Chapter 7

Fundamentals of Orbital Mechanics

Celestial mechanics began as the study of the motions of natural celestial bodies, such as the moon and planets. The field has been under study for more than 400 years and is documented in great detail. A major study of the Earth-Moon-Sun system, for example, undertaken by Charles-Eugene Delaunay and published in 1860 and 1867 occupied two volumes of 900 pages each. There are textbooks and journals that treat every imaginable aspect of the field in abundance.

Orbital mechanics is a more modern treatment of celestial mechanics to include the study the motions of artificial satellites and other space vehicles moving under the influences of gravity, motor thrusts, atmospheric drag, solar winds, and any other effects that may be present. The engineering applications of this field include launch ascent trajectories, reentry and landing, rendezvous computations, orbital design, and lunar and planetary trajectories.

The basic principles are grounded in rather simple physical laws. The paths of spacecraft and other objects in the solar system are essentially governed by Newton's laws, but are perturbed by the effects of general relativity (GR). These perturbations may seem relatively small to the layman, but can have sizable effects on metric predictions, such as the two-way round trip Doppler. The implementation of post-Newtonian theories of orbital mechanics is therefore required in order to meet the accuracy specifications of MPG applications.

Because it had the need for very accurate trajectories of spacecraft, moon, and planets, dating back to the 1950s, JPL organized an effort that soon became the world leader in the field of orbital mechanics and space navigation. In doing so, it developed the fundamental ephemerides of planets, moons, and asteroids now used by the International Astronomical Union (IAU) and appearing in the Astronomical Almanac.

The states of these and any other objects of interest in the solar system are tabulated in ephemerides, generated either by JPL Navigation or by spacecraft projects, and put into standardized forms for ready usage by any applications to which these ephemerides are made available. The standardization is made possible by software utilities provided by JPL's Navigation Ancillary Instrumentation Facility (NAIF).

NAIF also provides utilities for easy access to and manipulation of ephemerides, as well as tools for computation of characteristics of interest extracted from the ephemerides. For example, the `SPKEZ` function returns states of any object possessing a NAIF identifier (NAIFID) as observed by any other NAIFID at a given ephemeris time and specified aberration type, provided the needed underlying ephemerides have been supplied. The NAIF toolkit has become the international standard for ephemeris access and usage.

The reader should appreciate that, while the creation of ephemerides and the development of NAIF tools for accessing them require intensive knowledge of the principles of orbital mechanics, the users of these entities, such as the MPG developer, is spared much of this burden. The skills deemed necessary for MPG development and maintenance include a familiarity with the basic terms and fundamentals of orbital mechanics, awareness of the range of utilities contained in the SPICE toolkit, and appreciation of which SPICE functions may be effective in MPG applications.

The information contained in this chapter, of necessity, is therefore limited to an overview of orbital mechanics. The level of detail has been narrowed to that believed sufficient to permit future MPG personnel to understand its algorithms and to extend its capability. Those desiring fuller information are therefore directed to SPICE required reading, the commentary contained in the source code of NAIF utilities, any of a number of textbooks and journals in the field, and internet searches.

The reader is expected to have a basic familiarity with the concepts of time measurement, coordinate systems, mechanics, vector algebra, and numeric methods.

7.1 Coordinate Systems and Frames

The subject of coordinate systems and frames is treated more fully in another chapter of this work. This section provides a short summary as applicable to the needs of the current chapter.

In metric prediction generation, it is sometimes necessary to represent the states (i.e., positions and velocities) of objects in a number of different coordinate systems according to the contexts in which these are to be used. Each coordinate system corresponds to a way of expressing positions and velocities with respect to a particular frame of reference, such as a set of rectangular axes.

In principle, it is possible to obtain a standard celestial coordinate frame that is fixed in space by fixing the orientation of a chosen inertial coordinate frame at a specified instant, called the standard epoch. In practice, the axes may not be directly observable at the standard epoch, but they may be inferred by adopting a catalog of the positions and motions of a set of stars or other celestial objects that act as reference points in the sky. The International Celestial Reference Frame (ICFR) is commonly used in fundamental ephemerides of solar system objects.

In general, an object may be moving with respect to a coordinate system, and that coordinate frame may be moving or rotating with respect to other frames. Therefore, in order to be definite, it is necessary also to specify the time coordinate to which spatial coordinate values refer.

The apparent state of a celestial object also depends on the state of the observer, as well as the coordinate frame to which observation is referred. Such relative motions are governed by theories of relativity, discussed in the chapter on Spacetime. Coordinates may also include the effects of other distortions or aberrations that may require compensation.

The MPG standard epoch is J2000, defined by the positions of the Earth's equator and equinox on Julian Day 2451545.0, or January 1, 2000 at 12:00:00. Translation among other standard reference frames relies on standard models of precession and nutation to determine spatial coordinates at given epochs.

7.2 Two-Body Motion

Johannes Kepler observed from a study of the positions of planets that their motions in the solar system appeared to exhibit three behaviors that we now call Kepler's laws, published in 1609.

K-1: The orbits of the planets are ellipses, with the Sun at one focus of the ellipse.

K-2: The line joining the planet to the Sun sweeps out equal areas in equal times as the planet travels around the ellipse.

K-3: The ratio of the squares of the periods of revolution for two planets is equal to the ratio of the cubes of their semimajor axes.

Kepler's theory was later refined by Isaac Newton to account for the mutual perturbations among the bodies of the solar system. Newton published his three laws of motion and law of universal gravitation in 1687.

N-1: Objects at rest remain at rest and objects in uniform motion remain in uniform motion unless acted upon by an external net force.

N-2: An applied force on an object is equal to the time rate of change of momentum of that object.

N-3: For every action, there is an equal and opposite reaction.

N-UG: Every object in the universe attracts every other body in the universe with a force directed along the line of centers of the two objects that is proportional to the product of their masses and inversely proportional to the square of the distance between them,

$$F = \frac{G\, m_1\, m_2}{d^2} \tag{7-1}$$

In this relation, G is the Newton gravitational constant, m_1 and m_2 are the masses of the primary and secondary bodies, and d is the distance between them.

It is now understood that Kepler's laws only apply to the so-called two-body problem, where the number of objects interacting in space is limited to two. As will be discussed later in the chapter, the extension of Newton's laws to more than two bodies bears no simple solution, and generally requires numeric methods for determining positions over time.

Because the laws of Kepler and Newton agree in two-body theory, the orbits they describe are often referred to as *Keplerian*, with Newtonian theory reserved to describe multiple body interactions without relativistic effects applied.

There are a number of cases in which the effects of multiple bodies may be treated as slight perturbations superimposed on two-body theory. Such is the case, for example, in approximating positions of planets relative to the Sun, the Moon relative to Earth, and spacecraft in solar and planetary orbits.

7.2.1 Keplerian Motion

Let it now be assumed here that there are only two bodies whose motions are to be characterized. The more massive of these will be designated the primary, and the other, the secondary. If $\mathbf{r}_1$ is the position vector of the primary with respect to an arbitrary inertial origin, $\mathbf{r}_2$ is that of the secondary, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the vector from primary to secondary, and $r = |\mathbf{r}|$ is its magnitude, then the equations of force at the two bodies, by Newton's laws, are

$$\begin{aligned} m_1 \ddot{\mathbf{r}}_1 &= \frac{G\, m_1\, m_2}{r^3}\mathbf{r} \\ m_2 \ddot{\mathbf{r}}_2 &= -\frac{G\, m_1\, m_2}{r^3}\mathbf{r} \end{aligned} \tag{7-2}$$

The body mass on the left-hand side of each equation above divides out with that on the right-hand side. Consequently, subtracting the two gives

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = -\frac{G\,(m_1 + m_2)}{r^3}\mathbf{r} \tag{7-3}$$

Newton's law for a single body moving around a primary is therefore

$$\ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3} \tag{7-4}$$

where $= G(m_1 + m_2)$. All three of Kepler's laws follow from this expression.

Similarly, adding the two equations of Eq.(7-2) shows that the net force acting on the pair as viewed in an inertial frame is zero,

$$m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = \mathbf{0} \tag{7-5}$$

Integration of this equation therefore gives the net momentum of the pair, which is constant. A second integration gives the motion of the center of mass, or barycenter, which is located at

$$\mathbf{c} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \mathbf{r}_0 + v_c\, t \tag{7-6}$$

The barycenter is non-accelerating, and is therefore an inertial frame in which the position and velocity of the bodies and center of mass can be determined at all times from initial positions and velocities.

If the origin of the system is taken to be the barycenter, then

$$\mathbf{r}_1 = -\frac{m_2}{m_1}\mathbf{r}_2 \tag{7-7}$$

In this frame, the position vectors satisfy

$$\begin{aligned} \mathbf{r} &= -\frac{m_2 + m_2}{m_2}\mathbf{r}_1 \\ &= \frac{m_1 + m_2}{m_1}\mathbf{r}2 \end{aligned} \tag{7-8}$$

Substitution of Eq.(7-9) into Eq.(7-2) produces Newton's law for each body moving about the barycenter,

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= \frac{G\, m_2}{r^3}\mathbf{r} = -\frac{G\, m_2^3}{(m_1 + m_2)^2\, r_1^3}\mathbf{r}_1 \\ \ddot{\mathbf{r}}_2 &= -\frac{G\, m_1}{r^3}\mathbf{r} = -\frac{G\, m_1^3}{(m_1 + m_2)^2\, r_2^3}\mathbf{r}_2 \end{aligned} \tag{7-9}$$

Further, Eqs. (7-4) and (7-9) appear to be the same when the proper associations between radial vectors and gravitational constants are made.

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= -\frac{1}{r_1^3}\mathbf{r}_1 \qquad {}_1 = \frac{G\, m_2^3}{(m_1 + m_2)^2} \\ \ddot{\mathbf{r}}_2 &= -\frac{1}{r_1^3}\mathbf{r}_2 \qquad {}_2 = \frac{G\, m_1^3}{(m_1 + m_2)^2} \end{aligned} \tag{7-10}$$

They may each be solved separately, as if the system were uncoupled. But, of course, all three pertain to the same system, and orbit in synchrony.

It suffices, then to concentrate on only one of the forms, and the one chosen here is Eq.(7-4). The remainder of the treatment in this section focuses on characteristics of the trajectory of the secondary with the primary at the origin.

7.2.2 Angular Momentum

The angular momentum per unit mass is the vector product of the position and velocity vectors, or

$$\begin{aligned} \mathbf{h} &= \mathbf{r} \times \dot{\mathbf{r}} \\ h &= r^2 \,\dot{\ } \end{aligned} \tag{7-1}$$

where $\dot{\ }$ denotes the angular rate of the secondary as viewed at the primary. If the angular rate is negative, the motion is said to be *retrograde* in that reference frame.

In Newtonian two-body physics, the derivative of this vector is zero,

$$\dot{\mathbf{h}} = \mathbf{r} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \dot{\mathbf{r}} = -\frac{\ }{r^3}\mathbf{r} \times \mathbf{r} + \mathbf{0} = \mathbf{0} \tag{7-2}$$

since the cross product of a vector with itself is the zero vector. Consequently, the angular momentum is constant, and thus conserved along the trajectory[1]. The trajectory is therefore planar, since the $\mathbf{h}$ vector is perpendicular to both position and velocity and constant everywhere along the trajectory.

The magnitude of the momentum satisfies the relationship $h = |\mathbf{r}_p \times \mathbf{v}_p| = r_p v_p$, which follows since the position and velocity vectors are perpendicular at this point. The angular rate for a Keplerian trajectory satisfies the relation

$$\dot{\ } = \frac{h}{r^2} \tag{7-3}$$

As a further consequence of constant momentum, if is the angle between position and velocity vectors, then at any two points along the orbit,

[1] This characteristic is equivalent to Kepler's second law, since the differential area of between two arcs separated in time by dt and in angle by $d\ $ is equal to $r \cdot r\,d\ \ /2$. The time rate of change of area swept out by an arc is therefore $h/2$, which is constant.

$$r_1 v_1 \sin \ _1 = r_2 v_2 \sin \ _2 \tag{7-4}$$

Alternately, could be defined as the angle between the velocity vector and a line perpendicular to the position vector oriented such that $= \ / 2 - \ $. This angle, called the *flight angle*, satisfies

$$r_1 v_1 \cos \ _1 = r_2 v_2 \cos \ _2 \tag{7-5}$$

Both the **rv** angle and the flight angle are commonly denoted by the Greek letter *phi*, so it behooves the reader to determine from the context of usage which applies.

The total angular momentum about the center of gravity of the closed system is conserved, so

$$m_1 h_1 + m_2 h_2 = m h \tag{7-6}$$

where m is the equivalent mass in the primary-centered system and the h_i are the angular momentums per unit mass of each body.

$$m_1 r_1^2 \ \dot{} + m_2 r_2^2 \ \dot{} = m r^2 \ \dot{} \tag{7-7}$$

Substituting the distances values implied in Eq.(7-8), dividing out the common angular rate, and simplification produces the result

$$m = \frac{m_1 m_2}{m_1 + m_2} \tag{7-8}$$

This is the so called *reduced mass* of the system. It is less than either of the two masses comprising it, and is such that the motion of either body, with respect to the other as origin, is the same as a body having this mass moving with respect to the barycenter and acted upon by the same force.

7.2.3 Orbital Parameters

Although it is not done here, it is fairly straightforward[2] to show that the integration of Eq.(7-4) yields a conic trajectory in space (circle, ellipse, parabola, or hyperbola), in which the position of the secondary body at any given time can be

[2] See, for example, the solution of the Lagrange equations of motion given in [Irving & Mullineux1959]. Later in this chapter, this claim is validated by showing that a conic trajectory satisfies Eq.(7-4).

computed from and a set of six orbital parameters. One such set is comprised of the components of the position and velocity vectors at a given time. Another is the set of *osculating elements* that specify the position of the body at a reference epoch and the size, shape, and orientation of the orbit in space. A typical set of osculating elements is shown in Figure 7-1, and defined below:

$$\begin{aligned} r_p &= \text{pericenter distance} \\ e &= \text{eccentricity of trajectory} \\ i &= \text{inclination of trajectory plane to reference plane} \\ \Omega &= \text{longitude of ascending node} \\ &= \text{argument of pericenter} \\ t_p &= \text{epoch of pericenter} \end{aligned} \tag{7-9}$$

The *pericenter* is the point in a trajectory that is nearest to the center of attraction, and is synonymous with *periapsis* in this case. The distance from the primary to the pericenter is called the *perifocal distance*. *Eccentricity* is a parameter that specifies the shape of the conic section. *Inclination* is the angle between the reference plane and the orbital plane, positive when the pericenter is in the northern hemisphere of the reference frame. The *ascending node* is that intersection of the conic with the reference plane that carries the trajectory from the southern to northern hemisphere, and the *argument of pericenter* is the angle from the node to the pericenter, and is not defined for a circular orbit. The *semi-major axis* a of the conic is often given rather than the pericenter distance r_p, and the *longitude of pericenter* $= \Omega +$ is sometimes given instead of . The orbital period (of elliptical orbits) and *mean motion* are often of interest, but are not usually cited as one of the six elements needed to define the orbit.

