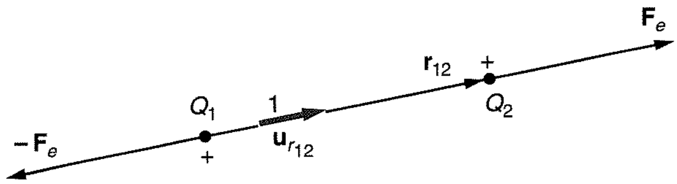


Figure 3.1 Notation in the vector form of Coulomb's law

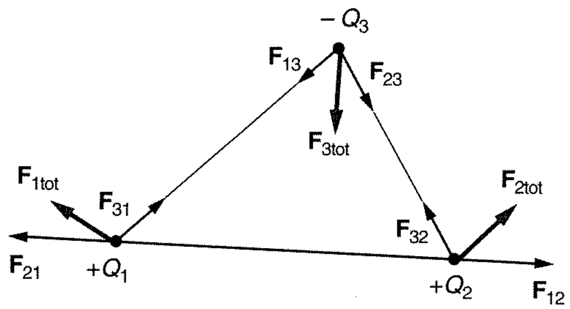


Figure 3.2 Example of vector addition of Coulomb forces

$$\epsilon_0 = 8.854 \cdot 10^{-12} \text{ farads per meter (F/m)} \approx \frac{1}{36\pi \cdot 10^9} \text{ F/m.} \tag{3.2}$$

(Permittivity of a vacuum)

The reason for writing $1/(4\pi\epsilon_0)$ in Coulomb's law instead of, say, simply k, is purely practical, removing the factor 4π from many other commonly used equations. Also, in many equations the permittivity of free space then appears as it is, ϵ_0, and not as its reciprocal.

Coulomb measured the electric force in air. We will see later that the electrical properties of air are very nearly the same as those of a vacuum, i.e., of space with no elementary particles of matter. Coulomb's law in Eq. (3.1) is therefore valid for charges that are strictly in a vacuum, but the presence of air does not change the result substantially. Therefore, the term "free space" usually implies vacuum or air.

Coulomb also measured the force on one point charge (e.g., Q) due to several point charges (e.g., $Q_1, Q_2, \ldots$). He concluded that the total force is obtained by *vector addition* of Coulomb's forces acting on Q by charges $Q_1, Q_2, \ldots$ individually. We know that mechanical forces on a body are summed in the same way, which is known as the *principle of superposition for forces*. An example of vector addition of Coulomb forces is sketched in Fig. 3.2.

How large are charges and electric forces we encounter around us? The charges rarely exceed a few nanocoulombs ($1\,\text{nC} = 10^{-9}\,\text{C}$). The largest electric forces around us do not exceed about 1 N, which is the weight of a small glass of water. Thus, measured by our standards, electric forces are very small.

Questions and problems: Q3.1 to Q3.10, P3.1 to P3.11

3.3 Electric Field Strength of Known Distribution of Point Charges

Let us repeat the definition of the electric field strength vector given in Eq. (1.5) in somewhat different notation. We first define the *test charge*, ΔQ, to be a very small body with negligibly small charge. (Such a test charge placed in an electric field will not affect the field, so that we can measure the electric field strength at a particular point of the field as it is in the absence of the test charge.) Then at any point in the electric field, the electric field strength vector is given by

$$\mathbf{E} = \frac{\mathbf{F}_{\text{on } \Delta Q}}{\Delta Q} \qquad \text{newtons per coulomb (N/C) = volts per meter (V/m).} \tag{3.3}$$

(Definition of the electric field strength vector)

The unit of the electric field strength is newtons per coulomb (N/C). For reasons to become clear in the next chapter, the equivalent unit, volts per meter (V/m), is used instead.

Combining this definition with the expression for the force exerted by one point charge on another point charge, Eq. (3.1), we see that the electric field vector due to a point charge Q is given by

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\mathbf{u}_r \qquad \text{(V/m)}, \tag{3.4}$$

(Electric field strength of a point charge)

where $\mathbf{u}_r$ is the unit vector directed *away* from charge Q. This is the electric field strength of a single point charge. It is a vector function of the distance from the point charge producing it and is directed away from a positive charge or toward a negative charge.

What if we have more than one charge producing the field? How is the electric field strength then obtained? The answer is fairly obvious: since the principle of superposition is valid for electric forces just as for mechanical forces, the equation follows directly from Eqs. (3.3) and (3.4). Assume that we have n point charges, $Q_1, Q_2, \ldots, Q_n$. The electric field strength at a point that is at distances $r_1, r_2, \ldots, r_n$ from the charges is simply

$$\mathbf{E} = \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0}\frac{Q_i}{r_i^2}\mathbf{u}_{ri} \qquad \text{(V/m)}. \tag{3.5}$$

Example 3.1—Superposition applied to the electric field strength. As an example, Fig. 3.3 shows how we obtain the electric field strength resulting from three point charges, Q, $2Q$, and $3Q$. Assume that the three charges are in air, in the plane of the drawing, and let us determine the total electric field at the point P which is at the same distance from the three charges. To obtain the total field, we first add up the field of the charges Q and $2Q$, and then add to this sum the field of the charge $3Q$, as indicated in the figure.

It is important to note that Eq. (3.5) can be used in this case because both vacuum and air, in which the charges are placed, are *linear media* (i.e., electrical properties of the medium, in this case of ϵ_0, do not depend on the electric field strength in the medium). This is not always the

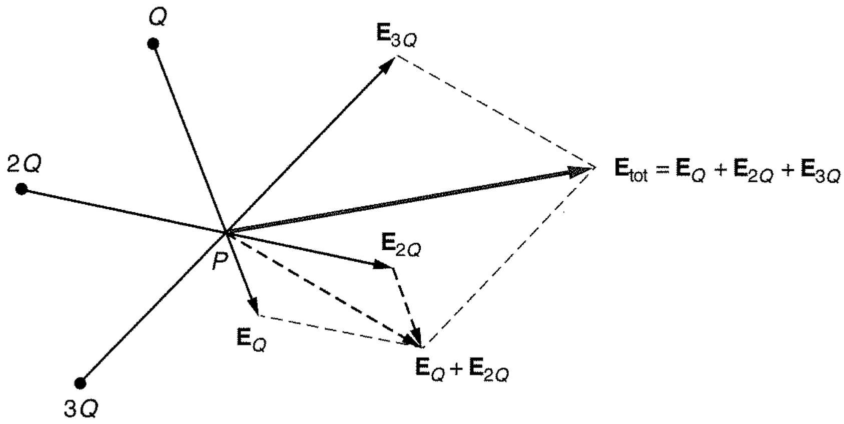


Figure 3.3 The electric field strength resulting from the three point charges Q, $2Q$, and $3Q$, situated in the plane of the drawing, at a point that is at the same distance from each of them

case—many important and practical media are nonlinear. For nonlinear media, superposition cannot be applied. Superposition allows us to break up a complicated problem into several easier ones and then add up their solutions to get the solution to the complicated problem. We will use it often.

Questions and problems: Q3.11 to Q3.14, P3.12 to P3.15

3.4 Electric Field Strength of Volume, Surface, and Line Charge Distributions

Elemental charges (electrons and protons) that create a field are always so small that they can be considered as point charges. Therefore Eq. (3.5) can be used, in principle, to calculate the electric field of any charge distribution. However, even for a tiny amount of charge, the number of elemental charges is very large. For example, -1 pC (-10^{-12} C) contains about 10^7 electrons. Therefore, in macroscopic electromagnetism, it is convenient to introduce the concept of *charge density,* and then to use integral calculus to evaluate the field of a charge distribution.

3.4.1 VOLUME CHARGE DENSITY

Consider first a cloud of static charges. (Of course, it must be kept in place by some means; otherwise it would move as a result of the electric forces the charges exert on each other.) Let them be packed so densely that even inside a small volume $\mathrm{d}v$ there are many charges, amounting to a total charge $\mathrm{d}Q_{\mathrm{in}\ \mathrm{d}v}$. We then define the *volume charge density,* ρ, at the point enclosed by that small volume:

$$\rho = \frac{\mathrm{d}Q_{\mathrm{in}\ \mathrm{d}v}}{\mathrm{d}v} \qquad \text{coulombs per cubic meter (C/m}^3\text{).} \tag{3.6}$$

(Definition of volume charge density)

Note that the unit of volume charge density is $\mathrm{C/m^3}$.

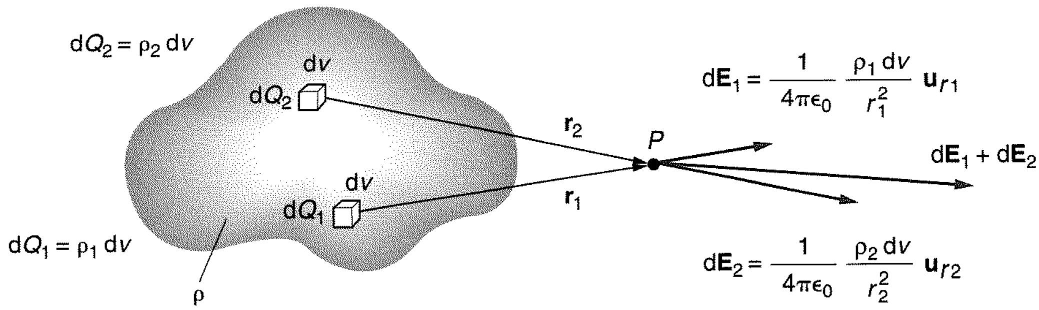


Figure 3.4 Calculating the electric field strength of a charged cloud of known charge density

According to this definition, inside a small volume dv where the charge density is ρ, there is a small charge

$$dQ = \rho\, dv. \tag{3.7}$$

This charge can be considered a point charge. If we have a charged cloud with known charge density ρ at all points, we can obtain the electric field strength at any point using, essentially, Eq. (3.5) but with a very large (theoretically infinitely large) number n of point charges. Such a sum is an integral (see Fig. 3.4):

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho\, dv}{r^2}\mathbf{u}_r \qquad (\text{V/m}). \tag{3.8}$$

(Electric field strength of volume distribution of charges)

Note that the charge density, ρ, and the ~~position vector, $\mathbf{r}$,~~ vary from one elemental volume dv to another. DISTANCE r FROM THE FIELD POINT & a_r

Suppose we know the shape of the cloud and volume charge density at all points of the cloud. It is possible only rarely to evaluate the integral in Eq. (3.8) analytically. However, we can approximately calculate the electric field strength at any point in space by dividing the volume charge into a finite number of very small volumes Δv, taking the value of ρ at the center of this small volume, and then summing all the vector electric field strengths resulting from these point charges. In this case, Eq. (3.8) becomes a sum over all the little volumes. This sum is not hard to evaluate on a computer, and the result will be more accurate with a greater number of small volumes.

3.4.2 SURFACE CHARGE DENSITY

Strictly speaking, volume charge is the only type of charge that appears in nature. However, in some cases this charge is spread in an extremely thin layer (of thickness on the order of a few atomic radii) and can be regarded as a *surface charge*. Such is, for example, the excess charge on a conducting body. To describe the surface charge distribution, we introduce the concept of *surface charge density*, σ. Figure 3.5 shows a body with surface charge. Consider a small area dS of the body surface. Let the charge on that small surface patch be $dQ_{\text{on } dS}$. The surface charge density at a point

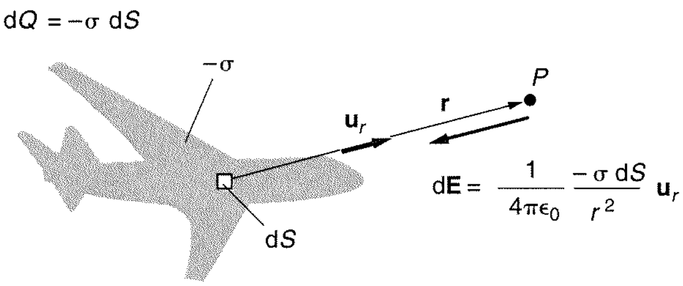


Figure 3.5 A body charged over its surface

of dS is then defined as

$$\sigma = \frac{dQ_{\text{on } dS}}{dS} \qquad \text{coulombs per square meter } (\text{C/m}^2). \tag{3.9}$$

(Definition of surface charge density)

(The symbol ρ_s is sometimes used instead of σ.) From this definition, it follows that if we know the surface charge density at a point on the body surface, the charge on a small patch of area dS enclosing this point is obtained as

$$dQ_{\text{on } dS} = \sigma\, dS. \tag{3.10}$$

The unit of surface charge density is C/m^2. Note that in general, the surface charge differs from one point of a surface to another.

The field of a given distribution of surface charge is obtained if in Eq. (3.8) we substitute the elemental charge, $\rho\, dv$, by $\sigma\, dS$:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma\, dS}{r^2} \mathbf{u}_r \qquad (\text{V/m}). \tag{3.11}$$

(Electric field strength of a surface distribution of charges)

3.4.3 LINE CHARGE DENSITY

Finally, we frequently encounter thin charged wires. Wires are usually conductors and the charge is distributed in a very thin layer on the wire surface. If the wire is thin compared to the distance of the observation point, we can consider the charge to be distributed approximately along a geometric line, for example along the wire axis. This type of charge is known as *line charge*. Its distribution along the line (i.e., along the wire the line approximates) is described by the *line charge density*, Q' (Fig. 3.6).

Let the charge on a very short segment dl of the line be $dQ_{\text{on } dl}$. The line charge density is defined as

$$Q' = \frac{dQ_{\text{on } dl}}{dl} \qquad \text{coulombs per meter (C/m).} \tag{3.12}$$

(Definition of line charge density)

(The symbol ρ_ℓ is sometimes used instead of Q'.) Thus if we know the line charge density at a point along a wire, the charge on a short segment dl of the wire containing

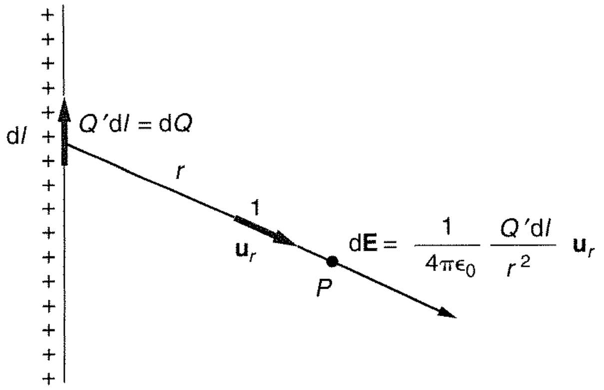


Figure 3.6 Electric field strength due to a thin charged wire

that point is simply

$$\mathrm{d}Q_{\text{on d}l} = Q'\,\mathrm{d}l. \tag{3.13}$$

The unit of line charge density is C/m. Note that Q' may differ from one point of the line to another.

The field of a given distribution of line charge along a line L is obtained as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_L \frac{Q'\,\mathrm{d}l}{r^2}\mathbf{u}_r \qquad \text{(V/m)}. \tag{3.14}$$

(Electric field strength of line distribution of charges)

The integrals in Eqs. (3.8), (3.11), and (3.14) usually cannot be evaluated analytically, but they can always be evaluated numerically, as described in connection with the volume charge distribution. We do not give any examples of analytical evaluation of these integrals because for those that can be evaluated, it is usually possible to obtain the result in a much simpler way, described in the next two chapters.

If we know the distribution of volume, surface, and line charges, i.e., if we know the volume, surface, and line charge density at all points, it is a simple matter to evaluate the electric field strength of these distributions at any point. As we shall see, however, the charge distribution is rarely known in advance. Instead, in practical problems we need to *determine* the charge distribution in order to calculate the electric field around it. Therefore, the formulas to determine the electric field strength from a known distribution of volume, surface, or line charges are mainly of academic interest. The concepts of volume, surface, and line charge are very useful, however, as we shall see later. Equations in the form of Eqs. (3.11) and (3.14) are also indispensable in determining unknown charge distributions numerically.

Questions and problems: Q3.15 to Q3.18, P3.16 to P3.27

3.5 Lines of the Electric Field Strength Vector

The electrostatic field is a vector field. A useful concept for visualizing vector fields is *field lines*. The lines of vector **E** are defined as imaginary, generally curved lines,

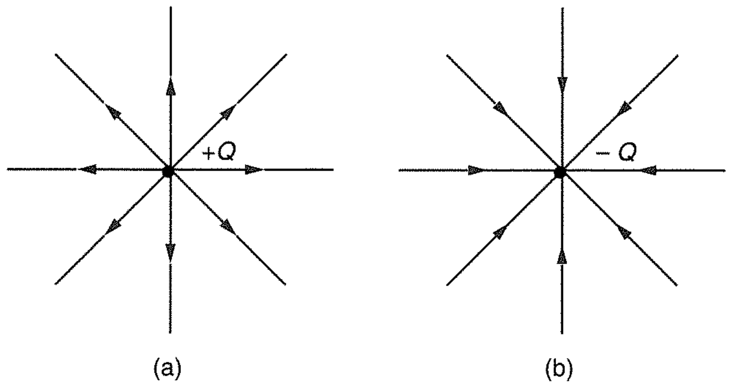


Figure 3.7 Electric field lines of (a) a positive and (b) a negative point charge

having the property that **E** is tangential to these lines at all points. For example, lines of vector **E** of the field of a point charge are straight lines emanating from the charge (Fig. 3.7). It is usual to add an arrow to the lines of vector **E** indicating the direction of **E** along the lines.

Example 3.2—Electric field lines of a very large, uniformly charged plate. As a further example of electric field lines, consider a very large, uniformly charged flat plate. Let the charge on the plate be positive. Since the plate is very large, if we consider the field close to the plate, the electric field strength vector must be normal to the plate and pointing away from it (why?). The electric field lines are as sketched in Fig. 3.8. This kind of electric field, which has in a region of space the same direction and magnitude, is called a *uniform electric field*.

For a negative plate, the lines are of the same form, only the arrowhead (indicating the direction of vector **E**) points toward the plate, i.e., the field is also uniform.

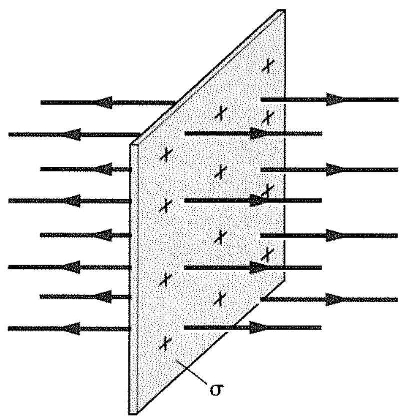


Figure 3.8 Electric field lines for a flat, large plate with uniform positive surface charge distribution

3.6 Chapter Summary

1. If written in vector form, Coulomb's law does not require additional explanations in words—all information is then contained in the formula. For formulating Coulomb's law in this manner, the positive and negative sign convention for charges is essential.
2. The electric field strength vector **E** of a point charge is defined from the vector form of Coulomb's law.
3. The expression for the electric field strength of a point charge can be used for obtaining vector **E** resulting from any distribution of point charges, or from volume, surface, and line distributions of charges. To do this, it is necessary to define volume, surface, and line charge distributions.
4. The charge distribution is usually not known in advance. Therefore, the formulas for the electric field strengths of known distributions of charges are of limited practical usefulness. However, they may be used for numerical evaluation of the vector **E** by means of integral equations.
5. We say that in a region of space the electric field is uniform if the electric field strength at all points of the region has the same direction and magnitude.

QUESTIONS

Q3.1. Discuss the statement that Eq. (3.1) indeed shows not only the magnitude but also the correct direction of the force $\mathbf{F}_{e12}$. Does Eq. (3.1) need an additional explanation in words? Explain.

Q3.2. Would the vector form of Coulomb's law (Eq. 3.1) be possible if plus and minus signs were not associated with the two types of charges? For example, suppose that they were denoted by subscripts A and B instead of plus and minus signs. Explain your answer. (This question is intended to show how important proper conventions are for simplifying the mathematical description of physical phenomena.)

Q3.3. Is it possible to *derive* the principle of superposition of Coulomb's forces, starting from Coulomb's law? Explain.

Q3.4. Prove that there can be no net electric force on an isolated charged body due to its charge only.

Q3.5. Similarly to the electric field, the gravitational field also acts "at a distance." But whereas we understand and accept that there is a downward force on an object we lift (e.g., a stone), with no visible reason, such an electric force with no visible reason is somewhat astonishing. Explain why this is so.

Q3.6. Of five equal conducting balls one is charged with a charge Q, and the other four are not charged. Find all possible charges the balls can obtain by touching one another, assuming that two balls are allowed to touch only once, and that while two balls are touching, the influence of the other three can be neglected.

Q3.7. If an electrified body (e.g., a plastic ruler rubbed against a wool cloth) is brought near small pieces of thin aluminum foil, you will see that the body first attracts, but after the contact repels, the small pieces. Perform this experiment and explain.

Q3.8. Imagine that you electrified a body, e.g., by rubbing it against another body. How could you determine the sign of the charge on the body? Try to perform the experiment.

Q3.9. You have two identical small metal balls. How can you obtain identical charges on them?

Q3.10. Two small balls carry charges of unknown signs and magnitudes. Experiment shows that there is no electric force on a third charged ball placed at the midpoint between the first two. What can you conclude about the charges on the first two balls?

Q3.11. An uncharged small ball is introduced into the electric field of a point charge. Is there a force on the ball? Explain.

Q3.12. Is it correct to write the following: (1) Q $(Q > 0)$; (2) $-Q$ $(Q < 0)$; (3) Q $(Q < 0)$; and (4) $-Q$ $(Q > 0)$? Explain.

Q3.13. To measure the electric field strength at a distance r from a small charge Q, a test charge ΔQ $(\Delta Q \ll Q)$ in the form of a sphere of radius $a = r/2$ is centered at that point. Discuss the correctness of the measurement.

Q3.14. What would the form of the expression in Eq. (3.4) be if $\mathbf{u}_r$ is *toward* the charge? What form does Eq. (3.4) take if we do not associate a sign with the charge Q?

Q3.15. Is ρ in Eq. (3.6) a function of coordinates, in general?

Q3.16. Assuming ρ in Eq. (3.8) to be known, explain in detail how you would numerically evaluate the *vector* integral to obtain $\mathbf{E}$.

Q3.17. Repeat question Q3.16 for a surface distribution of charges over a surface S, and for a line distribution of charges along a line L.

Q3.18. Why are the formulas in Eqs. (3.8), (3.11), and (3.14) only of limited practical value?

PROBLEMS

P3.1. What would be the charge of a copper cube, 1 cm on a side, if one electron were removed from all the atoms on the cube surface? A cubic meter of copper has about $8.4 \cdot 10^{28}$ atoms.

P3.2. Evaluate the force that would exist between two cubes as described in problem P3.1 when they are (1) $d = 1$ m, and (2) $d = 1$ km apart.

P3.3. Three small charged bodies arranged along a straight line are at distances a, b, and $(a + b)$ apart. Determine the conditions that the charges on the bodies have to satisfy so that the electric forces on all three are zero.

P3.4. Assume that the earth is electrified by a charge $2Q$, and the moon by a charge Q. How large does Q have to be so that the repulsive electric force between the earth and the moon becomes equal to the attractive gravitational force? The masses of the earth and the moon are $m_e = 5.983 \cdot 10^{24}$ kg and $m_m = 7.347 \cdot 10^{22}$ kg. The gravitational constant is $\gamma = 6.67 \cdot 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.

P3.5. Evaluate the specific charge of the electron (charge over mass). Estimate the charge of the book you are reading if it had the same specific charge. What would the force between two such charged books be if they were at a distance of 10 m?

P3.6. A given charge Q is divided between two small bodies, so that one has the charge Q', and the other has the rest. Determine the ratio Q/Q' resulting in the greatest electric force between them, assuming the distance between them is fixed.

BOTH CHARGES HAVE THE SAME SIGN.

P3.7. Three small charged bodies of charge Q are placed at three vertices of an equilateral triangle with sides of length a. What is the direction and magnitude of the electric force on each of them if $a = 3\,\text{cm}$ and $Q = 1.8 \cdot 10^{-10}\,\text{C}$?

P3.8. A charge Q exists at all vertices of a cube with sides of length a. Determine the direction and magnitude of the electric force on one of the charges.

P3.9. Two identical small, conducting balls with centers that are d apart have charges Q_1 and Q_2. The balls are brought into contact and then returned to their original positions. Determine the electric force if charges Q_1 and Q_2 are (1) of the same sign; (2) of opposite signs.

P3.10. Evaluate the velocity of an electron orbiting around the nucleus of a hydrogen atom along an approximately circular orbit of radius $a = 0.528 \cdot 10^{-10}\,\text{m}$. How many revolutions does the electron make in one second?

P3.11. Two small balls of mass m each have a charge Q and are suspended at a common point by separate thin, light, conducting filaments of length l. Assuming the charges are located approximately at the centers of the balls, find the angle α between the filaments. Suppose that α is small. (Such a system can be used as a primitive device for measuring charge, and is called an *electroscope*.)

P3.12. A small body with a charge $Q = 1.8 \cdot 10^{-10}\,\text{C}$ is situated at a point A in the electric field. The electric force on the body has an intensity $F = 5.4 \cdot 10^{-4}\,\text{N}$. Evaluate the magnitude of the electric field strength vector at that point.

P3.13. A point charge Q ($Q > 0$) is located at the point $(0, d/2)$, and a charge $-Q$ at the point $(0, -d/2)$, of a rectangular coordinate system. Determine and plot the magnitude of the total electric field strength vector at any point in the xy plane.

P3.14. An electric dipole consists of two equal and opposite point charges Q and $-Q$ that are a distance d apart, Fig. P3.14. (1) Find the electric field vector along the x axis in the figure. (2) Find the electric field vector along the y axis. (3) How does the electric field strength behave at distances $x \gg d$ and $y \gg d$ away from the dipole? How does this behavior compare to that of the field of a single point charge?

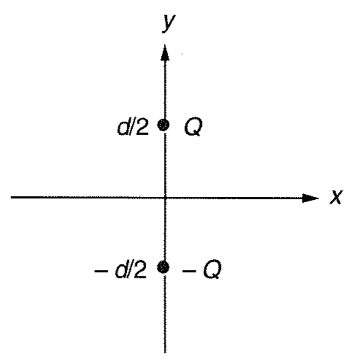


Figure P3.14 An electric dipole consists of two equal charges of opposite signs.

P3.15. Find the x and y components of the electric field vector at an arbitrary point in the field of the electric dipole from problem P3.14, assuming that the distance of the observation point from the dipole center is much greater than d. Plot your results.

P3.16. A thin, straight rod $a = 10\,\text{cm}$ long is uniformly charged along its length with a total charge $Q = 2 \cdot 10^{-10}\,\text{C}$. The rod extends from the point $(-a/2, 0)$ to the point $(a/2, 0)$ in an xy rectangular coordinate system. Evaluate the electric field strength vector at points $A(0, a/4)$ and $B(3a/4, 0)$.

P3.17. Solve problem P3.16 approximately, by dividing the rod into n segments. Compare the results with the exact solution for $n = 1, 2, 3, 4, 5, 6, 10$, and 20.

***P3.18.** An L-shaped rod with sides $a = 10\,\text{cm}$ extends from the origin of an xy rectangular system to the point $(a, 0)$, and from the origin to the point $(0, a)$. The rod is charged uniformly along its length with a total charge $Q = 2.6 \cdot 10^{-9}\,\text{C}$. Evaluate the electric field strength vector at points $A(a, a)$ and $B(3a/2, 0)$.

***P3.19.** Solve problem P3.18 approximately, by dividing the L-shaped rod into $2n$ segments. Compare the results with the exact solution for $n = 2, 3, 4, 5, 6$, and 20.

P3.20. A thin ring of radius a is uniformly charged along its length with a total charge Q. Determine the electric field strength along the ring axis.

P3.21. A thin circular disk of radius a is charged uniformly over its surface with a total charge Q. Determine the electric field strength along the disk axis normal to its plane and plot your result. What do you expect the expression for the electric field to become at large distances from the disk? What do you expect the expression to become if the radius of the disk increases indefinitely, and the surface charge density is kept constant?

P3.22. Calculate the electric field along the axis of the disk in problem P3.21 if the charge is not distributed uniformly but increases linearly along the disk radius, and it is zero at the disk center. Plot your result and compare it to those for problem P3.21.

P3.23. A dielectric cube with sides of length a is charged over its volume with a charge density $\rho(x) = \rho_0 x/a$, where x is the normal distance from one side of the cube. Determine the charge of the cube.

P3.24. The volume charge density in a spherical charged cloud of radius a is $\rho(r) = \rho_0(a-r)/a$, where r is the distance from the cloud center, and ρ_0 is a constant. Determine the charge of the cloud.

P3.25. Determine and plot the electric field strength at a distance r from a straight, very long, thin charged filament with a charge Q' per unit length.

P3.26. A wire in the form of a semicircle of radius a is charged with a total charge Q. Assuming the charge to be uniformly distributed along the wire, determine the electric field strength vector at the center of the semicircle.

P3.27. A hemispherical shell of radius a is charged uniformly over its surface by a total charge Q. Determine the electric field strength at the center of the sphere, one-half of which is the shell.

4

The Electric Scalar Potential

4.1 Introduction

You may recall from your physics courses that the gravitational field at any point can be described in two ways. One is by the force acting on a small mass located at that point. This is analogous to the electric force. The other is by specifying how large the *energy* at the point of that small mass is, per unit mass. This is known as the gravitational potential. The electric potential is analogous to the gravitational potential. It tells us how large the energy of a small charge at a point of the electric field is, per unit charge. Note that force is a *vector*, whereas energy is a *scalar*. Therefore the description of the field in terms of the potential is mathematically simpler than in terms of the field vector.

4.2 Definition of the Electric Scalar Potential

To understand the concept of the electric scalar potential, consider a test charge, ΔQ, at a point A in an electrostatic field. The electric force acting on ΔQ will tend to move the test charge. Let the force move the charge along a line a to a point B, as sketched in Fig. 4.1. How much work was done by the electric force in this case?

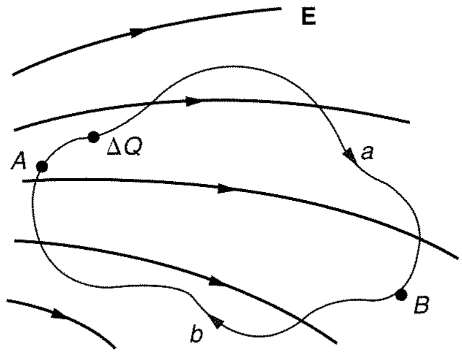


Figure 4.1 A test charge in an electrostatic field

We know from physics that if a force **F** moves a body along a small vector distance **dl**, the work done by the force is

$$\mathrm{d}A = F\,\mathrm{d}l \cos(\text{angle between vectors } \mathbf{F} \text{ and } \mathbf{dl}) \qquad \text{joules (J)}, \tag{4.1}$$

where F and $\mathrm{d}l$ are the magnitudes of the two vectors.

This type of product of two vectors occurs frequently in physics and engineering. (If necessary, before proceeding further please read Section A1.2 of Appendix 1.) It is known as the *scalar product*, or *dot product*. For any two vectors **X** and **Y**, the dot product is defined as

$$\mathbf{X} \cdot \mathbf{Y} = XY \cos(\text{angle between vectors } \mathbf{X} \text{ and } \mathbf{Y}). \tag{4.2}$$

Hence, instead of Eq. (4.1) we can use the shorthand

$$\mathrm{d}A = \mathbf{F} \cdot \mathbf{dl}. \tag{4.3}$$

Work is a *scalar* quantity. Therefore, to obtain the work done by the electric force in moving the test charge from point A to point B, we simply add all elemental works of the form as in Eq. (4.3), from A to B:

$$A_{\text{from } A \text{ to } B} = \int_A^B \mathbf{F} \cdot \mathbf{dl} \qquad (\mathrm{J}). \tag{4.4}$$

This type of integral (which is nothing but a sum of many very small terms) is known as a *line integral.*

The electric force on ΔQ is $\Delta Q\mathbf{E}$, and ΔQ is a constant that can be taken out of the integral sign. With this in mind, if we divide Eq. (4.4) by ΔQ, we get

$$\frac{A_{\text{from } A \text{ to } B}}{\Delta Q} = \int_A^B \mathbf{E} \cdot \mathbf{dl} \qquad (\mathrm{J/C} = \mathrm{V}). \tag{4.5}$$

Note that the right-hand side of this equation does *not* depend on ΔQ. It represents *the work that would be done by the electric field in moving the test charge from point A to point B,* per unit charge.

Imagine now that the field moves the test charge from A to B but along a different path, for example path b in Fig. 4.1. How much work is done by the electric forces in that case? It is easy to understand that the answer must be the same as for path a. Assume for a moment that the work that the electric field does when moving the charge along path b is larger than the work done along path a. We could then let the field move the test charge along b first. At point B, it would have a certain velocity, which means a certain kinetic energy. This energy would be greater than the work that needs to be done to return the test charge to point A along path a. So we would come back to A with extra energy, in spite of the system being again the same as in the beginning. Evidently, this is contrary to the law of conservation of energy. Therefore *the work done by the field or against the field in moving the test charge from one point of the field to another does not depend on the particular path between the two points.*

Since this is so, we can adopt the point B to be a *fixed point* and call it the *reference point*, R. We can next describe the field at all other points by specifying how large the expression in Eq. (4.5) is at these points. With the adoption of the fixed reference point $R = B$, we in fact have a *scalar* function of coordinates describing the field. It is known as the *electric scalar potential, V_A, at a point A of the electric field:*

$$V_A = \int_A^R \mathbf{E} \cdot \mathrm{d}\mathbf{l} \qquad \text{volts (V).} \tag{4.6}$$

(Definition of the electric scalar potential)

The unit of potential is the *volt* (abbreviated V), hence the unit V/m for **E**.

We have convinced ourselves that the line integral of **E** in an electrostatic field between two points does not depend on the path of integration. An important conclusion follows from this property: the line integral of the electric field strength along *any* closed contour C is zero:

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = 0. \tag{4.7}$$

(Law of conservation of energy in the electrostatic field)

Note that the contour C can be completely, but also only partly, in the field. The integral on the left side of this equation is known as a *contour integral.* Note also that Eq. (4.7) represents the mathematical expression of the law of conservation of energy for the electrostatic field. It is, therefore, a fundamental property of the electrostatic field.