Anomaly is the term that astronomers use to denote an angle. The angle measured at the primary body, from the pericenter to the orbiting body is called the *true anomaly*. The *mean anomaly* is the angle M from pericenter to a hypothetical body moving with a constant angular velocity n that is equal to the orbiting body's *mean orbital motion*.

$$M = (t - t_p)n \tag{7-10}$$

The mean anomaly of a circle or ellipse is simply an angle that marches uniformly in time from 0 to 360 degrees during one revolution. It is defined to be 0 degrees at periapsis, and therefore is 180 degrees at apoapsis. The mean motion is the constant rate at which the angle accumulates over time.

The *eccentric anomaly* is another angle that has an easily understood and physical interpretation for elliptical orbits that simplifies the relationship between mean and true anomalies, as described later in this chapter.

The concepts of mean and eccentric anomalies are also extended to parabolic and hyperbolic orbits, as they are useful mathematical concepts for interrelating orbital time and position. However, they bear no (easily assimilated) geometric significance otherwise in these cases.

The SPICE function `OSCELT` takes as its input the gravitational parameter , a time t, and the trajectory state vector $(\mathbf{r}, \mathbf{v})$ at this time; it returns a set of orbital elements similar to those of Eq.(7-9) above, but in which t_p is replaced the parameter M, the mean anomaly at this epoch. The remainder of this discussion relates to the methods by which the elements are calculated from the given state.

In 1710, Jacob Hermann discovered a vector which has since been rediscovered and refined many times since. Today it is referred to as the Laplace-Runge-Lenz vector, which in a particular normalized form, is called the *eccentricity vector*. It is given by

$$\mathbf{e} = \frac{\mathbf{v} \times \mathbf{h}}{} - \frac{\mathbf{r}}{r} \tag{7-11}$$

Its direction is along the major axis of the conic toward pericenter, and its magnitude is the eccentricity, $|\mathbf{e}| = e$. This expression makes it possible to determine the eccentricity and the direction of pericenter from any given state vector $(\mathbf{r}, \mathbf{v})$ along the orbit.

If the eccentricity is not unity, the semimajor axis is related to the state via a rearrangement of the vis-viva equation (given later in Eq.(7-31)), as

$$a = \left(\frac{2}{r} - \frac{v^2}{} \right)^{-1} \tag{7-12}$$

The periapsis and apoapsis distances are then likewise determined by

$$\begin{aligned} r_p &= a(1 - e) \\ r_a &= a(1 + e) \end{aligned} \tag{7-13}$$

The angular momentum is perpendicular to the plane of the new orbit. The inclination is the angle this vector makes with the reference frame z-axis, commonly denoted $\mathbf{k}$. It can be found using the SPICE function

$$= \mathrm{VSEP}(\mathbf{h},\mathbf{k}) \tag{7-14}$$

The node vector $\mathbf{n}$ is perpendicular to both the reference plane and the angular momentum vector, pointing toward the ascending node. It is given by

$$\mathbf{n} = \mathbf{k} \times \mathbf{h} \tag{7-15}$$

If the inclination is zero (or) this vector is of zero length; the SPICE convention in this case is to set $\mathbf{n} = (1,0,0)^T$.

The longitude of the ascending node is the angle between the x-axis and this vector, which may be computed by

$$\Omega = \arctan(n_x, n_y) \tag{7-16}$$

The arc tangent function used here computes the quadrant-corrected angle whose first component is in the x-direction, and the second, in the y-direction.

The argument of pericenter is the angle between the node vector and the eccentricity vector, positive if the pericenter lies in the direction of $\mathbf{h} \times \mathbf{n}$, which lies in the orbital plane.

$$= \mathrm{sgn}(\mathbf{e} \bullet \mathbf{h} \times \mathbf{n})\mathrm{VSEP}(\mathbf{e},\mathbf{n}) \tag{7-17}$$

If negative, this angle is sometimes augmented by 2 in order to provide a positive result. This element is undefined for circular orbits, by SPICE convention it is set to zero.

The final element is the true anomaly at the epoch of the given state. The true anomaly is the angle measured from the pericenter to the given position, positive in the clockwise direction about $\mathbf{h}$. In order to calculate this angle, two unit vectors are used:

$$\begin{aligned} \mathbf{u}_x &= \frac{\mathbf{e}}{e} \\ \mathbf{u}_y &= \frac{\mathbf{h} \times \mathbf{e}}{|\mathbf{h} \times \mathbf{e}|} \\ &= \arctan(\mathbf{r} \bullet \mathbf{u}_x, \mathbf{r} \bullet \mathbf{u}_y) \end{aligned} \tag{7-18}$$

The relationships between true, eccentric, and mean anomalies are discussed a little later on.

An alternate method of calculation of the true anomaly of non-circular orbits is by inverting the conic formula, appearing later, in Eq.(7-20), which gives

$$= \pm \cos^{-1}\left(\frac{(1+e)r_p - r}{e\,r}\right) \tag{7-19}$$

The method given in Eq.(7-18), however, avoids this ambiguity.

The process of determining the state of a trajectory at any given time using the gravitational parameter and orbital elements is called *propagation of the conic*. The SPICE utility `CONICS` embodies this process; given the gravitational constant, orbital elements, and time at which the state is to be computed, it returns the state vector. The method of propagation is different for the different types of conic sections, and is found in the discussions of each type.

7.2.4 Trajectories

As mentioned earlier, the motion of the secondary with respect to the primary takes the form of a conic section. This claim will now be validated. The claim is that, when viewed in the orbital plane with the primary at the origin and major axis along the line from primary to pericenter, the motion in polar coordinates is described by the familiar conic formula

$$r = \frac{(1+e)r_p}{1 + e\cos} \tag{7-20}$$

Typical plots of conic trajectories are illustrated in Figure 7-2. The numerator $(1+e)r_p$ of the polar equation is known as the *semi-latus rectum* of the conic section; it is the distance from the primary to the trajectory in a direction perpendicular to the major axis. When $e = 0$, the orbit is circular; when $0 < e < 1$, the orbit is an ellipse; when $e = 1$, the trajectory is a parabola; and when $e > 1$, the trajectory is hyperbolic.

The Cartesian vector form of the conic trajectory is

$$\mathbf{r} = r\begin{bmatrix} \cos \\ \sin \\ 0 \end{bmatrix} \tag{7-21}$$

The velocity vector is the derivative of this vector, which may be found using elementary calculus and the relation for $\dot{}$ given by Eq.(7-3) to be

$$\mathbf{v} = \frac{h}{(1+e)\,r_p}\begin{bmatrix} -\sin \\ e+\cos \\ 0 \end{bmatrix} = \sqrt{\frac{}{(1-e^2)\,a}}\begin{bmatrix} -\sin \\ e+\cos \\ 0 \end{bmatrix} \tag{7-22}$$

Differentiation of the velocity and reapplication of Eq. give the acceleration,

$$\mathbf{a} = \ddot{\mathbf{r}} = \frac{h^2}{(1+e)\,r_p\,r^2}\begin{bmatrix} -\cos \\ -\sin \\ 0 \end{bmatrix} \tag{7-23}$$

Substitution of this result into Eq.(7-4) indicates that the angular momentum must satisfy

$$h = \sqrt{(1+e)\,r_p} \tag{7-24}$$

Therefore, the secondary trajectory viewed by the primary is a conic section whose angular momentum is given above.

The orbital position and velocity in Cartesian coordinates are then

$$\mathbf{r} = \frac{r_p(1+e)}{1+e\cos}\begin{bmatrix} \cos \\ \sin \\ 0 \end{bmatrix} \qquad \mathbf{v} = \sqrt{\frac{}{r_p(1+e)}}\begin{bmatrix} -\sin \\ e+\cos \\ 0 \end{bmatrix} \tag{7-25}$$

7.2.5 Euler Angles

The trajectory plane frame of reference may be transformed to the reference frame of a given set of orbital elements by a rotation of coordinates. Leonard Euler showed that any rotation in 3-space can be decomposed into the product of three elemental rotations. Each rotation is quantified by an identified axis of the frame and an angle of rotation about this axis. The set of axes and the corresponding angles are called the *Euler angles* of the transformation. Euler angles provide a convenient means of representing the spatial orientation of any frame of the space as a composition of rotations from a reference frame.

Multiplying vectors in the trajectory plane frame by the following rotation matrix transforms them into vectors in the reference frame of the orbital elements.

$$\mathbf{R} = [-\Omega]_3 \cdot [- \ ]_1 \cdot, [- \ \ ]_3 \tag{7-26}$$

This formula may be found in the Explanatory Supplement to the Astronomical Almanac [Seidelmann1992]. The notation $[\ ]_n$ above denotes a rotation about the axis designated as n by an angle . It is implemented in the SPICE function `ROTATE`. The standard assignment of axes is x = 1, y = 2, z = 3. The combined rotation matrix implementing a given set of Euler angles is returned by the `EUL2M` function.

7.2.6 Total Energy and Orbital Shape

The total energy of the secondary in this configuration is the sum of its kinetic and potential energies, or

$$_{tot} = \frac{1}{2} m\, v^2 - \frac{G\, m_1\, m_2}{r} = \frac{1}{2} m\, v^2 - \frac{\ m}{r} \tag{7-27}$$

Here, r and v denote, respectively, the distance from the secondary to the primary and the velocity of the secondary in the trajectory plane relative to the primary. Note that the reduced mass m is used in computing the system kinetic energy.

By Newton's law. this total energy is constant, and may be equated to its value when the secondary is at pericenter. With substitution of $h = r_p\, v_p$, it is

$$\begin{aligned} _{tot} &= (\frac{v_p^2}{2} - \frac{}{r_p})\, m \\ &= (\frac{h^2}{2\, r_p^2} - \frac{}{r_p})\, m \end{aligned} \tag{7-28}$$

The total energy vanishes for an orbit having the same pericenter but whose angular momentum takes the value

$$h_{par} = \sqrt{2 \quad r_p} \tag{7-29}$$

The trajectory with zero total energy is parabolic, with $e = 1$. It represents the case in which the secondary's kinetic energy is just sufficient to escape the attraction of the primary. *Escape velocity*[3] for an object at pericenter is defined as the tangential rate required to achieve this state,

$$v_{esc} = \sqrt{\frac{2}{r_p}} \tag{7-30}$$

The escape velocity of an object at rest on a body is found using the above expression, but with the perifocal distance set equal to the body radius.

When the total energy is negative, the trajectory is bound and the orbit is elliptical. When positive, the trajectory is unbound and the orbit is hyperbolic.

By equating the relationships in Eqs.(7-27) and (7-28), dividing out the secondary mass, and rearranging terms, the following equation results:

$$v^2 = \left(\frac{2}{r} - \frac{1-e}{r_p} \right) \tag{7-31}$$

This equation, known as the *vis-viva equation* and also as the *orbital energy conservation* equation, relates the velocity v at any point of distance r along the trajectory to the excess orbital momentum, which has been related to eccentricity by the relationship

$$e = 2\left(\frac{h}{h_{par}} \right)^2 - 1 = \frac{h^2}{r_p} - 1 \tag{7-32}$$

This form is not obvious, but results when the magnitude of $\mathbf{e}$, as given in Eq.(7-11), is computed and then some vector algebra, the definition $\mathbf{h} = \mathbf{r} \times \mathbf{v}$, the vis-viva equation above, and Eq.(7-29) are applied. Eccentricity can also be written in the form

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[3] Since the quantity of reference is a scalar, the more proper term would be escape *speed*. The term is retained, however, for historical reasons. Escape velocity is often misunderstood to be the speed a powered vehicle (such as a rocket) must have in order to leave orbit. However, this is not the case. It is the speed a bullet fired from the surface would have to travel (ignoring the effects of drag) to leave orbit, but it is not the speed required for a rocket or other object in powered flight. An object under power could leave the Earth's gravity at any speed, assuming it had enough fuel.

$$e = \frac{v_p^2\, r_p}{\ } - 1 \tag{7-33}$$

The semimajor axis a of the trajectory is related to the pericenter distance by

$$a = \frac{r_p}{1-e} \tag{7-34}$$

The semimajor axis is thus positive for ellipses and circles, infinite for the parabola, and negative for the hyperbola. The vis-viva equation expressed using a is

$$v^2 = \left(\frac{2}{r} - \frac{1}{a}\right) \tag{7-35}$$

The difference in squared velocity at any two points on the trajectory is thus given by

$$v_2^2 - v_1^2 = 2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \tag{7-36}$$

In particular, the difference between squared velocities at periapsis and apoapsis of an elliptical orbit is

$$v_p^2 - v_a^2 = \frac{4\,e}{a(1-e^2)} \tag{7-37}$$

Further, the velocities at periapsis and apoapsis are

$$\begin{aligned} v_p &= \sqrt{\frac{\ }{a}\left(\frac{1+e}{1-e}\right)} \\ v_a &= \sqrt{\frac{\ }{a}\left(\frac{1-e}{1+e}\right)} \\ \frac{v_p}{v_a} &= \frac{1+e}{1-e} \end{aligned} \tag{7-38}$$

The semimajor axis and angular momentum are thus related by

$$h = r_p\, v_p = \sqrt{\ \ a\,(1-e^2)} \tag{7-39}$$

7.2.7 Relationship Between Mean and True Anomaly

The constant angular momentum $h = r^2\ \dot{}$ may be equated to its value as given in Eq.(7-20) to give

$$h = \sqrt{(1+e)\quad r_p} = r^2 \,\dot{}\, = \left(\frac{r_p(1+e)}{1+e\cos}\right)^2 \dot{} \tag{7-40}$$

which, upon rearranging terms, is

$$\sqrt{\frac{}{(1+e)^3 r_p^3}} = \frac{\dot{}}{(1+e\cos\ )^2} \tag{7-41}$$

This relation may be integrated directly to yield a linearly growing left-hand side,

$$n\,(t-t_p) = (t-t_p)\sqrt{\frac{}{(1+e)^3 r_p^3}} \tag{7-42}$$

where the constant of integration is chosen to make the quantity zero at pericenter. A constant parameter has been introduced as the proportionality to the mean motion, to be determined.

The integral of the right-hand side can also be calculated in closed form, by manual means or by a software tool such as *Mathematica*, to yield

$$M = -\frac{2\tanh^{-1}\left(\frac{(e-1)}{\sqrt{e^2-1}}\tan\left(\frac{}{2}\right)\right)}{(e^2-1)^{3/2}} - \frac{e\sin}{(e^2-1)(1+e\cos\ )} \tag{7-43}$$

No constant of integration of the right-hand side appears, as the expression is zero at pericenter.

This equation is the generalized Kepler equation that relates mean anomaly to true anomaly. The reader will note, however, that, as it is expressed above, there are imaginary terms when $e < 1$ and a possible singularity at $e = 1$. For elliptical and parabolic orbits, then, further manipulations of Eq.(7-43) are required in order to transform this result into real, nonsingular forms. Each case is considered separately in the paragraphs below.

It may also be noted that, while it may be simple enough in principle to determine the orbital time that corresponds to a given true anomaly, the reverse solution, for given t, involves solving a transcendental equation. Kepler's equation is therefore generally solved by iterative means. Fortunately there are methods for doing this that converge rather quickly.

7.2.8 Elliptic Orbit Relationships

The form of Eq.(7-43) for elliptical orbits $(e < 1)$ may be transformed by dint of circular and hyperbolic trigonometric transformations into

$$
\begin{aligned}
&(t - t_p)\sqrt{\frac{1}{(1+e)^3 r_p^3}} \\
&\quad = \frac{1}{(1-e^2)^{3/2}}\left(2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{}{2}\right)\right) - e\sqrt{1-e}\sin \quad \right)
\end{aligned}
\tag{7-44}
$$

(The author made this transformation using *Mathematica*, but laborious manual methods produce the same result.) This equation is sufficient to relate mean and true anomalies; however, some simplification can yet be made.

Motion in this case is periodic. As traverses an orbit, the parenthesized term of the right-hand side above varies from 0 to 2 radians, so the period T satisfies

$$
T\sqrt{\frac{}{(1+e)^3 r_p^3}} = \frac{2}{(1-e^2)^{3/2}} \tag{7-45}
$$

or

$$
T = 2\sqrt{\frac{rp^3}{(1-e)^3}} = 2\sqrt{\frac{a^3}{}} \tag{7-46}
$$

The mean motion is thus equal to

$$
n_E = \frac{2}{T} = \sqrt{\frac{(1-e)^3}{r_p^3}} = \sqrt{\frac{}{a^3}} \tag{7-47}
$$

The parameter appearing in Eq.(7-43) for this case is $= (1 - e^2)^{-3/2}$.

For elliptical orbits, the *eccentric anomaly* is defined as the angle E, shown in Figure 7-3, measured at the center of the ellipse, from the pericenter to the point on the circumscribing auxiliary circle of radius a from which a perpendicular to the major axis would intersect the orbiting body.

Since the ellipse pericenter is $r_p = a(1 - e)$, the distance from the ellipse center to the focus is $a\,e$. The geometry is thus such that the eccentric and true anomalies are related by

$$a\cos E = a\,e + r\cos \tag{7-48}$$

Substitution of Eq.(7-20) in the above and simplification produce the following relationship between true and eccentric anomalies:

$$\cos E = \frac{e + \cos}{1 + e\cos} \tag{7-49}$$

This may also be written in the inverted form

$$\cos \quad = \frac{\cos E - e}{1 - e\cos E} \tag{7-50}$$

Substitution of from Eq.(7-50) into Eq.(7-44), application of trigonometric identities, and simplification yield the familiar form of Kepler's equation for elliptical orbits:

$$\begin{aligned} M &= E - e\sin E \\ M &= n_E\,(t - t_p) \\ n_E &= \sqrt{\frac{1}{a^3}} \end{aligned} \tag{7-51}$$

To determine the position at a given a time t, the mean anomaly M is computed, from which the eccentric anomaly E must be found in order to determine the true anomaly via Eq.(7-50). Much effort has been expended over the last few hundred years in attempts to invert the Kepler equation in a closed form expression. One such expression, credited to Friedrich Wilhelm Bessel in 1817, is

$$E = M + \sum_{k=1}^{\infty} \frac{2}{k} J_k(k\,e)\sin(k\,M) \tag{7-52}$$

in which $J_k(x)$ is the Bessel function of the first kind and order *k*. However, this form converges so slowly at higher eccentricities that iterative root-finding methods are more desirable.

Newton-Raphson iteration is an efficient method for solving for the eccentric anomaly, as follows. Let $f(E) = E - e\sin E - M$ for given M and e be the function whose root is desired. The iteration begins using the first two terms of Eq.(7-52), or $E_0 = M + e\sin M$, and subsequent estimates of the root are given by the recursion

$$E_{k+1} = E_k - \frac{f(E_k)}{f'(E_k)} = E_k - \frac{E_k - e\sin E_k - M}{1 - \cos E_k} \tag{7-53}$$

Convergence is quadratic, and thus achieves high accuracy after only a few steps.

The orbital position and velocity in Cartesian coordinates relative to the *center* of the orbital ellipse, with x-axis in the direction of pericenter is then

$$\mathbf{r}_E = a\begin{bmatrix} \cos E \\ \sqrt{1-e^2}\sin E \\ 0 \end{bmatrix} \qquad \mathbf{v}_E = \frac{a^2\, n_E}{r}\begin{bmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{bmatrix} \tag{7-54}$$

The position equation follows from the geometry shown in Figure 7-3. The polar form of the position equation is

$$r = a(1 - e\cos E) \tag{7-55}$$

The velocity equation results upon differentiating the position equation and substituting $\dot{E}$ found by differentiating Eq.(7-51)

$$\dot{E} = \frac{n_E}{1 - e\cos E} = \frac{1}{r}\sqrt{\frac{}{a}} \tag{7-56}$$

The position and velocity vectors in Cartesian coordinates relative to the primary are easily related to Eq.(7-54),

$$\mathbf{r} = a\begin{bmatrix} \cos E - e \\ \sqrt{1-e^2}\sin E \\ 0 \end{bmatrix} \qquad \mathbf{v} = \frac{a^2\, n_E}{r}\begin{bmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{bmatrix} \tag{7-57}$$

7.2.9 Parabolic Orbit Relationships

For a parabolic orbit $(e = 1)$, the Kepler equation given in Eq.(7-43) is indeterminate. However, a tool such as *Mathematica*, or judicious application of L'Hospital's rule in taking the limit of this expression as $e \to 1$, profuse use of trigonometric identities, and algebraic simplification produce the reduced result

$$(t - t_p)\sqrt{\frac{1}{2^3 r_p^3}} = \frac{1}{2}\left(\tan\frac{\ }{2} + \frac{1}{3}\tan^3\frac{\ }{2}\right) \tag{7-58}$$

Again, this equation is totally sufficient to relate time and orbital position; however, it is traditional to introduce an intermediary term to simplify computation.

Unfortunately, the elliptical orbit eccentric anomaly does not translate at unit eccentricity to a meaningful intermediary formula. Instead, the *parabola eccentric anomaly* is defined to be

$$E = \tan\frac{\ }{2} \tag{7-59}$$

The same notation is used here as appeared in the elliptical case. However, no confusion should arise, since these cases are always separately considered. Under this convention, the parabolic trajectory Kepler equation is

$$\begin{aligned} M &= (t - t_p)\sqrt{\frac{1}{2\, r_p^3}} \\ &= E + \frac{E^3}{3} \end{aligned} \tag{7-60}$$

The time coefficient of the right-hand side is defined to be the *parabolic trajectory mean motion*,

$$n_P = \sqrt{\frac{1}{2\, r_p^3}} \tag{7-61}$$

The of Eq.(7-43) in this case is ½.

In this case, the Kepler equation Eq.(7-60) does have a closed-form solution, viz.,

$$E = \frac{\left(2^{-1/3}\left(3M + \sqrt{4 + 9M^2}\right)^{2/3} - 2^{1/3}\right)}{\left(3M + \sqrt{4 + 9M^2}\right)^{1/3}} \tag{7-62}$$

Newton-Raphson iteration may be used in this case, if desired. When applied to the function $f(E) = E + E^3 / 3 - M$ it produces the recursive, rapidly converging formula

$$E_{k+1} = E_k - \frac{E_k + E_k^3 / 3 - M}{1 + E_k^2} \tag{7-63}$$

Neither E nor n_P carry physical significance; they are significant merely as means to express the orbital time relationship. The tangent half-angle formula applied to Eq.(7-59) produces the polar form of the trajectory as a function of eccentric anomaly,

$$r = r_p\,(1 + E^2) \tag{7-64}$$

The state vectors are

$$\mathbf{r} = r_p \begin{bmatrix} 1 - E^2 \\ 2E \\ 0 \end{bmatrix} \qquad \mathbf{v} = \frac{2\, r_p\, n_P}{1 + E^2} \begin{bmatrix} E \\ 1 \\ 0 \end{bmatrix} \tag{7-65}$$

The velocity is found by differentiating the position and setting

$$\dot{E} = \frac{n_H}{1 + E^2} \tag{7-66}$$

which follows from Eq.(7-60).

7.2.10 Hyperbolic Orbit Relationships

For hyperbolic trajectories $(e > 1)$, it may be seen from Eq.(7-20) that the true anomaly range is limited to the interval $(-\ {}_{\max},\ {}_{\max})$ where ${}_{\max} = \cos^{-1}(-1/e)$. Analytic geometry teaches that the projection of incoming and outgoing asymptotes intersect at a distance of $e\,r_p\,/(e-1)$ from the focus in the pericenter direction, and that the angle between asymptotes is $2(\ \ -\ {}_{\max}) = 2\cos^{-1}(1/e)$. The *turning angle*, or angle between incoming and outgoing asymptotes, is $(\ \ - 2\cos^{-1}(1/e))$.

The time-vs.-anomaly relationship given in Eq.(7-43) may be used directly, although it is traditional to simplify it using a suitable definition of the *hyperbola eccentric anomaly* and mean motion. In this case, the eccentric anomaly is defined as

$$E = \cosh^{-1}\left(\frac{e + \cos}{1 + e\cos}\right) \tag{7-67}$$

This is similar in form to Eq.(7-49), but with hyperbolic cosine replacing the circular cosine. The true anomaly is then

$$\cos \quad = \frac{\cosh E - e}{1 - e\cosh E} \tag{7-68}$$

It is notable that E has singularities at true anomaly values satisfying $= \cos^{-1}(-1/e)$. This asymptotic behavior is familiar to those having studied hyperbola characteristics in descriptive geometry.

Substitution of Eq.(7-68) in Eq.(7-43) and reduction using trigonometric identities produce the classic relation

$$\begin{aligned} n_H\,(t - t_p) &= M = e\sinh E - E \\ n_H &= \sqrt{\frac{(e-1)^3}{r_p^3}} \end{aligned} \tag{7-69}$$

The parameter appearing in Eq.(7-43) for this case is $= (e^2 - 1)^{-3/2}$. Neither E nor n_H carry physical significance; they are significant merely as means to express the orbital time relationship.

Newton-Raphson iteration applied to the function $f(E) = e\sinh E - E - M$ produces the recursive rapidly converging formula

$$E_k = E_{k-1} - \frac{e\sinh E_{k-1} - E_{k-1} - M}{e\cosh E_{k-1} - 1} \tag{7-70}$$

Eq.(7-69) evaluated at small E gives $E \approx M/(e-1)$, and evaluated at large E gives $E \approx \sinh^{-1}(M/e)$. Thus, the iteration above, if begun with the starting value $E_0 = \min(M/(e-1), \sinh^{-1}(M/e))$, achieves high accuracy in only a few steps.

The polar form of the hyperbola expressed in terms of the eccentric anomaly is

$$r = r_p\frac{e\cosh E - 1}{e - 1} \tag{7-71}$$

The position and velocity vectors in the orbital plane, expressed in terms of eccentric anomaly are

$$\mathbf{r} = \frac{r_p}{e-1}\begin{bmatrix} e - \cosh E \\ \sqrt{e^2 - 1}\sinh E \\ 0 \end{bmatrix} \qquad \mathbf{v} = \frac{n_H}{r}\left(\frac{r_p}{e-1}\right)^2\begin{bmatrix} -\sinh E \\ \sqrt{e^2 - 1}\cosh E \\ 0 \end{bmatrix} \tag{7-72}$$

The velocity is found by differentiating the position and setting

$$\dot{E} = \frac{n_H}{e\cosh E - 1} \tag{7-73}$$

which follows from Eq.(7-69).

7.2.11 Higher-Order State Derivatives

SPICE toolkit functions, such as `SPKEZ`, typically return state vectors, i.e. position and velocity, of objects in the solar system. It is sometimes the case, however that an MPG algorithm needs to estimate a higher order derivative approximately to assist in its internal[4] processes. This section therefore presents methods for calculating higher order derivatives of Keplerian orbital functions.

Differentiation of a function $\mathbf{q}(\ )$ involving the true anomaly may apply the chain rule, replacing $\dot{}$ by h/r^2, so that

$$\frac{d\mathbf{q}(\ (\mathbf{t}))}{dt} = \frac{d\mathbf{q}(\ )}{d} \cdot = \frac{h}{r^2}\frac{d\mathbf{q}(\ )}{d} \tag{7-74}$$

Repeated applications of this rule will result in expressions having no derivatives of the true anomaly. It is therefore possible to express any time derivative of a function of purely as a function of .

Further, in foregoing three sections above, the velocity vector in each case was found by differentiating the position vector, which was expressed as a function of the eccentric anomaly of that trajectory type, using the chain rule and the time derivative of the eccentric anomaly found from Kepler's equation,

$$\begin{aligned} n(t - t_0) &= f(E) \\ n &= f'(E)\dot{E} \\ \dot{E} &= \frac{n}{f'(E)} \end{aligned} \tag{7-75}$$

The derivative $\dot{E}$ contains no derivatives of E, so the time rate of change in position depends only on angular quantities found via solution to Kepler's equation.