Questions and problems: Q4.1 to Q4.8

4.3 Electric Scalar Potential of a Given Charge Distribution

Let us determine the potential at a point A, which is a distance r away from a positive point charge Q. Assume that the reference point is a distance r_R away from the

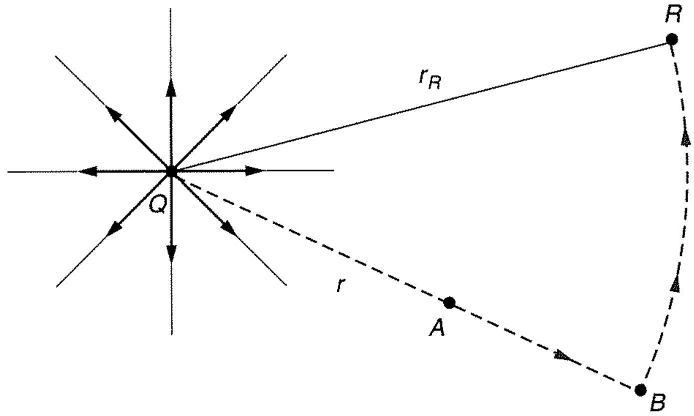


Figure 4.2 A field point, A, and the reference point, R, in the field of a point charge

charge, Fig. 4.2. We know that we can go from A to R along any path, so we adopt the simplest route: we first go from point A along a radius to the point B where it intersects with the circle of radius r_R. Along this path segment, vectors $\mathbf{E}$ and $\mathbf{dl}$ are parallel. Therefore the product $\mathbf{E} \cdot \mathbf{dl}$ is simply $E\,dl$ (cosine of zero is unity). We then continue to the point R along the arc of the circle, where the product $\mathbf{E} \cdot \mathbf{dl}$ is zero (cosine of $\pi/2$ is zero). We thus have

$$V_A = \int_A^B \mathbf{E} \cdot \mathbf{dl} + \int_B^R \mathbf{E} \cdot \mathbf{dl} = \frac{Q}{4\pi\epsilon_0} \int_r^{r_R} \frac{dr}{r^2}. \tag{4.8}$$

The integral is a standard one, and the result is

$$V_A = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{r_R} \right) \qquad \text{(V)}. \tag{4.9}$$

This is the formula for the potential of a point charge at a distance r from the charge, and with respect to the reference point a distance r_R from it.

So far, we have not discussed where the reference point should be. This can be *any* point. It is convenient to adopt it so that the expression for the potential is the simplest. In the case of Eq. (4.9), this is obtained if we assume that r_R is very large, theoretically infinite, i.e., if the reference point is at infinity. In that case the potential at point A of the field of a point charge Q becomes

$$V_A = \frac{Q}{4\pi\epsilon_0 r} \quad \text{(reference point at infinity)} \qquad \text{(V)}. \tag{4.10}$$

(Potential at a distance r from a point charge)

The reference point at infinity is the most convenient and is used most often. We shall see, however, that this point cannot be used if there are charges at infinity (e.g., for an infinitely long line charge).

How does the choice of the reference point influence the scalar potential function? For example, what happens if instead of R we adopt R_1 to be the new reference point? It is left to the reader to prove that in that case the potential at all points will be increased by the *same* amount:

$$\Delta V = \int_R^{R_1} \mathbf{E} \cdot d\mathbf{l}. \tag{4.11}$$

Once we know the potential of a point charge, it is quite simple to determine the potential of a given distribution of volume, surface, or line charges. Referring to Fig. 4.3, the potential of a volume charge distribution is given by

$$V_P = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho \, dv}{r} \quad \text{(reference point at infinity)} \quad \text{(V)}, \tag{4.12a}$$

(Potential of volume distribution of charges)

that of a surface charge distribution is obtained as

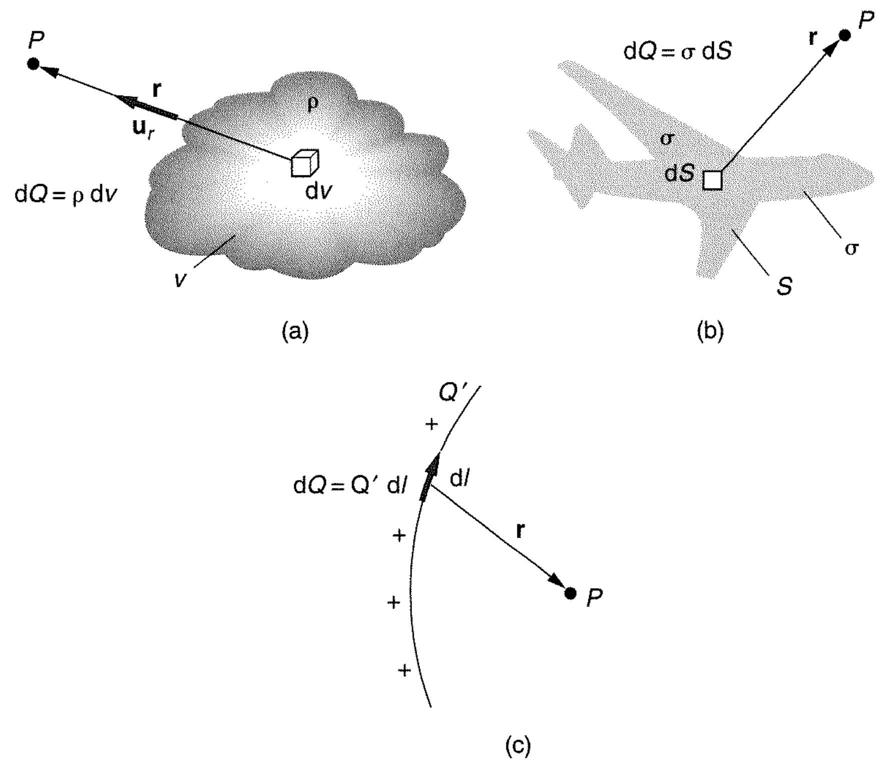


Figure 4.3 (a) A charged cloud, (b) a charged surface, and (c) a charged line, with a point P at which the electric scalar potential is calculated

$$V_P = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma \, \mathrm{d}S}{r} \quad \text{(reference point at infinity)} \quad \text{(V)}, \tag{4.12b}$$

(Potential of surface distribution of charges)

and the potential of a line charge distribution is, by analogy,

$$V_P = \frac{1}{4\pi\epsilon_0} \int_L \frac{Q' \, \mathrm{d}l}{r} \quad \text{(reference point at infinity)} \quad \text{(V)}. \tag{4.12c}$$

(Potential of line distribution of charges)

Example 4.1—Potential on the axis of a charged ring. Let us find the potential on the axis of a thin ring of radius R, uniformly charged along its length with a line charge density Q', Fig. 4.4. The element $\mathrm{d}l$ of the ring has a charge $\mathrm{d}Q = [Q/(2\pi R)]\,\mathrm{d}l$. The potential due to this charge is the same as that of a point charge, except that Q needs to be replaced by $\mathrm{d}Q$. The potential at a point P on the ring axis (Fig. 4.4) is therefore obtained as

$$V_P = \frac{1}{4\pi\epsilon_0} \int_C \frac{Q}{2\pi R} \frac{\mathrm{d}l}{r} = \frac{Q}{8\pi^2\epsilon_0 R r} \int_{\text{ring}} \mathrm{d}l.$$

Since the integral of $\mathrm{d}l$ around the ring equals its circumference, $2\pi R$, we finally obtain

$$V_P = \frac{Q}{4\pi\epsilon r} = \frac{Q}{4\pi\epsilon_0\sqrt{R^2 + x^2}}.$$

As already mentioned, we rarely know what the distribution of charges is. Therefore these formulas, similarly to those for the electric field strength of a given distribution of charges, do not have wide practical applicability. However, as in the case of the electric field strength, Eq. (4.12*b*), for example, can be used to calculate the charge distribution over a conducting body numerically.

Questions and problems: Q4.9 to Q4.13, P4.1 to P4.9

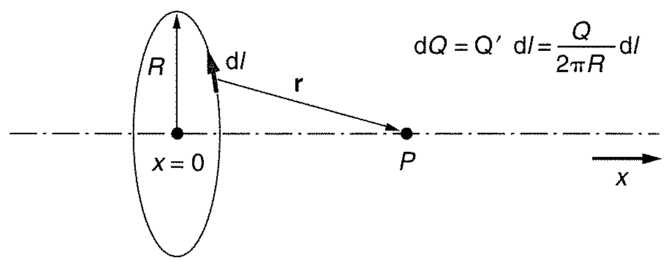


Figure 4.4 A thin ring uniformly charged along its length with a line charge density Q'

4.4 Potential Difference and Voltage

An important concept in circuit theory is the *potential difference* or *voltage* between two points in an electrostatic field. We shall see that voltage is a wider concept than just potential difference. *Only in the electrostatic field are the two concepts equivalent.*

We denote the voltage with the same letter V as the potential, either with two subscripts or with no subscripts at all (such as in the case of the potential difference between two terminals of a voltage source). The two subscripts tell us between which two points the potential difference is considered—for example, V_{12} is the voltage between points 1 and 2. In the case of the potential at a point, of course, there is only one subscript, for example V_1, although in electrostatics we could also write it as V_{1R}, where R denotes the reference point.

According to the definition of the potential in Eq. (4.6), the potential difference between points A and B is given by

$$V_{AB} = V_A - V_B = \int_A^R \mathbf{E} \cdot \mathrm{d}\mathbf{l} - \int_B^R \mathbf{E} \cdot \mathrm{d}\mathbf{l} \qquad (\mathrm{V}). \tag{4.13}$$

If in the second integral the upper and lower limits of integration are interchanged, the line element, $\mathrm{d}\mathbf{l}$, changes sign. Hence we can rewrite Eq. (4.13) as

$$V_{AB} = \int_A^R \mathbf{E} \cdot \mathrm{d}\mathbf{l} + \int_R^B \mathbf{E} \cdot \mathrm{d}\mathbf{l}. \tag{4.14}$$

So we have to integrate the dot product $\mathbf{E} \cdot \mathrm{d}\mathbf{l}$ from A to R, and then from R to B, i.e., from A to B over R. We know, however, that the path between A and B does not affect the result. Therefore we can calculate this integral along any path, not necessarily traversing point R. Thus we finally have

$$V_{AB} = \int_A^B \mathbf{E} \cdot \mathrm{d}\mathbf{l} \qquad (\mathrm{V}). \tag{4.15}$$

(Potential difference between points A and B)

Consequently, the position of the reference point does not influence the voltage between two points in an electrostatic field. This, of course, was to be expected—we know that a change in the position of the reference point changes the potential at all points by the same amount, ΔV in Eq. (4.11).

If we compare Eqs. (4.15) and (4.5), we see that the potential difference between two points can be given the following physical interpretation: it equals the work that would be done by the electric forces in moving a test charge from the first to the second point, per unit test charge.

What is the range of voltages encountered in practice? The smallest (time-varying) voltage we can measure is on the order of $1\,\mathrm{pV} = 10^{-12}\,\mathrm{V}$. The voltage of batteries for watches and calculators is about 1.5 V. The voltage in the plugs in our homes is, for example, 110 V in the United States and Canada, and 220 V in Europe. The largest voltage used in power transmission by high-voltage transmission lines is on the order of 1 MV = 10^6 V.

Questions and problems: Q4.14 to Q4.17, P4.10 to P4.14

4.5 Evaluation of Electric Field Strength from Potential

Here is the final basic question we may ask about the electric scalar potential V: we know how to determine V if we know $\mathbf{E}$ along any path from A to R, but can we determine $\mathbf{E}$ if we know V? This is quite simple to do.

Consider two *close* points, A and B, in an electrostatic field (Fig. 4.5). Let the potential at A be V_A, and at B be $V_B = V_A + \mathrm{d}V$. Assume that the vector line element from A to B is $\mathrm{d}\mathbf{l}$, and let it be along the x coordinate axis so that $\mathrm{d}l = \mathrm{d}x$. The potential difference between A and B is then simply $\mathbf{E}\cdot\mathrm{d}\mathbf{l} = E\mathrm{d}x\cos\alpha = E_x\mathrm{d}x$ [the integral in Eq. (4.15) consists of a single small term]. So we have

$$V_A - V_B = V_A - (V_A + \mathrm{d}V) = -\mathrm{d}V = E_x\,\mathrm{d}x. \tag{4.16}$$

In other words, the component E_x of vector $\mathbf{E}$ in the x direction is obtained by

$$E_x = -\frac{\mathrm{d}V}{\mathrm{d}x} \qquad (\mathrm{V/m}). \tag{4.17}$$

This is a very simple result. Assume that at a point in the electrostatic field we know V as a function of coordinate x along an x axis in any direction at that point. We can then determine the projection E_x of the vector $\mathbf{E}$ on the x axis at that point simply as the negative derivative of $V(x)$. The reference direction for the projection is the x axis.

Example 4.2—Electric field of a point charge found from the potential. Consider a point charge Q. Let the x axis be any radial line beginning at the charge. The potential is then

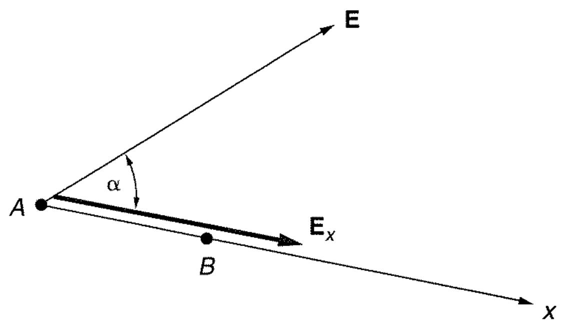


Figure 4.5 Determination of vector $\mathbf{E}$ from known V at two close points

given by Eq. (4.10), except that we have to replace r by x. According to Eq. (4.17), we have

$$E_x = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q}{4\pi\epsilon_0 x}\right) = \frac{Q}{4\pi\epsilon_0 x^2}, \tag{4.18}$$

as we know it should be from Coulomb's law.

Since we now know how to determine the projection of the vector **E** in *any* direction at a point, we can easily determine the complete vector **E** at that point. We simply define three coordinate axes at the point, calculate the projections of the vector **E** on all the three axes, and sum the three components as vectors. For example, let the three axes be the x, y, and z axes of a rectangular coordinate system. Then the vector **E** at any point is given by

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{u}_x + \frac{\partial V}{\partial y}\mathbf{u}_y + \frac{\partial V}{\partial z}\mathbf{u}_z\right) \qquad (\mathrm{V/m}), \tag{4.19}$$

where $\mathbf{u}_x$, $\mathbf{u}_y$ and $\mathbf{u}_z$ are unit vectors of the three coordinate axes. Partial derivatives must be used instead of ordinary derivatives because the potential $V = V(x, y, z)$ is a function of all three coordinates. To determine a projection of **E** on any one of the three coordinate axes, we have to differentiate $V(x, y, z)$ with respect to that coordinate only, considering the other two as constants. This is exactly the definition of the partial derivative of a function of several variables.

We know from mathematics that the expression in the parentheses on the right-hand side of Eq. (4.19) is called the *gradient* of the scalar function V. (If necessary, please read Section A1.4.1 of Appendix 1 before proceeding further.) It is sometimes written as grad V, but much more frequently we use the so-called *nabla operator* or *del operator*. The del operator in the rectangular coordinate system is defined as

$$\nabla = \left(\frac{\partial}{\partial x}\mathbf{u}_x + \frac{\partial}{\partial y}\mathbf{u}_y + \frac{\partial}{\partial z}\mathbf{u}_z\right) \qquad (1/\mathrm{m}), \tag{4.20}$$

(Definition of nabla or del operator)

with the assumption that the expression ∇V is a shorthand for the expression in parentheses on the right-hand side in Eq. (4.19). So we can write

$$\mathbf{E} = -\mathrm{grad} V = -\nabla V \qquad (\mathrm{V/m}), \tag{4.21}$$

(Evaluation of the electric field strength from potential)

where, in the rectangular coordinate system,

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{u}_x + \frac{\partial V}{\partial y}\mathbf{u}_y + \frac{\partial V}{\partial z}\mathbf{u}_z \qquad \underset{\text{m}}{\underline{\text{(V)}}}. \tag{4.22}$$

(Gradient of a scalar function V in rectangular coordinates)

Example 4.3—Vector E on the axis of a charged ring. As an example of the determination of **E** from the scalar potential, consider again the ring in Fig. 4.4. **E** is obtained as $-\nabla V$. The scalar potential along the ring axis is given at the end of Example 4.1. Note that it is a function of the coordinate x only. Therefore at a point x on the ring axis

$$\mathbf{E} = -\nabla\frac{Q}{4\pi\epsilon_0\sqrt{R^2+x^2}} = -\frac{\partial}{\partial x}\left(\frac{Q}{4\pi\epsilon_0\sqrt{R^2+x^2}}\right)\mathbf{u}_x = \frac{Qx}{4\pi\epsilon_0(R^2+x^2)^{3/2}}\,\mathbf{u}_x.$$

Questions and problems: Q4.18 to Q4.23, P4.15 to P4.18

4.6 Equipotential Surfaces

A surface in an electrostatic field having the same potential at all points is called an *equipotential surface*. This is an important concept. For example, we will see that in electrostatics, the surface of any conductor is always equipotential. It can also aid in visualizing the electric field, usually in combination with electric field lines.

Since all points of an equipotential surface are at the same potential, the potential difference between two close points A and B on the surface is zero. Let $d\mathbf{l}$ be the position vector of point B with respect to point A. Because $d\mathbf{l}$ is very small, the potential difference in Eq. (4.15) becomes simply $dV = \mathbf{E}\cdot d\mathbf{l}$. Since this potential difference dV is zero (we assumed A and B to be on the same equipotential surface), *the electric field strength vector at any equipotential surface is normal to that surface.*

Example 4.4—Equipotential surfaces in the field of a point charge. As an example, we know that the expression for the potential of a point charge is $V(r) = Q/(4\pi\epsilon_0 r)$. Therefore the equation of the equipotential surface at a potential V_0 is obtained from

$$V(r) = \frac{Q}{4\pi\epsilon_0 r} = V_0,$$

from which we obtain

$$r = \frac{Q}{4\pi\epsilon_0 V_0}.$$

For different V_0, equipotential surfaces are spheres centered at the charge, and vector **E** is normal to these spheres.

If plotted, equipotential surfaces usually have the same potential difference from one surface to the next. Let this potential difference be ΔV. For $V_0 = 0$ in the

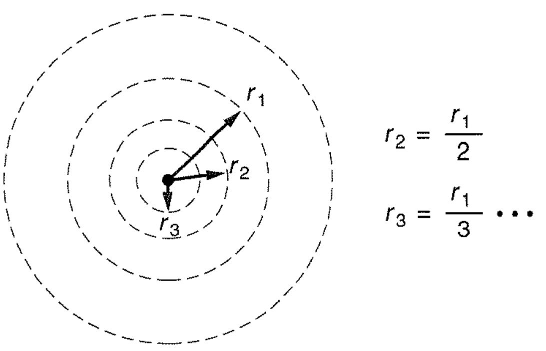


Figure 4.6 Equipotential surfaces of the field of a point charge

preceding equation we would then have $r_0 = \infty$, for $V_0 = 1 \times \Delta V$ the radius of the equipotential surface is $r_1 = Q/(4\pi\epsilon_0 V_0)$, and so on. With this convention, therefore, equipotential surfaces for a point charge are as in Fig. 4.6.

Questions and problems: Q4.24

4.7 Chapter Summary

1. The electric scalar potential is a scalar quantity that can be used instead of vector **E** for the description of the electrostatic field. It is defined as the line integral of **E** from any point of the field to an *arbitrary* reference point.
2. The electric scalar potential is not unique (it depends on the choice of the reference point), but for two reference points the potential at all points differs only by a constant. If there are no charges at infinity, the reference point is always adopted at infinity, but if the distribution of charges extends (theoretically) to infinity, this is not possible.
3. If we know the electric scalar potential as a function of coordinates, it is easy to obtain the component of **E** in any direction, and hence to obtain the complete vector **E**. For this, we need the mathematical concept of the gradient of a scalar function. In the rectangular coordinate system, the gradient of V is obtained by the ∇ operator acting on V, and $\mathbf{E} = -\nabla V$.
4. Being a scalar quantity, the electric scalar potential is more convenient than the vector **E** for the analysis of electrostatic fields.
5. An equipotential surface is defined as a geometrical surface with all points at the same potential. Lines of the electric field strength vector are normal to equipotential surfaces.

QUESTIONS

Q4.1. Consider a uniform electric field of electric field strength E, and two planes normal to vector **E**, that are a distance d apart. What is the work done by the field in moving a test charge ΔQ from one plane to another? Can the work be negative? Does it depend on the location of the two points on the planes? Explain.

Q4.2. Is it possible to have an electrostatic field with circular closed field lines, with the vector **E** in the same direction along the entire lines? Explain.

Q4.3. Is it possible to have an electrostatic field with parallel lines, but of different magnitude of vector **E** in the direction normal to the lines? Explain.

Q4.4. If the potential of the earth were taken to be 100,000 V (instead of the usual 0 V), would it be dangerous to walk around? What influence would this have on the potential at various points, and on the difference of the potential at two points?

Q4.5. If we know $\mathbf{E}(x, y, z)$, is the electric scalar potential $V(x, y, z)$ determined uniquely? Explain.

Q4.6. Equation (4.7) is satisfied by the electric field of a point charge. Does the expression for the electric field of a point charge *follow* from Eq. (4.7)?

Q4.7. Why does Eq. (4.7) represent the law of conservation of energy in the electrostatic field?

Q4.8. What is the potential of the reference point?

Q4.9. As we approach a point charge Q ($Q > 0$), the potential tends to infinity. Explain.

Q4.10. How much energy do you transfer to the electric field of a point charge when you move the reference point from a point at a distance r_R from the charge to a point at infinity?

Q4.11. Why do we usually adopt the reference point at infinity?

Q4.12. Is the potential of a positively charged body always positive, and that of a negatively charged body always negative? Give examples that illustrate your conclusions.

Q4.13. Why are the expressions for the potential in Eqs. (4.12*a*–*c*) valid for a reference point at infinity?

Q4.14. Does it make sense to speak about voltage between a point in the field and the reference point? If it does, what is this voltage?

Q4.15. A charge ΔQ is moved from a point where the potential is V_1 to a point where the potential is V_2. What is the work done by the electric forces? What is the work done by the forces acting against the electric forces?

Q4.16. A charge ΔQ ($\Delta Q < 0$) is moved from a point at potential V_1 to a point at potential V_2. What is the work done by the electric forces?

Q4.17. Is $V_{AB} = -V_{BA}$? Explain.

Q4.18. Why is the vector **E** at a point directed toward the adjacent equipotential surface of *lower* potential?

Q4.19. Why do we have $\mathbf{E} = -\text{grad}\, V$, and not $\mathbf{E} = +\text{grad}\, V$?

Q4.20. A cloud of positive and negative ions is situated in an electrostatic field. Which ions will tend to move toward the points of higher potential, and which toward the points of lower potential?

Q4.21. Suppose that $V = 0$ at a point. Does it mean that $\mathbf{E} = 0$ at that point? Explain.

Q4.22. Assume we know **E** at a point. Is this sufficient to determine the potential V at that point? Conversely, if we know V at that point, can we determine **E**?

Q4.23. The potential in a region of space is constant. What is the magnitude and direction of the electric field strength vector in the region?

Q4.24. Prove that **E** is normal to equipotential surfaces.

PROBLEMS

P4.1. Two point charges, $Q_1 = -3 \cdot 10^{-9}$ C and $Q_2 = 1.5 \cdot 10^{-9}$ C, are $r = 5$ cm apart. Find the potential at the point that lies on the line joining the two charges and halfway between them. Find the zero-potential point(s) lying on the straight line that joins the two charges.

P4.2. Two small bodies, with charges Q ($Q > 0$) and $-Q$, are a distance d apart. Determine the potential at all points with respect to the reference point at infinity. Is there a zero-potential equipotential surface? How much work do the electric forces do if the distance is increased to $2d$?

P4.3. A ring of radius a is charged with a total charge Q. Determine the potential along its axis normal to the ring plane with reference to the ring center.

P4.4. A soap bubble of radius R and very small wall thickness a is at a potential V with respect to the reference point at infinity. Determine the potential of a spherical drop obtained when the bubble explodes, assuming all the soap in the bubble is contained in the drop.

P4.5. A volume of a liquid conductor is sprayed into N equal spherical drops. Then, by some appropriate method, each drop is given a potential V with respect to the reference point at infinity. Finally, all these small drops are combined into a large spherical drop. Determine the potential of the large drop.

P4.6. Two small conducting spheres of radii a and b are connected by a very thin, flexible conductor of length d. The total charge of the system is Q. Assuming that d is much larger than a and b, determine the force F that acts on the wire so as to extend it. Charges may be considered to be located on the two spheres only, and to be distributed uniformly over their surfaces. (Hint: when connected by the conducting wire, the spheres will be at the same potential—see Chapter 6.)

P4.7. Two small conducting balls of radii a and b are charged with charges Q_a and Q_b, and are at a distance d ($d \gg a, b$) apart. Suppose that the balls are connected with a thin conducting wire. What will the direction of flow of positive charges through the wire be? Discuss the question for various values of Q_a, Q_b, a, and b. (Hint: when connected by the conducting wire, the balls will be at the same potential—see Chapter 6.)

***P4.8.** The source of an electrostatic field is a volume charge distribution of finite charge density ρ, distributed in a finite region of space. Prove that the electric scalar potential has a finite value at all points, including the points inside the charge distribution.

***P4.9.** Prove that the electric scalar potential due to a surface charge distribution of density σ over a surface S is finite at all points, including the points of S.

P4.10. The reference point for the potential is changed from point R to point R'. Prove that the potential of all points in an electric field changes by the voltage between R and R'.

P4.11. Four small bodies with equal charges $Q = 0.5 \cdot 10^{-9}$ C are located at the vertices of a square with sides $a = 2$ cm. Determine the potential at the center of the square, and the voltage between the square center and a midpoint of a square side. What is the work of electric forces if one of the charges is moved to a very distant point?

P4.12. An insulating disk of radius $a = 5$ cm is charged by friction uniformly over its surface with a total charge of $Q = -10^{-8}$ C. Find the expression for the potential of the points which lie on the axis of the disk perpendicular to its surface. Plot your result. What are the numerical values for the potential at the center of the disk, and at a distance

$z = a$ from the center, measured along the axis? What is the voltage between these two points equal to?

P4.13. The volume charge density inside a spherical surface of radius a is such that the electric field vector inside the sphere is pointing toward the center of the sphere, and varies with radial position as $E(r) = E_0 r/a$ (E_0 is a constant). Find the voltage between the center and the surface of the sphere.

P4.14. Two large parallel equipotential plates at potentials $V_1 = -10\,\mathrm{V}$ and $V_2 = 55\,\mathrm{V}$ are a distance $d = 2\,\mathrm{cm}$ apart. Determine the electric field strength between the plates.

P4.15. Determine the potential along the line joining two small bodies carrying equal charges Q. Plot your result. Starting from that expression, prove that the electric field strength at the midpoint between the bodies is zero.

P4.16. Two small bodies with charges $Q_1 = 10^{-10}\,\mathrm{C}$ and $Q_2 = -Q_1$ are a distance $d = 9\,\mathrm{cm}$ apart. Determine the potential along the line joining the two charges, and from that expression determine the electric field strength along the line. Plot your results.

P4.17. From the expression for potential found in problem P4.3, find the electric field strength vector along the ring axis. (See problem P3.20.)

P4.18. From the general expression for the potential along the axis of the disk from problem P4.12, determine the electric field strength along the disk axis. (See problem P3.21.)

5

Gauss' Law

5.1 Introduction

There is an important relation between the vector **E** in any electrostatic field and the static charge producing it. It is a consequence of the mathematical form of the electric field strength of a point charge, and is known as Gauss' law. Among other applications, Gauss' law enables a simple evaluation of the electric field in some simple but important cases.

To understand Gauss' law, we first need to understand an important mathematical concept, the *flux of a vector function through a surface*. The word "flux" originates from fluid mechanics and comes from the latin word "fluxus," which means "one that flows."

5.2 The Concept of Flux

Consider a uniform flow of a liquid of velocity **v** that is a function of coordinates but not of time. Imagine a net so fine that it does not disturb the flow of the liquid it is placed in. Let the surface of the net be S. We wish to determine the amount of the liquid that passes through the net (i.e., through S) in one second.

We can subdivide the surface S into a large number of small flat surface elements dS, as in Fig. 5.1. Obviously, the total amount of liquid passing through the net is obtained as a sum of the small amounts passing through all of the small elements.

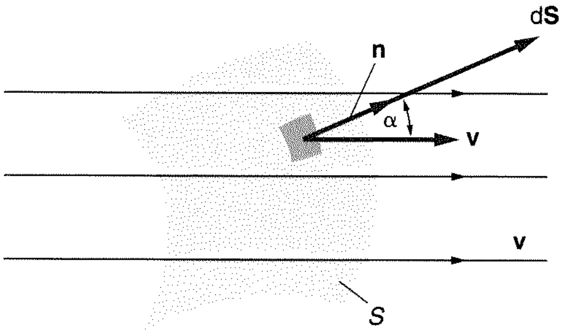


Figure 5.1 A fine net in a flow of liquid can be approximated by a large number of flat surface elements such as $d\mathbf{S}$

Consider a small flat surface element shown in the figure. The vector $\mathbf{n}$ denotes a unit vector normal to the element, and α is the angle between this unit vector and the local velocity $\mathbf{v}$ of the fluid. It is evident that if the velocity $\mathbf{v}$ is tangential to the element, there is no flow of fluid through it. Therefore only the component of the velocity *normal* to the element contributes to the flow of liquid through the element.

In one second, the fluid at that point moves by a distance normal to dS equal to $v \cos \alpha$. The quantity of fluid that passes through dS in one second is therefore $v \cos \alpha \, dS$. The quantity of fluid that passes through S in one second is a sum of all these infinitely small partial flows. It is therefore an integral (an infinite sum of infinitely small terms):

$$\text{Fluid flow through } S \text{ in one second} = \int_S v \cos \alpha \, dS. \tag{5.1}$$

The expression under the integral sign has a form of a dot product, but although v is the magnitude of a vector, dS is not. If, however, we *define* a vector surface element $d\mathbf{S}$ as

$$d\mathbf{S} = dS \, \mathbf{n}, \tag{5.2}$$

Eq. (5.1) can be written in the form

$$\text{Fluid flow through } S \text{ in one second} = \int_S \mathbf{v} \cdot d\mathbf{S}. \tag{5.3}$$

The integral on the right side of this equation is known as the *flux of vector* $\mathbf{v}$ *through the surface S*.

It is evident that the concept of flux can be used in connection with *any* vector function, not necessarily the velocity (in which case the flux has a clear physical meaning). It is evident as well that the surface S can be a closed surface. In that case, a small circle is added in the middle of the integral sign to indicate that the surface is closed.

The flux of a vector function through a closed surface is a very important concept in the theory of electromagnetic field. It is a convention to adopt the unit vector $\mathbf{n}$ normal to a closed surface *to be directed from the surface outward* (Fig. 5.2).

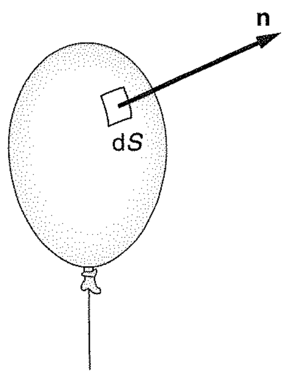


Figure 5.2 The unit vector normal to a closed surface is always adopted to be directed from the surface outward

5.3 Gauss' Law

Gauss' law is a very simple and important consequence of the mathematical form of the expression of the vector **E** of a point charge (i.e., of Coulomb's law). It states that the flux of the electric field strength vector through any closed surface in the electrostatic field equals the total charge enclosed by the surface, divided by ϵ_0:

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{\text{total in } S}}{\epsilon_0} \qquad (\mathrm{V} \cdot \mathrm{m}). \tag{5.4}$$

(Gauss' law)

Basically, Gauss' law is a relationship between the sources *inside a closed surface* and the field they produce *over this entire surface*. (For interested readers, the derivation of Gauss' law is given at the end of the chapter.)

Gauss' law in Eq. (5.4) is valid for free space (air, vacuum). We know, however, that elemental charges that are actual sources of the field (electrons, protons, ions) *are* situated in a vacuum. Using this fact, we are able to extend Gauss' law to electrostatic fields in the presence of conducting and dielectric materials.

Example 5.1—Gauss' law applied to point charges. Consider the closed surfaces S_1, S_2, and S_3 in Fig. 5.3. The flux of vector **E** through S_1 is $(Q_1 + Q_4)/\epsilon_0$, through S_2 is zero, and through S_3 is $(Q_2 + Q_3)/\epsilon_0$.

Questions and problems: Q5.1 to Q5.11, P5.1 to P5.4

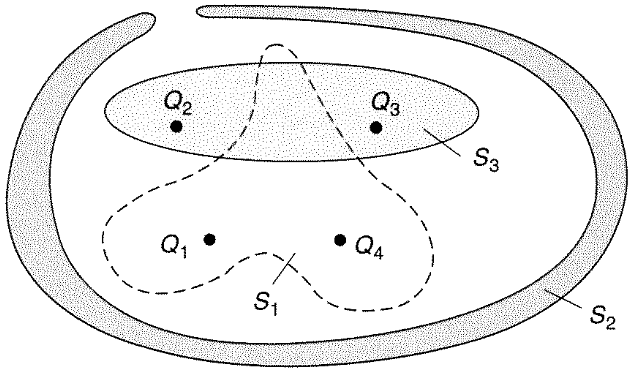


Figure 5.3 Three closed surfaces, S_1, S_2 and S_3, in the field of four point charges

5.4 Applications of Gauss' Law

The applications of Gauss' law are numerous. They are basically of two kinds: proofs of some general properties of the electrostatic field, and the evaluation of the vector **E** in some special cases with high degree of symmetry of charge distribution.

Example 5.2—Gauss' law applied to a surface of zero field. As an example of the first kind of application, assume that we have a surface S such that **E** is zero at all points of S. Gauss' law tells us that in *all* such cases the total enclosed charge must be zero. We will use this conclusion in the analysis of conductors in the next chapter.

Before giving further examples of Gauss' law, we note that it represents a *single* scalar equation. Therefore, in general it is not possible to determine a vector function from it (every vector function is defined by its *three* scalar components). It is possible to use Gauss' law to find **E** only if by symmetry we know everything about **E** except its magnitude.

Example 5.3—Electric field of an infinite, charged plate. Consider a large, theoretically infinite flat plate uniformly charged with a surface charge density σ (Fig. 5.4a). Due to symmetry, the lines of **E** are normal to the plate, and are directed from the plate if $\sigma > 0$ and toward the plate in the other case. What we do not know is the magnitude of **E** as a function of the distance x from the plate. We need one scalar equation for that, and Gauss' law can be used. Note that, from symmetry, we know that $E(-x) = E(x)$, and assume that $\sigma > 0$.

Imagine a cylinder of bases S parallel to the plate and of height $2h$, positioned symmetrically with respect to the plate, as in Fig. 5.4a. Let us apply Gauss' law to that closed surface.

On the curved surface, vector **E** is parallel to it, i.e., normal to the vector surface element. Therefore, the flux of **E** through the curved surface is zero. On the two bases, vector **E** is normal to them, i.e., it is parallel to the vector surface element, so the flux of **E** through each base is simply $E(x)\,S$. So we have

$$\oint_{\text{cylinder}} \mathbf{E} \cdot d\mathbf{S} = E(x)\,S + E(-x)\,S = 2E(x)\,S = \frac{\sigma\,S}{\epsilon_0},$$

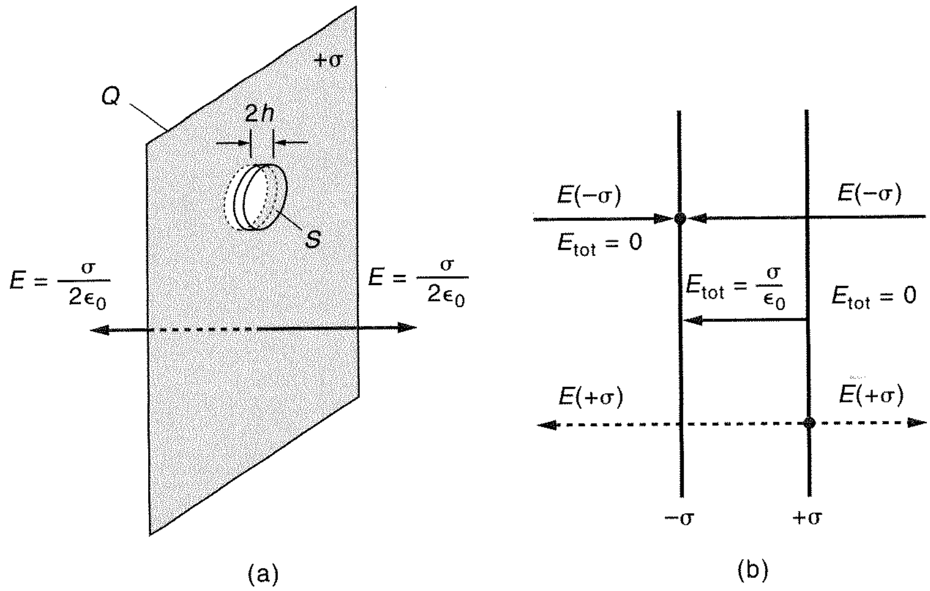


Figure 5.4 (a) A charged plate, and (b) two parallel plates charged with equal surface charges of opposite sign

because the charge enclosed by S is σS. We find that the magnitude E of the electric field strength *does not depend on the distance from the plate*:

$$E = \frac{\sigma}{2\epsilon_0} \qquad \text{(V/m)}. \tag{5.5}$$

(Electric field strength of uniformly charged plate)

How is it possible that E does not depend on x? The answer is simple. The plate being theoretically infinite, any finite distance from the plate measured with respect to the plate size is infinitely small; i.e., all points at a finite distance from the plate are equivalent.

Although we cannot have an infinite, uniformly charged plate, the result in Eq. (5.5) is nevertheless of significant importance. If we have a surface charge on a flat (or locally nearly flat) surface of any size and approach it sufficiently close and far from its edges, the field will also be given by Eq. (5.5). This is evident because from very close points the surface looks like a very large plane surface with uniform surface charge distribution (of density equal to the local surface charge density).

Example 5.4—Electric field between two parallel charged plates. Now consider two parallel flat plates charged with equal surface charge densities of opposite sign (Fig. 5.4b). If we have in mind the result of the preceding example, superposition yields immediately that between the plates

$$E = \frac{\sigma}{\epsilon_0} \qquad \text{(V/m)}, \tag{5.6}$$

(Electric field strength between two parallel plates with surface charges σ and $-\sigma$)

and that outside the plates there is no field ($\mathbf{E} = 0$). This formula may also seem unimportant for practical cases because it relates to two parallel *infinite* planes. However, this is a good approximation if the plates are of finite size but close to each other with respect to their size.

We will use Eq. (5.6) for the analysis of the parallel-plate capacitor, an important element in electrical engineering.

There are many more electrostatic systems where the magnitude of the electric field strength vector can be obtained by Gauss' law. We will consider several further important practical examples in the next chapters, when we include materials other than air (vacuum) in the analysis.

Questions and problems: Q5.12, P5.5 to P5.20

5.5 Proof of Gauss' Law

Recall that the electric field strength vector **E** of any distribution of charge is obtained as a vector sum of individual vectors **E** resulting from all point charges of which the charge distribution is composed. Therefore Gauss' law is proven for all cases if we can prove that Eq. (5.4) is valid for a single point charge.

Consider a point charge Q and let us determine the flux of vector **E** through a surface element $\mathbf{S} = \mathrm{d}S\,\mathbf{n}$ (Fig. 5.5). Let us denote this flux by $\mathrm{d}\Psi_E$. It is equal to

$$\mathrm{d}\Psi_E = \frac{Q}{4\pi\epsilon_0 r^2}\mathrm{d}S\cos\alpha = \frac{Q}{4\pi\epsilon_0}\frac{\mathrm{d}S_n}{r^2}, \quad = \frac{Q}{4\pi\epsilon_0}\;\frac{dS_n}{r^2} \tag{5.7}$$

where $\mathrm{d}S_n$ is the projection of the flat surface element $\mathrm{d}S$ on the plane normal to r.

The projection $\mathrm{d}S_n$ can be considered as the base of a cone with the apex at the charge. Let us cut this cone with another plane normal to r, for example at a distance r_1 from the charge, with a base of area $\mathrm{d}S_1$ (Fig. 5.5). From geometry we know that

$$\frac{\mathrm{d}S_1}{r_1^2} = \frac{\mathrm{d}S_n}{r^2}. \tag{5.8}$$

Note that r_1 is arbitrary. From Eq. (5.7) we conclude that the *flux through any cross-section of the cone is the same.*

Let us now enclose the charge Q in Fig. 5.5 by an arbitrary closed surface S, indicated in the figure. We can divide this surface into elemental surfaces by a very

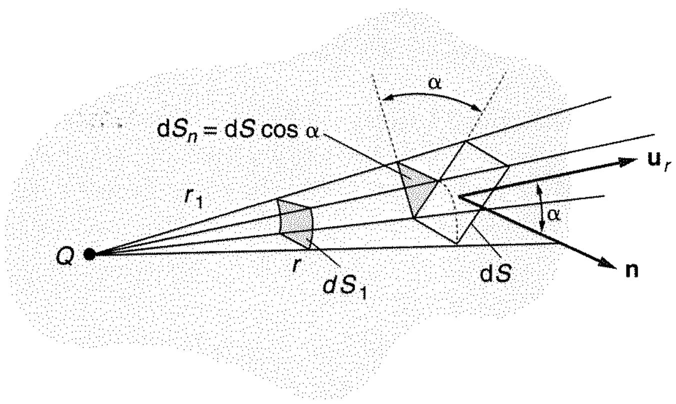


Figure 5.5 A point charge Q and a surface element d**S** a distance r from it

large number of cones with the common apex at the charge. To calculate the flux through any of these surfaces, we can take *any* cross-section of the cone. Therefore the flux through S is *exactly the same as that through the surface of any sphere centered at the charge.*

The flux through a sphere S of radius r centered at the charge is easy to find. Noting that the angle between the vector **E** and the vector surface element $d\mathbf{S}$ of the sphere is zero and that vector **E** has the same intensity at all points on S, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = E \oint_S dS = E4\pi r^2 = \frac{Q}{4\pi\epsilon_0 r^2} 4\pi r^2 = \frac{Q}{\epsilon_0}. \tag{5.9}$$

As explained, because superposition applies, this completes the proof of Gauss' law. Note that the right-hand side of Eq. (5.9) is zero if S does not enclose Q. Therefore the right-hand side in Gauss' law, Eq. (5.4), will be zero if the surface S encloses no charge.

Questions and problems: Q5.13 to Q5.15

5.6 Chapter Summary

1. Gauss' law in Eq. (5.4) is a direct consequence of Coulomb's law, i.e., of the mathematical form of vector **E** of a point charge resulting from it. It is, therefore, a fundamental law of electrostatics.
2. In this chapter, Gauss' law has been derived for a system of charges in a vacuum. We know that the elemental charges inside matter, which are the actual sources of the electrostatic field, *are* situated in a vacuum. Therefore Gauss' law, possibly modified, should be applicable to all electrostatic fields, not only in a vacuum.
3. Gauss' law has two important types of applications. One type is proofs of certain general properties of the electrostatic field. The other is the evaluation of the intensity of vector **E** of highly symmetrical charge distributions, where we know by symmetry the direction of **E**. In such cases Gauss' law, although being a single scalar equation of one ~~scalar~~ unknown, is sufficient to determine the magnitude of vector **E**.
VECTOR

QUESTIONS

Q5.1. Prove that in a uniform electric field the flux of the electric field strength vector through any closed surface is zero.

Q5.2. Can the closed surface in Gauss' law be infinitesimally small in the mathematical sense? Is the answer different for the case of a vacuum and some other material? Explain.

Q5.3. Assume we know that the vector **E** satisfies Gauss' law in Eq. (5.4), but we do not know the expression for the vector **E** of a point charge. Can this expression be *derived* from Gauss' law? Explain.

Q5.4. The center of a small spherical body of radius r, uniformly charged over its surface with a charge Q, coincides with the center of one side of a cube of edge length a ($a > 2r$). What is the flux of the electric field strength vector through the cube?

Q5.5. A dielectric cube of edge length a is charged by friction uniformly over its surface, with a surface charge density σ. What is the flux of the electric field strength vector through a slightly smaller and slightly larger imaginary cube? Do the answers look logical? Explain.

Q5.6. Is it possible to apply Gauss' law to a large surface enclosing a domain with a number of holes? If you think it is possible, explain how it should be done.

Q5.7. Inside an imaginary closed surface S the total charge is zero. Does this mean that at all points of S the vector **E** is zero? Explain.

Q5.8. A spherical rubber balloon is charged by friction uniformly over its surface. How does the electric field inside and outside the balloon change if it is periodically inflated and deflated to change its radius?

Q5.9. Assume that the flux of the electric field strength vector through a surface enclosing a point A is the same for any size and shape of the surface. What does this tell us about the charge at A or in its vicinity?

Q5.10. The electric field strength is zero at all points of a closed surface S. What is the charge enclosed by S?

Q5.11. An electric dipole (two equal charges of opposite signs) is located at the center of a sphere of radius greater than half the distance between the charges. What is the flux of vector **E** through the sphere?

Q5.12. Would it be possible to apply Gauss' law for the determination of the electric field for charged planes with nonuniform charge distribution? Explain.

Q5.13. What would be the form of Gauss' law if the unit vector normal to a closed surface were adopted to point into the surface, instead of out of the surface?

Q5.14. Gauss' law is a consequence of the factor $1/r^2$ in the expression for the electric field strength of a point charge (i.e., in Coulomb's law). At what step in the derivation of Gauss' law is this the condition for Gauss' law to be valid?

Q5.15. Try to derive Gauss' law for a hypothetical electric field where the field strength of a point charge is proportional to $1/r^k$, where $k \neq 2$.

PROBLEMS

P5.1. The flux of the electric field strength vector through a closed surface is $100\,\mathrm{V \cdot m}$. How large is the charge inside the surface?

P5.2. A point charge $Q = 2 \cdot 10^{-11}\,\mathrm{C}$ is located at the center of a cube. Determine the flux of vector **E** through one side of the cube using Gauss' law.

P5.3. A point charge $Q = -3 \cdot 10^{-12}\,\mathrm{C}$ is $d = 5\,\mathrm{cm}$ away from a circular surface S of radius $a = 3\,\mathrm{cm}$ as shown in Fig. P5.3. Determine the flux of vector **E** through S.

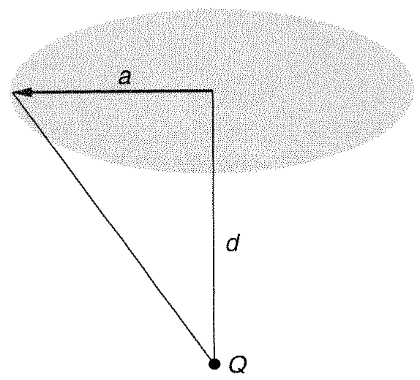


Figure P5.3 A circular surface near a point charge

P5.4. Determine the flux of vector **E** through a hemispherical surface of radius $a = 5\,\text{cm}$, if the field is uniform, with $E = 15\,\text{mV/m}$, and if vector **E** makes an angle $\alpha = 30°$ with the hemisphere axis. Use Gauss' law.

P5.5. Three parallel thin large charged plates have surface charge densities $-\sigma$, 2σ, and $-\sigma$. Find the electric field everywhere for all combinations of the relative sheet positions and $\sigma = 10^{-6}\,\text{C/m}^2$. Do the results depend on the distances between the plates? Determine the equipotential surfaces in all cases, and the potential difference between pairs of plates, if the distance between them is 2 cm.

P5.6. A very large flat plate of thickness d is uniformly charged with volume charge density ρ. Find the electric field strength at all points. Determine the potential difference between the two boundary planes, and between the plane of symmetry of the plate and a boundary plane.

P5.7. The volume charge density of a thick, very large plate varies as $\rho = \rho_0 x/d$ through the plate, where x is the distance from one if its boundary planes. Find the electric field strength vector everywhere. Plot your result. How large is the potential difference between the two boundary surfaces of the plate? d IS THE THICKNESS OF THE PLATE

P5.8. Two concentric spherical surfaces, of radii a and $b > a$, are uniformly charged with the same amounts of charge Q, but of opposite signs. Find the electric field strength at all points and present your expressions graphically.

P5.9. The spherical surfaces from the previous problem do not have the same charge, but are charged with $Q_{\text{inner}} = 10^{-10}\,\text{C}$ and $Q_{\text{outer}} = -5 \cdot 10^{-11}\,\text{C}$. The radii of the spheres are $a = 3\,\text{cm}$ and $b = 5\,\text{cm}$. Find the electric field strength and potential at all points and present your expressions graphically.

P5.10. A spherical cloud of radius a has a uniform volume charge of density $\rho = -10^{-5}\,\text{C/m}^3$. Find the electric field strength and potential at all points and present your expressions graphically.

P5.11. A spherical cloud shell has a uniform volume charge of density $\rho = 10^{-3}\,\text{C/m}^3$, an inner radius $a = 2\,\text{cm}$, and an outer radius $b = 4\,\text{cm}$. Find the electric field strength and potential at all points and present your expressions graphically.

P5.12. The volume charge density of a spherical charged cloud is not constant, but varies with the distance from the cloud center as $\rho(r) = \rho_0 r/a$. Determine the electric field strength and potential at all points. Present your results graphically.

P5.13. Find the expression for the electric field strength and potential between and outside two long coaxial cylinders of radii a and b $(b > a)$, carrying charges Q' and $-Q'$ per unit length. (This structure is known as a coaxial cable, or coaxial line.) Plot your results. Determine the voltage between the two cylinders.

P5.14. Repeat problem P5.13 assuming that the two cylinders carry unequal charges per unit length, when these charges are (1) of the same sign, and (2) of opposite signs. Plot your results and compare to problem P5.13.

P5.15. A very long cylindrical cloud of radius a has a constant volume charge density ρ. Determine the electric field strength and potential at all points. Present your results graphically. Is it possible in this case to adopt the reference point at infinity? Explain.

P5.16. Repeat problem P5.15 assuming that the charge density is not constant, but varies with distance r from the cloud axis as $\rho(r) = \rho_0 r/a$.

P5.17. Repeat problem P5.15 assuming that the cloud has a coaxial cavity of radius b $(b < a)$ with no charges.

***P5.18.** Prove that the electric scalar potential cannot have a maximum or a minimum value, except at points occupied by positive and negative charges, respectively.

***P5.19.** Prove ~~Earnshow's~~ EARNSHAW'S *theorem:* A stationary system of charges cannot be in a stable equilibrium without external nonelectric forces. (Hint: use the conclusion from problem P5.18.)

***P5.20.** Prove that the average potential of any sphere S is equal to the potential at its center, if the charge density inside the sphere is zero at every point.

6

Conductors in the Electrostatic Field

6.1 Introduction

Conductors are in all electric devices. They are as common in electrostatics as in other areas of electrical engineering. Nevertheless, it is important to understand how they behave in electrostatics. This behavior explains some useful electromagnetic devices. In addition, in many nonelectrostatic applications conductors behave similarly to the way they do in electrostatics. So this chapter is important beyond its application to electrostatics.

6.2 Behavior of Conductors in the Electrostatic Field

Conductors have a relatively large proportion of freely movable electric charges. The best conductors are metallic (silver, aluminum, copper, gold, etc.). They usually have one free electron per atom, an electron that is not bound to its atom, but moves freely in the space between atoms. Because of their small mass, these free electrons move in response to any electric field, however small, that exists inside a conductor. The same is true for all other conductors, e.g., liquid solutions and semiconductors, except that inside such conductors both positive and negative free charges can exist. The number of free charge carriers is smaller and their mass greater than in metals and electrons, but this has no influence on the behavior of conductors in the electrostatic field.

Let us make an imaginary experiment. Assume that this book is a conductor. Suppose that it has both free positive and negative charges in equal number. If the book is not situated in the electric field, the number of positive and negative free charges inside any small volume is the same, and there is no surplus electric charge at any point in the book. To be more picturesque, imagine that positive charges are blue, and negative yellow. If we mix blue and yellow we get green, so your book will look green both over its surface and at any point inside.

What would happen if we establish an electric field in the book, for example, by means of two electrodes on the two sides of the book, charged with equal charges of opposite sign? Let the positive electrode be on your left. The electric field in the book will then be directed from left to right. You would notice that blue (positive) charges move from left to right (repelled by the positive electrode), and that yellow (negative) charges move from right to left. Consequently, the right side of the book will become progressively more blue (positive), and its left side progressively more yellow (negative).

The surplus charges in the body created in this manner are known as *electrostatically induced charges*. They are, of course, the source of an electric field. Because the positive induced charge is on the right side of the book, and the negative on its left side, this electric field is directed from right to left, i.e., *opposite to the initial electric field that produced the charge*. As the amount of the induced charge increases, the total field inside the book becomes progressively smaller and the motion of charges inside the book decays. In the end, the electric field of induced charges at all points inside the book cancels out the initial electric field (due to the two charged electrodes). We thus reach electrostatic equilibrium, in which there can be no electric field at any point inside our conductive book.

From this simple imaginary experiment, we conclude the following: if we have a conducting body in an electrostatic field, and wait until the drift motion of charges under the influence of the field stops (in reality, an extremely rapid process), the electric field of induced charges will *exactly* cancel out the external field, and the total electric field at all points of a conductor will be zero. Thus the first fundamental conclusion is

In electrostatics, $\mathbf{E} = 0$ inside conductors.	(6.1)

With this knowledge, let us apply Gauss' law to an arbitrary closed surface S that is completely inside the conductor. Because vector $\mathbf{E}$ is zero at all points on S, the total charge enclosed by S must be zero. This means that *all the excess charge (if any) must be distributed over the surfaces of conductors:*

In electrostatics, a conductor has charges only on its surface. (6.2)

Because there is no field inside conductors, the tangential component of the electric field strength, **E**, on the very surface of conductors is also zero (otherwise it would produce organized motion of charge on its surface):

In electrostatics, $\mathbf{E}_{\text{tangential}} = 0$ on conductor surfaces. (6.3)

Because the tangential component of **E** is zero on conductor surfaces, the potential difference between any two points of a conductor is zero. This means that the surface of a conductor in electrostatics is equipotential. Because there is no **E** inside conductors either, it follows that all points of a conductor have the same potential:

In electrostatics, the surface and volume of a conductor are equipotential. (6.4)

Finally, a simple relation exists between the normal component, E_n, of **E** on a conductor surface, and the local surface charge density, σ. To derive this relation, consider a small cylindrical surface, similar to a coin, with a base ΔS and a height $\Delta h \to 0$. One base is in the conductor and the other in air (Fig. 6.1). Let us apply Gauss' law to the closed surface of the cylinder. There is no flux of **E** through the base inside the conductor (zero field) and through the infinitely narrow strip connecting the two bases (zero area). The flux of **E** through the cylinder is thus equal only to

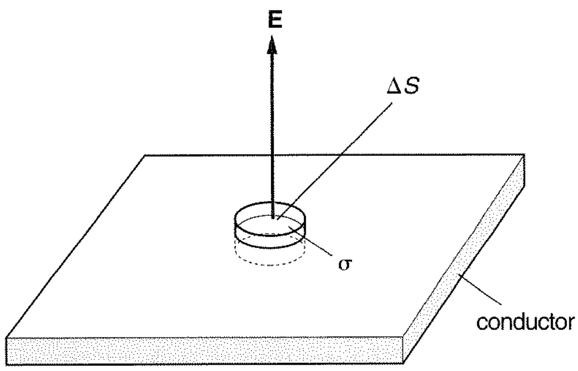


Figure 6.1 A small cylinder of negligible height with one base in the conductor and the other in air

$E_n \Delta S$. Because the charge enclosed is $\sigma \Delta S$, using Gauss' law we obtain that on the air side of a conductor surface,

$$E_n = \frac{\sigma}{\epsilon_0} \qquad \text{(V/m)}. \tag{6.5}$$

(Normal component of electric field strength close to conductor surface)

The simple conclusions in Eqs. (6.1) through (6.5) are all we need to know to understand the behavior of conductors in the electrostatic field.

Example 6.1—Charged Metal Ball. Suppose that a metal ball of radius a is situated in a vacuum and has a charge Q. How will the charge be distributed over its surface? [We know from Eq. (6.2) that Q exists only over the conductor surface.] Because equal charges repel, due to symmetry the charge distribution over the surface of the ball must be uniform. The surface charge density is therefore simply $\sigma = Q/(4\pi a^2)$. Let us determine **E** and V due to this charge.

Due to the uniform charge distribution, vector **E** is radial and has the same magnitude on any spherical surface concentric to the ball. (Is such a surface an equipotential surface?) We can use Gauss' law to find the magnitude of vector **E** on any of these surfaces:

$$\oint_{\text{sphere}} \mathbf{E}(r) \cdot d\mathbf{S} = E(r)\, 4\pi r^2 = \frac{Q}{\epsilon_0}.$$

Note that the sphere encloses no charge if $r < a$. Thus

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{\sigma 4\pi a^2}{4\pi\epsilon_0 r^2} = \frac{\sigma a^2}{\epsilon_0 r^2} \quad (r > a), \qquad E(r) = 0 \quad (r < a). \tag{6.6}$$

This expression is the same as the one for the field of a point charge Q at the center of the ball. On the surface of the ball ($r = a$), $E(a) = \sigma/\epsilon_0$, as predicted by Eq. (6.5).

It follows that outside the ball, the potential is the same as that of a point charge Q placed at the center of the ball. Inside the ball the potential is constant, equal to that on the ball surface, that is,

$$V(a) = \frac{Q}{4\pi\epsilon_0 a} = \frac{\sigma a}{\epsilon_0}. \tag{6.7}$$

Example 6.2—Charged Metal Wire. Consider a very long (theoretically, infinitely long) straight metal wire of circular cross section of radius a. Let it be charged with Q' per unit length. What are the field and potential everywhere around the wire?

Due to symmetry, the charge will be distributed uniformly over the wire surface. It is not difficult to conclude that, as the result of this symmetrical charge distribution, vector **E** is radial. Its magnitude depends only on the normal distance r from the wire axis and can be determined by Gauss' law.

For the application of Gauss' law, we adopt the cylindrical surface shown in Fig. 6.2. There is no flux through the cylinder bases because vector **E** is tangential to them. The total flux through the closed surface is therefore equal to the flux through its cylindrical part. Applying Gauss' law gives

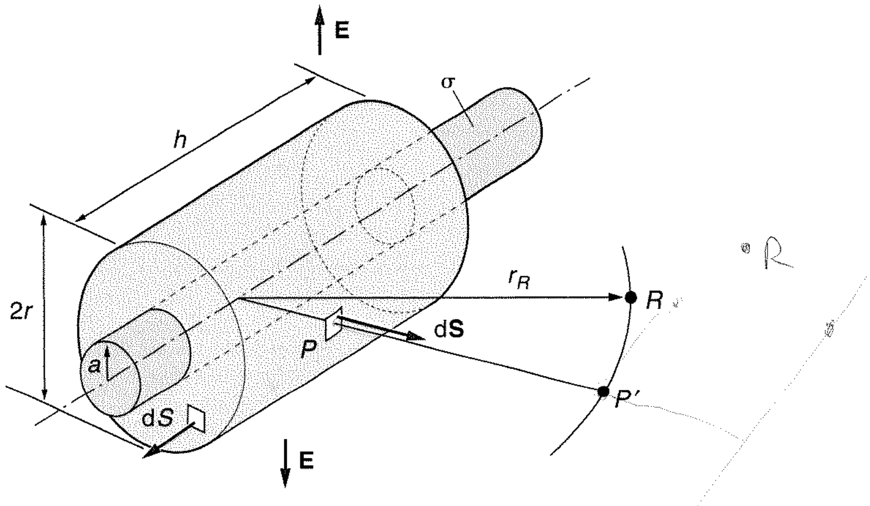


Figure 6.2 Segment of an infinitely long straight wire of circular cross section of radius a

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_{\text{belt}} \mathbf{E} \cdot d\mathbf{S} = E(r)\, 2\pi r h = \frac{Q_{\text{in cylinder}}}{\epsilon_0} = \frac{Q' h}{\epsilon_0}.$$

Note that if $r < a$ the surface encloses no charge. Thus,

$$E(r) = \frac{Q'}{2\pi\epsilon_0 r} \quad (r > a), \qquad E(r) = 0 \quad (r < a). \tag{6.8}$$

(Electric field of straight, infinitely long, uniformly charged thin wire)

Because the surface charge density on the cylinder is $\sigma = Q'/(2\pi a)$, $E(r)$ on the wire surface can be written in the form $E(a) = \sigma/\epsilon_0$. This, of course, is the same result as obtained by applying Eq. (6.5).

The determination of potential is slightly more complicated. Consider a point P at a distance r from the wire axis. Let the reference point, R, be at r_R from the axis, in the plane containing P and the wire axis. Recall that we can adopt any path from P to R in determining the potential. We choose the simplest: first a radial line from P to the distance r_R from the wire axis, and then a line parallel to the axis to R, Fig. 6.2. Along the first path segment, $\mathbf{E}$ and the line element are parallel, so $\mathbf{E} \cdot d\mathbf{l} = E(r)\, dl = E(r)\, dr$, because the line element, dl, becomes the differential increase in r, dr. Along the second path segment $\mathbf{E} \cdot d\mathbf{l} = 0$. Thus we have

$$V(r) = \int_P^R \mathbf{E} \cdot d\mathbf{l} = \int_r^{r_R} E(r)\, dr = \frac{Q'}{2\pi\epsilon_0} \int_r^{r_R} \frac{dr}{r},$$

or

$$V(r) = \frac{Q'}{2\pi\epsilon_0} \ln\frac{r_R}{r}. \tag{6.9}$$

(Potential of straight, infinitely long, uniformly charged thin wire)

We see that in this case we *cannot* adopt the reference point at infinity, because $\log\infty \to \infty$.

The expressions in Eqs. (6.8) and (6.9) are also useful for noninfinite wires, as long as we are interested in the field at points close to the wire and away from the ends. Because metallic wires are used often in electrical engineering, these equations are important.

Questions and problems: Q6.1 to Q6.4

6.3 Charge Distribution on Conductive Bodies of Arbitrary Shapes

Only for symmetrical isolated conductors is the charge distribution on their surface known—actually, inferred from symmetry. For conducting bodies of arbitrary shape the determination of charge distribution is one of the most important—and the most difficult—problems in electrostatics. Except in a few relatively simple cases, it can be determined only numerically. For many applications, it is useful to have a rough idea what the charge distribution is like. In estimating the charge distribution, the following simple reasoning can be of significant help.

We know that on an isolated metal sphere the charge is distributed uniformly. We also know that if the radius of the sphere is a and the surface charge density on it is σ, then the potential of the sphere is $V(a) = \sigma a/\epsilon_0$ (Eq. 6.7). Let us use this expression to estimate the charge distribution on a more complex conducting body.

Consider a charged metal body sketched in Fig. 6.3. It consists of a larger sphere of radius a, onto which are pressed parts of two smaller spheres of radii b and c.

Close to points A, B, and C indicated in the figure, the surface charge density is not the same. These three points are, however, at the same potential, V, because the body is conductive. Because charges that are close to a certain point predominantly contribute to the potential at that point, roughly speaking the surface charge density σ_A is approximately that of a sphere of radius a at the potential V. Therefore, according to Eq. (6.7), $\sigma_A \simeq \epsilon_0 V/a$. Similarly, $\sigma_B \simeq \epsilon_0 V/b$, and $\sigma_C \simeq \epsilon_0 V/c$. Thus, for the conducting body shown in Fig. 6.3,

$$a\sigma_A \simeq b\sigma_B \simeq c\sigma_C. \tag{6.10}$$

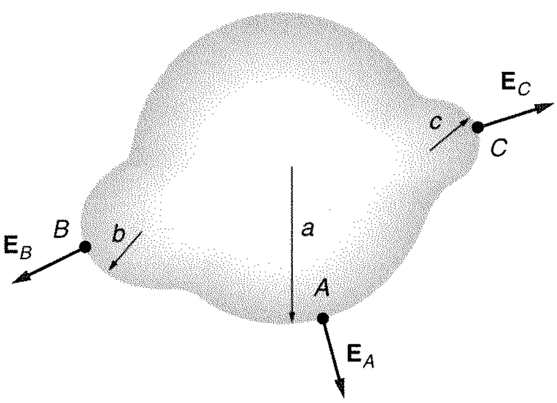


Figure 6.3 A charged metal body

Because the surface charge density is proportional to the local electric field strength,

$$aE_A \simeq bE_B \simeq cE_C. \tag{6.11}$$

These are simple but important approximate results. They tell us that the surface charge density at different points on a metal body is approximately inversely proportional to the curvature of the surface of the body at these points. This means that the *largest charge density and electric field strength on charged conductive bodies is around sharp parts of the body.*

An application of Eq. (6.11), for example, is a simple method for discharging aircraft. During flight, the plane becomes charged due to air friction. This charge could produce large fields during landing that in turn could produce a spark resulting in fire. However, if we place conducting spikes on the wings and other pointed plane parts, the charge density and, consequently, the electric field at these points become very high and the air ionizes (i.e., becomes conductive). A large portion of the charge "leaks" through these conducting channels into the atmosphere. We will see later that the principle behind lightning arresters is quite similar.

Questions and problems: Q6.5 to Q6.8, P6.1 to P6.7

6.4 Electrostatic Induction

Let us reconsider the electrostatically induced charges introduced in section 6.2 from a slightly different viewpoint. Assume that the metal body A shown in Fig. 6.4a is charged. The charge is distributed approximately as shown. What happens if we bring an *uncharged* conductive body B close to body A, and do it very quickly (theoretically, at infinite speed)?

In slow motion, the electric field of body A will first move free charges inside body B. Assume that there are both positive and negative free charges in body B. The force on the positive charges will be in the direction of vector **E**, and on the negative ones in the opposite direction. Charges of opposite sign will crowd up on two sides of B. We know from section 6.2 that their electric field is opposite to the field of charges

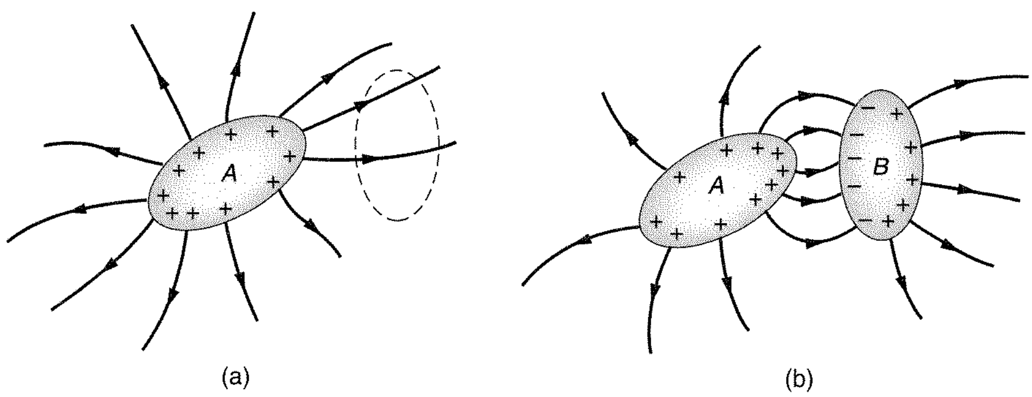


Figure 6.4 (a) A charged conducting body, and (b) approximate distribution of charge and field when an *uncharged* body, B, is brought near the first body, A

on body A. Once electrostatic equilibrium is reached, this field exactly cancels the field due to the charge on A. Of course, *the field of the charge distribution on B will change to some extent the original charge distribution on body A*. The final charge distribution and electric field lines are sketched in Fig. 6.4b. As already mentioned, the entire process of charge redistribution is very fast, practically instantaneous.

Because body B in the beginning was not charged, the total charge in body B must remain zero. However, equal charges of opposite sign do appear over the body surface. This is called *electrostatic* or *electric induction*, and we say that the charges on body B are *induced*. If body B had been charged previously, a similar process would have taken place. The charge would have redistributed itself so that the total electric field inside the body is zero.

Electrostatic induction is of great importance in electrical engineering. For example, so-called electric coupling between elements or wires in an electronic circuit (including traces on a printed-circuit board) is a result of electrostatic induction. In the examples that follow we describe some of the effects and applications of electrostatic induction.

Example 6.3—Electrostatic induction for a conductive body in a uniform electric field. Consider a metal sphere placed in a uniform electrostatic field, as shown in Fig. 6.5 (for example, between two very large, oppositely charged, parallel metal plates). The induced charge on the sphere will distribute itself to cancel out the uniform electric field inside it. We know that the resulting electric field on the air side is perpendicular to the surface of the sphere. The field lines will therefore "bend," as indicated in Fig. 6.5. Note that the resulting electric field is much stronger than the original at points A and B. (It can be proved that for the sphere, it is exactly three times stronger.) This is important for understanding the influence of the presence of water drops and metal particles on the electrical properties of liquid dielectrics. We will later come back to this example when we study the processes of xerography, electrostatic exhaust gas purification, and industrial electrostatic separation in Chapter 11.

Example 6.4—Faraday's cage. Consider an uncharged metal shell, as in Fig. 6.6. Let the shell be situated in an electrostatic field. We know that the electric field inside the conducting

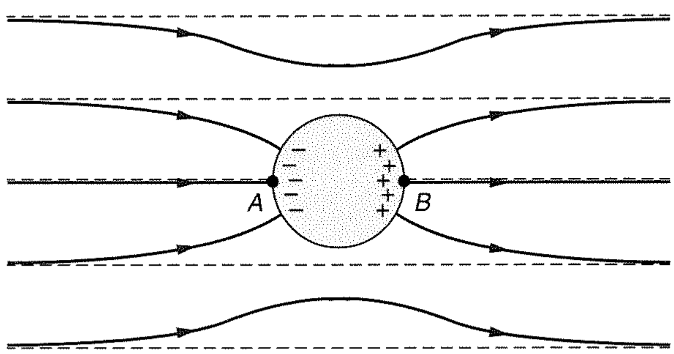


Figure 6.5 When an uncharged metal sphere is brought into a uniform electric field, the field becomes nonuniform. The strength of the field becomes significantly greater at points A and B of the sphere.

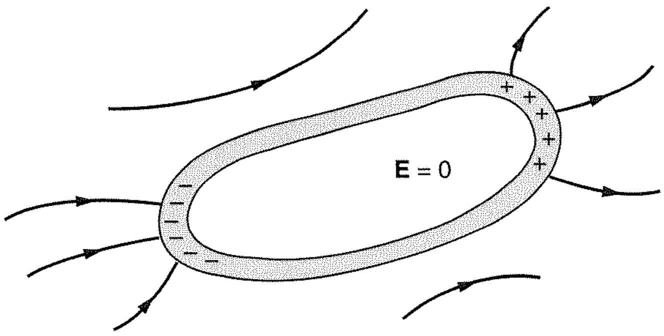


Figure 6.6 A metal shell shields its inside from electrostatic fields.

walls of the shell is zero (Eq. 6.1). It is not possible that the field penetrates through a field-free region (that would be analogous to attempting to pull something with a rope, one part of which is cut off). Therefore *there is no field inside the shell cavity either, no matter how thin the shell may be.* This means that we can shield a part of space from electrostatic fields perfectly.