---

[4] Such approximations are not allowed to propagate into output predictions when accuracy requirements would be compromised.

The same process as used to find velocity extends to any function, vector or otherwise, as

$$\frac{d}{dt}\mathbf{q}(E) = \frac{d\mathbf{q}(E)}{dE}\dot{E} = \frac{d\mathbf{q}(E)}{dE}\frac{n}{f'(E)} \tag{7-76}$$

Again, no derivatives of the eccentric anomaly appear in the result. Therefore, any derivative of a function of E may be expressed purely as a function of E.

Still further, all functions involving higher-order derivatives of the position vector may be expressed in terms of position and velocity because of Eq.(7-4). For example, in order to find the acceleration vector $\ddot{\mathbf{r}}$, one merely notes that

$$\begin{aligned} \ddot{\mathbf{r}} &= \frac{d}{dt}\ddot{\mathbf{r}} = -\frac{d}{dt}\left(\frac{}{r^3}\mathbf{r}\right) \\ &= \left(\frac{3\,\dot{r}}{r^4}\mathbf{r} - \frac{1}{r^3}\dot{\mathbf{r}}\right) \\ &= \left(\frac{3\,\dot{r}}{r^4}\mathbf{r} - \frac{1}{r^3}\mathbf{v}\right) \end{aligned} \tag{7-77}$$

Extension of this practice to higher order derivatives follows by induction. By such means, then, all higher-order derivatives of conic section positions can be expressed as functions of positions and velocities only.

Positions and velocities may further be expressed only as functions of input orbital parameters and time, as related to either true or eccentric anomalies.

7.3 Perturbed Conic Representations

The orbital elements describing Keplerian motion provide an excellent reference for describing orbital characteristics. However, there are generally other effects that perturb a body's motion relative to a reference frame, which necessitate adjustments to the orbital theory governing the object's motion.

Approximations of the positions and velocities of trajectories may often be approximated using Kepler's laws and associated osculating elements evolving slowly over time. Kepler's laws are presumed to apply at a given instant to instantaneous element values. Instances of such trajectories are embodied in several SPICE ephemeris types discussed later in this chapter. The discussion below treats the method by which solutions are generated.

Perturbations may be classified according to how they affect the Keplerian elements. *Secular* variations represent deviations that are long-term, steady non-periodic changes over time. *Short-term* periodic effects are deviations whose extents are much less than the orbital period. *Long-term* periodic effects are deviations whose period is much longer than the orbital period. Typical analyses treat secular and long-term periodic orbital perturbations, but precise orbit determination requires that short-term effects be included as well.

7.3.1 Other-Body Perturbations

The presence other bodies in an orbital system upsets the characteristics of the Keplerian trajectory. If the Keplerian elements are only slightly altered, the effects due to each body may often be treated separately and superimposed on those elements affected. In such cases, the gravitational effect of each perturbing body may be treated as a force acting on the angular momentum vector of the orbit. This force generally causes periodic variations in all of the orbital elements and secular variations in longitude of the ascending node Ω, argument of periapsis ω, and mean anomaly M.

7.3.2 Non-Spherical Primary Perturbation

In the foregoing two-body theory, it was assumed that the gravitational potential of the primary body was spherically symmetric. In actuality, this hypothesis fails for satellites in proximity to a primary that is neither spherical nor homogeneous in composition. For Earth and many of the planets, the most dominant features are a bulge at the equator, a slight pear shape, and flattening at the poles. The gravitational potential in such cases is expressed in terms of a spherical harmonic series, given by

$$U(r,\phi) = \frac{\mu}{r}\left[1 - \sum_{k=2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k(\sin\phi)\right] \tag{7-78}$$

Here, $\mu = GM$ is the gravitational parameter of the primary, R is its equatorial radius, r is the distance to the orbiting object, ϕ is the primary-centric latitude, and $P_k(x)$ is the Legendre polynomial of degree k. The coefficients J_k in this expansion are called *zonal harmonics*.

The non-spherical gravitational potential of an oblate primary, in which only J_2 is assumed present, causes a periodic variation in all of the orbital elements, but the dominant effects are secular variations in longitude of the ascending node and argument of perigee. The secular rates of change have been calculated to be

$$\begin{aligned} \dot{\Omega} &= -\frac{3}{2} n\, J_2 \left(\frac{R}{r_p(1+e)} \right)^2 \cos i \\ \dot{\ } &= \frac{3}{4} n\, J_2 \left(\frac{R}{r_p(1+e)} \right)^2 (4 - 5\sin^2 i) \end{aligned} \qquad \text{(7-79)}$$

These characteristics are sometimes found useful in mission design, where the rates of these parameters may be chosen to enhance performance, as discussed later in this chapter (see the Orbital Types section).

7.3.3 Atmospheric Drag Perturbation

Drag is the resistance encountered by an object that impedes its progress. In space this resistance is typically due to a planetary atmosphere through which the object is moving. This drag, of course, is greatest when leaving or entering the atmosphere, as at launch and reentry, the intensity decreasing as the atmosphere thins out. Since drag depletes the kinetic energy of the object in space over time, the orbit will decay, spiraling inward, and, eventually, will undergo a reentry path, possibly disintegrating and burning up.

If an Earth satellite is within about 120-160 km of the surface, atmospheric drag will cause reentry within a few days, with destruction occurring at about 80 km. However, if its apogee reaches 600 km, drag is so week that orbits usually last in excess of 10 years.

The drag force $\mathbf{F}_D$ on an object acts in the direction opposite to the velocity vector. It is given by the equation

$$\mathbf{F}_D = -\frac{1}{2} C_D \quad v\, A\, \mathbf{v} \qquad \text{(7-80)}$$

In this equation, C_D is the drag coefficient, is the atmospheric density at the object's altitude, $\mathbf{v}$ is the object velocity relative to the atmosphere, v is its magnitude, and A is the object's cross sectional area perpendicular to the velocity. The drag coefficient depends on the geometric form of the object, and is usually determined by experimental means. The drag coefficient of a sphere is only about unity at Mach I, but it ranges from about 2 to 4 for typical Earth satellites.

Atmospheric densities at various altitudes and temperatures are available from standard tables. The density changes are not uniform in altitude and even vary by about two orders of magnitude during periods of high solar activity at altitudes in the range of about 500 to 800 km. Satellites thus tend to decay more rapidly during periods of solar maxima than during solar minima.

Approximate changes in semi-major axis, period, and velocity for a satellite of mass m in nearly circular orbit have been found to be

$$
\begin{aligned}
\dot{a} &= -\frac{2\quad C_D A \quad a^2}{m} = -2 \quad B \quad a^2 = -4 \quad B^* (\ \ /\ \ _0)\, a^2 \\
\dot{P} &= -\frac{6\quad ^2 C_D A \quad a^2}{m v} = -\frac{12\quad ^2 B^* (\ \ /\ \ _0) a^2}{v} \\
\dot{v} &= -\frac{C_D A \quad a v}{m} = -2 \quad B^* (\ \ /\ \ _0)\, a v
\end{aligned}
\qquad (7\text{-}81)
$$

The units of these changes are per revolution, not per second.

The term $B = C_D A/m$ in aerodynamic theory is called the *ballistic coefficient*. It represents how susceptible an object is to drag—the higher the number, the higher the drag. The B^* term (called *B-star*) is an adjusted value of B using the reference value of atmospheric density, $_0$, given by $B^* = B\ \ _0/2$, which has units of (Earth radii)$^{-1}$. B-star coefficients are given as constants for satellites listed in the NORAD Two-Line Element Sets, available from the *Center for Space Standards & Innovation*. See the discussion of SPICE ephemeris Type 10 for further information.

It is interesting to note that the term "ballistic coefficient," as used in the above context, is redefined as the inverse of this quantity when used in other contexts; similarly, object weight is sometimes used in place of object mass in this inverse definition. The reader should thus take care to determine the context of usage in any in application of interest.

7.3.4 Radiation Pressure Perturbation

Radiation pressure is the force per unit area exerted on any surface by electromagnetic radiation. For an object in the solar system, the radiation present is almost all due to sunlight. The solar radiation pressure at Earth is about 1366 watts/meter^2, varying about 7% over the year, being higher in early January than in early July. Since solar radiation is isotropic, its intensity follows the inverse square law with distance from the Sun.

The acceleration of a body in space due to solar radiation will be

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{\Phi_E \, c_R \, A}{m} \left(\frac{r}{AU} \right)^2 \mathbf{u} \tag{7-82}$$

Here Φ_E is the solar flux at Earth, A is the cross section area of the body, c_R is the its coefficient of reflection, m is its mass, r is its distance from the Sun, and $\mathbf{u}$ is a unit vector in the direction of the flux.

There is also a small force that may be generated from the body's own blackbody radiation, which, if isotropic, amounts to a pressure of

$$\Phi_B = \frac{T^4}{3c} f \tag{7-83}$$

where $= 5.67 \times 10^{-8} \quad (W \, m^{-2} \, K^{-4})$ is the Stephan-Bolzmann constant, T is the body temperature, c is the speed of light, and f is a form factor that directs this radiation in a particular direction. If the body were at $20°K$ and the directivity were 6, the pressure would only amount to about 3×10^{-6} watts/meter^2. The body would have to be beyond thousands of AU before this pressure would become significant with respect to solar radiation.

While these effects are small, they are constantly present and capable of significant orbital change when integrated over time. The average energy per second (wattage) figure is multiplied by a factor of 3×10^7 over a year's time.

7.3.5 Planetary Positions

Lower accuracy formulas for planetary positions have a number of important applications when the full accuracy of an integrated ephemeris, discussed in the next section, is not needed. These may often be useful in scheduling of observations, antenna pointing, prediction of certain view period phenomena, and planning and design of spacecraft missions. They are not currently used in MPG applications, as ephemerides are available for all of its missions. However, they have been found useful in MPG development, where simulated trajectories were sufficient for validating algorithms under evaluation.

E. Myles Standish, of JPL's Solar System Dynamics Group, published [Yeomans2006] a list of such elements and their rates for the nine major planets, in which the values given were adjusted to provide the best fit with actual positions derived from fundamental ephemerides. Element values are provided at the J2000 epoch, with variation rates in units per century. The interested reader may full details and data in the reference. The method used to calculate a planet's position is outlined below, and is typical of other modified Kepler methods.

For a given UTC time of interest, compute the corresponding ephemeris time in seconds past J2000, and convert this to the number of centuries of 36525 days past J2000, *T*.

Compute each of the planet's six osculating elements using a linear formula, as $a = a_0 + \dot{a}_0 T$. The parameters given in Standish's tables are

$$
\begin{aligned}
a_0, \dot{a}_0 &= \text{semimajor axis (AU, AU/century)} \\
e_0, \dot{e}_0 &= \text{eccentricity (,/century)} \\
i_0, \dot{i}_0 &= \text{inclination (degrees, degrees/century)} \\
L_0, \dot{L}_0 &= \text{mean longitude (degrees, degrees/century)} \\
{}_0, \dot{}_0 &= \ + \Omega = \text{longitude of perihelion (degrees, degrees/century)} \\
\Omega_0, \dot{\Omega}_0 &= \text{longitude of ascending node (degrees, degrees/century)}
\end{aligned}
\tag{7-84}
$$

Compute the argument of perihelion $= \ - \Omega$ and mean anomaly $M = L - \ + bT^2 + c\cos fT + s\sin fT$, where coefficients for the final three terms are provided only for the planets Jupiter through Pluto. Reduce the modulus of M to line in $(-180°, 180°)$.

Solve Kepler's equation $M\left(\frac{}{180}\right) = E - e\sin E$ for E in radians.

Compute the state vector in the orbital plane using Eq.(7-57), then convert it to ecliptic coordinates using Eq.(7-26), and thence into the ICRF (or J2000) frame using the rotation matrix $[\ ]_1$, in which is the J2000 obliquity $= 23.43928°$.

Standish provides tables of elements valid over the period -3000 BC to 3000 AD.

7.4 Multiple-Body Motion

The SPICE system accommodates ephemerides generated by any of a number of different models. Some are numerically integrated, as will be described below. Some are based on analytic theories, and some are produced by unspecified methods, being within the purview of a particular space flight project.

As of this writing, SPICE recognizes 18 different formats of ephemerides. Some of these are polynomial fits to integrated states, while others are based on various orbital theories, perturbation theories, and modified conic propagation. Those desiring more details may consult [SPK.REQ2004] for further information.

The MPG does not discriminate the method by state vectors over time are produced, but, rather, generates its predictions faithfully based on the values it is given. For this reason, MPG algorithms are completely oblivious to the theory by which the paths of interest are generated. Those interested in the MPG rationale, however, require an appreciation of more than the simple models discussed so far. Completing this chapter with only a discussion of the two-body problem and some simple perturbations of it would short-change the subject of orbital mechanics drastically. Yet, the full theory is too extensive for inclusion here. A thumbnail sketch is warranted and is probably sufficient to engender an appreciation of the domain knowledge and technology spanning the subject, as well as the level of difficulty in computing the general solution.

A precise description of the paths of objects in the solar system must not only account for Newtonian interactions among simple point-mass multiple bodies, but also non-point-mass objects and post-Newtonian[5] perturbations of Einstein and others. There have been several approaches to this problem, and the interested reader is referred to [Moyer1971] for history and details.

This section touches briefly, though, on the method used to generate JPL's high-accuracy ephemerides. In a nutshell, the multiple-body dynamical equations of motion that describe the gravitational physics of the solar system are a set of sec-

[5] Post-Newtonian modeling includes the effects of general relativity, with parameters added to allow least-squares regression for theoretical hypotheses concerning nonlinearity in superposition of gravity and space curvature produced by unit rest mass. JPL ephemerides, however, are generated with these parameters set to values consistent with general relativity.

ond-order interrelated differential equations that must solved subject to given assumed values for initial distances, masses, and other parameters of the solar system. As a result of many years spent in developing this system of equations and the software tools for solving them, in collecting and analyzing spacecraft tracking data, planetary radar data, and other solar system observations, and in subsequent regression of the input parameter values to obtain least-squares best-fit observations, JPL has produced fundamental ephemerides of solar system objects and spacecraft that are highly accurate.