It turns out that the shielding is efficient (although not any more perfect) if we use a metal grid instead of the metal wall, and that there are shielding effects even when the field is not electrostatic. Therefore such a shield can be used to protect a domain of space (for example, a small room) from external fields. Such a shielded space is known as a *Faraday cage.* It is a standard piece of equipment in many electrical engineering laboratories, both in electronics and power engineering. For example, your microwave oven door has a metal mesh, and when it is closed it forms a Faraday cage with the metal walls.

Example 6.5—Electrostatic induction due to a point charge inside a hollow conducting sphere. Let a point charge be inside a spherical uncharged metal shell, as in Fig. 6.7. It will induce some charge on the inside shell wall. To determine the induced charge, we apply Gauss' law on a spherical surface S inside the metal wall. The field at all points of S being zero, the total enclosed charge is zero as well. This means that the induced charge on the inside shell wall amounts to exactly $-Q$. Because the total shell charge is zero, and we know that inside conductors there can be no charge, a charge Q appears on the outside shell wall. How is this charge distributed?

Because there is no field inside the wall, there is no connection whatsoever between the field inside the shell cavity and the field outside the shell. The outer charge, therefore, is distributed as if there were no charge inside the cavity, which means *uniformly, irrespective of the position of the charge Q inside the cavity.*

Conversely, what would happen with the field inside the cavity if we change the outer charge, remove it, or bring an electrified object near the shell? Because there is no field in the shell wall, nothing of the kind can have any effect on the field inside the cavity. Therefore we can perform any electromagnetic experiment *inside* the cavity knowing it cannot be detected from outside. This is an example of the "reverse" application of the Faraday cage.

Example 6.6—Induced charges on the surface of the earth due to charged clouds. Let a charged cloud be above the earth's surface, as shown in Fig. 6.8. The soil is always conductive to some extent, so charges of opposite sign to those of the cloud will be induced on the earth's surface below the cloud. (We know that somewhere far away on the planet's surface the same amount of charge, of opposite sign, will appear.)

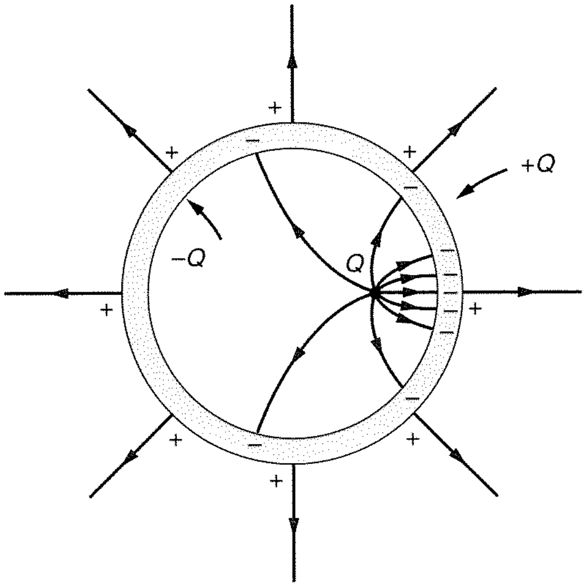


Figure 6.7 A point charge Q inside an uncharged metal spherical shell induces a total charge $-Q$ on the inside walls of the shell. The charge on the shell outside wall is distributed uniformly irrespective of the position of the point charge.

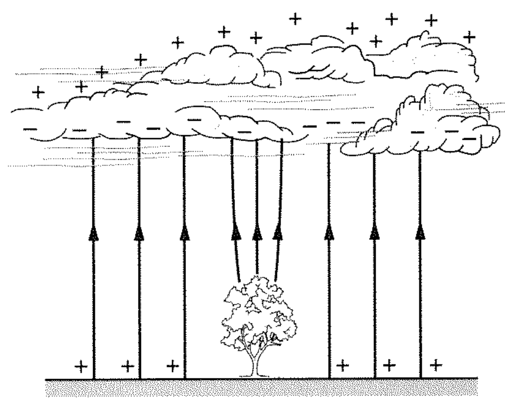


Figure 6.8 A charged cloud above the surface of the earth

The electric field on the earth's surface below the cloud will be the strongest at higher, sharper points on it—for example, at the top of a tree or at the top of a tall building. It is frequently so strong that it provokes local ionization of air, which may extend up to the cloud in the form of lightning (the cloud discharge to the earth). To protect an object from lightning strikes, a metal spike (or a system of spikes) on the top of the roof is connected to the ground with a wire. In this way the lightning is purposely attracted to strike at a desired point, and the cloud charge is taken to the ground through the wire.

Example 6.7—The effect of connecting a charged body to ground. Consider a charged metallic body of charge Q situated above ground. What happens if we decrease the height of the body until it touches the ground (which, as mentioned, is always conductive)? An example of such a case is shown in Fig. 6.9. An airplane gets charged by friction when flying through clouds, say with a positive charge Q. (The cloud remains negatively charged.) As the plane is landing, the induced charge on the ground redistributes. When the plane is at a height h_1 (Fig. 6.9a), there is an induced negative charge under it, spread over a large area. The remaining positive charge of the neutral earth is distributed over far areas of the earth. As the plane is landing (Fig. 6.9b,c), the induced charge becomes more and more localized below the plane. When the plane touches down, its charge Q neutralizes the local induced charge and the plane is discharged. There is still leftover induced positive charge, which now redistributes itself uniformly over the entire earth, but due to the enormous size of the earth, its surface density is negligible. The overall charge of the atmosphere-earth system is still the same—the original cloud carries the negative of the remaining induced positive charge. (This charge could become part of a lightning stroke and eventually neutralize the positive induced charge.)

Questions and problems: Q6.9 to Q6.17, P6.8 to P6.12

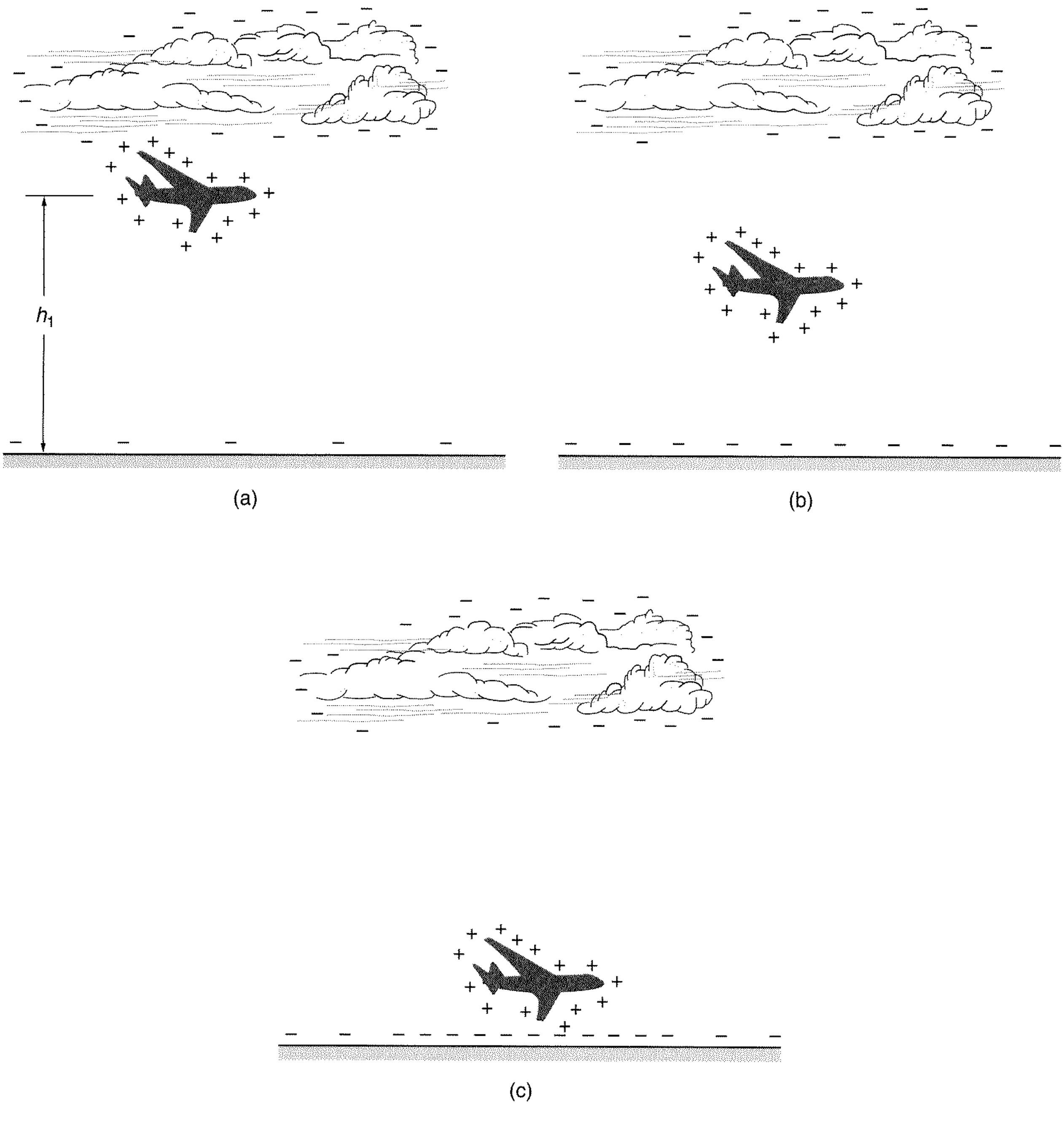


Figure 6.9 Induced charge when a charged airplane is landing

6.5 Image Method for Charges Above a Conducting Plane

The often-used *method of images* is a special case of a general theorem in electromagnetics known as the *equivalence theorem*. (Thévenin's and Norton's theorems in circuit theory are also special cases of the equivalence theorem.) The fundamental concept behind this theorem is the following.

It turns out that there are an infinite number of sources that can be placed *inside* any region of space, such that they would all produce the same field *outside* that region. For example, the field outside a spherically symmetrical cloud of radius a and total charge Q is the same as that due to a point charge Q at its center, or to a uniform surface charge Q over *any* sphere of radius less than or equal to a. These three sources are shown in Fig. 6.10. They are said to be equivalent with respect to the region where we are interested in the field, in this case the outside of the sphere. (Note that inside the sphere, the field is different in the three cases.) In some cases

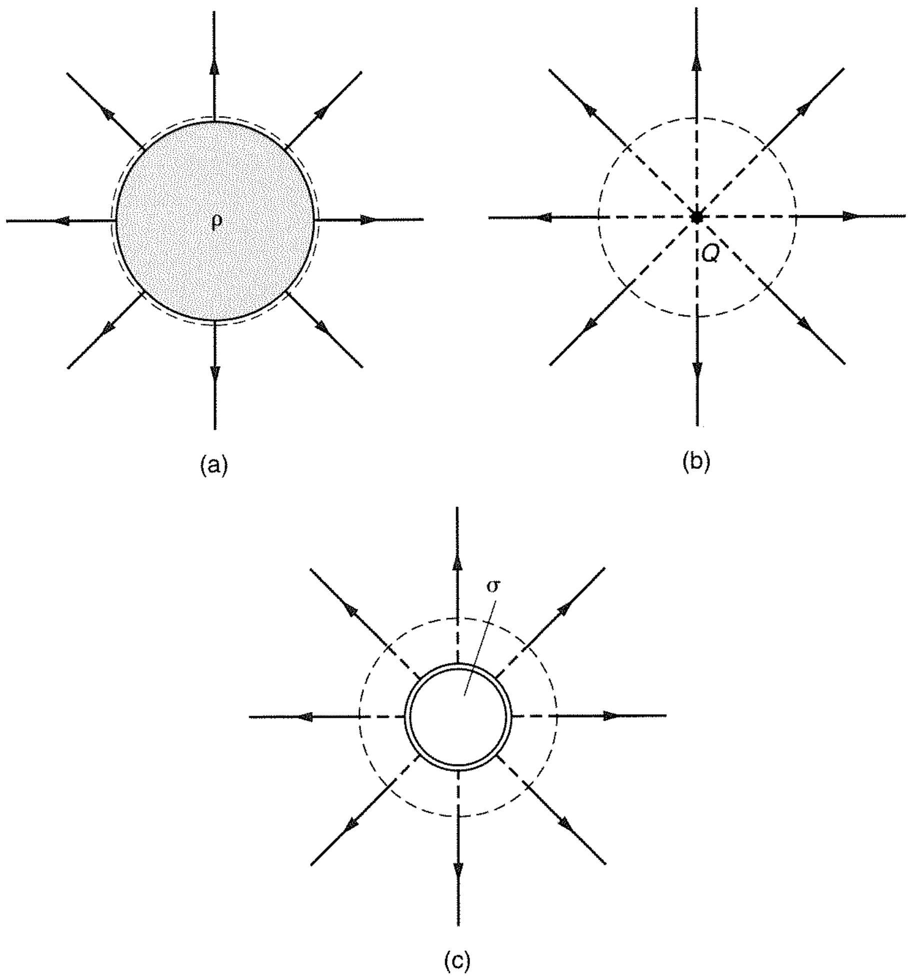


Figure 6.10 Equivalent charge distributions for the field outside a spherical surface of radius a: (a) charged ball of radius a; (b) point charge; and (c) surface-charged spherical shell of radius $r \leq a$

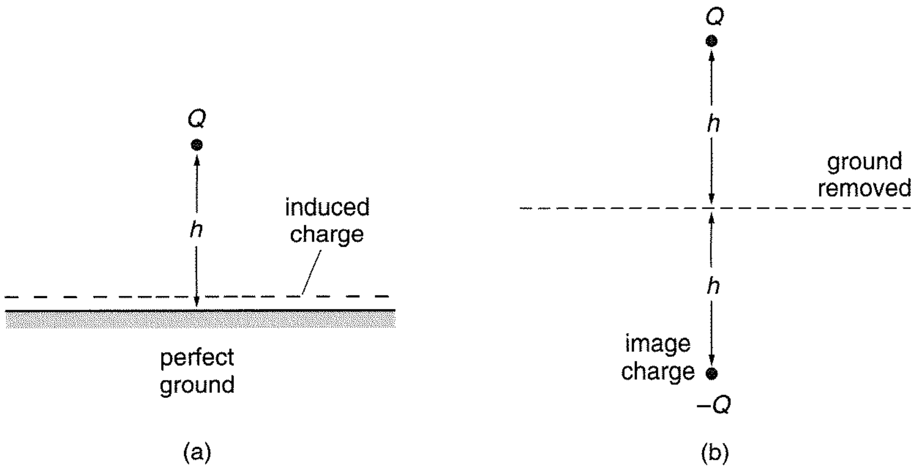


Figure 6.11 (a) Point charge above a large, grounded conducting sheet, and (b) induced charges on the sheet replaced by an equivalent point charge

it is possible to find equivalent sources that are much simpler than the actual ones. The method of images of charges in a conducting plane is one such example of great practical usefulness.

Let a point charge Q be located above a very large, flat conducting sheet that is grounded, as in Fig. 6.11a. The sheet is equipotential. (For example, it can be the surface of the earth, which is usually adopted as the potential reference.) According to Gauss' law, a charge $-Q$ is induced on the upper surface of the sheet. (It is advised that the reader prove this statement as an exercise.) We know that the induced charge is distributed in such a way as to cancel the electric field inside the sheet and make the tangential component of **E** equal to zero on the surface. We do not know, however, what this distribution is like, and therefore we cannot evaluate the field it produces above the sheet.

Although it is possible to determine this distribution starting from an integral equation, there is a much simpler way of doing it. Note that two charges of the same magnitude and opposite sign result in zero tangential electric field on the plane of symmetry of the two charges. This leads us to the conclusion that a source equivalent to all these unknown induced charges, with respect to the space above the plane, is a *single point charge* $-Q$, positioned symmetrically with respect to the plane. (Of course, once the equivalent charge is in place, we remove the induced charges.) This equivalent system is sketched in Fig. 6.11b. The equivalent source $-Q$ in this case is usually referred to as the *image of the charge Q in the conducting plane.* Once the ground plane is replaced by the image, the field *below* the ground plane is different than in the original system. Note that, knowing the image, we can also find the actual surface charge distribution over the metal plane (see problem P6.17).

Since superposition applies, images of any distribution of charges above a conducting plane are found in the same way. An important example is a wire at a height h above ground, such as a conductor of a power line or a phone cable, with a charge Q' per unit length (Fig. 6.12a). The equivalent source to the charges induced on the

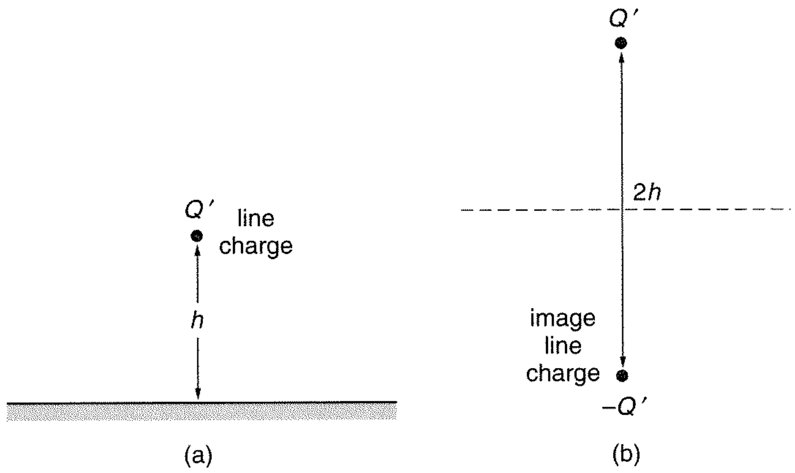


Figure 6.12 (a) Line charge above a large ground plane, and (b) induced charges on the plane replaced by an equivalent line charge

ground is simply a wire with a charge $-Q'$ per unit length situated at a depth h below the former surface of the ground, with the ground removed, as in Fig. 6.12b.

Questions and problems: P6.13 to P6.18

6.6 Chapter Summary

1. Most properties of the electrostatic field in the presence of conducting bodies can be deduced from the following:
   a. In electrostatics, the charge distribution must be such that $\mathbf{E} = 0$ inside conductors.
   b. In electrostatics, all excess charge in a conductor is spread over its surface.
   c. In electrostatics, $\mathbf{E}_{\text{tang}} = 0$ on the surface of conductors; that is, $\mathbf{E}$ is perpendicular to the surface of conductors.
   d. In electrostatics, conductor surfaces (and all points inside conductors) are equipotential.
   e. The normal component of the vector $\mathbf{E}$ on the surface of a conductor in a vacuum is equal to σ/ϵ_0.

   Actually, the properties b to d are consequences of a.
2. With these facts, we are able to understand the phenomenon of electrostatic (or electric) induction, the physics behind electrostatic screens (Faraday cages), the functioning of lightning rods, and the physical meaning of grounding a metal object.
3. They also help us understand the concept of replacing a ground plane with an image of a charge above it. This image, situated below the original ground plane, produces the same field above the ground plane as the original induced

charges on the plane. It provides a simple way of finding the fields due to charges above a perfectly conducting ground plane.

QUESTIONS

Q6.1. Prove that all points of a conducting body situated in an electrostatic field are at the same potential.

Q6.2. Two thin aluminum foils of area S are pressed onto each other and introduced into an electrostatic field, normal to the vector $\mathbf{E}$. The foils are then separated while in the field, and moved separately out of the field. What is the charge of the foils?

Q6.3. An uncharged conducting body has four cavities. In every cavity there is a point charge, $-Q_1$, $-Q_2$, $-Q_3$, and $-Q_4$. What is the induced charge on the surfaces of the cavities? What is the charge over the outer surface of the body?

Q6.4. We know that there is no electrostatic field inside a conductor. Assume that we succeeded in producing an electric field that is tangential to a conducting body just above its surface. Is this physically possible? If you think it is not, which law do you think would be violated in that case?

Q6.5. If *uncharged* pieces of aluminum foil are brought close to an electrified metal body, you will notice that they will be attracted, and then some of them repelled. Explain.

Q6.6. If an uncharged body (e.g., your finger) is brought near a small charged body, you will notice that the body is attracted by the uncharged body (your finger). Explain.

Q6.7. A very thin short conducting filament is hanging from a large conducting sphere. If the sphere is charged with a charge Q, is the charge on the filament greater or less than that which remains on the sphere? Explain.

Q6.8. An uncharged conducting flat plate is brought into a uniform electrostatic field. In which position of the plate will its influence on the field distribution be minimal, and in which maximal?

Q6.9. Assume that the room in which you are sitting is completely covered by thin aluminum foil. To signal to a friend outside the room, you move a charge around the room. Is your friend going to receive your signal? Explain.

Q6.10. Assume that in question Q6.9 your friend would like to signal you by moving a charge. Would you receive his signal? Explain.

Q6.11. A small charged conducting body is brought to a large uncharged conducting body and connected to it. What will happen to the charge on the small body? Is this the same as if a charged conducting body is connected to the ground?

Q6.12. A point charge Q is brought through a small hole into a thin uncharged metallic spherical shell of radius R, and fixed at a point that is a distance d ($d < R$) from its center. What is the electric field strength outside the shell?

Q6.13. A very thin metal foil is introduced exactly on a part of the equipotential surface in an electrostatic field. Is there any change in the field? Are there any induced charges on the foil surfaces? Explain.

Q6.14. A closed equipotential surface enclosing a total charge Q is completely covered with very thin metal foil. Is there any change in the field inside and outside the foil? What is the induced charge on the inner surface of the foil, and what on the outer surface?

Q6.15. A thin wire segment is introduced in the field and placed so that it lies completely on an equipotential surface. Is there any change in the field? Are there any induced charges on the wire surface? Explain.

Q6.16. Repeat question Q6.15 assuming that the wire segment is made to follow a part of the line of vector **E**.

Q6.17. Describe what happens as an airplane, charged negatively by friction with a charge $-Q$, is landing and finally touches down.

PROBLEMS

P6.1. A small conducting sphere of radius $a = 0.5\,\text{cm}$ is charged with a charge $Q = 2.3 \cdot 10^{-10}\,\text{C}$, and is at a distance $d = 10\,\text{m}$ from a large uncharged conducting sphere of radius $b = 0.5\,\text{m}$. The small sphere is then brought into contact with the large sphere, and moved back into its original position. Determine approximately the charges and potentials of the small and the large spheres in the final state. Take into account that $a \ll b$.

P6.2. A large charged conducting sphere of radius $a = 0.4\,\text{m}$ is charged with a charge $Q = -10^{-9}\,\text{C}$. A small uncharged conducting sphere of radius $b = 1\,\text{cm}$ is brought into contact with the large sphere, and then taken to a very distant point. Determine approximate charges and potentials of the large and small spheres in the end state, as well as the potential of the large sphere in the beginning.

P6.3. Two conducting spheres of equal radii $a = 2\,\text{cm}$ are far away from each other, and carry charges $Q_1 = -4 \cdot 10^{-9}\,\text{C}$ and $Q_2 = 2 \cdot 10^{-9}\,\text{C}$. The spheres are brought to each other, touched, and moved back to their positions. Determine the charges of the spheres in the final state, as well as the potentials of the spheres in the initial and final states.

P6.4. The electric field strength at a point A on the surface of a very thin charged conducting shell is **E**. Determine the electric field strength in the middle of a small round hole made in the shell and centered at point A.

* WITH RESPECT TO A POINT AT INFINITY

P6.5. Inside a spherical conducting shell of radius b is a conducting sphere of radius a ($a < b$), charged with a charge Q_a. What is the potential* of the shell: (1) if it is uncharged? (2) if it is charged with a charge Q_b? Does the potential depend on the position of the sphere inside the shell? Will it change if we move the sphere into contact with the inner surface of the shell?

P6.6. Suppose that the shell in problem P6.5 is connected by a thin conducting wire to the reference point of the potential. Determine its charge, and determine the electrostatic potential function outside the shell.

P6.7. A conducting sphere of radius a carries a charge Q_1. Concentric with the sphere is a spherical shell of inner radius b ($b > a$) and outer radius c, carrying a charge Q_2. Determine the electric field intensity and the electric scalar potential at every point of the system. Plot the dependence of E and V on the distance r from the common center.

P6.8. Twenty small charged bodies each carrying a charge $Q = 10^{-10}\,\text{C}$ are brought into an uncharged metallic shell of radius $R = 5\,\text{cm}$. Evaluate the potential of the shell and the electric field strength on its surface.

P6.9. How large an electric charge must be brought into the shell from problem P6.8 to achieve a field of $30\,\text{kV/cm}$ at its surface? (This is approximately the greatest electric field strength in air; for larger fields, the air ionizes and becomes a conductor, or breaks down.)

P6.10. A metal shell with a small hole is connected to ground with a conducting wire. A small charged body with a charge Q ($Q > 0$) is periodically brought through the hole into the shell without touching it, then taken out of it, and so on. Determine the charge that passes through the conducting wire from the shell to ground.

P6.11. Three coaxial conducting hollow cylinders have radii $a = 0.5\,\text{cm}$, $b = 1\,\text{cm}$, and $c = 2\,\text{cm}$, and equal lengths $d = 10\,\text{m}$. The middle cylinder is charged with a charge $Q = 1.5 \cdot 10^{-10}\,\text{C}$, and the other two are uncharged. Determine the voltages between the middle cylinder and the other two. Neglect effects at the ends of the cylinders.

P6.12. A charged conducting sphere of radius $b = 1\,\text{cm}$ and with a charge $Q = 2 \cdot 10^{-12}\,\text{C}$ is located at the center of an uncharged conducting spherical shell of outer radius $a = 10\,\text{cm}$. The inner sphere is moved to touch the shell, and returned to its initial position. Calculate the potential of the spheres in the initial and end states for the following values of the wall thickness of the large sphere: $d = 0$ (i.e., vanishingly small), $d = 1\,\text{cm}$, and $d = 5\,\text{cm}$.

P6.13. A line charge Q' is at a height h above a large flat conducting surface. Determine the electric field strength along the conducting surface in the direction normal to the line charge.

P6.14. A point charge Q is at a point $(a, b, 0)$ of a rectangular coordinate system. The half-planes ($x \geq 0$, $y = 0$) and ($x = 0$, $y \geq 0$) are conducting. Determine the electric field at a point $(x, y, 0)$, where $x > 0$ and $y > 0$.

P6.15. Repeat problem P6.14 for a line charge parallel to the z axis.

P6.16. A thunderstorm cloud can be represented as an electric dipole with $\pm 10\,\text{C}$ of charge. The bottom part of the cloud is at $h_1 = 5$ km above the ground, and the top is $h_2 = 8$ km above the ground (Fig. P6.16). The soil is wet and can be assumed to be a good conductor. (1) Find the potential and the electric field at the surface of the earth right under the cloud. (2) Find the surface charge density at points A and B on the surface (Fig. P6.16), for $x = 5$ km.

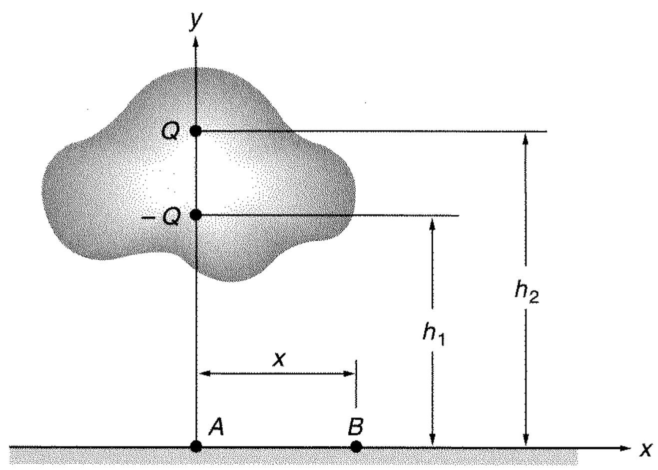


Figure P6.16 A thunderstorm cloud

P6.17. Find the induced charge distribution $\sigma(r)$ on the ground when a point charge $-Q$ is placed at a height h above ground, assuming the ground is an infinite flat conductor. Plot your results.

P6.18. Repeat problem P6.17 for the case of a dipole such as the one shown in Fig. P6.16.

7

Dielectrics in the Electrostatic Field

7.1 Introduction

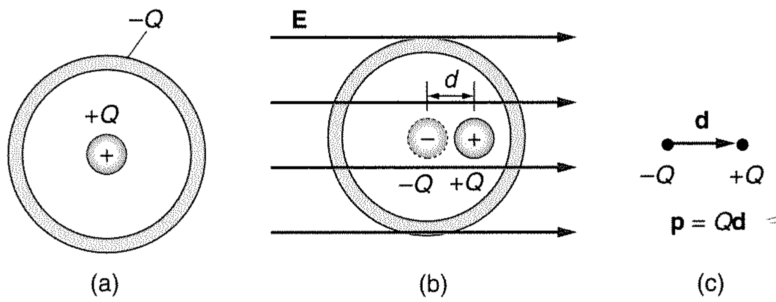
We now know that conductors change the electrostatic field by a mechanism called electrostatic induction, because any conductor has a large number of free charges that move in response to even the slightest electric field.

A wide class of substances known as *dielectrics* or *insulators* do not have free charges inside them. We might expect that, consequently, they can have no effect on the electrostatic field. This is not correct, although the mechanism by which dielectrics affect the electric field is different than in the case of conductors.

Dielectrics or insulators have many applications in electric engineering. Just as there is no electrical device without conductors, there is also no device without insulators. Therefore the analysis of dielectrics in an electrostatic field is as important as that of conductors.

7.2 Polarization of Dielectrics in the Electrostatic Field

Molecules of most substances behave as if electrically neutral when they are not in an electric field. We can imagine a molecule as a positive central point charge Q surrounded by a spherical cloud of negative charges of total charge $-Q$ (Fig. 7.1a). This



ELECTRIC DIPOLE MOMENT

Figure 7.1 (a) Model of a nonpolar molecule, (b) the molecule in an external electric field, and (c) the electric dipole that produces the same field as the molecule in (b)

is an acceptable model, for in reality, at distances larger than a few molecular diameters, the fields of the positive and negative charges cancel out and there is no net electric field. In this rough model of a molecule, some nonelectric forces that keep the molecule spherical and symmetrical must also be present.

Assume now that we move the molecule in Fig. 7.1a into an electric field with electric field strength **E**. The field acts by a force $Q\mathbf{E}$ on the central positive charge, and by the same force, in the opposite direction, on the negatively charged cloud. Due to the forces keeping the molecule together, this will only slightly displace the central positive charge with respect to the center of the negatively charged cloud, as in Fig. 7.1b. The cloud produces the same field at points far away as if the total charge were at its center. Therefore, if we are interested in the electric field produced by the deformed molecule, we can consider it as two point charges, Q and $-Q$, displaced by a small distance d, as in Fig. 7.1c. Two such point charges are known as an *electric dipole*.

H_2O

In some substances, such as water, the molecules are electric dipoles even with no applied electric field. Such molecules are known as *polar molecules*. Those that are not dipoles in the absence of the field are termed *nonpolar molecules*. In the absence of the electric field, polar molecules are oriented at random and no electric field due to them can be observed. If a polar molecule is brought into an electric field, there are forces on the two dipole charges that tend to align the dipole with the field lines (Fig. 7.2). This alignment is more pronounced for stronger fields.

Thus for dielectrics consisting of any of the two types of molecules, the external electric field makes the substances behave as huge arrays of oriented electric dipoles. We say in such a case that the dielectric is *polarized*. The process of making a dielectric polarized is known as *polarization*.

POLARIZATION

7.3 The Polarization Vector

According to our model, a polarized dielectric is a vast collection of electric dipoles situated in a vacuum. If we knew the charges Q and $-Q$ of the dipoles and their positions, we could evaluate the electric field strength and the scalar potential at any point. This, however, would be practically impossible due to the extremely large

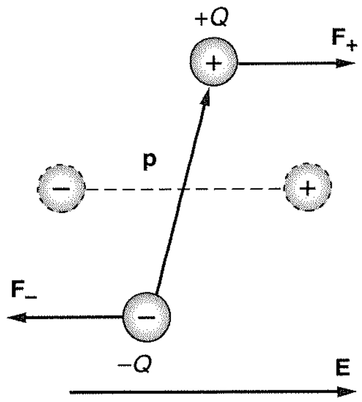


Figure 7.2 Model of a polar molecule in an external electric field

number of dipoles. For this reason we define a kind of average dipole density, a vector quantity known as the *polarization vector*.

We first need to characterize a single dipole by a vector quantity. Let $\mathbf{d}$ be the position vector of the charge Q of the dipole with respect to the charge $-Q$. We define the *electric dipole moment* of the dipole (Fig. 7.3) as

$$\mathbf{p} = Q\mathbf{d} \qquad (\mathrm{C \cdot m}). \tag{7.1}$$

(Definition of dipole moment)

The unit of $\mathbf{p}$ is $\mathrm{C \cdot m}$.

Consider now a small volume $\mathrm{d}v$ of a polarized dielectric. Let N be the number of dipoles per unit volume inside $\mathrm{d}v$, and $\mathbf{p}$ be the moment of the dipoles. The polarization vector, $\mathbf{P}$, at a point inside $\mathrm{d}v$ is defined as

$$\mathbf{P} = \frac{\sum_{\mathrm{d}v} \mathbf{p}}{\mathrm{d}v} = N\mathbf{p} \qquad (\mathrm{C/m^2}). \tag{7.2}$$

(Definition of the polarization vector)

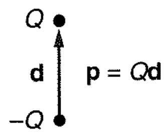


Figure 7.3 The dipole moment of an electric dipole is defined as the product $Q\mathbf{d}$. Note that the vector distance $\mathbf{d}$ between the two charges is adopted to be directed from the negative to the positive dipole charge.

Because the unit for the dipole moment, $\mathbf{p}$, is C · m, the unit of $\mathbf{P}$ is C/m^2. Note that this is the same unit as that of the surface charge density σ.

From this definition it follows that if we know the polarization vector at a point, we can replace a small volume dv (which contains a large number of dipoles) enclosing that point by a single dipole of moment

$$\mathrm{d}\mathbf{p} = \mathbf{P}\,\mathrm{d}v \qquad (\mathrm{C}\cdot\mathrm{m}). \tag{7.3}$$

(Dipole moment of a small domain dv *with polarization* **P***)*

This expression allows us to express the scalar potential and electric field strength of a polarized dielectric as an *integral.*

Equation (7.3) can be used for the evaluation of V and $\mathbf{E}$ of a polarized dielectric, but for that we need to know the expressions for V and $\mathbf{E}$ of a single dipole. Consider the dipole shown in Fig. 7.4. The scalar potential at a point P in the field of a dipole is obtained as the sum of potentials of the two dipole point charges:

$$V_P = \frac{Q}{4\pi\epsilon_0 r_+} + \frac{-Q}{4\pi\epsilon_0 r_-} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right). \tag{7.4}$$

Because the distance d between the dipole charges is always much smaller than the distance r of the point P from the dipole, the line segments r, r_+, and r_- are practically parallel. Therefore (Fig. 7.4)

$$\frac{1}{r_+} - \frac{1}{r_-} = \frac{r_- - r_+}{r_+ r_-} \simeq \frac{d\cos\theta}{r^2}, \tag{7.5}$$

so that the scalar potential at point P has the form

$$V_P = \frac{Qd\,\cos\theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p}\cdot\mathbf{u}_r}{4\pi\epsilon_0 r^2} \qquad (\mathrm{V}), \tag{7.6}$$

where $\mathbf{u}_r$ is the unit vector directed from the dipole toward point P (see Fig. 7.4). The potential of a point in the field of the dipole does not depend on Q and $\mathbf{d}$ separately,

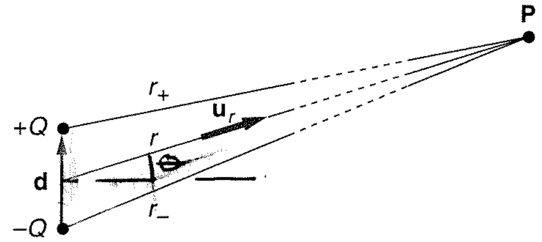


Figure 7.4 A point P in the field of an electric dipole. The distance r between P and the dipole is much larger than the dipole size d

but on their product, $\mathbf{p}$, the dipole moment. The electric field strength therefore also depends only on $\mathbf{p}$, and not on Q and $\mathbf{d}$ separately. (It is a simple matter to obtain $\mathbf{E}$ from the relation $\mathbf{E} = -\nabla V$, which is left as an exercise for the reader.)

The electric scalar potential of a polarized dielectric of volume v is now obtained from Eqs. (7.3) and (7.6) as

$$V = \frac{1}{4\pi\epsilon_0}\int_v \frac{\mathbf{P}\cdot\mathbf{u}_r}{r^2}\mathrm{d}v \qquad \text{(V)}. \tag{7.7}$$

(Potential of a polarized dielectric body)

When polarized, a dielectric is a source of an electric field. Consequently, the polarization of a dielectric body depends on the primary field, but also on its own polarization. It can be determined only if we know the dependence of the polarization vector on the *total* electric field strength, $\mathbf{E}$. Experiments show that for most substances

$$\mathbf{P} = \chi_e\epsilon_0\,\mathbf{E} \qquad (\mathbf{P}\text{ is in C/m}^2,\ \ \chi_e\text{ is dimensionless}), \tag{7.8}$$

i.e., $\mathbf{P}$ at every point is proportional to $\mathbf{E}$ at that point. The constant χ_e is referred to as the *electric susceptibility* of the dielectric. If it is the same at all points, the dielectric is said to be *homogeneous*, and if it varies from point to point, the dielectric is *inhomogeneous*. Dielectrics for which Eq. (7.8) applies are known as *linear dielectrics*, and they are *nonlinear* if such a relation does not hold. For all dielectrics, $\chi_e > 0$. Only for a vacuum, $\chi_e = 0$.

Questions and problems: Q7.1 to Q7.13, P7.1 to P7.3

7.4 Equivalent Charge Distribution of Polarized Dielectrics

A polarized dielectric can always be replaced by an equivalent volume and surface charge distribution in a vacuum. This is a very useful equivalence because we know how to determine the potential and field strength of such a charge distribution. This equivalent charge distribution can be derived from the polarization vector, $\mathbf{P}$.

uncompensated volume and surface charge.

Qualitatively, when a dielectric body is brought into an electric field, as we said earlier, all the molecules become dipoles oriented in the direction of the electric field. Inside a homogeneous dielectric the fields of all the dipoles cancel out on average, because the negative part of one dipole comes close to the positive part of its identical neighbor. However, at the surface of the dielectric there will be ends of dipoles that are uncompensated. This is the extra charge that appears at the surface of a dielectric when brought into an electric field. In the case of homogeneous dielectrics, this is the *only* uncompensated charge due to polarization. Inside an inhomogeneous dielectric, there will be some net volume charge as well, because all the individual dipoles are not identical and their field does not cancel out on average anymore. Both surface and volume polarization charges can now be considered to be in a vacuum, as the rest of the dielectric does not produce any field.

The relationship between the polarization charge inside a closed surface and the polarization vector on the surface can be derived by counting the charge that passes through a surface during the polarization process (the derivation is not given in this text). The resulting expression for the polarization charge in terms of **P** is

$$Q_{\mathrm{p\ in}\ S} = -\oint_S \mathbf{P} \cdot \mathbf{dS} \qquad \text{(C)}. \tag{7.9}$$

(Polarization (excess) charge in a closed surface enclosing a polarized dielectric)

Example 7.1—Proof that the volume polarization charge density is zero inside a homogeneous polarized dielectric. Consider a polarized *homogeneous* dielectric of electric susceptibility χ_e, *with no volume distribution of free charges,* and a small closed surface ΔS in it. Because we have replaced the dielectric with equivalent charges in a vacuum, Gauss' law applies and the *total* charge, free and polarization, enters on the right-hand side of the formula for Gauss' law. By assumption, there are no free charges in ΔS, and therefore

$$\epsilon_0 \oint_{\Delta S} \mathbf{E} \cdot \mathbf{dS} = Q_{\mathrm{p\ in}\ \Delta S}. \tag{7.10}$$

According to Eq. (7.9), $Q_{\mathrm{p\ in}\ \Delta S}$ can also be expressed as

$$Q_{\mathrm{p\ in}\ \Delta S} = -\oint_{\Delta S} \mathbf{P} \cdot \mathbf{dS} = -\chi_e \epsilon_0 \oint_{\Delta S} \mathbf{E} \cdot \mathbf{dS}. \tag{7.11}$$

Since $\chi_e > 0$, Eqs. (7.10) and (7.11) can both be satisfied only if the flux of **E** through ΔS is zero. The flux of **P** through ΔS is therefore also zero. This means that *inside a homogeneous dielectric there can be no volume distribution of polarization charges, i.e., polarization charges reside only in a thin layer on the dielectric surface.*

Questions and problems: Q7.14 and Q7.15

7.5 Density of Volume and Surface Polarization Charge

Consider now an *inhomogeneous* polarized dielectric. We will show that inside such a dielectric there *is* a volume distribution of polarization charges. To determine the density of these charges, ρ_p, we start from Eq. (7.9). Imagine a small closed surface ΔS enclosing the point at which we wish to determine ρ_p. The left-hand side of Eq. (7.9) can be written as a product of ρ_p and the volume Δv enclosed by ΔS. Consequently,

$$\rho_\mathrm{p} = -\left(\frac{\oint_{\Delta S} \mathbf{P} \cdot \mathbf{dS}}{\Delta v}\right)_{\Delta v \to 0} \qquad (\mathrm{C/m^3}). \tag{7.12}$$

The expression in parentheses is known as the *divergence* of vector **P**. (For additional explanations of the concept of divergence, read Section A1.4.2 of Appendix 1 before proceeding.) It can always be evaluated from this definition in any coordinate system. In a rectangular coordinate system, the divergence of a vector **P** has the form

$$\text{div}\mathbf{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \qquad (\text{C/m}^3), \tag{7.13}$$

(Divergence in a rectangular coordinate system)

where P_x, P_y, and P_z are scalar rectangular components of the vector $\mathbf{P}$. Using the del operator, Eq. (4.20), the expression for the divergence on the right side of this equation can formally be written in a short form,

$$\text{div}\mathbf{P} = \nabla \cdot \mathbf{P}. \tag{7.14}$$

Thus the volume density of polarization charges can be written as

$$\rho_{\text{p}} = -\text{div}\mathbf{P} = -\nabla \cdot \mathbf{P} \qquad (\text{C/m}^3). \tag{7.15}$$

(Volume density of polarization charges)

Note that Eq. (7.15) is but a shorthand of Eq. (7.12), and that in a rectangular coordinate system, which we will use frequently, $\nabla \cdot \mathbf{P}$ is given by Eq. (7.13).

To determine the density of surface polarization charges, consider Fig. 7.5, showing the interface between two polarized dielectrics, 1 and 2. Apply Eq. (7.9) to the closed surface that looks like a coin, shown in the figure. There is no flux of vector $\mathbf{P}$ through the curved surface because its height approaches zero. Therefore the flux through the closed surface ΔS is given by

$$\oint_{\Delta S} \mathbf{P} \cdot d\mathbf{S} = \mathbf{P}_1 \cdot \Delta\mathbf{S}_1 + \mathbf{P}_2 \cdot \Delta\mathbf{S}_2 \qquad (\text{C}).$$

Let us adopt the reference unit vector, $\mathbf{n}$, normal to the interface, to be directed into dielectric 1 (Fig. 7.5). Then we can write $\Delta\mathbf{S}_1 = \Delta S_1\,\mathbf{n}$ and $\Delta\mathbf{S}_2 = -\Delta S_1\,\mathbf{n}$. The

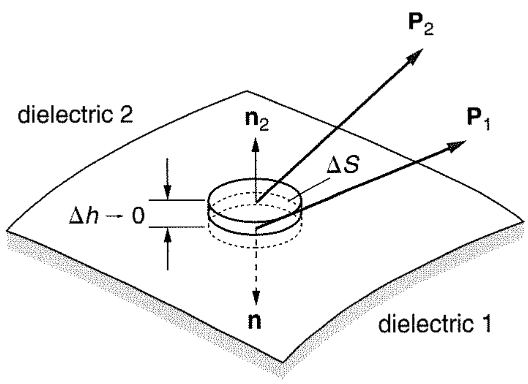


Figure 7.5 Interface between two polarized dielectrics

surface charge density is obtained if we divide the charge enclosed by ΔS by the area ΔS_1 cut out of the interface by ΔS. So we have

$$\sigma_\mathrm{p} = \mathbf{n} \cdot (\mathbf{P}_2 - \mathbf{P}_1) \qquad (\mathrm{C/m^2}). \tag{7.16}$$

(Surface density of polarization charges on the interface between two dielectrics)

If we know the polarization vector at all points of a dielectric, from Eq. (7.15) we can find the density of volume polarization charges (if they exist), and from the last equation we can find the density of surface polarization charges (which *always* exist). Because there are no excess charges in the rest of the dielectric, it can be disregarded. The problem of dielectric bodies in electrostatic fields is therefore reduced to that of a *distribution of charges in a vacuum*, a problem we know how to solve. What remains to be done is the determination of the polarization vector at all points. In most instances this is hard to do, but in many important cases it can be done using numerical methods.

Questions and problems: Q7.16 to Q7.19, P7.4 to P7.9

7.6 Generalized Form of Gauss' Law: The Electric Displacement Vector

With the knowledge from the preceding section, Gauss' law can be extended to electrostatic fields with dielectric bodies.

We know that from the electrostatic-field point of view, a polarized dielectric body can be considered as a distribution of volume and surface polarization charges in a vacuum. Gauss' law is valid for a vacuum. Therefore it is straightforward to extend Gauss' law to the case of fields with dielectrics: simply add the polarization charge to the free charge enclosed by S. Consequently, Gauss' law in Eq. (5.4) becomes

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q_{\text{free in } S} + Q_{\text{polarization in } S}}{\epsilon_0}. \tag{7.17}$$

Usually, this generalized Gauss' law is written in a different form. First, the polarization charge in S is represented as in Eq. (7.9). Note that the surface S is the same for the integral on the left-hand side of Eqs. (7.17) and (7.9). We can, therefore, multiply Eq. (7.17) by ϵ_0, move the integral representing $Q_{\text{polarization in } S}$ to the left-hand side of Eq. (7.17), and use just one integral sign. The result of this manipulation is

$$\oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot \mathrm{d}\mathbf{S} = Q_{\text{free in S}} \qquad (\mathrm{C}). \tag{7.18}$$

This is a very interesting result: the flux of the sum of the vectors $\epsilon_0\mathbf{E}$ and $\mathbf{P}$ through any closed surface S is equal to the total *free* charge enclosed by S. The form of Gauss' law (7.18) is more convenient than that of (7.17) because the only charges we can influence directly are free charges.

To simplify Eq. (7.18), we define the *electric displacement vector*, $\mathbf{D}$, as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2}). \tag{7.19}$$

(Definition of the electric displacement vector)

With this definition, the generalized Gauss' law takes the final form:

$$\oint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = Q_{\text{free in S}} \qquad (\mathrm{C}). \tag{7.20}$$

(Generalized Gauss' law)

The expression in Eq. (7.19) is the most general definition of the electric displacement vector. If the dielectric is linear (as most, but not all, dielectrics are), vector $\mathbf{D}$ can be expressed in terms of the electric field strength, $\mathbf{E}$:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\mathrm{C/m^2}), \tag{7.21}$$

(Electric displacement vector in linear dielectrics)

where

$$\epsilon_r = (1 + \chi_e) \qquad \text{(dimensionless)} \tag{7.22}$$

(Definition of relative permittivity—linear dielectrics only)

is known as the *relative permittivity* of the dielectric, and

$$\epsilon = \epsilon_r \epsilon_0 \qquad (\mathrm{F/m}). \tag{7.23}$$

(Definition of permittivity—linear dielectrics only)

as the *permittivity* of the dielectric.

Because the electric susceptibility, χ_e, is always greater than zero, the relative permittivity, ϵ_r, is always greater than unity. The most frequent values of ϵ_r are between 2 and about 10, but there are dielectrics with much higher relative permittivities. For example, distilled water (which is a dielectric) has relative permittivity of about 80 (this is because its molecules are polar molecules). A table of values of relative permittivities for some common dielectrics is given in Appendix 4. // End class

SKIP **Example 7.2—Electric field in a *pn* diode.** A *pn* diode, sketched in Fig. 7.6, is a fundamental semiconductor device and is a part of all bipolar transistors. Unlike in a metal, where electrons are the only charge carriers, in a semiconductor diode both negative and positive free charges are responsible for current flow when the diode is biased. The semiconductor material

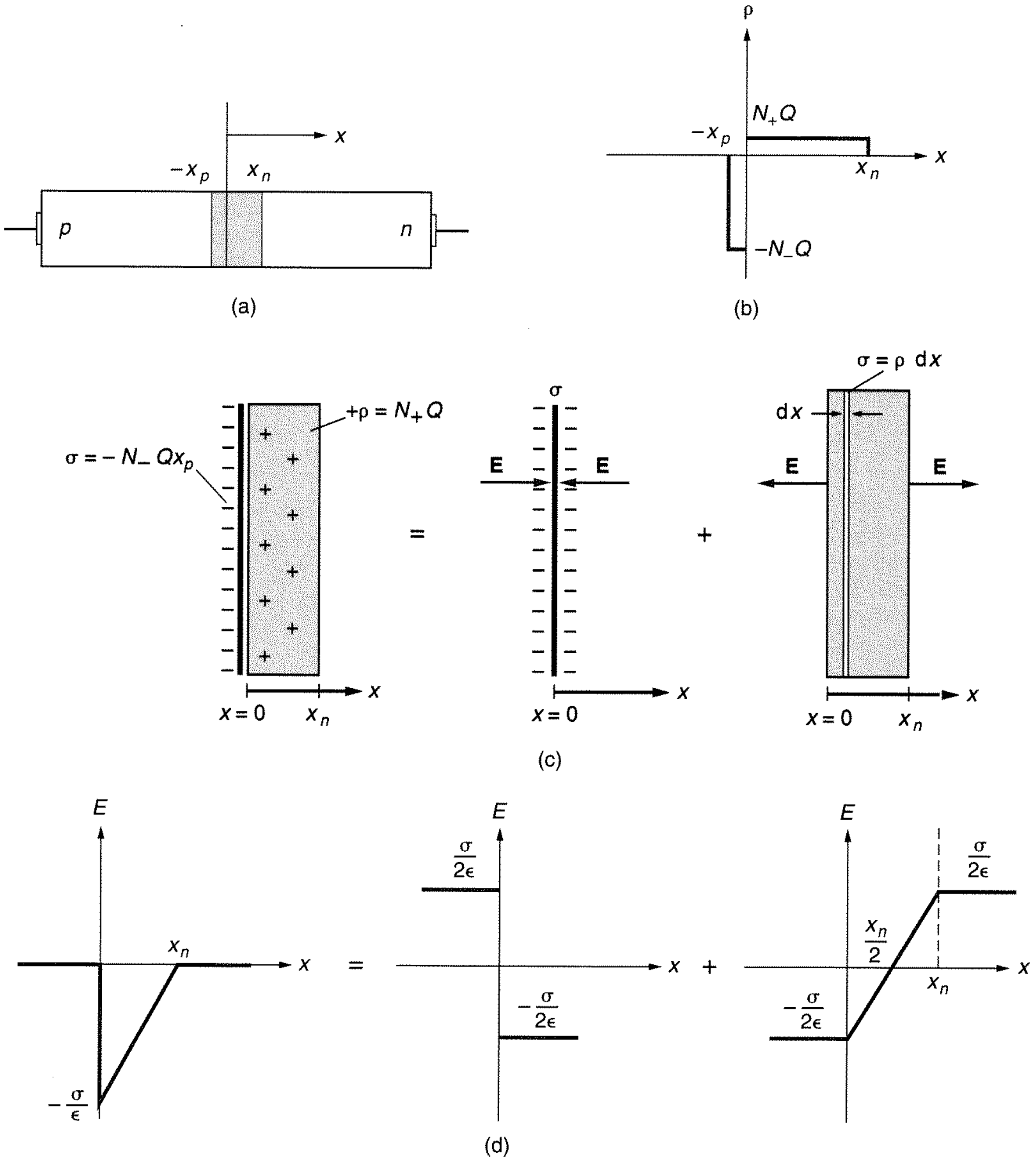


Figure 7.6 (a) Sketch of a *pn* diode and (b) its approximate charge density profile. (c) A diode can be approximated by a sheet of negative surface charge and a bulk of positive volume charge. (d) Superposition of the individual fields of the two charge distributions from (c) gives the final field distribution in the diode.

has a permittivity ϵ (for silicon $\epsilon_r = 11$, and for gallium arsenide $\epsilon_r = 13$), and if it is pure it behaves as a dielectric. When certain impurities called *dopants* are added to the material, it becomes conductive. The p region of the diode is a doped semiconductor material that has p *positive* free charge carriers per unit volume. This part is in physical contact with the n region, which has n *negative* free charge carriers per unit volume.

When the two parts are put together but not biased, the negative charge carriers (electrons) diffuse into the neighboring p region. Positive charge carriers ("holes" with a charge equal to that of an electron) diffuse into the neighboring n region. (The diffusion process is similar to the diffusion of two different gases through a thin membrane, except that the diffused charge carriers remain in the immediate vicinity of the boundary surface.) Because the negative charge carriers move into the region from which positive charge carriers partly left, leaving behind negatively charged atoms, there will be a surplus of negative charge in this thin layer of the p region. Similarly, there will be a surplus of positive charges in the adjoining thin layer of the n region.

These two charged layers produce an electric field (as in a parallel-plate capacitor), resulting in an electric force on free charge carriers that opposes the diffusion process. This electric force eventually (actually, in a very short time) stops the diffusion of free charge carriers. Thin layers on both sides of the boundary surface are thus depleted of their own free charge carriers. These two layers are known as the *depletion region*. Consequently, the depletion region finds itself between the p and n undepleted regions, and contains two layers of equal and opposite charges. Let the number of *positive* charges per unit volume in the n region be N_+, and the number of *negative* charges in the p region be N_-. The volume densities of charge in the two layers of the depletion region are $\rho_+ = N_+Q$ (in the n part), and $\rho_- = -N_-Q$ (in the p part), where Q is the absolute value of the electron charge.

If the diode is not biased (its two terminals are left open), the opposite charges on the two sides of the junction are of equal magnitude. Therefore the thicknesses of the two charged layers, x_p and x_n, are connected by the relation $N_-x_p = N_+x_n$. Usually the diode is made so that the n side of the junction has a much larger concentration of diffused negative free charge carriers than the other, that is, $N_- \gg N_+$. This means that $x_n \gg x_p$. Such a junction is called a one-sided step junction, and its charge concentration profile is sketched in Fig. 7.6b. This tells us that the width of the depletion layer on the p side can be neglected to the first order, i.e., this charged layer can be approximated by a negatively charged sheet of a surface charge density $\sigma = N_-Q/x_p$, Fig. 7.6c. On the n side, the depletion layer is effectively a uniform volume charge density (that is, N_+ is coordinate-independent). We already know from Example 5.3 what the field of the negative surface-charge sheet is, and it is shown in the middle of Fig. 7.6c.

What is the electric field of a volume charge, such as the one on the right in Fig. 7.6c? Outside the charged layer, it is equal to the field of a charged sheet of the same *total* charge:

$$E_{\text{outside}} = \frac{\sigma}{\epsilon} = \frac{\rho x_n}{2\epsilon} = \frac{N_+Qx_n}{2\epsilon}. \tag{7.24}$$

Inside the volume charge, we can apply Gauss' law to a thin slice $\mathrm{d}x$ wide, as indicated on the right in Fig. 7.6c, which contains $\rho\,\mathrm{d}x$ surface charge. It is left to the reader to show that integration of the field resulting from all the slices between 0 and x_n gives the following expression for the electric field inside the volume charge density as a function of the x coordinate:

$$E_{\text{inside}} = \frac{\rho}{\epsilon}\left(x - \frac{x_n}{2}\right) = \frac{N_+Qx_n}{\epsilon}\left(x - \frac{x_n}{2}\right). \tag{7.25}$$

DIRECTED IN THE +X DIRECTION IF X>X$_n$, AND IN THE -X DIRECTION IF X < X$_n$

This expression is shown graphically on the ~~left~~ RIGHT in Fig. 7.6d. Using the principle of superposition, we can now add the field of the negative surface charge (in the middle of Fig. 7.6d) to the field of the positive volume charge we found (on the ~~left~~ RIGHT in Fig. 7.6d) to get the field profile of a *pn* diode, shown on the ~~right~~ LEFT in Fig. 7.6d. It is left to the reader as an exercise to sketch the potential distribution inside a diode.

Questions and problems: Q7.20 and Q7.21, P7.10

SKIP

7.7 Electrostatic Boundary Conditions

start class

In inhomogeneous media consisting of several homogeneous parts there is, obviously, an abrupt change in some quantities describing the field on the two sides of boundaries. For example, if on such a boundary there is a surface polarization charge, it is a source of the electric field component directed in opposite directions on the two sides of the boundary; consequently, the total electric field must have a different direction and magnitude on the two sides of the boundary.

Such abrupt changes of any quantity describing the field must satisfy basic field equations and definitions. Specialized field equations describing this behavior, more precisely connecting the values of any field quantity on two sides of a boundary surface, are known as *boundary conditions*. What are boundary conditions needed for? Note that they represent, in fact, fundamental equations of the electrostatic field specialized to boundary surfaces. Therefore in a medium consisting of several dielectric bodies, the field transition from one body to the adjacent body through a boundary surface *must* be as dictated by the boundary conditions. Otherwise this could not be a real electric field because it would not satisfy the field equations *everywhere*. Note that this is true for all boundary conditions we introduce in later chapters.

$\oint_C \vec{E} \cdot d\vec{l}$

Let us apply first the law of conservation of energy of the electrostatic field, Eq. (4.7), to the narrow rectangular contour ΔC in Fig. 7.7. Because the length of the shorter sides approaches zero, the contribution to the line integral of **E** along them is

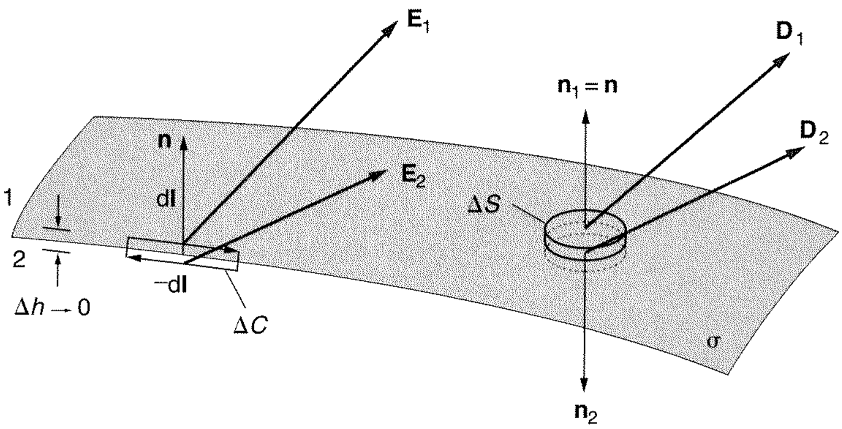


Figure 7.7 Boundary between two media. A narrow rectangular contour is used in the law of conservation of energy and a coinlike closed surface is used in Gauss' law for deriving boundary conditions for vectors **E** and **D**, respectively.

zero. Along the two longer sides, the contribution is $(\mathbf{E}_1 \cdot d\mathbf{l}_1 + \mathbf{E}_2 \cdot d\mathbf{l}_2)$. The scalar products are simply tangential components of the two electric field strength vectors, which we denote by the subscript "t." Because $d\mathbf{l}_2 = -d\mathbf{l}_1$, the boundary condition for the tangential components of vector **E** is

$$E_{1t} = E_{2t} \quad \text{(valid in general).} \tag{7.26}$$

(Boundary condition for tangential components of vector **E***)*

Note that no other assumptions are needed to derive this condition except Eq. (4.7). Consequently, it is valid for all cases of the electrostatic field. We will see that it is valid also for the general case of a time-varying electromagnetic field.

Now let us apply Gauss' law, Eq. (7.20), to the small cylindrical coinlike surface in Fig. 7.7. Let there be a surface charge σ on the boundary inside the surface. There is no flux of vector **D** through the curved surface because its height is vanishingly small. The flux through the two cylinder bases is $D_{1n}\,\Delta S$ (the outward flux) and $-D_{2n}\,\Delta S$ (the inward flux), both with respect to the reference unit vector **n** directed into dielectric 1, where the subscript "n" denotes the normal component. The enclosed charge being $\sigma\,\Delta S$, the generalized Gauss' law yields

$\oint_S \bar{D} \cdot d\bar{S} = Q_{encl}$

$$\mathbf{D}_1 \cdot \mathbf{n} - \mathbf{D}_2 \cdot \mathbf{n} = \sigma, \quad \text{or } D_{1n} - D_{2n} = \sigma \quad \text{(valid in general).} \tag{7.27}$$

(Boundary condition for normal component of vector **D***; unit vector normal,* **n***, directed into medium 1)*

In the special case when there is no surface charge on the boundary, this becomes

$$D_{1n} = D_{2n} \quad \text{(no free surface charges on boundary).} \tag{7.28}$$

Another important case is the boundary between a conductor and a dielectric. Let the dielectric be medium 1, and the conductor be medium 2. We know that there is no field inside a conductor. Therefore $D_{2n} = 0$, and Eq. (7.27) becomes

$\left(E = \frac{\sigma}{\varepsilon_0}\right)$

$$D_n = \sigma \quad \text{(on boundary of dielectric and conductor).} \tag{7.29}$$

Note that this is essentially the same equation as Eq. (6.5). We will see that Eqs. (7.27) to (7.29) are also valid in general, and not only for electrostatic fields.

Questions and problems: Q7.22 to Q7.24, P7.11 to P7.15

7.8 Differential Form of Generalized Gauss' Law

The generalized Gauss' law in Eq. (7.20) can be transformed into a differential equation, known as the differential form of Gauss' law. To obtain this differential equation, let us apply Eq. (7.20) to a small volume Δv enclosed by a surface ΔS, and divide both sides of the equation by Δv. The right side then becomes simply the volume charge density, ρ, inside ΔS. The left side becomes the same as the expression in Eq. (7.12), with **P** substituted by **D**. We know that this expression is the divergence of vector **D**. So we obtain

one of Maxwell Eq.

$$\boxed{\operatorname{div}\mathbf{D} = \rho. \qquad (7.30)}$$

(Differential form of generalized Gauss' law)

Since the divergence of **D** is a combination of derivatives of the components of **D**, this is indeed a differential equation in three unknowns, the three scalar components of vector **D**. It is known as a partial differential equation because partial derivatives, with respect to individual coordinates, enter into the equation. We will see that the basic equations of the electromagnetic field, Maxwell's equations, are a set of four partial differential equations. Equation (7.30) is one of these four equations.

7.9 Poisson's and Laplace's Equations: The Laplacian

The potential at a point is related to the volume charge density at that point by a differential equation known as *Poisson's equation*. A special case of Poisson's equation for the case when the volume charge density is zero is called *Laplace's equation*. The derivation of these equations is quite simple.

We know that we can always represent vector **E** as $\mathbf{E} = -\operatorname{grad} V = -\nabla V$. For linear media, therefore, $\mathbf{D} = -\epsilon \operatorname{grad} V = -\epsilon \nabla V$, so that from the generalized form of Gauss' law, Eq. (7.13), we obtain

$$\operatorname{div}(\epsilon \operatorname{grad} V) = \nabla \cdot (\epsilon \nabla V) = -\rho. \qquad (7.31)$$

This is the most general form of Poisson's equation. For the frequent case of a homogeneous dielectric (ϵ the same at all points), Eq. (7.31) becomes

$\nabla^2 V = -\frac{\rho}{\varepsilon}$

$$\boxed{\operatorname{div}(\operatorname{grad} V) = \nabla \cdot (\nabla V) = -\frac{\rho}{\epsilon}. \qquad (7.32)}$$

(Poisson's equation)

Laplace's equation is obtained from Eqs. (7.31) and (7.32) if we set $\rho = 0$:

$$\operatorname{div}(\epsilon \operatorname{grad} V) = \nabla \cdot (\epsilon \nabla V) = 0 \qquad (7.33)$$

for a general, inhomogeneous dielectric with no free charges, and

$$\operatorname{div}(\operatorname{grad} V) = \nabla \cdot (\nabla V) = 0 \tag{7.34}$$

(Laplace's equation)

for a homogeneous dielectric with no free charges.

The operator div(grad) $= \nabla \cdot \nabla$ is known as *Laplace's operator*, or the *Laplacian*, and is denoted briefly as Δ or ∇^2. It is a simple matter to show that, in a rectangular coordinate system, Laplace's operator has the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{7.35}$$

(Laplacian operator in rectangular coordinate system)

As an important example, if the volume charge distribution in a region is a function of a single rectangular coordinate, for example of x, V is then also a function of x only. Poisson's equation becomes

$$\frac{d^2 V(x)}{dx^2} = -\frac{\rho(x)}{\epsilon}. \tag{7.36}$$

This equation is used often, for example, in the analysis of semiconductor devices including diodes, transistors, and capacitors.

Example 7.3—The *pn* Diode Revisited. In this example, we use Poisson's equation to find the potential distribution in a pn diode, using the one-sided step junction approximation from Example 7.2. Poisson's equation for the p side of the junction can be written as

$$\frac{d^2 V}{dx^2} = -\left(-\frac{QN_-}{\epsilon_0 \epsilon_r}\right) = \frac{QN_-}{\epsilon_0 \epsilon_r}, \tag{7.37}$$

and for the n side as

$$\frac{d^2 V}{dx^2} = -\frac{QN_+}{\epsilon_0 \epsilon_r}. \tag{7.38}$$

However, in the one-sided step approximation, the width of the depletion layer on the p side is negligible, so we only need to solve Eq. (7.38). We first integrate once with respect to x from 0 to x. We need one boundary condition to determine the integration constant in this step. We know that there is no electric field outside of the depletion region, so the boundary condition is $dV/dx = 0$ at $x = x_n$. Integrating Eq. (7.38) once therefore yields

$$\frac{dV}{dx} = -\frac{QN_+}{\epsilon_0 \epsilon_r}(x - x_n). \tag{7.39}$$

Because we know that $\mathbf{E} = -(\mathrm{d}V/\mathrm{d}x)\mathbf{u}_x$, we can rearrange terms in Eq. (7.39) to obtain the same expression for the electric field as the one shown graphically in Fig. 7.6d. To get the potential, we integrate another time. Let us adopt as the boundary condition that the potential is zero at $x = x_n$ (we know that we can adopt it to be zero at any point). We thus obtain

$$V(x) = -\frac{QN_+x_n^2}{2\epsilon_0\epsilon_r}\left(1 - \frac{x}{x_n}\right)^2. \tag{7.40}$$

As this potential exists inside the diode even when its terminals are not connected to an external voltage source, it is called the *built-in potential*.

When a bias is applied to a diode, it changes the width of the depletion layer. If we connect the diode p region to the positive output of a voltage source and the n side to the negative one, the depletion layer gets narrower, making it easier for free charges to flow through it. This is called *forward bias*. If the diode terminals are connected the other way, the depletion layer becomes thicker and current flow is disabled. This is called *reverse bias*. If an ac voltage is applied to the diode, in one half of the cycle the diode will conduct and in the other half there will be no current. Therefore a diode is a *rectifier*.

Questions and problems: P7.16 to P7.22

7.10 Some Practical Electrical Properties of Dielectrics

Applications of dielectrics in electrical engineering are hardly possible without knowing their electrical properties. We briefly mention here some of these properties.

strength

In addition to relative permittivity, two more properties need particular attention. The first is the *dielectric strength* of a dielectric. This is the largest magnitude of the electric field that can exist in a dielectric without damaging it. If the field magnitude is greater than the dielectric strength of the dielectric, *dielectric breakdown* occurs (the dielectric burns, cracks, ionizes, and becomes conductive, becomes very lossy, etc.).

The typical value of the dielectric strength for air is about $3 \cdot 10^6$ V/m, or 30 kV/cm. For liquid and solid dielectrics the electric field strength ranges from about $15 \cdot 10^6$ V/m to about $40 \cdot 10^6$ V/m. Values of the dielectric strength of some common dielectrics are given in Appendix 4.

loss

Another important property of dielectrics is loss that produces heat. Most dielectrics have a very small number of free charges, so that resistive (Joule's) losses in them due to time-constant fields (except for very large field magnitudes) are usually negligible. In time-varying fields, however, there is a new type of loss, known as the *polarization loss*, that is much larger than Joule's losses. Qualitatively, the time-varying electric field induces time-varying dipoles in the dielectric, which start to vibrate more vigorously due to these oscillations. This vibration is heat, i.e., it represents losses to the field polarizing the dielectric.

Questions and problems: Q7.25 to Q7.27, P7.23

7.11 Chapter Summary

1. If introduced into an electrostatic field, all dielectrics can be visualized as a vast ensemble of small electric dipoles situated in a vacuum. We say that such a dielectric is polarized.
2. The polarization of a dielectric at any point is described by the polarization vector, $\mathbf{P}$, representing a vector density of dipole moments at that point. The dipole moment of a dipole of charges Q and $-Q$ separated by a distance $\mathbf{d}$ (directed from $-Q$ to Q) is defined as $\mathbf{p} = Q\mathbf{d}$.
3. The polarized dielectric can further be considered as an equivalent distribution of volume and surface charges, known as *polarization charges*. These two charge densities are determined in terms of the polarization vector, $\mathbf{P}$. The rest of the dielectric has no effect whatsoever on the field and can be removed. The polarization charges must, therefore, be considered to be situated in a vacuum.
4. The vector quantity $\mathbf{D} = (\epsilon_0\mathbf{E} + \mathbf{P})$ has a simple and useful property: its flux through any closed surface equals the total free charge inside the surface. This equation is known as the *generalized Gauss' law*, and vector $\mathbf{D}$ as the *electric displacement vector*.
5. The generalized Gauss' law can also be written in the form of a differential equation, $\nabla \cdot \mathbf{D} = \rho$. This is known as the differential form of Gauss' law.
6. There is a simple differential relationship between the potential function at a point and volume charge density at that point, known as the Poisson equation, $\nabla \cdot \epsilon\nabla V = -\rho$. Its special form, when there are no volume charges, is known as Laplace's equation, $\nabla \cdot \epsilon\nabla V = 0$.