The fundamental planetary and lunar ephemerides of the Astronomical Almanac, starting in the year 2003, were generated by JPL and designated DE405 and LE405. They replaced JPL's DE200 and LE200, which had been used in the Almanac since 1984. Previous to 1984, fundamental ephemerides of the Almanac were based upon analytical theories. DE405 and LE405 were the results of least-squares adjustments to previously existing ephemerides using a variety of observed measurements, followed by a numerical integration of the dynamical equations of motion. The reference frame for these ephemerides is the International Celestial Reference Frame (ICRF), discussed in the chapter on Coordinate and Reference Frames.

In this exposition, the reader will no doubt become aware that the number of computations required in generating accurate ephemerides of solar system objects is immense. The store containing the results is also immense. It may be some consolation, then, to realize that once generated, the ephemerides never have to be recomputed (for the applicable time spans) unless more accurate observations of the universe are found.

7.4.1 Point-Mass Interactions

The n-body equations were derived by JPL's Frank Estabrook in 1971, with various refinements within the astronomical community continuing since. Details of the original derivation appear in [Moyer1971], Appendix C.

Numerical integration of the equations of motion is the only known method capable of computing fundamental ephemerides to accuracy comparable to that of the available observations. Analytical theories have so far not been able to attain accuracy this high. The numerical integration method used in generating the fundamental ephemerides was a variable step size, variable order Adams method developed by JPL's Fred Krogh [Krogh1972], called DIVA.

The equations of motion and the methods used to integrate them are considered to be so precise that the accuracies of the initial conditions and assumed dynamical constants are now the dominant factors that determine ephemeris accuracy.

The equations governing object positions and accelerations include the n-body general relativistic (GR) metric tensor values defined in the earlier Spacetime chapter. The short form of the that solution is

$$\ddot{\mathbf{r}}_i = \sum_{j \neq i} \frac{m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \left[1 + O\left(\frac{1}{c^2} \right) \right] + O\left(\frac{1}{c^2} \right) \tag{7-85}$$

There is one such equation for each of body i of the solar system[6] considered to have a significant enough effect on the trajectories of the bodies included. The first term of this relationship may be recognized as the Newtonian acceleration of body i. The terms involving $O(1/c^2)$ are GR perturbations deriving from elements of the metric tensor $\mathbf{g}$ discussed in the Spacetime chapter. They are sizable in number, and are suppressed here for brevity.

The number of significant terms retained in the summand above will depend on the position of the object i relative to the other perturbing influences. When the object is a spacecraft in orbit about a planet, then that planet becomes the primary gravitational influence. More terms may be added in, depending on the degree of influence, such as the moons of the planet in the spacecraft case.

The system of equations solved in generating fully accurate ephemerides for a given set of objects in the solar system thus require judicious adaptation to those objects. The long-form system of equations for point mass interactions appears below [Standish2007].

[6] The masses of spacecraft or other objects of low mass are not included when generating fundamental ephemerides of the planets, moons, and asteroids. Ephemerides for low-mass objects are treated separately, after the fundamental ephemerides have been generated, using the positions of bodies in the fundamental ephemerides.

$$
\begin{aligned}
\ddot{\mathbf{r}}_{i(ptmass)} = \sum_{j\neq i} \frac{{}_{j}(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \Bigg\{ 1 - \frac{2(\quad + \quad)}{c^2} \sum_{k\neq i} \frac{{}_{k}}{r_{ik}} - \frac{2 \quad - 1}{c^2} \sum_{k\neq j} \frac{{}_{k}}{j_{ik}} \\
\left(\frac{v_i}{c}\right)^2 + (1 + \quad)\left(\frac{v_j}{c}\right)^2 - \frac{2(1+\quad)}{c^2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_j \\
- \frac{3}{2c^2}\left[\frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot \dot{\mathbf{r}}_j}{r_{ij}}\right]^2 + \frac{1}{2c^2}(\mathbf{r}_j - \mathbf{r}_i)\cdot \ddot{\mathbf{r}}_j \Bigg\} \qquad (7\text{-}86) \\
+ \frac{1}{c^2} \sum_{j\neq i} \frac{{}_{j}}{r_{ij}^3} \Big\{ [\mathbf{r}_i - \mathbf{r}_j] \cdot \big[(2 + 2 \quad)\dot{\mathbf{r}}_i - (1 + 2 \quad)\dot{\mathbf{r}}_j\big] \Big\} (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) \\
+ \frac{(3 + 4 \quad)}{2c^2} \sum_{j\neq i} \frac{{}_{j}\ddot{\mathbf{r}}_j}{r_{ij}} + \sum_{m=1}^{3} \frac{{}_{m}(\mathbf{r}_m - \mathbf{r}_i)}{r_{im}^3} + \sum_{c,s,m} \mathbf{F}
\end{aligned}
$$

This is almost the same as Eq.(5.211-1) appearing in the Explanatory Supplement to the Astronomical Almanac, except for the last term above, which does not appear, and the next-to-last, which sums over five asteroids.

In the above equation, the principal gravitational forces on the nine planets, Sun, and Moon are modeled as point masses in an isotropic, parameterized post-Newtonian (PPN) *n*-body metric, such as that discussed in the Spacetime chapter. Also included are Newtonian gravitational perturbations from 300 asteroids, chosen because they have the most pronounced effect on the Earth–Mars distance over the time span for which accurate spacecraft ranging observations exist.

In this equation, $\mathbf{r}_i$, $\dot{\mathbf{r}}_i$ and $\ddot{\mathbf{r}}_i$ represent the solar system barycentric position, velocity, and acceleration, respectively of the body indexed by i; $\quad_i = G\, m_i$ is its gravitational constant; and $v_i = |\, \dot{\mathbf{r}}_i \,|$ is its velocity. The coefficients and are post-Newtonian parameters, respectively measuring nonlinearity in superposition of gravity and curvature of space produced by unit rest mass. The JPL ephemerides set these values to their GR-theoretic values of unity.

The astute reader will note the appearance of two terms on the right-hand side of the equation containing second derivatives, which denote barycentric accelerations of each body due to effects of the remaining bodies and the asteroids. Strictly speaking, the right-hand side of the equation is dependent upon the left-hand side, and so, to be rigorous, the computation should be iterated at each step. However, use of Newtonian accelerations for these terms is deemed sufficiently accurate, and highly efficient. These terms therefore are replaced by the corresponding results of Eq.(7-86) with all relativistic terms (those involving c) omitted.

The next-to-last term of the equation reflects the effects of asteroids, and, in this version, is limited to Ceres, Pallas, and Vesta. The ESAA [Seidelman1992] version of this formula, however, included 5 asteroids. The last term represents forces upon the Earth, Moon, and Mars, only, from 297 other asteroids, grouped according to three taxonomic classes (C, S, M).

7.4.2 Solar System Barycenter

The solar system barycenter definition is modified from the usual Newtonian formulation, which, according to the ESAA [Seidelman1992], is

$$\mathbf{c} = \sum_i \; {}^{*}_{i}\mathbf{r}_i = 0 \tag{7-87}$$

where summation extends to all bodies of sufficient import, with the modified gravitational constants

$$ {}^{*}_{i} = \; {}_{i}\left\{1 + \frac{v_i^2}{2c^2} - \frac{1}{2\,c^2}\sum_{j \neq i}\frac{{}_{j}}{r_{ij}}\right\} \tag{7-88}$$

It is noted that the barycentric position depends on the set of positions of all objects in the summation, which, in turn, depend on the position of the barycenter. Iteration is thus required in order to assure that the barycenter satisfies Eq.(7-87).

7.4.3 Lunar Ephemerides

In addition to the point-mass interactions outlined above, the integrated lunar ephemerides require the inclusion of the figures of the Earth, Moon, and Sun in the mathematical model. The figure of a body is a characterization of its shape, in this case, relative to a spheroid. The figure of importance in this case is the representation of non-spherical gravitational potentials, such as the near-fields of the Earth, Moon, Venus, and Sun, since this potential is proportional to the force of attraction of a point mass at a given location.

A short introduction into these effects is given below.

7.4.3.1 Figure Effects

The common representation of the gravitational potential at a point expressed in spherical coordinates $(r, \;, \;)$, where r is distance from the body barycenter, is colatitude from the polar axis, and is longitude, of a body having semimajor axis a and gravitational constant is in the form of a *spherical harmonics* expansion,

$$U = \frac{\ }{r}\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^{n}\sum_{m=0}^{n} P_{n,m}(\sin\ )\Big[C_{n,m}\cos m\ \ + S_{n,m}\sin m\ \ \Big] \tag{7-89}$$

In this representation, $C_{n,m}$ and $S_{n,m}$ are the spherical harmonic coefficients of degree n and order m and $P_{n,m}$ is the so-called Associated Legendre Function of degree n and order m. A gravitational model of the body consists of a set of constants that specify the semimajor axis, gravitational constant, and spherical harmonic coefficients.

Such models are usually divided into three sets of terms, here denoted U_0, U_1, and U_2, in which the cases for $n = 0$ and $m = 0$ are separated, as

$$\begin{aligned}
U_0 &= \frac{\ }{r} \\
U_1 &= -\frac{\ }{r}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n} J_n\, P_{n,0}(\sin\ ) \\
U_2 &= \frac{\ }{r}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n}\sum_{m=1}^{n} P_{n,m}(\sin\ )\Big[C_{n,m}\cos m\ \ + S_{n,m}\sin m\ \ \Big]
\end{aligned} \tag{7-90}$$

The first of these is recognized as the point-mass potential, and its effects are already included in Eq.(7-86). No further consideration of it is warranted here.

The second, whose coefficients have been redefined for historical purposes, is independent of longitude; its constituents are referred to as *zonal harmonics*; these terms quantify how much the body is "out-of-round" about the equator. The even-degree zonal harmonics are symmetric, and the odd-degree ones are asymmetric about the equator. The degree-2 zonal harmonic models the oblateness of the body. The zonal harmonic of degree n has this many zeros from pole to pole; thus, the set of zonal harmonics map out the latitudinal fine structure that is independent of longitude.

The harmonic coefficients in the third potential term characterize variations in potential that depend on longitude. Those terms with $n = m$ are called *sectorials*, and the others are called *tesserals*.

So, having gravitational models of the bodies of interest, it remains to compute the accelerations due to their U_1 and U_1 terms. Since the force field at the given point is the negative of the gradient of the potential function, the acceleration due to the two terms is

$$\ddot{\mathbf{r}}_{fig} = -\mathbf{G}\,\nabla(U_1 + U_2) \tag{7-91}$$

The matrix $\mathbf{G}$ is necessary to transform the reference frame from body-centric to solar system barycentric. Evaluation of Eq.(7-91) is reasonably straightforward, but tedious and relatively uninformative. Details may be found in [Moyer1971].

The JPL ephemerides include the force of attraction between the known zonal harmonics (through fourth degree) of the Earth and the point-mass Moon, Sun, Venus, and Jupiter, together with the force of attraction between the zonal harmonics (through fourth degree) and the second- through fourth-degree tesseral harmonics of the Moon and the point-mass Earth, Sun, Venus, and Jupiter, plus the J_2 dynamical form-factor of the Sun.

7.4.3.2 Earth Tides

The tides on Earth raised by the Moon and Sun appear as a bulge on the Earth that, in turn, affects the lunar orbit. The model for the differential acceleration of the Moon due to these tides used in generating the JPL lunar ephemerides is found in [Standish2007].

7.4.3.3 Lunar Librations

In order to express coordinates in a Moon-centered frame, it is necessary to translate the reference frame of the ephemerides (i.e., the ICRF) into the Moon-centered system, called the *selenographic frame*. The Moon rotates relative to the inertial frame and its surface is distorted by Earth-induced and figure-induced torques; nevertheless, its mean principal axes are well defined. The transformation is characterized by a set of Euler angles (, ,), in which is the angle from the x-axis of the ephemeris frame along the xy-plane to the intersection of the lunar equator, is the inclination of the lunar equator upon the xy-plane, and is the longitude from that intersection along the lunar equator to the prime meridian. Derivation of the Euler angles and related computations are found in [Standish2007].

7.5 Types of Orbits

Trajectories of space objects are not only classified with regard to the theory by which their ephemerides are generated—Keplerian, perturbed Keplerian, or integrated equations of motion—but also by the usages intended for the objects in these trajectories and by salient characteristics of their paths relative to given reference frames. This section discusses some of these usages and characteristics.

7.5.1 Orbital Transfer

An object at rest on a planet or moon requires a rocket-powered launch to put it into orbit about that planet. An object in orbit about a planet or moon requires a rocket-powered deceleration to land on the intended surface. An object in orbit about one body may be sent toward another body, where it may fly by, orbit, or land on that body. All of these trajectories may be viewed as changes from one set of orbital elements to another. The process of changing from one set of orbital parameters to another is called *orbital transfer.*

Since orbital transfers require energy, a commodity always at a premium in space flight, all means for carrying out low energy transfers are rigorously sought by mission designers. On the other hand, the time to reach an intended mission state, such as a planetary encounter or injection into orbit, usually requires the expenditure of fiscal resources during the flight, which is also a concern. So mission expense and rocket fuel are correlated, sometimes competing mission design factors.

Insofar as the MPG is concerned, the final, post-design-phase spacecraft trajectories are supplied in the form of precision ephemerides. It is not concerned with orbital transfer, *per se*. However, ephemerides incorporating orbital changes have, in the past, caused MPG problems to arise, usually during the intervals surrounding the transfer. Aside from orbital changes due to solar pressure or ion engines, transfers are usually relatively short term[7] phenomena, as compared to the overall orbital duration.

[7] The orbit transfers here are considered instantaneous ones. In practice, they may be extended over a few seconds (as in motor thrusts) to a few minutes (planetary landings), to hours (aero braking). Typical spacecraft ephemerides during such periods are characterized by short, perhaps discontinuous records that create challenges to MPG generation of polynomial profiles of data types during these extents.

In order to conform to these short-term orbital adjustments, the MPG design was able to incorporate methods to detect and adjust its operation, without operator intervention. An understanding of the character of orbital transfer often proved useful in such cases to adapt MPG performance to the ephemeris phenomena.

Further, by simulating certain orbital characteristics in a *Mathematica* environment that gave rise to problems, MPG development was able to delve into the mathematical reasons for the problems causes and to devise correct responses to the errant behavior.