QUESTIONS

Q7.1. At a point of a polarized dielectric there are N dipoles per unit volume. Each dipole has a moment $\mathbf{p}$. What is the polarization vector at that point?

Q7.2. A body is made of a linear, homogeneous dielectric. Explain what this means.

Q7.3. What is the difference between an inhomogeneous linear dielectric and a homogeneous nonlinear dielectric?

Q7.4. Why is $\chi_e = 0$ for a vacuum?

Q7.5. Are there substances for which $\chi_e < 0$? Explain.

Q7.6. An atom acquires a dipole moment proportional to the electric field strength $\mathbf{E}$ of the external field, $\mathbf{p} = \alpha\mathbf{E}$ (α is often referred to as the *polarizability*). Determine the electric force on the atom if it is introduced into a *uniform* electric field of intensity $\mathbf{E}$.

Q7.7. Answer question Q7.6 for the case in which the atom is introduced into the field of a point charge Q. Determine only the direction of the force, not its magnitude.

Q7.8. A small body—either dielectric or conducting—is introduced into a nonuniform electric field. In which direction (qualitatively) does the force act on the body?

Q7.9. Two point charges are placed near a piece of dielectric. Explain why Coulomb's law cannot be used to determine the *total* force on the two charges.

Q7.10. A small charged body is placed near a large dielectric body. Will there be a force acting between the two bodies? Explain.

Q7.11. A closed surface S situated in a vacuum encloses a total charge Q *and* a polarized dielectric body. Using a sound physical argument, prove that in this case also the flux of the electric field strength vector **E** through S is Q/ϵ_0.

Q7.12. Arbitrary pieces of dielectrics and conductors carrying a total charge Q are introduced through an opening in a hollow, uncharged metal shell. The opening is then closed. Using a physical argument and Gauss' law for a vacuum, prove that the charge appearing on the outer surface of the shell is exactly equal to Q.

Q7.13. A positive point charge is placed in air near the interface of air and a liquid dielectric. Will the interface be deformed? If you think it will be deformed, then will it raise or sink? What if the charge is negative?

Q7.14. Explain in your own words why Eqs. (7.10) and (7.11) imply that the flux of **E** through a closed surface ΔS is zero.

Q7.15. Electric dipoles are arranged along a line (possibly curved) so that the negative charge of one dipole coincides with the positive charge of the next. Describe the electric field of this arrangement of dipoles.

Q7.16. Write Eq. (7.16) for the interface of a dielectric and a vacuum. For case (1) assume the dielectric to be medium 1, and for case (2) medium 2.

Q7.17. Is there a pressure of electrostatic forces acting on a boundary surface between two different dielectrics situated in an electrostatic field? Explain.

Q7.18. Prove that the total polarization charge in any piece of a dielectric material is zero.

Q7.19. A point charge Q is placed inside a spherical metal shell, a distance d from its center. In addition, the shell is filled with an inhomogeneous dielectric. Determine the electric field strength outside the shell.

Q7.20. Does Eq. (7.18) mean exactly the same as Eq. (7.17)? Explain.

Q7.21. Can the relative permittivity of a dielectric be less than one, or negative? Explain.

Q7.22. Can you find an analogy between properly connecting sleeves to a jacket, and using boundary conditions in solving electrostatic field problems? Describe.

Q7.23. Prove that a charged conductor situated in an inhomogeneous but linear dielectric has a potential proportional to its charge. [*Hint:* consider the polarized dielectric as an aggregate of dipoles situated in a vacuum.]

Q7.24. Discuss question Q7.23 for a case in which the dielectric is not linear.

Q7.25. What is the unit of dielectric strength of a dielectric?

Q7.26. Explain how 30 kV/cm is the same as $3 \cdot 10^6$ V/m.

Q7.27. Are polarization losses in a dielectric the same as resistive Joule's losses? Explain.

PROBLEMS

P7.1. Using the relation $\mathbf{E} = -\nabla V$, determine the spherical components E_r, E_θ, and E_ϕ of the electric field strength of the electric dipole in Fig. 7.4.

P7.2. Determine the electric force on a dipole of moment **p** located at a distance r from a point charge Q_0, if the angle between **p** and the direction from the charge is arbitrary.

P7.3. An atom acquires a dipole moment proportional to the electric field strength **E** of the external field, $\mathbf{p} = \alpha\mathbf{E}$. Determine the force on the dipole if it is introduced into the field of a point charge Q at a distance r from the charge.

P7.4. A homogeneous dielectric sphere is polarized uniformly over its volume. The polarization vector is **P**. Determine the distribution of the polarization charges inside and on the surface of the sphere.

P7.5. A thin circular dielectric disk of radius a and thickness d is permanently polarized with a dipole moment per unit volume **P**, parallel to the axis of the disk that is normal to its plane faces. Determine the electric field strength and the electric scalar potential along the disk axis. Plot your results.

P7.6. Determine the density of volume polarization charges inside a linear but inhomogeneous dielectric of permittivity $\epsilon(x, y, z)$ at a point where the electric field strength is **E**. There is no volume distribution of free charges inside the dielectric.

P7.7. The permittivity of an infinite dielectric medium is given as the following function of the distance r from the center of symmetry: $\epsilon(r) = \epsilon_0(1 + a/r)$. A small conducting sphere of radius R, carrying a charge Q, is centered at $r = 0$. Determine and plot the electric field strength and the electric scalar potential as functions of r. Determine the volume density of polarization charges.

P7.8. A conducting sphere of radius a carries a charge Q. Exactly one half of the sphere is pressed into a dielectric half-space of permittivity ϵ. Air is above the dielectric. Determine the free and polarization surface charge density on the sphere and in the dielectric.

P7.9. Repeat problem P7.8 for a circular cylinder of radius a with charge Q' per unit length.

P7.10. A small spherical charged body with a charge $Q = -1.9 \cdot 10^{-9}$ C is located at the center of a spherical dielectric body of radius a and relative permittivity $\epsilon_r = 3$. Determine the vectors **E**, **P**, and **D** at all points, volume and surface density of polarization charges, and the potential at all points. Is it possible to determine the field and potential outside the dielectric body without solving for the field inside the body? Explain.

P7.11. What is **E** equal to in a needlelike air cavity inside a homogeneous dielectric of permittivity ϵ if the cavity is parallel to the electric field vector $\mathbf{E}_d$ inside the dielectric (Fig. P7.11)?

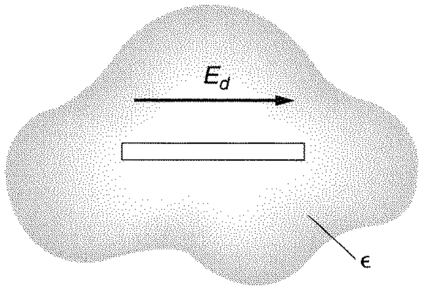


Figure P7.11 A needlelike cavity

P7.12. What is **E** equal to in a disklike air cavity with faces normal to the electric field vector $\mathbf{E}_d$ inside a homogeneous dielectric of permittivity ϵ (Fig. P7.12)?

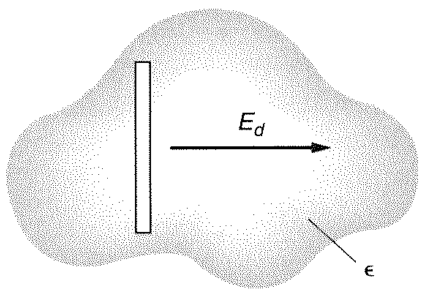


Figure P7.14 A disklike cavity

P7.13. At a point of the boundary surface between dielectrics of permittivities ϵ_1 and ϵ_2, the electric field strength vector in medium 1 makes an angle α_1 with the normal to the boundary, and that in medium 2 an angle α_2. Prove that $\tan\alpha_1/\tan\alpha_2 = \epsilon_1/\epsilon_2$.

P7.14. A dielectric slab of permittivity $\epsilon = 2\epsilon_0$ is situated in a vacuum in an external uniform electric field **E** so that the field lines are perpendicular to the faces of the slab (Fig. P7.14). Sketch the lines of the resulting vectors **E** and **D**.

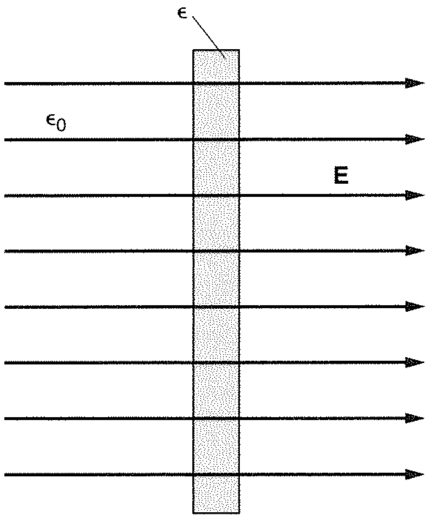


Figure P7.14 Field lines normal to dielectric slab

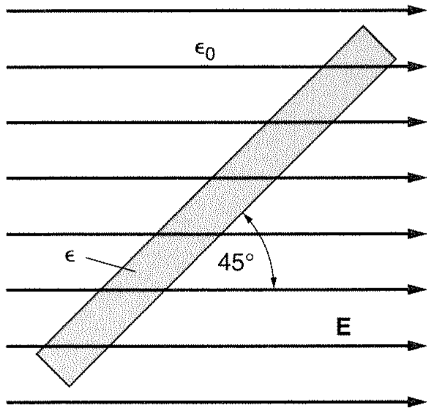


Figure P7.15 Field lines oblique to dielectric slab

P7.15. Repeat problem P7.14 assuming that the dielectric slab is at an angle of 45 degrees with respect to the lines of the external electric field (Fig. P7.15).

P7.16. One of two very large parallel metal plates is at a zero potential, and the other at a potential V. Starting from Laplace's equation, determine the potential, and hence the electric field strength, at all points.

P7.17. Two concentric spherical metal shells, of radii a and b ($b > a$), are at potentials V (the inner shell) and zero. Starting from Laplace's equation in spherical coordinates, determine the potential, and hence the electric field strength, at all points. Plot your results.

P7.18. The charge density at all points between two large parallel flat metal sheets is ρ_0. The sheets are d apart. One of the sheets is at a zero potential, and the other at a potential V. Find the potential at all points between the plates starting from Poisson's equation. Plot your result.

P7.19. Repeat problem P7.18 if the charge density between the plates is $\rho(x) = \rho_0 x/d$, x being a coordinate normal to the plates, with the origin at the zero-potential plate. Plot your result and compare to problem P7.18.

P7.20. Repeat problem P7.19 if the origin is at the plane of symmetry of the system.

P7.21. Two long coaxial cylindrical thin metal tubes of radii a and b $(b > a)$ are at potential zero (the outer tube) and V. Starting from Laplace's equation in cylindrical coordinates, determine the potential between the cylinders, and hence the electric field strength.

P7.22. Prove that if V_1 and V_2 are solutions of Laplace's equation, their product is not generally a solution of that equation.

P7.23. The radii of conductors of a coaxial cable with air dielectric are a and b $(b > a)$. Determine the maximum value of the potential difference between the conductors for which a complete breakdown of the air dielectric does not occur. The dielectric strength of air is E_0.

8

Capacitance and Related Concepts

8.1 Introduction

Capacitors consist of two metal bodies, known as the *capacitor electrodes*, charged with equal charges of opposite sign. They are characterized by a quantity known as the *capacitance*. Capacitors are of fundamental importance in electrical engineering and are commonly used by most engineers. Other important concepts related to the capacitance, however, are less widely understood. In this chapter, we examine electric coupling and shielding as phenomena closely related to the topic of capacitance.

8.2 Capacitors and Capacitance

Consider first a conductive body with charge Q far away from any other charges. Assume that the potential of the body is V. If the charge on the body is changed to kQ, where k is any real number, what does the potential of the body become?

$Q' = kQ$
$\downarrow$
$V' = kV$

The surface charge density on the body, σ, is such that the electric field is zero inside the body. The electric field at any point on the surface is given by $\mathbf{E} = (\sigma/\epsilon)\mathbf{n}$. When the body is charged to kQ, the electric field inside the body has to remain zero, so the surface charge density at every point *must* be $k\sigma$. This means that the new electric field strength at all points will be $k\mathbf{E}$. The potential then also has to increase by a factor of k, so the new potential is kV. The conclusion is that *the charge of a conductive body and its potential are proportional to each other.* This proportionality is written as

$$Q = CV, \qquad \text{or} \qquad C = \frac{Q}{V} \qquad [C \text{ is in farads (F)}]. \tag{8.1}$$

(Definition of capacitance of an isolated body)

The constant C does not depend on Q or V, but only on the shape and size of the body and on the materials surrounding it. It is called the *capacitance* of an isolated body. To determine it, we need to know its potential in terms of a given charge. For example, for a metal ball of radius a in a vacuum, we get

$$C = \frac{Q}{V_{\text{ball}}} = 4\pi\epsilon_0 a \qquad \text{(F)}. \tag{8.2}$$

The unit for capacitance is the *farad* (abbreviated F). It is equal to coulomb/volt.

Example 8.1—Capacitance of the earth. Let us use the formula for the capacitance of an isolated conducting sphere to find the capacitance of the earth:

$$C_{\text{earth}} = 4\pi\epsilon_0 R_{\text{earth}} = \frac{1}{9\cdot 10^9} 6.37\cdot 10^6 \simeq 0.708\cdot 10^{-3}\,\text{F} < 1\,\text{mF}. \tag{8.3}$$

The capacitance of the whole earth is much less than a farad. Obviously, the farad is a very large unit. The values of capacitance of commonly used capacitors are from a few pF to a few μF.

2 conductor bodies

As already mentioned, a system consisting of two conductive bodies charged with equal charges of opposite sign is called a *capacitor*, shown in Fig. 8.1. The metal

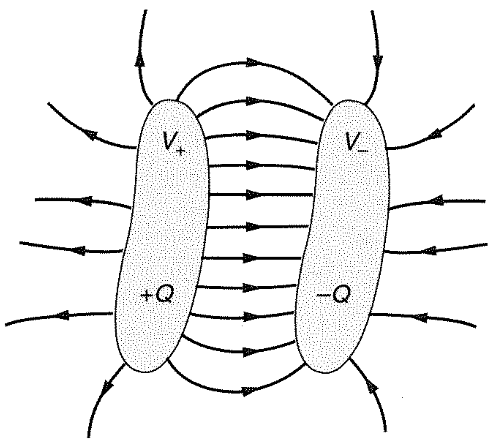


Figure 8.1 A capacitor consists of two bodies charged with equal amounts of charge, but of opposite sign.

bodies are the capacitor electrodes. Capacitors can have different shapes. Following the same reasoning as for an isolated metal body, we can conclude that the charge on the reference capacitor electrode, e.g., Q, is proportional to the potential difference between the two electrodes (the proof is left as an exercise for the reader; see Q8.3). This is written as

$$Q = C(V_Q - V_{-Q}), \qquad \text{or} \qquad C = \frac{Q}{V_Q - V_{-Q}} \qquad [C \text{ is in farads (F)}]. \tag{8.4}$$

(Definition of the capacitance of a capacitor)

The constant C is the capacitance of the capacitor. It depends on the shape, size, and position of the electrodes and on the properties of the dielectric between them. Usually the capacitance does not depend on the charge Q on the electrodes, nor does it depend on the voltage $V_{+-} = V_Q - V_{-Q}$ between them. Such capacitors are *linear capacitors*. For *nonlinear capacitors* these conditions are not satisfied. For example, in a semiconductor device known as a varactor diode, the capacitance of the diode depends on the voltage applied to its terminals.

LINEAR CAPACITORS

Although parallel and series connections of capacitors are familiar from circuit theory, we repeat them here from the field-theory point of view. We will see that some conditions implicit in the definitions of the equivalent capacitor in the two cases cannot be seen from circuit theory.

Example 8.2—Parallel connection of capacitors. Consider a parallel connection of capacitors as in Fig. 8.2. Although it might seem a bit strange, this is just a form of the capacitor shown in Fig. 8.1. We have two terminals connected to two electrodes with equal but opposite charges; the charge is just distributed over electrodes of more complicated shape. So we use the same definition for capacitance as in Eq. (8.4). The potential difference between any pair of electrodes of any of the capacitors is the same, equal to $(V_Q - V_{-Q})$. The total charge is simply the sum of the charges on the reference electrodes, i.e., $Q_{tot} = Q_1 + Q_2 + \cdots + Q_n$. From Eq. (8.4) it follows that the capacitance of such a connection of capacitors is

$$C_{equiv} = C_1 + C_2 + \cdots + C_n \quad \text{(F)}. \tag{8.5}$$

(Equivalent capacitance of a parallel connection of capacitors)

See chart

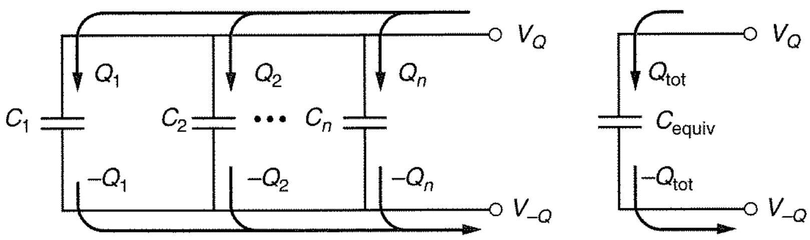


Figure 8.2 A parallel connection of capacitors

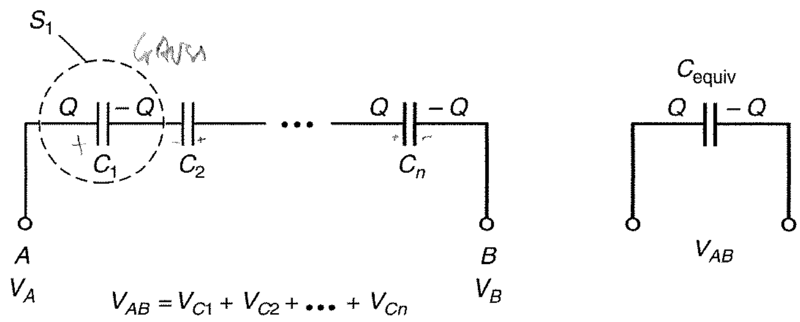


Figure 8.3 A series connection of capacitors

Example 8.3—Series connection of capacitors. The series connection of capacitors, Fig. 8.3, is a little more complicated to analyze than the parallel case. It is now not obvious at all that such a structure is equivalent to the capacitor model of Fig. 8.1. This is indeed a different kind of structure. We have only two electrodes we can charge (the leftmost and the rightmost); the other electrodes are not accessible.

Note that the pairs of "internal" electrodes form conductive bodies with zero total charge. Assume that we charge the outermost electrodes with charges Q and $-Q$. The left outer electrode will then induce a charge on the nearest electrode, and theoretically on all the others.

sum voltage

How large is this charge? Normally capacitors are made so that there is no field outside them if they are charged with equal but opposite charges. Assuming all the capacitors in Fig. 8.3 are of this type, and *only* in that case, if we enclose the first capacitor with a surface S_1 and apply Gauss' law, the total enclosed charge must be zero. This means that the induced charge on the second electrode from the left must be exactly $-Q$. This leaves Q on the third electrode, which induces $-Q$ on the fourth one, etc. We see that *all the capacitors are charged with equal charge*, Q and $-Q$.

Knowing the charge of all the capacitors, we know the voltage between their terminals:

$$V_{C1} = \frac{Q}{C_1}, \; V_{C2} = \frac{Q}{C_2}, \; \ldots, \; V_{Cn} = \frac{Q}{C_n}. \tag{8.6}$$

The total voltage, i.e., the voltage between the two outermost electrodes of the series connection, is the sum of these voltages. Since the charges corresponding to this voltage are Q and $-Q$, the capacitance of this combined capacitor is given by

$$\frac{1}{C_{\text{equiv}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n} \quad \left(\frac{1}{\text{F}}\right). \tag{8.7}$$

(Equivalent capacitance of a series connection of capacitors)

Note that in a parallel connection of capacitors with greatly differing capacitances the dominant one is the one with the *largest* capacitance. If we have a series connection of such capacitors, the dominant one is the one with the *smallest* capacitance.

Example 8.4—Parallel-plate capacitor filled with a homogeneous dielectric. A parallel-plate capacitor consists of two parallel metal plates of areas S charged with $Q_1 = Q$ and $Q_2 =$

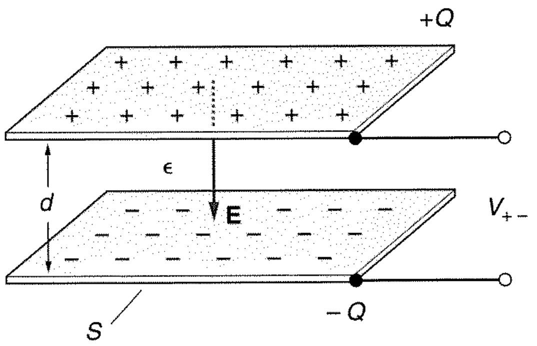


Figure 8.4 A parallel-plate capacitor

$-Q$ (Fig. 8.4). Assume that a homogeneous dielectric of permittivity ϵ is between the plates and that the distance between them, d, is small compared to the plate dimensions.

Under these assumptions, the field between the plates is very nearly the same as that between two uniformly charged planes. The electric displacement vector, **D**, is normal to the plates. The surface charge density is $\sigma = Q/S$. Using the generalized Gauss' law we find that the intensity of the electric displacement vector $\mathbf{D} = \sigma = Q/S$. The electric field strength is hence $E = Q/(\epsilon S)$.

The voltage between the two plates corresponding to the given charge is now obtained easily. Since vector **E** between the plates is normal to them and constant,

$$V_Q - V_{-Q} = \int_{+}^{-} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = Ed = \frac{Qd}{\epsilon S},$$

so that

$$C = \epsilon \frac{S}{d} \quad \text{(F)}. \tag{8.8}$$

(Capacitance of a parallel-plate capacitor)

Example 8.5—Parallel-plate capacitor with two dielectric layers. Figure 8.5 shows a parallel-plate capacitor with two dielectrics, with the interface parallel to the plates. What is the capacitance in this case? The electric field is normal to the boundary between the two dielectrics, so we need to use the boundary condition for the displacement vector **D**. Consequently, in dielectric 1, next to the left plate, $E_1 = \sigma/\epsilon_1 = Q/(\epsilon_1 S)$, and in the second dielectric, next to the right plate, $E_2 = Q/(\epsilon_2 S)$, where S is the plate area. The vector **D** is normal to all the boundary surfaces. It is therefore continuous across the entire capacitor. The capacitance is given by

$$C = \frac{Q}{V_Q - V_{-Q}} = \frac{Q}{E_1 d_1 + E_2 d_2},$$

(where d_1 and d_2 are thicknesses of the two layers (Fig. 8.5)). That is,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}, \tag{8.9}$$

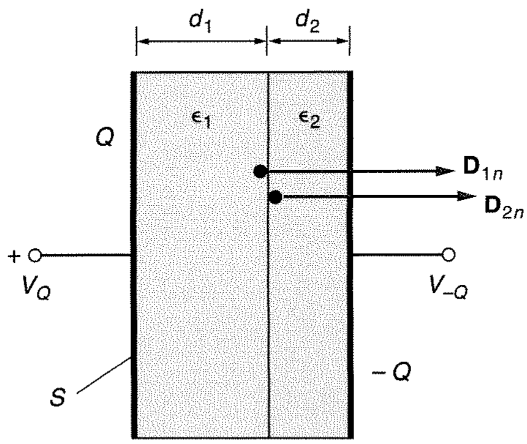


Figure 8.5 Parallel-plate capacitor with two different dielectrics between the plates

where C_1 and C_2 are the capacitances of parallel-plate capacitors with homogeneous dielectrics ϵ_1 and ϵ_2. This means that this capacitor looks like two capacitors in series.

Example 8.6—Some other kinds of capacitors. The expression $C = \epsilon S/d$ for the capacitance of the parallel-plate capacitor is often used even if the capacitor does not consist of two metal plates. For example, it is used for calculating the capacitance of the variable capacitor shown in Fig. 8.6a, where the capacitance is changed by turning one set of plates to overlap with the other set.

When variable capacitance is not needed and relatively large capacitance is required, capacitors like the one in Fig. 8.6b are used. Between two long ribbons made of aluminum foil is an insulating ribbon (for example, oily paper), and an insulating ribbon is also on the outside of one of the ribbons. The ribbons are tightly wrapped. The capacitance of such a capacitor can be precisely determined by the parallel-plate capacitor formula. (Note that because of the two insulating ribbons, the capacitance of the wrapped capacitor is *twice* that of the unwrapped capacitor.) Its capacitance can vary in a broad range, from about 10 pF to about 100 μF.

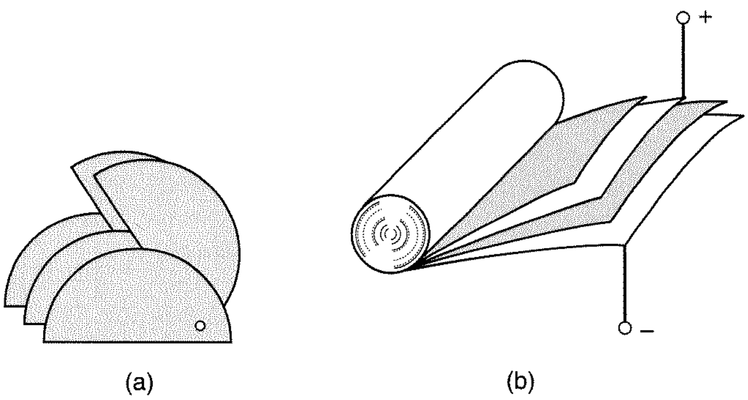


Figure 8.6 (a) A variable parallel-plate capacitor, and (b) a paper-insulator capacitor

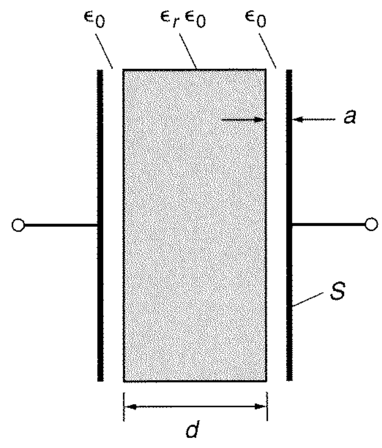


Figure 8.7 A parallel-plate capacitor with a solid dielectric and a thin layer of air between the dielectric and the electrodes

We see from the formula for the capacitance of the parallel-plate capacitor that very large capacitances can be obtained when d is very small. In so-called electrolytic capacitors, the dielectric is a very thin layer of aluminum oxide (about 10^{-5} cm) deposited on the inside surface of a metal cap, and the capacitor is then filled with a conducting fluid. The fluid is one electrode, and the metal cap the other. In this way, the contact between the dielectric and the electrodes is very good. Electrolytic capacitors can have capacitances as large as hundreds and even thousands of microfarads.

In capacitors with solid insulators, the insulator and the metal electrodes might not have tight contact. In that case a thin layer of air is between them, as shown in Fig. 8.7. The capacitance of such a capacitor is

$$C = \frac{C_0 C_\epsilon}{C_0 + C_\epsilon}, \qquad C_0 = \epsilon_0 \frac{S}{2a}, \qquad C_\epsilon = \epsilon \frac{S}{d}. \tag{8.10}$$

Usually $a \ll d$ and $C_0 \gg C_\epsilon$, so the thin layer of air does not affect the capacitance too much. However, the electric field in the air layer is $E_0 = \epsilon_r E_d$, so it is larger than in the dielectric. Usually the dielectrics used in capacitors have a higher breakdown field than the 30 kV/cm for air. This air layer is a weak spot for high-voltage capacitors because breakdown would first occur in that layer.

Example 8.7—Capacitance per unit length of a coaxial cable. A coaxial cable, or coaxial transmission line, is used for guiding electromagnetic energy, especially at high frequencies. It consists of an inner wire conductor and an outer tubular conductor, coaxial with the wire; hence the cable name (Fig. 8.8). The coaxial cable is frequently nicknamed "coax."

Let the inner conductor have a radius a, and the outer conductor have an inside radius b. Usually the inner conductor is connected to the positive terminal of a voltage source, and the outer conductor is grounded. As a result, the inner conductor is charged along its length with Q' coulombs/m (conditionally $Q' > 0$), and the outer conductor with $-Q'$ coulombs/m. Let the permittivity of the dielectric filling the cable be ϵ.

Using Gauss' law on the surface S_2 for $b < r < c$, we find that all the charge on the outer conductor is distributed over its inside surface. The electric field is zero outside the coax. We

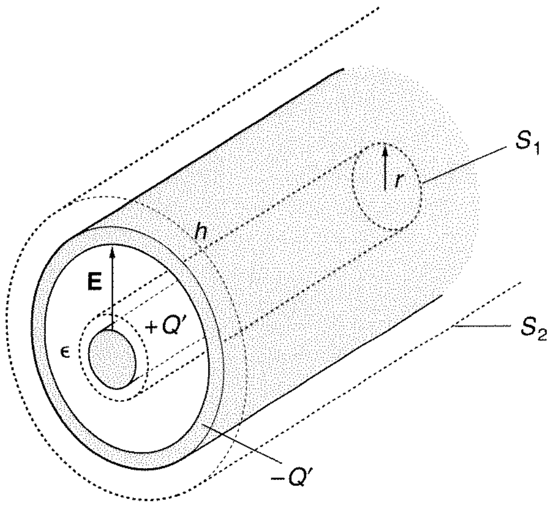


Figure 8.8 A coaxial cable with a dielectric of permittivity ϵ between the two conductors. The inner conductor is charged positively, and the outer negatively (connected to ground).

want to find the capacitance per unit length of the coax. To do this, we need to find the electric field inside the cable, because $(V_{Q'} - V_{-Q'}) = \int \mathbf{E} \cdot \mathrm{d}\mathbf{l}$ from the inner to the outer conductor.

For determining $\mathbf{E} = \mathbf{D}/\epsilon$, we use Gauss' law on the surface S_1, which is a cylinder of radius r and height h. The flux through the cylinder bases is zero, so

$$\oint_{\text{cylinder}} \mathbf{D} \cdot \mathrm{d}\mathbf{S} = D 2\pi r h = Q' h.$$

From here, we have

$$E = \frac{D}{\epsilon} = \frac{Q'}{2\pi \epsilon r}. \tag{8.11}$$

(Electric field at a distance r from a long line charge)

Note that the radius a does not come into this expression, so it is valid for any radius of the inner conductor.

The voltage between the two conductors is now obtained as follows:

$$(V_{Q'} - V_{-Q'}) = \int_a^b E\,\mathrm{d}r = \frac{Q'}{2\pi\epsilon}\int_a^b \frac{\mathrm{d}r}{r} = \frac{Q'}{2\pi\epsilon}\ln\frac{b}{a}.$$

The capacitance per unit length of a coax is thus

$$C' = \frac{Q'}{V_+ - V_-} = \frac{2\pi\epsilon}{\ln(b/a)}. \tag{8.12}$$

(Capacitance per unit length of a coax)

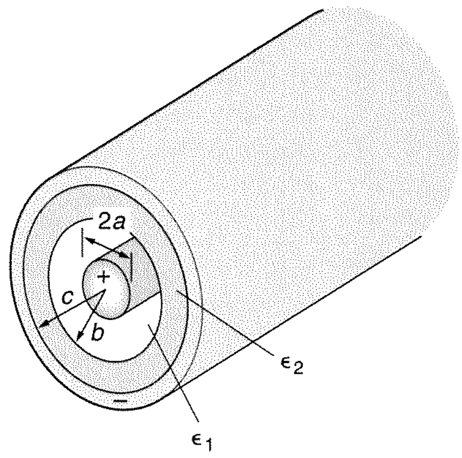


Figure 8.9 A high-voltage coaxial cable

We will see that all transmission lines are characterized by a so-called characteristic impedance. Coaxial cables are made most frequently with characteristic impedances of 50 Ω and 75 Ω. A typical value of the capacitance per unit length for a 50-Ω coaxial cable is about 100 pF/m, or 1 pF/cm.

As an example, a coaxial cable commonly used at high frequencies (e.g., in satellite receivers) is called RG-55/U and has the following parameters: $a = 0.5\,\mathrm{mm}$, $b = 2.95\,\mathrm{mm}$, $\epsilon_r = 2.25$. What is its capacitance per unit length?

Example 8.8—Capacitance of a high-voltage coaxial cable. In the expression for the electric field inside a coaxial cable, Eq. (8.11), we see that the electric field is the strongest right next to the inner conductor. In cables used for high-voltage applications, there is a danger of dielectric breakdown inside the coax. Therefore its inner conductor (where the field is the strongest) is frequently coated with a dielectric that has a high dielectric strength. The cross-section of a high-voltage coaxial cable is shown in Fig. 8.9. The electric field strength and displacement vectors are perpendicular to the boundary between the two dielectrics. Thus D in the two dielectrics is given by the same expression, $D = Q'/(2\pi r)$, and E in the two dielectrics is given by $E_1 = Q'/(2\pi\epsilon_1 r)$ and $E_2 = Q'/(2\pi\epsilon_2 r)$. The capacitance per unit length of this cable is

$$C' = \frac{Q'}{\int_a^b E_1\,\mathrm{d}r + \int_b^c E_2\,\mathrm{d}r} = \frac{C_1' C_2'}{C_1' + C_2'},$$

where $C_1' = 2\pi\epsilon_1/\ln(b/a)$ and $C_2' = 2\pi\epsilon_2/\ln(c/b)$.

Example 8.9—Capacitance of a MOS capacitor. The metal-oxide-semiconductor (MOS) capacitor is part of every metal oxide field-effect transistor (MOSFET), and millions of transistors are in every piece of electronic equipment. Figure 8.10 shows a MOS capacitor, which consists of a piece of n semiconductor with a layer of dielectric (usually silicon dioxide) and a metal electrode deposited on top of the oxide. Similarly to a pn diode, a depletion layer forms on the semiconductor side because the metal has many free electrons. The width of the depletion layer can be controlled by a voltage applied to the metal, and this effect is used in transistors, where the control electrode (gate) is essentially a MOS capacitor. However, due to

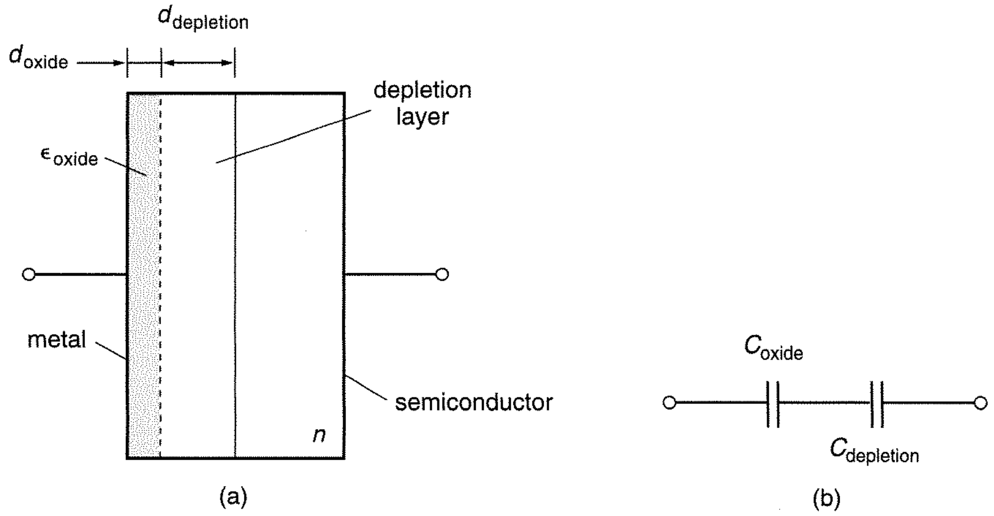


Figure 8.10 (a) A MOS capacitor, and (b) its equivalent series capacitor connection

the presence of the oxide, the current flowing through the capacitor is essentially zero, and this makes the input impedance of a transistor very large.

The capacitance of the oxide is in series with the capacitance of the depletion layer, and the total capacitance is given by

$$\frac{1}{C_{\text{MOS}}} = \frac{1}{C_{\text{oxide}}} + \frac{1}{C_{\text{depletion}}},$$

where $C_{\text{oxide}} = \epsilon_{\text{oxide}} S / d_{\text{oxide}}$. We have seen in Examples 7.2 and 7.3 that the depletion layer is a uniform volume charge and that the electric field inside it is a linear function of the x coordinate. Therefore, to find the capacitance we find the voltage by integrating the electric field from one end of the depletion layer to the other end. This is left as an exercise for the reader. Another useful exercise is to use superposition, as in Example 7.2, to find the electric field profile in a MOS capacitor.

Questions and problems: Q8.1 to Q8.13, P8.1 to P8.23

8.3 Electrostatic Coupling in Multibody Systems

So far, we have considered an isolated conducting body and two bodies with equal but opposite charges. In practical applications we often have more than one or two conducting bodies. A multiconductor transmission line (bus) for connecting different parts of a computer is an example. We now consider this more general case, an electrostatic system consisting of an arbitrary number of charged conducting bodies.

We can adopt the reference point arbitrarily. To enable the analysis to apply to infinite structures as well (e.g., parallel, infinitely long wires), let us adopt as the reference *one of the conductors*. (We know that all points of a conductor in electrostatics are equipotential, and therefore we can adopt the entire body as the reference, not just

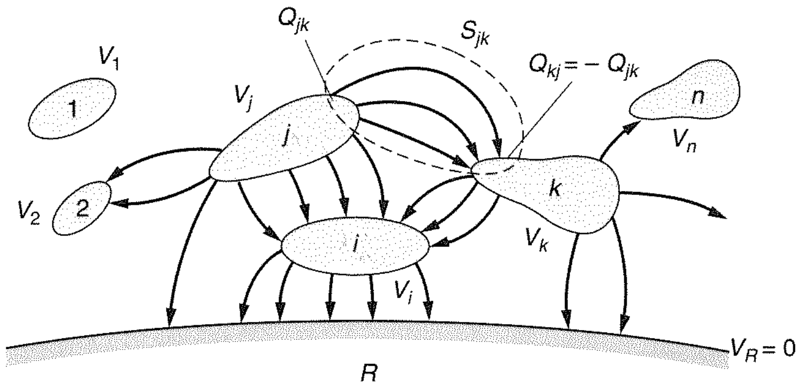


Figure 8.11 A system of n charged conducting bodies with the $(n+1)$th body, R, being the reference for potential

one of its points.) Most often, this reference body will be the earth or a convenient metallic part of the structure, such as the casing of an electronic device.

Consider a system of n charged bodies with charges $Q_1, Q_2, \ldots, Q_n$, in addition to the reference body (Fig. 8.11). Let the dielectric be linear (but it need not be homogeneous). Is there a relationship between the charges on the bodies and their potentials, $V_1, V_2, \ldots, V_n$?

Because the system is linear, the principle of superposition applies. The potential of any of the bodies is obtained as

$$V_i = a_{i1}Q_1 + a_{i2}Q_2 + \cdots + a_{in}Q_n, \qquad i = 1, 2, \ldots, n. \tag{8.13}$$

(Definition of coefficients of potential, a_{ij})

The coefficients a_{ij} are termed, logically, the *coefficients of potential*. Note that their unit is 1/farad.

Provided that we know the coefficients a_{ij}, Eqs. (8.13) represent, in fact, a system of n linear equations in n unknowns. These unknowns can be the charges Q_j of the bodies, but also their potentials, V_i. If the charges are known, Eqs. (8.13) represent the solution for the potentials. If the potentials are known, Eqs. (8.13) need to be solved for the charges, resulting in

$$Q_i = c_{i1}V_1 + c_{i2}V_2 + \cdots + c_{in}V_n, \qquad i = 1, 2, \ldots, n. \tag{8.14}$$

(Definition of coefficients of electrostatic induction, c_{ij})

The coefficients c_{ij} have several names. The most common is probably the *coefficients of electrostatic induction*. Their unit is the same as for capacitance, the farad.

Eqs. (8.14) can be rewritten in the following form:

$$\begin{aligned} Q_i = -c_{i1}(V_i - V_1) - c_{i2}(V_i - V_2) - \cdots + (c_{i1} + c_{i2} + \cdots + c_{in})V_i - \cdots \\ - c_{in}(V_i - V_n), \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{8.15}$$

Introducing new coefficients,

$$C_{ij} = -c_{ij} \text{ if } i \neq j, \text{ and } C_{ii} = c_{i1} + \cdots + c_{in}, \tag{8.16}$$

Eqs. (8.15) can be rewritten as

$$Q_i = C_{i1}(V_i - V_1) + C_{i2}(V_i - V_2) + \cdots + C_{ii}V_i + \cdots + C_{in}(V_i - V_n), \; i = 1, 2, \ldots, n. \quad (8.17)$$

(Definition of coefficients of capacitance, C_{ij})

The coefficients C_{ij} are known as the *coefficients of capacitance*. Their unit is also the farad.

If we know any of the three sets of coefficients, a_{ij}, c_{ij}, or C_{ij}, for a given system of conducting bodies, we can calculate mutual electrostatic effects in diverse circumstances. For example, we can assume that a body, instead of being at a desired potential, is unexpectedly grounded (at potential zero) and calculate the consequences of such an event. As another example, we can analyze the relative charge per unit length that one conductor of a multiconductor transmission line induces on the others, for given potentials of all the conductors. This, in fact, is an analysis of *electrostatic coupling*.

The only problem that needs to be solved is to determine in any way (analytically or experimentally) all the coefficients of one of the three sets, since they are derivable from each other. Let us explain, for example, how we can obtain the a_{ij} coefficients. It is left as an exercise for the reader to imagine how to obtain in principle the coefficients c_{ij} and C_{ij}.

Theoretically, we can find the coefficients a_{ij} as follows. Assume that all the bodies except body i are discharged, and that the charge on the i-th body is Q_i. Eqs. (8.13) then show that if we can measure the potentials V_j, $j = 1, 2, \ldots, n$ of the n bodies, we can calculate n potential coefficients a_{ij}. We repeat this procedure for all the n bodies and obtain the complete set of the a_{ij} coefficients.

Example 8.10—Electrostatic coupling between a two-wire line and a parallel grounded wire. Figure 8.12 shows a two-wire line at a height d above ground. A wire at potential zero is parallel to the line, and all wires are in the same plane. Let conductors 1 and 2 of the line be charged with charges Q' and $-Q'$, as indicated. Let the reference plane for potential be at the ground (Fig. 8.12). We wish to determine the charge per unit length induced on the grounded wire, Q'_1. It can be determined from the condition that the potential of the wire due to the three line charges, Q', $-Q'$, and Q'_1, is zero.

The potential of a line charge is given by Eq. (6.9). Note that the distance of the three line charges from the reference plane for potential in this case is the same, equal to d. The poten-

$V_3 = a_{31}Q' - a_{32}Q' + a_{33}Q'_1 = 0$

$V_1 =$

$\frac{1}{2\pi\varepsilon_0} \ln \frac{\sqrt{2}2d}{2d}$

$\ln \frac{R}{2d}$

$-\frac{R}{2\sqrt{2}d}$

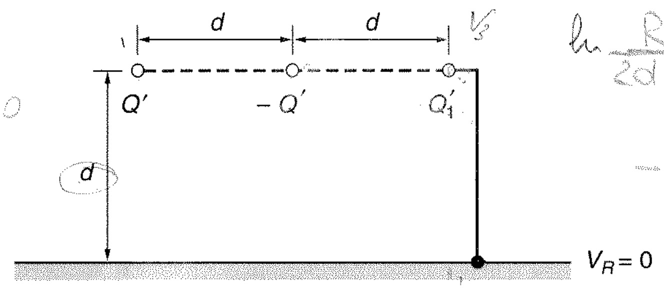


Figure 8.12 A two-wire line and a grounded wire running parallel to it

tial of the grounded wire is due to all three charges. Since it is zero, we obtain the following equation for the unknown charge per unit length Q_1' of the grounded wire:

$$V_{\text{grounded wire}} = \frac{Q'}{2\pi\epsilon_0}\ln\frac{d/2}{2d} - \frac{Q'}{2\pi\epsilon_0}\ln\frac{d/2}{d} + \frac{Q_1'}{2\pi\epsilon_0}\ln\frac{3d/2}{a} = 0, \tag{8.18}$$

from which we easily find Q_1'.

If charges Q' and $-Q'$ are time-varying, the induced charge on the wire will also be time-varying. This means that we would have an induced time-varying current in the wire due to electric coupling with the two-wire line. Such electric coupling is present in every multiconductor cable (such as a computer bus), between phone lines and power lines, and so on.

Questions and problems: Q8.14 to Q8.16, P8.24 to P8.27

8.4 Chapter Summary

1. Capacitance can be defined only under certain conditions. In the case of a single body, the condition is that it should be far from other bodies. For a two-body problem (the capacitor), the bodies should have equal but opposite charges, and the field of these charges should be restricted to the domain of the capacitor. Only in these circumstances are the familiar formulas for the capacitance of parallel and series connections of capacitors valid.
2. The ideal capacitor is an example of perfect electrostatic coupling between two bodies (by definition, there is no field outside the capacitor).
3. In a multibody system, mutual electrostatic coupling can be analyzed by means of any of three sets of coefficients, known as the coefficients of potential, coefficients of electrostatic induction, and coefficients of capacitance. These coefficients can (at least in principle) always be measured, but in many cases they can be calculated by numerical methods.

QUESTIONS

Q8.1. A conducting body is situated in a vacuum. Prove that the potential of the body is proportional to its charge.

Q8.2. Repeat the preceding question if the body is situated in a linear (1) homogeneous or (2) inhomogeneous dielectric.

Q8.3. Two conducting bodies with charges Q and $-Q$ are situated in a homogeneous linear dielectric. Prove that the potential difference between them is proportional to Q. Does the conclusion remain true if the dielectric is inhomogeneous (but still linear)?

Q8.4. Two conducting bodies with charges Q and $-Q$ are situated in a homogeneous, but nonlinear, dielectric. Is the potential difference between them proportional to Q?

Q8.5. The capacitance of a diode is a function of the voltage between its terminals. Is this a linear or nonlinear capacitor?

Q8.6. Prove in your own words that a parallel connection of capacitors is indeed just a single unconventional capacitor.

Q8.7. Four metal spheres of radii R are centered at corners of a square of side length $a = 3R$. Two pairs of the spheres are considered to be the electrodes of two capacitors, and are connected "in series." Is it possible to calculate the equivalent capacitance exactly using Eq. (8.7)? Explain.

Q8.8. A parallel-plate capacitor is connected to a source of voltage V. A dielectric slab is periodically introduced between the capacitor electrodes and taken out. Explain what happens with the capacitor charge.

Q8.9. Explain in your own words why the capacitance of a capacitor filled with a dielectric is larger than the capacitance of the same capacitor without the dielectric.

Q8.10. A negligibly thin metal foil is introduced between the plates of a parallel-plate capacitor, parallel to the plates. Is there any change in the capacitor capacitance? Can it be regarded as a series connection of two capacitors? Explain.

Q8.11. Repeat question Q8.10 assuming that the foil is not parallel to the plates.

Q8.12. Repeat question Q8.10 assuming the foil is thick.

Q8.13. A metal foil of thickness a is introduced between and parallel to the plates of a parallel-plate capacitor that are a distance d ($d > a$) apart. If the area of the foil and the capacitor plates is S, what is the capacitance of the capacitor without, and with, the foil?

Q8.14. Describe the procedure for measuring the coefficients of potential, a_{ij}, in Eq. (8.13).

Q8.15. Describe the procedure for measuring the coefficients of electrostatic induction, c_{ij}, in Eq. (8.14).

Q8.16. Describe the procedure for measuring the coefficients of capacitance, C_{ij}, in Eq. (8.17).

PROBLEMS

P8.1. Two large parallel metal plates of areas S are a distance d apart, have equal charges of opposite sign, Q and $-Q$, and the dielectric between the plates is homogeneous. Using Gauss' law, prove that the field between the plates is uniform. Calculate the capacitance of the capacitor per unit area of the plates.

P8.2. The permittivity between the plates of a parallel-plate capacitor varies as $\epsilon(x) = \epsilon_0(2 + x/d)$, where x is the distance from one of the plates, and d the distance between the plates. If the area of the plates is S, calculate the capacitance of the capacitor. Determine the volume and surface polarization charges if the plate at $x = 0$ is charged with a charge Q ($Q > 0$), and the other with $-Q$.

P8.3. A parallel-plate capacitor with plates of area $S = 100\,\text{cm}^2$ has a two-layer dielectric, as in Fig. 8.5. One layer, of thickness $d_1 = 1\,\text{cm}$, has a relative permittivity $\epsilon_r = 3$, and a dielectric strength five times that of air. The other layer is air, of thickness $d_0 = 0.5\,\text{cm}$. How large a voltage will produce breakdown of the air layer, and how large does the voltage need to be to cause breakdown of the entire capacitor?

P8.4. A capacitor with an air dielectric was connected briefly to a source of voltage V. After the source was disconnected, the capacitor was filled with transformer oil. Evaluate the new voltage between the capacitor terminals.

P8.5. A capacitor of capacitance C, with a liquid dielectric of relative permittivity ϵ_r, is connected to a source of voltage V. The source is then disconnected and the dielectric drained from the capacitor. Determine the new voltage between the capacitor electrodes.

P8.6. Two conducting bodies with charges Q and $-Q$ are situated in a linear, but inhomogeneous, dielectric. Prove that the potential difference between them is proportional to the charge Q.

P8.7. A parallel-plate capacitor has plates of area S and a dielectric consisting of n layers as in Fig. 8.5, with permittivities $\epsilon_1, \ldots, \epsilon_n$, and thicknesses $d_1, \ldots, d_n$. Evaluate the capacitance of the capacitor.

P8.8. Repeat problem P8.7 assuming that the layers are normal to the capacitor plates and that each layer takes the same amount of the capacitor plate area.

P8.9. Evaluate the maximal capacitance of the capacitor sketched in Fig. 8.6a if the plates are semicircular, of radius R, and the distance between adjacent plates is d. The dielectric is air.

P8.10. Evaluate the capacitance of the capacitor in Fig. 8.6b if the dielectric and aluminum ribbons are $a = 5\,\text{cm}$ wide, $b = 2\,\text{m}$ long, and $d = 0.1\,\text{mm}$ thick. Assume the dielectric has a relative permittivity $\epsilon_r = 2.7$.

P8.11. Determine the polarization charges on all surfaces in Fig. 8.5.

P8.12. Determine the polarization charges on all dielectric surfaces in Fig. 8.9. Are there volume polarization charges anywhere? If so, where?

P8.13. One of two long, straight parallel wires is charged with a charge Q' per unit length, and the other with $-Q'$. The wires have radii a and are d ($d \gg a$) apart. (1) Find the expression for the voltage between the wires and the capacitance per unit length of the line. Plot the magnitude of the electric field in a cross section of this two-wire line along the straight line joining the two wires. (2) At which points is it likely that the surrounding air will break down and ionize, given that a high-voltage generator is connected to the two wires? (3) If the wire radius is $a = 0.5\,\text{mm}$, and the wires are $d = 1$ cm apart, how large is the voltage of a voltage generator connected to the wires if the air at the wire surfaces breaks down?

P8.14. A spherical capacitor with two dielectrics is shown in Fig. P8.14. The inner radius is a, the outer radius is b, and the outer radius of the shell is c. The inner sphere is charged with Q ($Q > 0$), and the outer shell with $-Q$. (1) Find the expression for the electric field everywhere and present your result graphically. (2) Find the expression for the

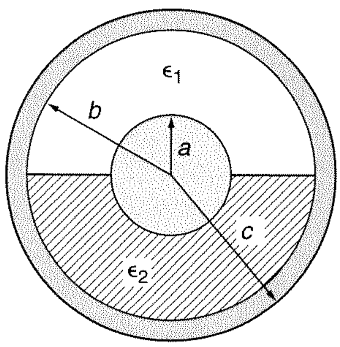


Figure P8.14 A two-dielectric spherical capacitor capacitance of the capacitor. (3) If the outer shell is made to be much larger than the inner shell, what does the capacitance become and what does this mean physically?

P8.15. Two flat parallel conductive plates of surfaces $S = 0.05\,\mathrm{m}^2$ are charged with $Q_1 = 5 \cdot 10^{-8}\,\mathrm{C}$ and $Q_2 = -Q_1$. The distance between the plates is $D = 1\,\mathrm{cm}$. Find the electric field strength vector at all points if a third, uncharged metal plate, $d = 5\,\mathrm{mm}$ thick, is placed between the two plates $a = 2\,\mathrm{mm}$ away from one of the charged plates and parallel to it. Plot the electric field strength before and after the third plate is inserted. Compare and explain. Find the capacitance between the charged plates without and with the third plate between them.

P8.16. The dielectric in a parallel-plate capacitor of plate area $S = 100\,\mathrm{cm}^2$ consists of three parallel layers of relative permittivities $\epsilon_{1r} = 2$, $\epsilon_{2r} = 3$, and $\epsilon_{3r} = 4$. All three layers are $d = 1\,\mathrm{mm}$ thick. The capacitor is connected to a voltage $V = 100\,\mathrm{V}$. (1) Find the capacitance of the capacitor. (2) Find the magnitude of the vectors $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$ in all dielectrics. (3) Find the free and polarization charge densities on all boundary surfaces.

P8.17. The surface area of each plate of a parallel-plate capacitor is $S = 100\,\mathrm{cm}^2$, the distance between the plates is $d = 1\,\mathrm{mm}$, and it is filled with a liquid dielectric of unknown permittivity. In order to measure the permittivity, we connect the capacitor to a source of voltage $V = 200\,\mathrm{V}$. When the capacitor is connected to the source, it charges up, and the amount of charge is measured as $Q = 5.23 \cdot 10^{-8}\,\mathrm{C}$ (the instrument that can measure this is called a ballistic galvanometer). Find the relative permittivity of the liquid dielectric.

P8.18. We wish to make a coaxial cable that has an electric field of constant magnitude. How does the relative permittivity of the dielectric inside the coaxial cable need to change as a function of radial distance in order to achieve this? The radius of the inner conductor is a and the value of the relative permittivity right next to the inner conductor is $\epsilon_r(a)$. Find the capacitance per unit length of this cable.

P8.19. A capacitor in the form of rolled metal and insulator foils, Fig. 8.6b, needs to have a capacitance of $C = 10\,\mathrm{nF}$. Aluminum and oily paper foils $a = 3\,\mathrm{cm}$ wide are available. The thickness of the paper is $d = 0.05\,\mathrm{mm}$, and its relative permittivity is $\epsilon_r = 3.5$. The thickness of the aluminum foil is also $0.05\,\mathrm{mm}$. Find the needed length of the foil strips, as well as the maximum voltage to which such a capacitor can be connected. (Note that when rolled, the capacitance of the capacitor is twice that when the strips are not rolled.)

P8.20. A coaxial cable has two dielectric layers with relative permittivities $\epsilon_{1r} = 2.5$ and $\epsilon_{2r} = 4$. The inner conductor radius is $a = 5\,\mathrm{mm}$, and the inner radius of the outer conductor is $b = 25\,\mathrm{mm}$. (1) Find how the dielectrics need to be placed and how thick they need to be so that the maximum electric field strength will be the same in both layers. (2) What is the capacitance per unit length of the cable in this case? (3) What is the largest voltage that the cable can be connected to if the dielectrics have a breakdown field of $200\,\mathrm{kV/cm}$?

P8.21. Figure P8.21 shows what is known as a capacitor bushing, which is used to insulate a high-potential conductor A at its passage through the grounded wall W. The shaded surfaces represent thin dielectric sheets of permittivity ϵ, and the thicker lines represent conducting foils placed between these sheets. Referring to Fig. P8.21, prove that the electric field intensity throughout the bushing is approximately the same, provided that $a_1 d_1 = a_2 d_2 = \ldots = a_4 d_4$.

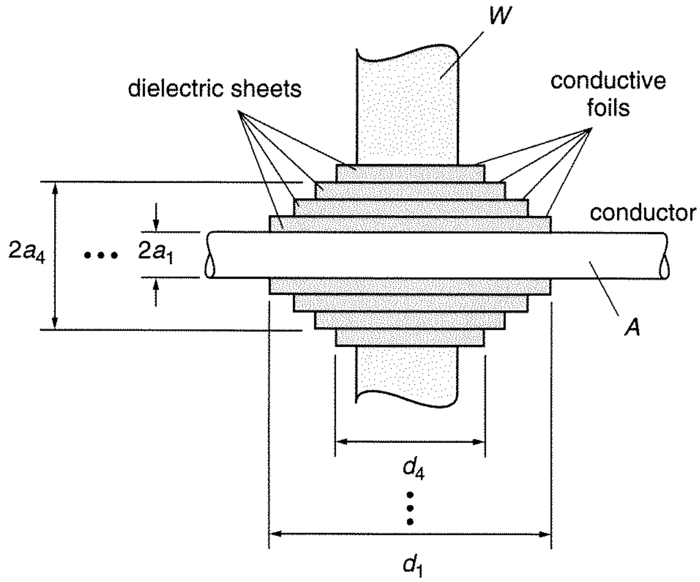


Figure P8.21 A capacitor bushing

***P8.22.** Find the capacitance of a *pn* diode with a linear charge gradient, i.e., when the charge distribution on the *p* and *n* sides is as shown in Fig. P8.22. As in Example 7.2, you can assume that the charge on one side is much denser than that on the other side, and can therefore be assumed to be a charge sheet.

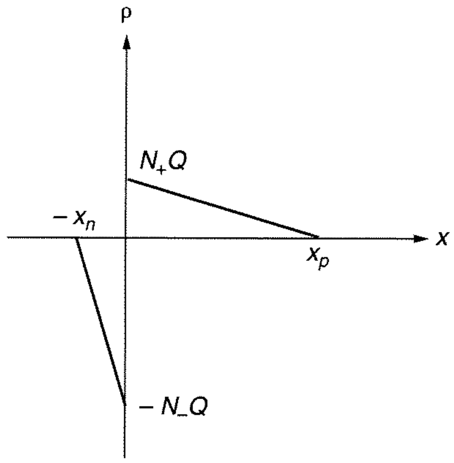


Figure P8.22 Linear charge profile in *pn* diode

P8.23. Plot the capacitance of a varactor diode as a function of the voltage across the diode. The capacitance of this diode is nonlinear and can be approximated with the following function of the voltage across the diode:

$$C_d(V) = \frac{C_0}{\sqrt{1+(V/V_d)}}, \tag{8.19}$$

where C_0 is the built-in capacitance (given) and V_d is the built-in voltage of the diode (given). (This diode is used as an electrically variable capacitor because its capacitance can change significantly with applied voltage.)

P8.24. Find the expression for the capacitance C_{12} between two bodies in terms of the coefficients of potential a_{ij} defined by Eqs. (8.13). The two bodies have potentials V_1 and V_2, and the reference potential is the ground potential, as in Fig. P8.24.

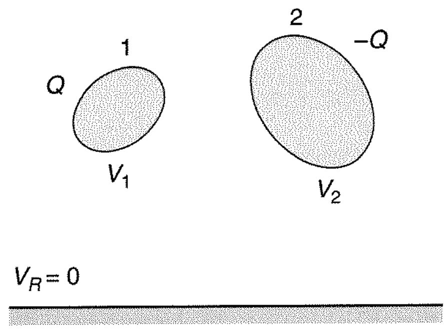


Figure P8.24 A capacitor above ground

P8.25. A two-wire line with charges Q' and $-Q'$ runs parallel to the ground, with the two wires at different heights. The positively charged wire is h_1 above ground, and the negatively charged wire is h_2 above ground. The radii of both wires are a. Find the capacitance per unit length of such a line directly (from the definition of capacitance, and making use of images), and via the coefficients of potential defined by Eqs. (8.13), as follows:

(a) Assuming the earth is at zero potential, that the left wire is charged with Q', and that the other is uncharged, find the potential of both wires.

(b) Repeat part a if the right wire is charged with $-Q'$ and the left wire is uncharged.

(c) From the preceding and Eqs. (8.13), write down the expressions for the coefficients of potential a_{ij} of the system.

(d) Find the capacitance per unit length of the line.

***P8.26.** Prove that Eqs. (8.15) follow from Eqs. (8.14).

***P8.27.** Prove that from Eqs. (8.16) it follows that $c_{ij} = -C_{ij}$, and $c_{ii} = C_{i1} + \cdots + C_{in}$.

9

Energy, Forces, and Pressure in the Electrostatic Field

9.1 Introduction

Measured by average human standards, electric energy, forces, and pressures are small. For example, it is virtually impossible to have an electric force of magnitude greater than a few newtons, or electric systems with energy exceeding a few thousand joules. Nevertheless, electric forces have surprisingly wide engineering applications. For example, purification of some ores, extraction of solid particles from smoke or dusty air, spreading of the toner in xerographic copying machines, and efficient and economical painting of car bodies are all based on electric forces.

Electrostatic energy is of equal engineering importance. For example, sufficient energy to destroy virtually any semiconductor device can easily be created in the field of a person charged by walking on a carpet. This is the meaning of the commonly used warning "static sensitive."

Questions and problems: Q9.1

9.2 Energy of a Charged Capacitor

In the preceding chapter, we defined *capacitance* and described and analyzed several types of capacitors. It is easy to understand that every charged capacitor contains a certain amount of energy. For example, the plates of a charged parallel-plate capacitor attract each other. If we let them move, they will perform a certain amount of work. In order for a system to do work, it must contain energy. Since the capacitor plates do not attract each other if they are not charged, it follows that some energy is stored in a *charged* capacitor. We can find how much energy there is by looking at what happens while a capacitor is being charged.

Consider a capacitor of capacitance C that is initially not charged. We wish to charge its electrodes with Q and $-Q$. To do this, we take small positive charges $\mathrm{d}q$ from the negative electrode and take them over to the positive electrode. To move the charge against the electric forces ($\mathrm{d}q$ is attracted by the negative electrode, and repelled by the positive electrode), we must do some work. Suppose that, at an instant during this process, the capacitor electrodes are charged with charges q and $-q$ ($0 < q \leq Q$). This means that the potential difference between them is $v = q/C$. By definition of the potential difference between the electrodes, the work we have to do against the electric forces in moving the next $\mathrm{d}q$ from the negative to the positive electrode equals $\mathrm{d}A = v\,\mathrm{d}q = q\,\mathrm{d}q/C$. So the total work that needs to be done to charge the capacitor electrodes with the desired charges, Q and $-Q$, is

$$A = \int_0^Q \frac{q}{C}\,\mathrm{d}q = \frac{Q^2}{2C} \qquad (\mathrm{J}). \tag{9.1}$$

Since there were no losses in charging the capacitor, this work was transformed into potential energy of the capacitor. This energy we call the *electric energy*. Noting that $Q = CV$, the electric energy of a charged capacitor is thus given by the following equivalent expressions:

$$W_e = \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2 \qquad (\mathrm{J}). \tag{9.2}$$

(Energy of a charged capacitor)

Let us look at a few examples. The largest possible energy of an air-filled parallel-plate capacitor with plate area $S = 1\,\mathrm{dm}^2$ and with a distance between plates of $d = 1$ cm is

$$(W_e)_{\max} = \frac{1}{2}\epsilon_0\frac{S}{d}E_{\max}^2 d^2 = \frac{1}{2}\epsilon_0 E_{\max}^2 Sd \simeq 4\,\mathrm{mJ}, \tag{9.3}$$

since $E_{\max} \simeq 30\,\mathrm{kV/cm}$. (The maximum energy corresponds to the maximum voltage, i.e., to the maximum electric field.) This is not very much energy from a human viewpoint (although it can destroy practically any semiconductor device).

If we consider a high-voltage capacitor, for example one where $V = 10\,\mathrm{kV}$ and $C = 1\,\mu\mathrm{F}$, we obtain instead

$$W_e = \frac{1}{2}CV^2 = 50\,\mathrm{J}. \tag{9.4}$$

This is roughly equivalent to the potential energy of a 1-kg coconut that is 5 m above ground. The energy of high-voltage capacitors is clearly quite large, and touching their electrodes can be fatal.

Questions and problems: Q9.2 to Q9.8, P9.1 to P9.4

9.3 Energy Density in the Electrostatic Field

The expression for the energy of a parallel-plate capacitor can be rewritten as

$$W_e = \frac{1}{2}CV^2 = \frac{1}{2}\epsilon\frac{S}{d}V^2 = \frac{1}{2}\epsilon E^2 Sd, \tag{9.5}$$

since $V/d = E$. The product Sd is equal to the volume of the capacitor dielectric (i.e., the volume of the domain with the field). Therefore, no error will be made in computing the capacitor energy if we assume that it is distributed in the *entire* field, with a density

$$w_e = \frac{W_e}{v} = \frac{1}{2}\epsilon E^2. \tag{9.6}$$

(Energy density, J/m^3, in an electrostatic field)

We will now show that this result is valid in general and not just for a parallel-plate capacitor. Let us look at a system of charged bodies in an arbitrary dielectric, as shown in Fig. 9.1. When we place thin aluminum foil exactly over an equipotential surface, we do not change the electric field. This is because we place a conducting surface, which must be equipotential, on an equipotential surface.

We can therefore place many thin aluminum foils on many equipotential surfaces, very close to each other, without changing the field. However, in this way we have divided up the space around the charged bodies into a very large number of small parallel-plate capacitors. The total energy of this system is given by the sum of all the little capacitor energies. The energy density of each of the capacitors is equal to $w_e = \epsilon E^2/2$, where E is the electric field at that point, and ϵ the permittivity at that point. Consequently the energy of the whole system is given by

$$W_e = \int_v \frac{1}{2}\epsilon E^2 \mathrm{d}v. \tag{9.7}$$

(Energy of an electric field)

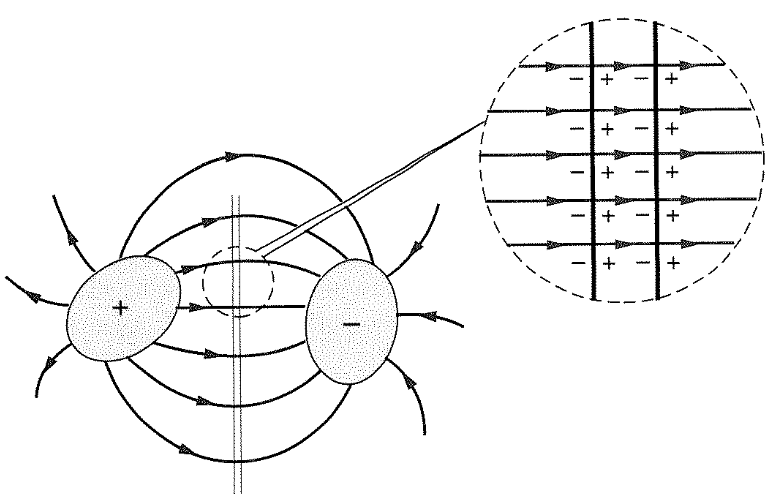


Figure 9.1 The electric field does not change when two aluminum foils are placed exactly at two close equipotential surfaces. Charges are induced on the surface of the foils (as shown in the enlarged circle), and the field between the foils is approximately uniform.

The integral in the equation is a volume integral *over the entire volume in which the electric field exists.*

Example 9.1—Energy of a high-voltage coaxial cable. In Example 8.8, we saw that a high-voltage coaxial cable consists of two dielectric layers and that the electric field in the two layers is given by

$$E_1 = \frac{Q'}{2\pi\epsilon_1 r} \qquad a < r < b$$

$$E_2 = \frac{Q'}{2\pi\epsilon_2 r} \qquad b < r < c.$$

The energy per unit length of the cable is the sum of the energies contained in the two dielectric layers:

$$W'_e = \int_{\text{layer1}} \frac{1}{2}\epsilon_1 E_1^2 \mathrm{d}v' + \int_{\text{layer2}} \frac{1}{2}\epsilon_1 E_2^2 \mathrm{d}v'.$$

Now $\mathrm{d}v' = 2\pi r\mathrm{d}r$, and E_1 and E_2 are given by the expressions at the beginning of the example. With respect to r, the first integral has limits from a to b, and the second one from b to c. After integrating, we get

$$W'_e = \frac{Q'^2}{2}\left[\frac{\ln(b/a)}{2\pi\epsilon_1} + \frac{\ln(c/b)}{2\pi\epsilon_2}\right].$$

If we use $W'_e = \cancel{Q^2/2}C$, we get the same expression.

$= (Q')^2/2C$

We have concluded that energy contained in an electrostatic system can be determined if we assume that it is distributed throughout the field, with a density given in Eq. (9.6), even if the dielectric is a vacuum. In the case of dielectrics in the field, obviously at least some of the energy must be stored throughout the dielectric: to polarize the dielectric, the electric field needs to do some work *at the very point* where a dielectric molecule is, and *this molecule* acquires some energy. This means that the energy used to polarize a dielectric is distributed throughout the dielectric, just like the energy used in stretching a spring is distributed inside the entire spring.

In the case of a vacuum, however, such a physical explanation does not exist. How can we then state that the field in a vacuum also contains energy? In electrostatics, such a proof is not possible, but we shall see that in time-varying fields, the field *does* have energy distributed in a vacuum. For example, we know that a radio wave, which is but a combination of electric and magnetic fields, is able to carry a signal from the earth to Jupiter and back. This is a vast distance, and for a significant time the signal is neither on earth nor on Jupiter. It travels through a vacuum in between. It certainly carries some energy during this travel, because we are able to detect it.

Questions and problems: Q9.9 to Q9.11, P9.5 to P9.12

9.4 Forces in Electrostatics

We started discussing electrostatics with Coulomb's law for the electric force between two point charges. Because the principle of superposition applies, it can be used as a basis for determining the electric force on any body in a system where we know the distribution of charges.