This section discusses some of the techniques used to achieve orbital transfers and their associated energy costs. Keplerian orbital descriptions will suffice to illustrate the principles involved.

Energy costs are typically cited in terms of the change in velocity required, at a given point in a trajectory, to effect the needed change. Termed *delta-v*, or $\Delta\mathbf{v}$, the change is comprised of in-plane and out-of-plane components. Coplanar changes can affect all elements except inclination, which is only affected by out-of-plane delta-v ,

If, at a given state $(\mathbf{r}_0, \mathbf{v}_0)$ of a Keplerian orbit, a given $\Delta\mathbf{v}$ is instantly applied, then the orbital angular momentum changes to

$$\begin{aligned} \mathbf{h} &= \mathbf{r}_0 \times (\mathbf{v}_0 + \Delta\mathbf{v}) = \mathbf{h}_0 + \Delta\mathbf{h} \\ \Delta\mathbf{h} &= \mathbf{r}_0 \times \Delta\mathbf{v} \end{aligned} \tag{7-92}$$

The remainder of the object's trajectory (until a new orbital adjustment) is then determined by the new state,

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_0 \\ \mathbf{v} &= \mathbf{v}_0 + \Delta\mathbf{v} \end{aligned} \tag{7-93}$$

The SPICE utility `OSCELT` can be used, given this state vector, to determine the new orbital parameters. However, while designing the orbital transfer, the reverse process may be required: given goals for the altered orbit, determine the needed delta-v to achieve them. In this case, one works from the given information to find the orbital elements required, using the element equations given in an earlier section, and then solves for the needed delta-v. However, the nonlinearity of the problem generally bodes against this procedure, except in idealized situations, so some iteration is usually required.

The delta-v required to change the semimajor axis, however, may be readily found by differencing the squared velocities just before and just after the transfer (where the position vector is yet unchanged), using the vis-viva equation that applies to each case, and solving for a. The result is that

$$a = \frac{a_0}{(v^2 - v_0^2)a_0 +} \tag{7-94}$$

Notice that the magnitude of the transfer velocity determines the semimajor axis, independently of the direction of velocity change.

By applying (7-39), the change in eccentricity is found to follow the relationship

$$e^2 - e_0^2 = \frac{(v^2 - v_0^2)\,h^2}{} - \frac{h^2 - h_0^2}{a_0} \tag{7-95}$$

Like semimajor axis, changes in eccentricity are determined by overall magnitudes of change in velocity and angular momentum, not by their directions.

7.5.1.1 In-Plane Orbital Adjustments

Applying an instantaneous coplanar delta-v at a given point in an orbit changes only the magnitude of the angular momentum, and thus affects potentially all of the orbital parameters except inclination and longitude of the ascending node, which are functions of the orientation of the angular momentum, and not its magnitude.

In particular, the application of a coplanar $\Delta\mathbf{v}$ produces an angular momentum $\mathbf{h}$ of the subsequent trajectory that is in the direction of the initial angular momentum, with magnitude

$$\begin{aligned} h &= h_0 + \Delta h \\ \Delta h &= r\,\Delta v \cos \end{aligned} \tag{7-96}$$

where is the flight angle (angle between $\Delta\mathbf{v}$ and tangent to $\mathbf{r}$). The subsequent apsis distances and velocities are related via

$$r_p\,v_p = r_a\,v_a = r_{0,p}\,v_{0,p} + \Delta h \tag{7-97}$$

The eccentricity, for example, changes by an amount

$$e - e_0 = [(v_p - v_{p,0})\, h_0 + v_p\, r\, \Delta v \cos \quad]/ \qquad \text{(7-98)}$$

7.5.1.2 Inclination Changes

Consider the instantaneous delta-v required to change only the inclination of the orbital plane by an amount Δ . In this case, the magnitudes of the initial and final velocity vectors are equal, but differ in direction. The difference vector $\Delta\mathbf{v}$ is the base of an isosceles triangle, whose magnitude Δv is, from elementary geometry,

$$\Delta v = 2\, v \sin\left(\frac{\Delta}{2}\right) \qquad \text{(7-99)}$$

As may be seen, inclination changes may require significant energy to complete. Such transfers in a mission are therefore made only when necessary and at points in the orbit where it is most economical to do so. Additional discussion of this topic appears in the next section.

7.5.2 Hohmann Transfer Orbits

The Hohmann transfer orbit is one half of an elliptic orbit that touches both the circular orbit that one wishes to leave and the circular orbit that one wishes to reach. The transfer is initiated by firing the spacecraft's engine in order to give it the required delta-v that will cause it to follow the elliptical orbit. When the spacecraft reaches its destination orbit, a new orbital transfer is initiated to enter the final circular orbit. Orbital transfers of this sort are almost always the most energy-efficient way to get from one circular orbit to another.

This maneuver is named after the German scientist Walter Hohmann, who published it in 1925, although the Russian mathematician Vladimir Vetchinkin is also known to have presented public lectures on this subject in 1921-1925. The technique, so important to modern space travel, was thus invented long before the space age.

These transfer orbits work to bring a spacecraft from a higher orbit into a lower one, as well as from a lower orbit into a higher one. In the former case, the spacecraft's engine is fired in the opposite direction to its current path, decelerating the spacecraft and causing it to drop into the lower-energy elliptical transfer orbit. The engine is then fired again in the lower orbit to decelerate the spacecraft into its final orbit.

The technique can be generalized somewhat, to relax the condition, that the spacecraft at the initiation of transfer, and the target at the completion of the transfer, be circular orbits. The orbit between, however, is required to be an elliptical path from the initial state to the final position. However, the treatment here will stick to the classic formulation, and, in particular, the elliptic trajectory between a point on an inner orbit to a point on an outer orbit such that the line connecting the two passes through the center of mass of the primary.

7.5.2.1 Coplanar Hohmann Trajectory

One of the assumptions of the Hohmann theory is that the transfer orbit between the two circular orbits actually touches each orbit; that is, that the point of encounter actually lies in the transfer orbit plane. This case will be first. The case in which this is not true is considered in the next subsection.

The point on the inner orbit will be the periapsis $\mathbf{r}_p$ of the orbit and that on the outer will be the apoapsis $\mathbf{r}_a$. The ratio of the magnitudes of these two vectors determines the orbital eccentricity, via Eq.(7-13),

$$\begin{aligned} \frac{r_p}{r_a} &= \frac{1-e}{1+e} \\ e &= \frac{1-r_p/r_a}{1+r_p/r_a} = \frac{r_a-r_p}{r_a+r_p} \end{aligned} \tag{7-100}$$

The semimajor axis is the average

$$a = \frac{r_p + r_a}{2} \tag{7-101}$$

The time required to traverse the Hohmann orbital segment is one-half the period of the ellipse given by Eq.(7-46), or

$$T_{p,a} = \sqrt{\frac{a^3}{}} \tag{7-102}$$

The velocity required at the starting point and that achieved at the ending point, i.e., at the two ends of the orbit, are given by Eq.(7-38). If, as the Hohmann theory assumes, the pre- and post-transfer orbits are circular, then the velocities before and after the maneuver are

$$\begin{aligned} v_{p,0} &= \sqrt{\frac{}{r_p}} \\ v_{a,0} &= \sqrt{\frac{}{r_a}} \end{aligned} \tag{7-103}$$

A subscript 0 is used on both, to designate that the theory applies to circular orbits. Therefore, the delta-v components required for orbital transfer from circular, to elliptic, and then back to circular at each end are

$$\begin{aligned} \Delta v_p &= \sqrt{\frac{}{r_p}}\left(\sqrt{1+e}-1\right) \\ \Delta v_a &= \sqrt{\frac{}{r_a}}\left(1-\sqrt{1-e}\right) \end{aligned} \tag{7-104}$$

Both of these are expressed as positive values, as the energy required to effect each is positive. The direction, however, must be chosen to boost or retard at one end and to do the same at the other. When transferring form an inner trajectory to an outer one, additional velocity is required to inject the spacecraft into the transfer orbit, and when it arrives, it still does not have the required orbital velocity, so an additional boost will be required. The opposite is true of an inward bound transfer. The spacecraft must be retarded in order for it to fall inward along the transfer orbit, and, when it reaches the inner circle, it must be slowed further to transfer into that circular orbit.

As an example, the Hohmann trajectory between Earth and Mars requires a launch delta-v of about 2945 m/s, and an encounter delta-v of about 2649 m/s.

7.5.2.2 Orbit Inclination Adjustment

While most of the planets lie near the ecliptic, their orbital planes all differ. In such cases, a Hohmann orbit chosen, for example, to leave Earth toward Mars would reach its destination only to find Mars perhaps too many kilometers distant for an economical delta-v to conform the two orbits. In general, when the target body at the terminus of the Hohmann orbit does not lie in the orbital plane, then a mid-course maneuver somewhere along the route is necessary.

The position of the midcourse correction can be chosen to be any point along the trajectory between injection and encounter; however, tremendous expense is incurred if made very near either of the endpoints, for the required inclination change in these cases is about 90º. The delta-v required at the end of an outward-bound Hohmann trajectory would be

$$\Delta v_a = 2\sqrt{\frac{(1-e)}{(1+e)\,a}} \tag{7-105}$$

If the correction is made at the semi-latus rectum point ($=$ $/2$), the velocity given by Eq.(7-22)) put into the formula of Eq.(7-99) sets the required delta-v at

$$\Delta v = 2\sqrt{\frac{}{a}\left(\frac{1+e^2}{1-e^2}\right)}\sin\left(\frac{}{2}\right) \tag{7-106}$$

where is the inclination of the target orbit relative to the Hohmann orbit. The ratio of the two delta-v requirements is

$$\frac{\Delta v}{\Delta v_a} = \frac{\sqrt{1+e^2}}{1-e}\sin(\ /2) \tag{7-107}$$

As an example, the Mars orbit is tilted relative to the Earth orbit by about $= 0.03$ radians, so a Mars-bound trajectory requires a maximum Δv at the semi-latus rectum of about 850 m/s. Since velocity is decreasing in the outward journey, the optimal burn, at a true anomaly of 0.57 , reduces the requirement somewhat, to 832 m/s. The ratio in Eq.(7-107) is 0.02, a factor of about 50!

7.5.3 Launch Trajectories

In order for an object at rest on the surface a body to enter into a orbit about the body, it must be launched to an elevation above the body's atmosphere (if there is one), accelerated to a sufficient velocity, and given a direction such that it does not (soon) reenter the atmosphere or impact upon its surface.

If energy efficiency is an issue, as it usually is, the path requiring the least propellant will make use of the body's rotation velocity by launching in the direction of the body rotation from a location as near the equator as feasible. The launch phase typically ends when the payload reaches a design state $(\mathbf{r},\mathbf{v})$, whose orbital elements may be found using `OSCELT`, or the formulas it embodies. If the orbit achieved at this point is not the intended final one, then subsequent orbital maneuvers are applied to attain the mission goals.

Trajectories and energy requirements of vehicles in powered flight are not described by Keplerian elements, except on an instantaneous basis, and are beyond the scope of discussion here.

Launch phase ephemerides are susceptible to the segmentation and dynamic behavior imposed by the assumed force and mass models of the launch process. The MPG is required to adapt to such ephemerides, however, without any external information about the mission itself or its dynamics.

7.5.4 Reentry and Landing Trajectories

Trajectories of vehicles in atmospheric reentry and landing on a planetary or moon surface are subject to counter thrusts required to initiate and control reentry, atmospheric drag (if there is an atmosphere present), and landing dynamics (such as bouncing off the surface, if aboard a balloon). Reentry does not follow Keplerian theory, but generally requires more sophisticated propagation methods, such as integration of the equations of motion.

Reentry phase ephemerides, like launch ephemerides, are susceptible to segmentation and dynamic behavior imposed by the assumed force and mass models of the reentry process. The MPG is again required to adapt to such ephemerides without any external information about the mission itself or its dynamics.

7.5.5 Synchronous Orbits

The MPG does not generally deal with spacecraft in geosynchronous orbit, but some characteristics of the orbit may be of relevance, as a topic of general interest. The geosynchronous orbit is ideally circular, at zero inclination, with period exactly one sidereal day. The satellite distance from the geocenter follows from Eq.(7-46),

$$a = \left[\left(\frac{T}{2} \right)^2 \right]^{1/3} = 42{,}164.2\text{km} \tag{7-108}$$

or 35,786 km above the Earth surface. This orbit is attained in two stages: first, a Hohmann orbit places the satellite into an orbit with the apogee above, and then, an adjustment is made to circularize the orbit by the appropriate thrust at apogee (sometimes humorously referred to as a *kick in the apogee*). If the original apogee cannot be located in the equatorial plane, a third maneuver may be necessary to zero the inclination.

7.5.6 Polar Orbits

Orbits whose inclinations are (nearly) 90º are useful for mapping and surveillance operations, because, as the body rotates, the entire surface area comes into view.

7.5.7 Walking Orbits

A walking orbit, also known as a precessing orbit, is one in which the orbital plane moves slowly with respect to inertial space. The rate of precession can be intentionally controlled by choosing the orbital parameters to take advantage of some or all of the gravitational influences that cause orbital precession. Such factors include the non-spherical distribution of mass of the primary body, as well as the gravitational attractions of other bodies, such as the Sun or nearby moons.

Orbital precession is due to the secular motion of the orbital plane, and, when oblateness of the primary is its chief causal agent, its rate is given by Eq.(7-79),

$$\dot{\Omega} = -\frac{3}{2} n\, J_2 \left(\frac{R}{r_p(1+e)} \right)^2 \cos i \tag{7-109}$$

For a given primary equatorial radius R and J_2 factor, orbital parameters may be chosen to provide the precession desired.

7.5.8 Sun Synchronous Orbits

Sun synchronous orbits are walking orbits whose orbital plane precesses at the same rate as the primary's solar orbit. Such orbits are useful for satellites that may require a constant angle of illumination from the Sun. Orbital adjustments may be occasionally warranted in order to maintain the correct orientation.

7.5.9 Molniya Orbits

The Molniya (Russian for "lightening") orbit is a highly eccentric Earth orbit (HEO) of period of about 12 hours, whose orbital parameters have been chosen to provide long periods of operation over the northern hemisphere and to set the rate of change in perigee to zero. As indicated in Eq.(7-79), this requirement will be fulfilled if the inclination is chosen to satisfy

$$4 - 5\sin^2 i = 0 \tag{7-110}$$

The inclination in this case is 63.4º or 116.6º.

The eccentric anomaly of the orbit at the semi-latus rectum point $(\ = \pm \ /2)$ is, by Eq.(7-49), equal to $E = \cos^{-1} e$, so the mean anomaly at this point is

$$\begin{aligned} M_{\ /2} &= \cos^{-1} e - e\sin(\cos^{-1} e) = \cos^{-1} e - e\sqrt{1-e^2} \\ &= \sqrt{\frac{\ }{a^3}}\, t_{\ /2} \end{aligned} \tag{7-111}$$

The time from perigee may thus be calculated. The total time from one semi-latus rectum point to the next, passing through apogee, is then found to be

$$T_{visible} = T - 2t_{\ /2} = 2\sqrt{\frac{a^3}{\ }}\left(1 - \frac{\cos^{-1} e - e\sqrt{1-e^2}}{\ }\right) \tag{7-112}$$

The semimajor axis that gives a half-sidereal-day orbit is $a = 26{,}658\,\mathrm{km}$, and the eccentricity that keeps the trajectory in the northern hemisphere for 11/12 (i.e., 11 hours) of this time will be $e = 0.724$. The perigee and apogee are

$$\begin{aligned} r_p &= 7{,}358\,km = 980\,kmalt \\ r_a &= 45{,}959\,km = 39{,}582\,kmalt \end{aligned} \tag{7-113}$$

Actual Soviet Molniya orbits averaged $1507 \times 39{,}305$ km, which correspond to

$$\begin{aligned} a &= 2{,}6784\mathrm{km} \\ e &= 0.706 \\ T &= 0.503\mathrm{day} \\ T_{above} &= 11.009\mathrm{hours} \end{aligned} \tag{7-114}$$

7.5.10 Gravity Assisted Orbits

A gravity assisted orbital maneuver is an example of the so-called *restricted three-body problem*, in which one body of insignificant mass is influenced two others having more significant masses. The two larger-mass bodies, say the Sun and Jupiter, orbit their barycenter, while the other, say a spacecraft, takes an orbit influenced by the other two. By directing the smaller mass toward a close encounter with the secondary mass body, a net change in orbital energy can be attained, sending the spacecraft toward an intended target.

Much work has been done over the years toward solving this problem without resorting to numerical methods, but all rigorous attempts so far have failed. The method was considered in modern space navigation by the Russian Academy of Sciences in 1959. Michael Mintovitch, however, who came to JPL in 1961, is credited with being the first person to thoroughly explore the concept of gravity assist transfers between multiple bodies in any order.

The technique was first used in the Pioneer 10 mission, in which the spacecraft was boosted beyond the Sun's escape velocity by a flyby of Jupiter in 1973. In 1974, Mariner 10 fly by of Venus in 1974 produced a transfer orbit toward Mercury. Since then, it has been regularly applied to many other missions. The method is discussed below.

While the precise path of the spacecraft during the time it is within the transition region between the two gravitational influences requires solution by rigorous means, the gravitational assist mechanism itself may be explained in rather simpler terms. If the spacecraft state $(\mathbf{r}_0, \mathbf{v}_0)$ at entry into the planet's sphere of influence and the planet's state $(\mathbf{r}_p, \mathbf{v}_p)$ at this time are given, then the exit state $(\mathbf{r}_1, \mathbf{v}_1)$ can be estimated as follows:

Since the spacecraft is presumed to be in the planet's sphere of influence, the flight in the planetocentric reference frame may be approximated by a conic trajectory, and, in this case, a hyperbolic orbit. The planetocentric frame is parallel to the inertial frame, but is centered at the planet, rather than the SSB. The state of the spacecraft in this frame at entry will be

$$(\mathbf{r}_{0,p}, \mathbf{v}_{0,p}) = (\mathbf{r}_0, \mathbf{v}_0) - (\mathbf{r}_p, \mathbf{v}_p) \tag{7-115}$$

This state determines the path's set of orbital elements in the planetocentric frame. These parameters may then be used to propagate the conic state to a point

$(\mathbf{r}_{1,p}, \mathbf{v}_{1,p})$ having the equal true anomaly, but beyond periapsis. This is the point at which the spacecraft is exiting the planet's sphere of influence, and is such that

$$\begin{aligned} |\mathbf{r}_{1,p}| &= |\mathbf{r}_{0,p}| \\ |\mathbf{v}_{1,p}| &= |\mathbf{v}_{0,p}| \end{aligned} \tag{7-116}$$

In planetocentric terms, then, the maneuver has changed the spacecraft velocity in orientation, but not in magnitude. The velocity turning angle for the hyperbolic orbit is $(\quad - 2\cos^{-1}(1/e))$.

The exit state in inertial space is then

$$(\mathbf{r}_1, \mathbf{v}_1) = (\mathbf{r}_{1,p}, \mathbf{v}_{1,p}) + (\mathbf{r}_p + 2\,\Delta t\, \mathbf{v}_p, \mathbf{v}_p) \tag{7-117}$$

where Δt is the time to periapsis found from the osculating element set. The velocity of the planet is presumed to be relatively constant over the spacecraft flyby period and equal to $\mathbf{v}_p$.

The delta-v incurred in this maneuver can now be estimated to be

$$\begin{aligned} \Delta\mathbf{v} &= \mathbf{v}_1 - \mathbf{v}_0 = \mathbf{v}_{1,p} + \mathbf{v}_p - \mathbf{v}_{0,p} + \mathbf{v}_p \\ &= \mathbf{v}_{1,p} - \mathbf{v}_{0,p} + 2\,\mathbf{v}_p \end{aligned} \tag{7-118}$$

The spacecraft kinetic energy has changed by the factor

$$\frac{\ _1}{\ _0} = \frac{v_1^2}{v_0^2} \tag{7-119}$$

This phenomenon presents a seeming paradox. Physics teaches that the total energy, potential plus kinetic, of a test particle is constant as it moves through the gravitational field of a far more massive body. Its speed increases as it nears the other body and the direction of its velocity vector changes as it passes the body. As it recedes from the encounter, its speed decreases again. The total energy in this situation (the two body problem) is constant.

However, as seen from the above analysis, there is a net change in kinetic energy over a relatively short period during which the potential energy relative to a third, even more massive body, stays relatively constant. As the spacecraft recedes from the planet, its momentum has been changed by an amount $\Delta\mathbf{p} = m\,\Delta\mathbf{v}$ and the planet's momentum is then $M\,\mathbf{v}_p - \Delta\mathbf{p}$. Here m represents the mass of the spacecraft, and M, that of the planet. The planet's velocity immediately after encounter thus becomes $\mathbf{v}_p - \Delta\mathbf{p}/M$, so the change in its kinetic energy is

$$\begin{aligned}\Delta_{p,k} &= \frac{1}{2}M(\mathbf{v}_p - \Delta\mathbf{p}/M)\bullet(M\,\mathbf{v}_p - \Delta\mathbf{p}/M) - \frac{1}{2}M\,v_p^2 \\ &\approx -\mathbf{v}_p\bullet\Delta\mathbf{p} = -m\,\mathbf{v}_p\bullet\Delta\mathbf{v}\end{aligned} \tag{7-120}$$

The planet therefore loses a slight amount of energy that is proportional to the spacecraft mass, its change in velocity, the planet's speed, and the cosine of the flight angle. This loss is the energy gained by the spacecraft.

7.6 NAIF Ephemeris Types

Ephemerides that are accessed by the MPG are not all of the same format, although the information retrieved from them using the SPICE toolkit makes them appear to be. Although the differences between accessed ephemeris data is hidden from the MPG as a user, a knowledge of the different data types has been found useful when developing or debugging a particular application. For this reason, and to display the range of forms of representation, the various SPICE ephemeris types are enumerated and briefly discussed below. The interested reader may consult the SPICE required reading file SPK.REQ for further details.

The SPICE file types are labeled numerically, currently 1 to 18, and are mainly differentiated in the way data are represented rather than by the method by which data were generated.

The trajectories of objects found in Spacecraft and Planetary Kernel (SPK) files are represented in pieces called segments. A segment represents some arc of the full trajectory of a specified object. Each segment contains information that specifies the geometric state of an identified target object relative to an identified center of motion in a designated reference frame over some particular interval of time. Either body may be a spacecraft, a planet or planet barycenter, a satellite, a comet, an asteroid, a tracking station, a roving vehicle, or an arbitrary point for which an ephemeris has been calculated. From the point of view of the SPK system, segments are the atomic portions of a trajectory.

The epochs corresponding to the states are barycentric dynamical times (TDB), expressed as seconds past J2000.

SPK files are binary files. The data in each segment are stored as an array of double precision numbers. The format of these binary files is based upon a more abstract file architecture called Double precision Array File, or DAF. However, being binary, they must be produced or translated using SPICE utilities into forms suitable for each particular MPG host in use.

Each segment contains a summary descriptor, which contains the initial and final epochs of the interval; the NAIFIDs of the target, center, and reference frame; the ephemeris type number; and locators for beginning and end of the segment.

The SPICE toolkit has a number of utilities that are used to access ephemeris states. Perhaps those that are most used in the MPG are `SPKEZ` and `SPKEZR`, which return states of any target object possessing a NAIFID as observed by any other NAIFID at a given ephemeris time for a specified aberration type, provided the needed underlying ephemerides have been supplied. The former utility requires the NAIFID in numeric form, while the latter also accommodates text name equivalents of the NAIFID numbers.

7.6.1 Modified Difference Arrays

Type 1 ephemerides are created by the JPL Orbit Determination Program[8] (ODP), and used primarily for integrated spacecraft ephemerides. The file records in this representation are polynomial profiles of position and velocity in a form referred to as "modified difference arrays," or MDAs. These are a revised form of Newton's divided-difference coefficients, as presented in an earlier chapter. Unfortunately, the method of interpolation is largely undocumented, and a mystery to those who ported the pre-existing code into the SPICE library.

MDA files were the principle form of ephemerides used by the MPG predecessor, the Network Support Subsystem Metric Prediction application. They are often referred to as "P-files" or "PV-files" or "NIO files." The SPICE versions of these ephemerides organizes the polynomial records into SPICE's standard file format, and little else.

[8] Often referred to at JPL as the DPODP, or double precision ODP, to differentiate it from an earlier incarnation. As used here, the term applies to a system of programs, including PV, REGRES, and DPTRAJ.

Unfortunately, the method of interpolation is undocumented, and a mystery to those who ported the pre-existing code into the SPICE library. For this reason, only the ODP is capable of generating output that can be made into Type 1 files.

Ephemerides of spacecraft during launch and reentry or landing phases are typically composed of a series of records in which the ODP has modeled force effects and mass changes, such as are due to motor thrusts, solar wind, aerodynamic braking, fuel depletion, and rocket staging. The integrator in the ODP appears to restart at each time boundary where thrust or mass models abruptly change. The ephemeris records in these regions typically appear to become very short, growing longer as the model becomes better established.

The NAIF utility SPKBRK locates the boundaries of ephemeris records at which possible discontinuities appear, so that the MPG curve fitting object may also observe these boundaries.

7.6.2 Chebyshev Polynomials, Position Only

Type 2 SPICE files contain sets of coefficients of Chebyshev polynomials for the position of a body as a function of time. The coefficients for each position polynomial are presumed to apply to the same time span. All polynomials in Type 2 ephemerides have the same degree.

These files typically describe the paths of planet barycenters and satellites whose ephemerides are integrated. Since only the position of the body is modeled by the polynomial profile, the body's velocity is found by differentiating the position polynomial.

The method for evaluation follows that given in the chapter on Interpolation.

7.6.3 Chebyshev Polynomials, Position and Velocity

Type 3 SPICE files contain sets of coefficients of Chebyshev polynomials for both position and velocity of a body as a function of time. This data type is typically used for satellites for which the ephemeris is computed from analytical theories.

The structure of these files is much like that of Type 2 files, except for the added velocity coefficients. As in Type 2 files, the coefficients for each position and velocity polynomial are presumed to apply to the same time span. All polynomials in Type 3 ephemerides have the same degree.

The method for evaluation follows that given in the chapter on Interpolation.

7.6.4 Reserved for future use

There are, as of this writing, no Type 4 SPICE files. The type was reserved for future implementation of a style referred to as "TRW elements for TDRSS and Space Telescope," which is not further described.

7.6.5 Discrete States

Type 5 SPICE files contain sets of discrete states of the body over a specified period of time. The representation of the body state at any other time is found using weighted two-body propagation. Files of this type typically are used for comets and asteroids, whose ephemerides are integrated from some given initial state or osculating elements.

The evaluation method calculates the state of a body at a given time that lies between the instants of two recorded states by propagating the initial and final states to the given time using two-body theory, and then taking a weighted average of the two results. The weighing function is nonlinear, giving a proportionately higher preference to the nearest recorded state's estimate and an equal weighting in the middle of the interval.

7.6.6 Reserved

The Type 6 SPICE file format is reserved, intended for future implementation of an analytic model for the orbits of Phobos and Deimos.

7.6.7 Reserved

The Type 7 SPICE file format is reserved, intended for future implementation of precessing classical elements such as used by the Space Telescope Science Institute.

7.6.8 Equally Spaced Discrete States

Type 8 SPICE files contain sets of state vectors at equally spaced times over a specified interval. The geometric state at any given time within the overall interval is evaluated using Lagrange interpolation of the component samples. Each segment in the file contains the degree of the polynomial to be used in performing the interpolation. SPICE makes special use of the equal spacing of samples to improve run-time speed and efficiency over Type 9 files.

7.6.9 Unequally Spaced Discrete States

Type 9 SPICE files contain sets of state vectors at sample times that may be irregularly spaced over the specified coverage interval. The geometric state at any given time within the overall interval is evaluated using Lagrange interpolation of the component samples in the vicinity of the given time. Each segment in the file contains the degree of the polynomial to be used in performing the interpolation. Because of the possible non-uniformity in spacing of sample states, interpolation suffers somewhat in run-time speed and efficiency as compared to Type 8 files.

7.6.10 Space Command Two-Line Elements

The SPICE data Type 10 uses SPICE generic segments to store a collection of packets each of which models the trajectory of some Earth satellite using the Space Command Two-Line Element (TLEs) format, formerly known as the North American Air Defense (NORAD) SGP4/SDP4 model.