As an example, consider the two charged conducting bodies shown in Fig. 9.2, with a known surface charge distribution. Let us find the expression for the force $\mathbf{F}_{12}$ with which body 1 acts on body 2. To find this force, we divide body 2 into small patches dS_2 and determine the electric field strength $\mathbf{E}_1$ at all these patches due to the charge on body 1. The force is then obtained as

$$\mathbf{F}_{12} = \oint_{S_2} \sigma_2 \, dS_2 \mathbf{E}_1. \tag{9.8}$$

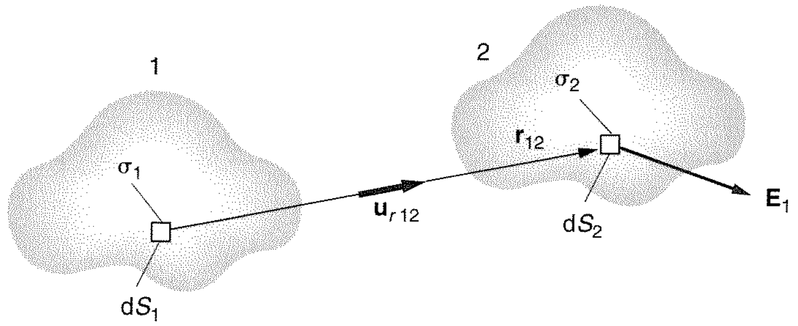


Figure 9.2 Finding the electric force between two large charged conducting bodies

In this equation, the field $\mathbf{E}_1$ is given by

$$\mathbf{E}_1 = \oint_{S_1} \frac{1}{4\pi\epsilon_0} \frac{\sigma_1 \, \mathrm{d}S_1}{r_{12}^2} \mathbf{u}_{r12}, \tag{9.9}$$

where $\mathbf{r}_{12}$ is the vector directed from an element of body 1 toward an element of body 2, and $\mathbf{u}_{r12}$ is the unit vector along this direction.

Example 9.2—Force between the plates of a parallel-plate capacitor. Let us find the electric force that the electrodes of a parallel-plate capacitor of plate area S exert on each other. We know that the charge is distributed practically uniformly on the electrode surface, i.e., the charge distribution is known. Let the capacitor be connected to a source of voltage V. The charge on the positive plate is then $Q = CV = \epsilon_0 SV/d$. We have found by Gauss' law that the electric field strength of the charge on the positive plate at the negative plate is $E_Q = Q/(2\epsilon_0 S)$. So using Eq. (9.8) we have

$$\mathbf{F}_{12} = \int_S (-\sigma) \mathrm{d}S \mathbf{E}_Q = -Q\mathbf{E}_Q.$$

The force is attractive, as it should be, and its intensity is given by

$$F_{12} = QE_Q = \frac{Q^2}{2\epsilon_0 S} = \frac{1}{2}\epsilon_0 S \frac{V^2}{d^2}.$$

Example 9.3—Magnitude of electric force in some typical devices. What is the maximal electric force in a parallel-plate capacitor filled with air, with $S = 1\ \mathrm{dm}^2$? The air breakdown field is $V/d \simeq 30\,\mathrm{kV/cm}$, so we obtain $F_{12} \simeq 0.4\,\mathrm{N}$, which amounts to the weight of about one quarter of a glass of water. Note that this is the *largest possible force.*

Another example is the force between the two wires of a two-wire line connected to a source of voltage V. The charge per unit length on the wires is $Q' = C'V = \pi\epsilon_0 V/\ln(d/a)$. At the place of the negatively charged wire, the positively charged wire produces a field $E_{Q'}$ equal to

$$E_{Q'} = \frac{Q'}{2\pi\epsilon_0 d}.$$

The force per unit length on the negatively charged wire is then

$$F'_{12} = -Q'E_{Q'} = -\frac{Q'^2}{2\pi\epsilon_0 d} = -\frac{\pi\epsilon_0 V^2}{2d[\ln(d/a)^2]}.$$

The minus sign tells us that the force is attractive, which it should be. Its maximal value for $a = 2.5$ mm, $d = 1$ m, and $E_{\max} = 30\,\mathrm{kV/cm}$ is $F'_{12} \simeq 0.00313\,\mathrm{N/m}$. This is again quite a small force.

The two preceding examples illustrate the statement in the chapter introduction that in normal circumstances, electric forces acting between charged bodies are very small. Therefore, they can be neglected most of the time. There are nevertheless many applications of the electrostatic forces, as will be discussed in Chapter 11.

Questions and problems: Q9.12 to Q9.19, P9.13 to P9.16

9.5 Determination of Electrostatic Forces from Energy

We saw that we can find electric forces between charged bodies only if we know the charge distribution on them, which is rarely the case. Moreover, the previously discussed method cannot be used to determine forces on polarized bodies except in a few simple cases.

For example, suppose that a parallel-plate capacitor is partially dipped in a liquid dielectric, as in Fig. 9.3a. If the capacitor is charged, polarization charges exist only on the two vertical sides of the dielectric inside the capacitor. The electric force acting on them has only a horizontal component, if any. Yet experiment tells us that when we charge the capacitor, *there is a small but noticeable rise in the dielectric level between the plates*. How can we explain this phenomenon?

The answer lies in what happens not at the top of the dielectric but near the bottom edge of the capacitor. In that region, the dipoles in the dielectric orient themselves as shown in Fig. 9.3b. The net force on the dipoles points essentially upward and pushes the dielectric up between the plates. Although we can explain the nature of this force, based on what we have learned so far we have no idea how to calculate it. The method described next enables us to determine the electric forces in this and many other cases where the direct method fails. In addition, conceptually the same method is used for the more important determination of magnetic forces in practical applications.

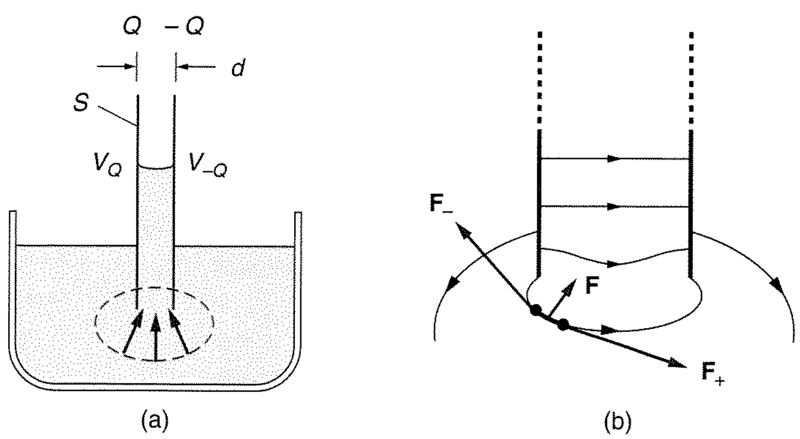


Figure 9.3 (a) When a parallel-plate capacitor dipped in a liquid dielectric is charged, the level between the plates rises due to electric forces acting on dipoles in the dielectric in the region around the edge of the capacitor, where the field is not uniform. (b) Enlarged domain of the capacitor fringing field in the dielectric, indicating the force on a dipole in a nonuniform field.

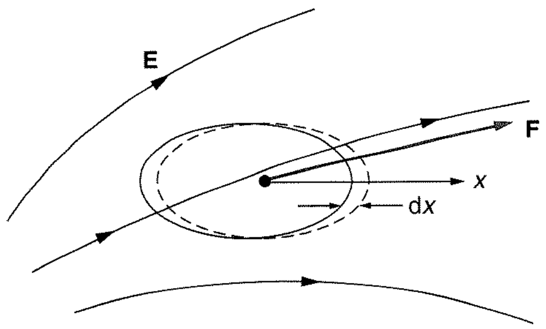


Figure 9.4 A body in an electrostatic system moved a small distance dx by the electric force

Consider an arbitrary electrostatic system consisting of a number of charged conducting and polarized dielectric bodies. We know that there are forces acting on all these bodies. Let us concentrate on one of the bodies, for example the one in Fig. 9.4, that may be either a conductor or a dielectric. Let the *unknown* electric force on the body be **F**, as indicated in the figure.

Suppose we let the electric force move the body by a small distance dx in the direction of the x axis indicated in the figure. The electric force would in this case do work equal to

$$dA_{\text{el.force}} = F_x\, dx, \tag{9.10}$$

where F_x is the projection of the force **F** on the x axis.

At first glance we seem to have gained nothing by this discussion: we do not know the force **F**, so we do not know the work $dA_{\text{el.force}}$ either. However, we will now show that if we know how the electric energy of the system depends on the coordinate x, we can determine the work $dA_{\text{el.force}}$, and then from Eq. (9.10), the component F_x of the force **F**. In this process, either (1) the charges on all the bodies of the system can remain unchanged or (2) the potentials of all the conducting bodies can remain unchanged.

Let us consider case (1) first. The charges can remain unchanged in spite of the change in the system geometry only if *none of the conducting bodies is connected to a source that could change its charge* (for example, a battery). Therefore, by conservation of energy, the work in moving the body can be done only at the expense of the electric energy contained in the system.

Let the system energy as a function of the coordinate x of the body, $W_e(x)$, be known. The increment in energy after the displacement, $dW_e(x)$, is negative because some of the energy has been used for doing the work. Since work has to be a positive number, we have in this case $dA_{\text{el.force}} = -dW_e(x)$. Combining this expression with Eq. (9.10), the component F_x of the electric force on the body is

$$F_x = -\frac{dW_e(x)}{dx} \qquad \text{(charges kept constant).} \tag{9.11}$$

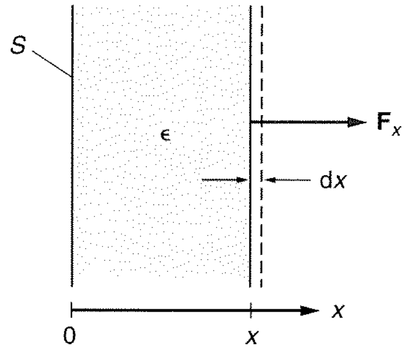


Figure 9.5 Determination of the force on the electrodes of a parallel-plate capacitor using Eq. (9.11)

Example 9.4—Force acting on one plate of a parallel-plate capacitor. In this example, we will find the electric force acting on one plate of a parallel-plate capacitor. The dielectric is homogeneous, of permittivity ϵ, the area of the plates is S, and the distance between them is x. One plate is charged with Q and the other with $-Q$ (Fig. 9.5). Let the electric force move the right plate by a small distance dx. The energy in the capacitor is given by $W_e(x) = Q^2/2C(x) = Q^2x/(2\epsilon S)$, so the force that tends to *increase* the distance between the plates is

$$F_x = -\frac{\mathrm{d}W_e(x)}{\mathrm{d}x} = -\frac{Q^2}{2\epsilon S}.$$

This is the same result as in Example 9.2, except for the sign. The minus sign tells us that the force tends to *decrease* the coordinate x, i.e., that it is attractive.

Example 9.5—Force per unit length acting on a conductor of a two-wire line. The wires of a two-wire line of radii a are x apart, and are charged with charges Q' and $-Q'$. The energy per unit length of the line is

$$W_e'(x) = \frac{Q'^2}{2C'} = \frac{Q'^2}{2\pi\epsilon_0}\ln\frac{x}{a},$$

using C' as calculated in problem P8.13. From Eq. (9.11) we obtain the force per unit length on the right conductor, tending to *increase* the distance between them, as

$$F_x = -\frac{\mathrm{d}W_e}{\mathrm{d}x} = -\frac{Q'^2}{2\pi\epsilon_0 x}$$

This is the same as in Example 9.3, except for the minus sign. We know that this means only that the force tends to *decrease* the distance x between the wires, i.e., that it is attractive.

Example 9.6—Force acting on a dielectric partly inserted into a parallel-plate capacitor. Let us find the electric force acting on the dielectric in Fig. 9.6. Equation (9.11) allows us to do this in a simple way. The capacitance of a capacitor such as this one is given by

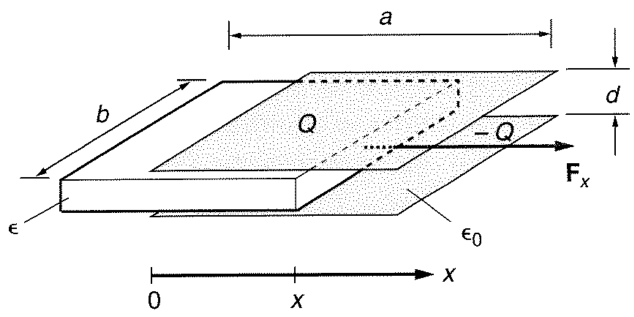


Figure 9.6 Determination of the force on the dielectric partly inserted between the electrodes of a parallel-plate capacitor using Eq. (9.11)

$$C = C_1 + C_2 = \epsilon\frac{bx}{d} + \epsilon_0\frac{b(a-x)}{d}$$

(see problem P8.8). The energy in the capacitor is

$$W_e(x) = \frac{Q^2}{2C} = \frac{Q^2}{2(C_1+C_2)} = \frac{Q^2 d}{2b[\epsilon x + \epsilon_0(a-x)]}.$$

The derivative $dW_e(x)/dx$ in this case is a bit more complicated to calculate, and it is left as an exercise. The force is found to be

$$F_x = \frac{V^2}{2}\frac{b}{d}(\epsilon - \epsilon_0).$$

Note that this force is *always positive* because $\epsilon > \epsilon_0$. This means that the forces tend to pull the dielectric further in between the plates.

Example 9.7—Rise of level of liquid dielectric partly filling a parallel-plate capacitor. As a final example of the application of Eq. (9.11), let us determine the force that raises the level of the liquid dielectric between the plates of the capacitor in Fig. 9.3. Assume the dielectric is distilled water with $\epsilon_r = 81$, the width of the plates is b, their distance is $d = 1\,\text{cm}$, and the capacitor was charged by being connected to $V = 1000\,\text{V}$. The electric forces will raise the level of the water between the plates until the weight of the water between the plates becomes equal to this force. The weight is equal to

$$G = \rho_m xbdg,$$

where ρ_m is the mass density of water and $g = 9.81\,\text{m/s}^2$. By equating this force to the force that we found in Example 9.6, we get

$$\rho_m xbdg = \frac{V^2}{2}\frac{b}{d}(\epsilon - \epsilon_0)$$

$$x = \frac{V^2}{2d^2\rho_m g}(\epsilon - \epsilon_0) = 1.44\,\text{mm}.$$

So far, we have discussed examples of case (1), where the charges in a system were kept constant. Case (2) is finding forces from energy when the voltage, not the charge, of the n conducting bodies of the system is kept constant (for example, we connect the system to a battery). When a body is moved by electric forces again by $\mathrm{d}x$, some changes must occur in the charges on the conducting bodies, due to electrostatic induction. These changes are made at the expense of the energy in the sources (battery). So we would expect the energy contained in the electric field to increase in this case. It can be shown in a relatively straightforward way that the expression for the component F_x of the electric force on the body in this case is

$$F_x = +\frac{\mathrm{d}W_e(x)}{\mathrm{d}x} \qquad \text{(potentials kept constant).} \tag{9.12}$$

Of course, this formula in all cases leads to the same result for the force as Eq. (9.11), but in some cases it is easier to calculate $\mathrm{d}W_e/\mathrm{d}x$ for constant potentials than for constant charges, and conversely.

Example 9.8—Example 9.6 revisited. Let us compute the force from Example 9.6 using Eq. (9.12) instead of Eq. (9.11), which we used in Example 9.6. Now we assume the potential of the two plates to be constant, and therefore express the system energy in the form

$$W_e(x) = \frac{1}{2}CV^2 = \frac{V^2}{2}\left[\epsilon\frac{bx}{d} + \epsilon_0\frac{b(a-x)}{d}\right],$$

so that

$$F_x = +\frac{\mathrm{d}W_e}{\mathrm{d}x} = \frac{V^2}{2}\frac{b}{d}(\epsilon - \epsilon_0).$$

The result is easier to obtain than in Example 9.6.

Questions and problems: P9.17 to P9.20

9.6 Electrostatic Pressure on Boundary Surfaces

In an electrostatic field there is pressure on all boundary surfaces. Although it is always small in terms of the pressure values we encounter around us (e.g., pressure of air in tires, pressure on pistons of combustible engines), it has interesting applications. Therefore we will derive the general expression for pressure on the boundary surface between two dielectrics, and estimate its magnitude.

Assume first that the boundary surface is tangential to the lines of the electric field strength vector (Fig. 9.7a). Let the electric forces push the surface by a small distance $\mathrm{d}x$ from dielectric 2 toward dielectric 1, as in the figure. Since the lines of the vector $\mathbf{E}$ are tangential to the surface, the boundary conditions have not changed, so $\mathbf{E}$ remains the same. Therefore, the potential difference between any two bodies in the system remains the same as well. This means we have to use Eq. (9.12) to determine the force per unit area on the boundary surface.

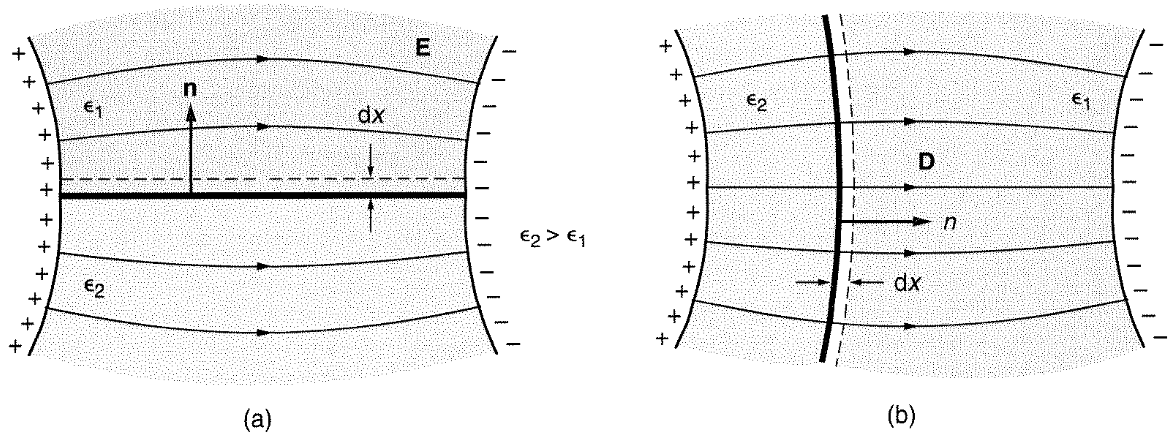


Figure 9.7 Boundary surfaces between two dielectrics. (a) The lines of the electric field strength **E** are tangential to the boundary. (b) The lines of the electric displacement vector **D** are normal to the boundary.

The energy in the system did change, since in the thin layer of thickness $\mathrm{d}x$ the energy density before the displacement was $\epsilon_1 E_{\mathrm{tang}}^2/2$, and after the displacement it became $\epsilon_2 E_{\mathrm{tang}}^2/2$. If we consider a small patch of the boundary surface of area ΔS, Eq. (9.12) yields

$$(F_x)_{\mathrm{on}\ \Delta S} = +\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(\epsilon_2 E_{\mathrm{tang}}^2 - \epsilon_1 E_{\mathrm{tang}}^2\right)\mathrm{d}x\,\Delta S\right], \tag{9.13}$$

from which the pressure on the boundary surface is

$$p_{E\mathrm{tang}} = \frac{(F_x)_{\mathrm{on}\ \Delta S}}{\Delta S} = \frac{1}{2}(\epsilon_2 - \epsilon_1)E_{\mathrm{tang}}^2. \tag{9.14}$$

Note that the pressure acts *toward the dielectric of smaller permittivity*.

Consider now the case in Fig. 9.7b, where the boundary surface is such that the lines of the electric displacement vector **D** are normal to it. Assume again that due to electric forces, the surface is displaced by a small distance $\mathrm{d}x$. The boundary conditions for vector **D** are satisfied, so it will not change. According to generalized Gauss' law, the charges on conducting bodies will therefore not be changed either. Hence this case corresponds to the formula in Eq. (9.11).

Again, in this case the energy density changed in the thin layer of thickness $\mathrm{d}x$, so the force on a small patch of the boundary surface of area ΔS is found as

$$(F_x)_{\mathrm{on}\ \Delta S} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(\frac{D_{\mathrm{norm}}^2}{\epsilon_2} - \frac{D_{\mathrm{norm}}^2}{\epsilon_1}\right)\mathrm{d}x\,\Delta S\right]. \tag{9.15}$$

The electrostatic pressure in this case is thus

$$p_{D\mathrm{norm}} = \frac{1}{2}\left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right)D_{\mathrm{norm}}^2. \tag{9.16}$$

Note that in this case also the pressure acts *toward the dielectric of smaller permittivity*.

The lines of vector **E** are rarely tangential, and lines of vector **D** rarely normal, to boundary surfaces. When they are at an arbitrary angle with respect to the surface, the energy density in either of the two dielectrics can be expressed as

$$\frac{1}{2}\epsilon E^2 = \frac{1}{2}(\epsilon E_{\text{tang}}^2 + \epsilon E_{\text{norm}}^2) = \frac{1}{2}(\epsilon E_{\text{tang}}^2 + D_{\text{norm}}^2/\epsilon). \tag{9.17}$$

This means that the pressure due to the electrostatic field in the general case is given as the sum of the pressures in Eqs. (9.14) and (9.16):

$$p = \frac{1}{2}(\epsilon_2 - \epsilon_1)\left(E_{\text{tang}}^2 + \frac{D_{\text{norm}}^2}{\epsilon_1\epsilon_2}\right). \tag{9.18}$$

It is interesting that from Eq. (9.16) we can also obtain the pressure on the surface of a charged conductor. Let the conductor be medium 2, and assume that $\epsilon_2 \to \infty$, which implies that it is "infinitely polarizable," an electrostatic equivalent to a conductor. Replacing ϵ_1 by ϵ, Eq. (9.16) yields

$$p_{\text{on conductor surface}} = \frac{1}{2}\frac{D_{\text{norm}}^2}{\epsilon} = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}. \tag{9.19}$$

The pressure is directed toward the dielectric.

Example 9.9—Pressure on a liquid dielectric between plates of a parallel-plate capacitor. Consider again the parallel-plate capacitor dipped into a liquid dielectric, Fig. 9.3a. Eq. (9.14) tells us immediately that there is an upward pressure on the upper surface of the dielectric. It is left as an exercise for the reader to show that the same result is obtained as before, but in a much simpler way.

Example 9.10—Force acting on a plate of a parallel-plate capacitor. The force on one of the plates of the parallel-plate capacitor (from Example 9.4) can now be obtained easily using Eq. (9.19). Note that we know the field on the plate surface if we know either the voltage between the plates or the charge of a plate (assumed to be distributed uniformly over it). The completion of this example is also left to the reader.

Example 9.11—Magnitude of electrostatic pressure on a dielectric surface. Let us now do a simple calculation that will tell us how strong electrostatic pressures can be. Imagine a slab of dielectric of $\epsilon_r = 4$ (say, quartz) is placed in an electric field perpendicular to the field lines (Fig. 9.8). Let us find the pressure on the front side of the slab for the strongest possible field in air, $E_0 = 30\,\text{kV/cm}$. Using Eq. (9.16), we obtain

$$p_{\text{Dnorm}} = \frac{1}{2\epsilon_0}\left(1 - \frac{1}{4}\right)D_0^2 = \frac{3}{8}E_0^2\epsilon_0 \simeq 30\,\text{Pa}.$$

In comparison, typical pressure inside a car tire is 200 kPa (30 psi), or four orders of magnitude larger. [A pascal (Pa) is the SI unit for pressure equal to N/m^2. The psi stands for "pounds per square inch."]

Questions and problems: Q9.20 to Q9.22, P9.21 and P9.22

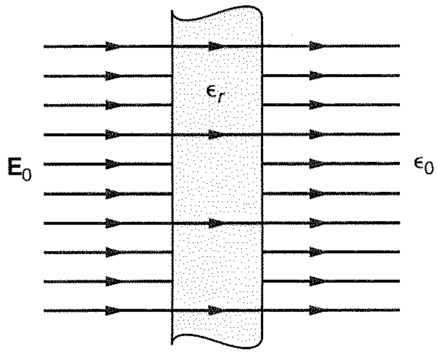


Figure 9.8 A dielectric slab in an electric field. Electrostatic pressure on the slab side can be calculated using Eq. (9.16.)

9.7 Chapter Summary

1. Electrostatic energy, forces, and pressures are small when compared with the usual magnitude of these quantities around us. However, we will later show that they have considerable practical significance.
2. Electrostatic energy can be considered as a potential energy of a system of charges, or as distributed throughout the field with a density equal to $\frac{1}{2}\epsilon E^2$.
3. Electric forces can be obtained directly only if the charge distribution is known, which is rarely the case. Therefore a method for determining the forces based essentially on the law of conservation of energy has been derived. It enables the forces to be found from energy.
4. There is a pressure acting on all boundary surfaces in an electrostatic field. It is always directed toward the medium of lower permittivity.

QUESTIONS

Q9.1. What force drives electric charges that form electric current through circuit wires?

Q9.2. Capacitors of capacitances $C_1, C_2, \ldots, C_n$ are connected (1) in parallel, or (2) in series with a source of voltage V. Determine the energy in the capacitors in both cases.

Q9.3. A parallel-plate capacitor with an air dielectric, plate area S, and distance d between plates is charged with a fixed charge Q. If the distance between the plates is increased by dx ($dx > 0$), what is the change in electric energy stored in the capacitor? Explain the result.

Q9.4. Repeat question Q9.3 assuming that $dx < 0$.

Q9.5. A parallel-plate capacitor with an air dielectric and capacitance C_0 is charged with a charge Q. The space between the electrodes is then filled with a liquid dielectric of permittivity ϵ. Determine the change in the electrostatic energy stored in the capacitor. Explain the result.

Q9.6. Can the density of electric energy be negative? Explain.

Q9.7. If you charge a 1-pF capacitor by connecting it to a source of 100 V, do you think the energy contained in the capacitor can damage a semiconductor device if discharged through it? Explain.

Q9.8. If you touch your two hands to the electrodes of a charged high-voltage capacitor, what do you think are the principal dangers to your body?

Q9.9. Explain in your own words why a polarized dielectric contains energy distributed throughout the dielectric.

Q9.10. Discuss whether a system of charged bodies can have zero total electric energy.

Q9.11. Can the electric energy of a system of charges be negative?

Q9.12. Explain in detail how you would calculate approximately the force $\mathbf{F}_{12}$ in Eq. (9.8), assuming that you know the charge distribution on the two bodies in Fig. 9.2.

Q9.13. If the field induces a dipole moment in a small body, it will also tend to move the body toward the region of stronger field. Sketch an inhomogeneous field and the dipole, and explain.

Q9.14. Under the influence of electric forces in a system, a body is rotated by a small angle. The system consists of charged, insulated conducting bodies. Is the energy of the system after the rotation the same as before, larger than before, or smaller than before? Explain.

Q9.15. If we say that dW_e is negative, what does this mean?

Q9.16. Is weight a force? If it is, what kind of force? If it is not, what else might it be?

Q9.17. Is it possible to have a system of three point charges that are in equilibrium under the influence of their own mutual electric forces? If you can find such a system, is the equilibrium stable or unstable?

Q9.18. A soap bubble can be viewed as a small stretchable conducting ball. If charged, will it stretch or shrink? Do you think the change in size can be observed?

Q9.19. Explain why a charged body attracts *uncharged* small bodies of any kind.

Q9.20. Explain why, in Eq. (9.13), we subtracted the energy density in the first medium from the energy density in the second medium, and not the other way around.

Q9.21. A glass of water is introduced into an arbitrary inhomogeneous electric field. What is the direction of the pressure on the water surface?

Q9.22. Derive Eq. (9.19) from Eq. (9.18).

PROBLEMS

P9.1. A bullet of mass 10 g is fired with a velocity of 800 m/s. How many high-voltage capacitors of capacitance 1 μF can you charge to a voltage of 10 kV with the energy of the bullet?

P9.2. A coaxial cable h long, of inner radius a and outer radius b, is first filled with a liquid dielectric of permittivity ϵ. Then it is connected for a short time to a battery of voltage V. After the battery is disconnected, the dielectric is drained out of the cable. (1) Find the voltage between the cable conductors after the dielectric is drained out of the cable. (2) Find the energy in the cable before and after the dielectric is drained.

P9.3. A spherical capacitor with an air dielectric, of electrode radii $a = 10$ cm and $b = 20$ cm, is charged with a maximum charge for which there is still no air breakdown around the inner electrode of the capacitor. Determine the electric energy of the system.

P9.4. Repeat the preceding problem for a coaxial cable of length $d = 10\,\text{km}$, of conductor radii $a = 0.5\,\text{cm}$ and $b = 1.2\,\text{cm}$.

P9.5. Calculate the largest possible electric energy density in air. How does this energy density compare with a $0.5\,\text{J/cm}^3$ chemical energy density of a mixture of some fuel and compressed air?

P9.6. Show that half of the energy inside a coaxial cable with a homogeneous dielectric, of inner conductor radius a and outer conductor radius b, is contained inside a cylinder of radius $a < r < \sqrt{ab}$.

P9.7. A metal ball of radius $a = 10\,\text{cm}$ is placed in distilled water ($\epsilon_r = 81$) and charged with $Q = 10^{-9}\,\text{C}$. Find the energy that was used up to charge the ball.

P9.8. A dielectric sphere of radius a and permittivity ϵ is situated in a vacuum and is charged throughout its volume with volume density of free charges $\rho(r) = \rho_0 a/r$, where r is the distance from the sphere center. Determine the electric energy of the sphere.

P9.9. Repeat the preceding problem if the volume density of free charges is constant, equal to ρ.

P9.10. Inside a hollow metal sphere, of inner radius b and outer radius c, is a metal sphere of radius a. The centers of the two spheres coincide (concentric spheres), and the dielectric is air. If the inner sphere carries a charge Q_1 and the outer sphere a charge Q_2, what is the energy stored in the system?

***P9.11.** Prove *Thomson's theorem:* the distribution of static charges on conductors is such that the energy of the system of charged conductors is minimal.

***P9.12.** Prove that if an uncharged conductor, or a conductor at zero potential, is introduced into an electrostatic field produced by charges distributed on conducting bodies, the energy of the system decreases.

P9.13. An electric dipole of moment $\mathbf{p}$ is situated in a uniform electric field $\mathbf{E}$. If the angle between the vectors $\mathbf{p}$ and $\mathbf{E}$ is α, find the torque of the electric forces acting on the dipole. What do the electric forces tend to do?

P9.14. An electric dipole of moment $\mathbf{p} = Q\mathbf{d}$ is situated in an electric field of a negative point charge Q_0, at a distance $r \gg d$ from the point charge. If the vector $\mathbf{p}$ is oriented toward the point charge, find the total electric force acting on the dipole.

P9.15. A two-wire line has conductors with radii $a = 3\,\text{mm}$ and the wires are $d = 30\,\text{cm}$ apart. The wires are connected to a voltage generator such that the voltage between them is on the verge of initiating air ionization. (1) Find the electric energy per unit length of this line. (2) Find the force per unit length acting on each of the line wires.

P9.16. A conducting sphere of radius a is cut into two halves, which are pressed together by a spring inside the sphere. The sphere is situated in air and is charged with a charge Q. Determine the force on the spring due to the charge on the sphere. In particular, if $a = 10\,\text{cm}$, determine the force corresponding to the maximal charge of the sphere in air for which there is no air breakdown on the sphere surface.

P9.17. Find the electric force acting on the dielectrics labeled 1 and 2 in the parallel-plate capacitor in Fig. P9.17. The capacitor plates are charged with Q and $-Q$. Neglect edge effects.

P9.18. The inner conductor of the coaxial cable in Fig. P9.18 can slide along the cylindrical hole inside the dielectric filling. If the cable is connected to a voltage V, find the electric force acting on the inner conductor.

P9.19. One end of an air-filled coaxial cable with inner radius $a = 1.2\,\text{mm}$ and an outer radius of $b = 1.5\,\text{mm}$ is dipped into a liquid dielectric. The dielectric has a density of mass equal to $\rho_m = 0.8\,\text{g/cm}^3$, and an unknown permittivity. The cable is connected to a voltage

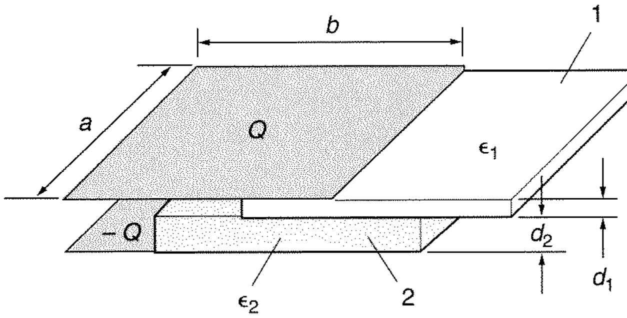


Figure P9.17 Three-dielectric capacitor

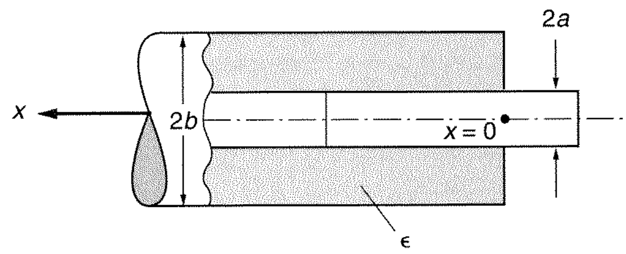


Figure P9.18 Coaxial cable with sliding conductor

$V = 1000\,\text{V}$. Due to electric forces, the level of liquid dielectric in the cable is $h = 3.29\,\text{cm}$ higher than the level outside the cable. Find the approximate relative permittivity of the liquid dielectric, assuming the surface of the liquid in the cable is flat.

P9.20. The end of a coaxial cable is closed by a dielectric piston of permittivity ϵ and length x. The radii of the cable conductors are a and b, and the dielectric in the other part of the cable is air. What is the magnitude and direction of the axial force acting on the dielectric piston, if the potential difference between the conductors is V?

P9.21. One branch of a U-shaped dielectric tube filled with a liquid dielectric of unknown permittivity is situated between the plates of a parallel-plate capacitor (Fig. P9.21). The voltage between the capacitor plates is V, and the distance between them d. The cross section of the U-tube is a very thin rectangle, with the larger side parallel to the electric field intensity vector in the charged capacitor. The dielectric in the tube above the liquid dielectric is air, and the mass density of the liquid dielectric is ρ_m. Assume that h is the measured difference between the levels of the liquid dielectric in the two branches of the U-tube. Determine the permittivity of the dielectric.

P9.22. A soap bubble of radius $R = 2\,\text{cm}$ is charged with the maximal charge for which breakdown of air on its surface does not occur. Calculate the electrostatic pressure on the bubble.

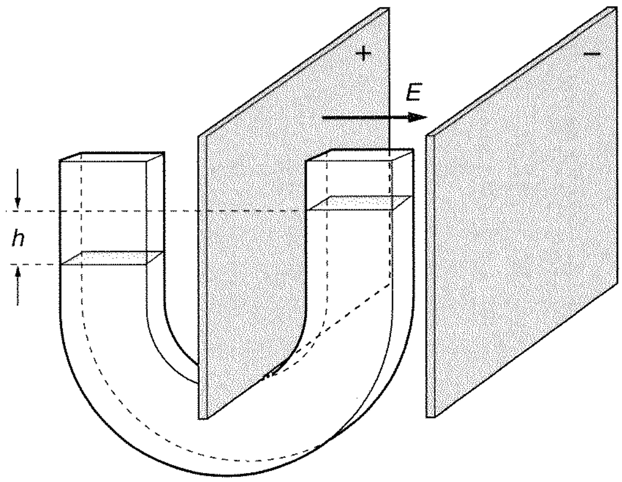


Figure P.9.21 Dielectric tube that is partially between the plates of a parallel-plate capacitor.