Algorithms for TLE propagation are described in the Spacetrack 3 Report (see [Hoots & Roehrich 1980]). However, that implementation contained several programming errors, which are reported as being corrected in the SPICE toolkit.

The TLE format contains 14 numeric values on the first line and 10 on the second. Together, these identify and classify the satellite and its launch, ephemeris type and number, the epoch of the elements, the orbital elements themselves, derivatives of some orbital elements, and a ballistic drag coefficient that represents the susceptibility of the object to orbital decay. The six orbital elements provided are inclination, ascending node, eccentricity, argument of perigee, mean anomaly at epoch, and mean motion. The ephemeris designation indicates the complexity of the model and whether the body is considered a near-Earth or deep-space trajectory.

The SPICE Type 10 record format additionally contains the gravitational constants that apply to the orbit, such as GM, J_2, J_3, etc., the planet's equatorial radius and atmospheric model heights.

The most important point to be noted about the TLE format is that the parameters it contains are so-called "mean" values obtained by removing periodic variations in a particular way peculiar to the ephemeris model. *In order to obtain good predictions, these periodic variations must be reconstructed by that model which reinserts these components in exactly the same way they were removed by NORAD.* Using the given parameters in a different model will result in degraded predictions, even though that model may be based on a higher accuracy theory.

7.6.11 Reserved

The intended use of the SPICE Type 11 format is unspecified.

7.6.12 Hermite Interpolation, Uniform Spacing

The twelfth SPICE data type represents a continuous ephemeris by a discrete set of states and a sliding window Hermite interpolation method. The epochs, or *time tags*, that are associated with the states are evenly spaced by a positive constant STEP such that each time tag differs from its predecessor by STEP seconds.

For any requested epoch, the corresponding state is found by interpolating a set of consecutive states, or window, centered as closely as possible about the requested epoch. Interpolated position values are obtained for each coordinate by fitting a Hermite polynomial to the window's set of position and velocity values for that coordinate. The velocity that is returned is obtained by differentiating the position polynomials.

The SPICE system also represents ephemerides using unequally spaced discrete states and Hermite interpolation in the SPICE Type 13 format. Ephemerides of Type 13 sacrifice some run-time speed and economy of storage in order to achieve greater flexibility.

Each segment also has a polynomial degree associated with it. This is the degree of the interpolating polynomials to be used in evaluating states based on the data in the segment. The identical degree is used for interpolation of each state component.

The interpolation method uses Newton's divided difference formula, as interpreted for function and derivative samples, as described in the Interpolation chapter of this work. The interested reader may consult the SPICE `HRMESP` (Hermite Interpolation, Equally Spaced samples) function commentary for specifics.

7.6.13 Hermite Interpolation, Nonuniform Spacing

The thirteenth SPICE ephemeris type, as was also the Type 12 format, a representation of a continuous trajectory using a discreet set of states and a sliding window Hermite interpolation method. However, in this case, the time tags associated with the set of states are not necessarily evenly spaced. The rest of the functionality is the same as given for Type 12 files, however.

The interpolation method in this case also uses Newton's divided difference formula, as interpreted for function and derivative samples, as described in the Interpolation chapter of this work. The interested reader may consult the SPICE `HRMINT` (Hermite Interpolation, non-equally spaced samples) function commentary for specifics.

7.6.14 Chebyshev Polynomials, Nonuniform Spacing

The Type 14 ephemeris representation stores a collection of packets each of which models the trajectory of the given object with respect to another over some interval of time. Each file packet contains coefficients for Chebyshev polynomials that approximate the position and velocity components of the identified object over the interval of time spanned by the start and end times of the segment. The time intervals corresponding to each packet are non-overlapping.

Unlike Types 2 and 3, the time spacing between sets of coefficients for a Type 14 segment may be non-uniform. Further, Type 14 segments contain more meta data than do Types 2 and 3.

Each partition contains a numeric value that defines the degree of the Chebyshev representation. The maximum degree that can (currently) be accommodated is 18.

The method of evaluation follows that given in the Interpolation chapter of this work.

7.6.15 Precessing Conic Propagation

The Type 15 ephemeris is a continuous trajectory assumed to follow a compact analytic model. The specified object's trajectory is modeled as two body motion, in which the object orbits the identified central body under the influence of the central body's mass, perturbed by first order secular effects of the J_2 term in harmonic expansion of the central body gravitational potential. These secular effects were described earlier in this chapter.

Each segment in the file defines orbital parameters, including the epoch of periapsis, the orbital pole vector, the unit vector in the direction of periapsis, the semi-latus rectum, and eccentricity, along with parameters of the central body, including its GM, equatorial radius, J_2, and pole vector. The secular effects of J_2 are not applied if the orbital eccentricity is not less than unity.

At any requested epoch, the body's position is computed using conic propagation with secular corrections as described earlier in this chapter.

7.6.16 Reserved (European Space Agency ISO)

The SPICE Type 16 format is reserved for ephemeris elements of European Space Agency ISO spacecraft.

7.6.17 Equinoctial Elements

The SPICE Type 17 format represents a continuous ephemeris using a compact analytic model, in which the identified object is presumed to be following an elliptic orbit with a precessing line of nodes and argument of periapsis relative to the equatorial frame of an identified central body. The orbit is modeled using equinoctial elements.

Each segment of the Type 17 ephemeris contains orbital information that includes the epoch of periapsis; mean longitude at epoch; semimajor axis; mean longitude rate; the h, k, p, and q equinoctial elements (described below); and the secular rates of change in longitude of periapsis, longitude of the ascending node, assumed constant over the time interval between segments. The body information includes the right ascension and declination of the equatorial pole.

In an earlier section of this chapter, the six osculating elements, $(r_p, e, i, \Omega, \ , t_p)$ were introduced, all of which had physical interpretations, and which could be used to propagate conic motion to any requested epoch. It was shown that the six values present in a given state vector were another set of values that could be used to extract these osculating elements and to propagate the conic motion. The six quantities that define a conic are thus not unique.

As discussed earlier, the ascending node in an orbit of zero inclination and the periapsis of a circular orbit are indeterminate quantities. To avert difficulties in describing orbits having low inclinations and/or eccentricities, other parametric orbital descriptors have been proposed.

In 1970, J. L. Arsenault introduced a set he termed *equinoctial elements* that have since been studied and modified by a number of authors since. Four of these elements do not have easily identified physical interpretations, but all are stable at low inclinations and eccentricities. Moreover, JPL's Roger Broucke was able in 1971 to show [Broucke & Cefola 1972] that propagation using these elements could be extended to include secular perturbation terms affecting the motions of the periapsis and ascending node.

The set of equinoctial elements defined by Arsenault and Broucke are listed below, along with their osculating element equivalents.

$$\begin{aligned} a &= a = r_p/(1-e) \\ &= M + \quad + \Omega \\ h &= e\sin(\quad + \Omega) \\ k &= e\cos(\quad + \Omega) \\ p &= \tan(i/2)\sin\Omega \\ q &= \tan(i/2)\cos\Omega \end{aligned} \tag{7-121}$$

The osculating elements may be written in terms of equinoctial elements as

$$
\begin{aligned}
e &= \sqrt{h^2 + k^2} \\
r_p &= a(1 - e) \\
i &= 2\tan^{-1}(\sqrt{p^2 + q^2}) \\
\Omega &= \tan^{-1}(q, p) \\
&= \tan^{-1}(kq + hp, hq - kp) \\
M &= \quad - \tan^{-1}(k, h)
\end{aligned}
\tag{7-122}
$$

In this notation, the two-argument arc tangent function is the quadrant-corrected computation $\tan^{-1}(x,y) = \tan^{-1}(y/x)$.

The method of propagation follows that given in the reference and is embodied in the SPICE utility `EQNCPV` (Equinoctial Elements to Position and Velocity).

7.6.18 MEX/Rosetta Hermite/Lagrange Interpolation

The SPICE Type 18 format is meant to be flexible enough to accommodate multiple mathematical representations of ephemerides as may evolve over time, in such a way as to avoid further proliferation of SPK data types. Originally meant to support ephemerides used by the European Space Agency (ESA) on the Mars Express (MEX) and Rosetta missions, the SPK Type 18 architecture was made very general, so applicability is not limited to these missions.

Each Type 18 mathematical representation of an ephemeris data set is identified as a *subtype*. As of this writing, SPICE supports two Type 18 subtypes, designated 0 and 1.

Subtype 0 implements separate sliding-window Hermite interpolations of position and velocity. Each ephemeris segment is represented as a series of 12 extended state vectors and their associated time tags, which may irregularly spaced. Each extended state vector contains position, velocity, and acceleration vectors in Cartesian coordinates. Position interpolation uses position and velocity samples to form a Hermite polynomial, from which the position is calculated. Similarly, velocity interpolation uses velocity and acceleration samples to form a Hermite polynomial, from which the velocity is computed. The same interpolation degree is used for each position and velocity component. An interpolated velocity found by differentiating the position polynomial may not agree with that interpolated using the velocity polynomial, except at the given time tags.

Subtype 1 implements separate sliding-window Lagrange interpolations of position and velocity. Each ephemeris segment in this case is represented as a sequence of 6 state vectors in Cartesian coordinates and their associated time tags, which may be irregularly spaced. Position components are interpolated separately from velocity components, and thus are, in principle, independent. Interpolated velocities found by differentiating the position polynomial may not be equal to the corresponding values found by interpolating the velocity polynomial. The same interpolation degree is used for each position and velocity component.

The sliding-window interpolation works by defining a set of consecutive time tags, or window, centered as closely as possible on the request epoch. The window size is somewhat variable, but in any case never includes more than half the specified nominal number of points on either side of the requested time.

Lagrange and Hermite interpolations are carried out using SPICE functions `LGRINT` and `HRMINT`, respectively.

7.7 MPG Special Ephemeris Utility `SPKBRK`

The MPG curve fitting object generates a polynomial profile that describes a required DSN data type within a prespecified error over a requested time interval. Experience has shown that special care is required not to generate any of the polynomials making up the profile using ephemeris data that derive from separate segments for the same body in the chain between observer and target. When this condition is not met and discontinuities appear at a segment boundary, the resulting polynomial exhibits an error pattern that is referred to as “ringing.”

For example, suppose that a particular DSS begins observing, at the TAI time $_1$, the state of a spacecraft about Mars relative to the DSS at the spacecraft time $_2$ that is one light-time earlier. In computing the observed state vector, SPICE accesses ephemerides of the DSS relative to Earth, Earth relative to the SSB, Mars relative to the SSB, and the spacecraft relative to Mars. The epoch at which each ephemeris is accessed is found in a particular segment of that ephemeris file. In order to avoid ringing, it is sufficient that all other accesses used to generate that polynomial have times that are covered by these same segments. The maximum ending time that can apply to a polynomial is the least time boundary among the segments involved.

The NAIF special routine `SPKBRK` determines these boundaries. The example above corrected for light-time (planetary) aberration. The same conditions apply to other forms of aberration (including none). For a specified state vector defined by time, target, observer, type of aberration correction, light-travel direction, and reference frame, `SPKBRK` finds the maximal interval containing the requested time, over which the state is continuous. It also finds the upper bound of the previous neighboring continuity interval and the lower bound of the subsequent neighboring continuity interval.

Under certain circumstances `SPKBRK` reports small gaps in coverage between segments. These gaps are generally small, but since MPG predictions must cover an entire given time interval, these situations require that some bridge be generated to cover the breech. The cubic splines discussed in the Interpolation chapter are applied in these cases.

Since the beginning time of a polynomial is, in some sense, random with respect to segment boundaries, some polynomial intervals may be arbitrarily short when the beginning time corresponds to a segment time that is very near the end of the segment. It is also typical of trajectories undergoing launch, midcourse correction, and landing maneuvers that ephemeris segments can be very short.

Not all segment boundaries produce discontinuities in the accessed state vectors. But some do, and to assure that fidelity requirements could always be met, MPG development came to observe all segment boundaries as polynomial boundaries.

It would perhaps have been of advantage to augment the `SPKBRK` utility to check for continuity of the returned state vector at segment boundaries. However, no such adaptation was made in the first operational version of the MPG. Subsequent studies have shown, if no discontinuities in position or velocity are present, that the effects of ringing are not readily discernable. It would be possible, in such cases, to extend polynomial fit intervals to limits imposed by interpolation error.

`SPKBRK` is capable of operating correctly as advertised with all of the SPICE ephemeris types that were accommodated as of its completion date. However, should the SPICE toolkit later be expanded to include other ephemeris types, `SPKBRK` may require alteration if the MPG is required also to accommodate these types.

References

[Abramowitz & Stegun] Abramowitz, Milton, and Stegun, Irene A., *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series 55, U.S. Government Printing Office, Washington, DC, June 1964, 25.1.4 and 25.2.26.

[Broucke & Cefola 1972] Broucke, R. A., and Cefola, P. J., "On the Equinoctial Elements," Celestial *Mechanics* **5** (1972) pp. 303-310.

[Hoots & Roehrich 1980] Hoots, F. R., and Roehrich, R. L., *Spacetrack Report No. 3, Models for Propagation of NORAD Element Sets*, December, 1980, Compiled by T. S. Kelso, December 1988, U.S. Dept. of Commerce NTIS, 5285 Royal Rd., Springfield, VA, 22161.

[Irving & Mullineux1959] Irving, J., and Mullineux, N., *Mathematics in Physics and Engineering*, Academic Press, New York, 1959.

[Krogh1972] Krogh, Fred T., *Lecture Notes in Mathematics*, 362, Springer-Verlag, New York, 1972, p. 22.

[Lass1950] Lass, Harry, *Vector and Tensor Analysis*, McGraw-Hill Book Company, Inc., New York, 1950.

[Moyer1971] Moyer, T. D., "Mathematical Formulation of the Double Precision Orbit Determination Program,", Technical Report 32-1527, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA, May 15, 1971.

[Moyer2000] Moyer, T. D., *Formulation for Observed and Computed Values of Deep Space Network Data Types for Navigation*, JPL Publication 00-7, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA, October 2000.

[Seidelman1992] Seidelmann, K. P. (ed), *Explanatory Supplement to the Astronomical Almanac*, University Science Books, Mill Valley, CA,1992.

[SPK.REQ2004] *SPK File Format Description*, Navigation and Ancillary Information Facility Required Reading SPK.REQ, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA, October 2004.

[Standish2007] Standish, E. Myles, and Williams, James G., "Orbital Ephemerides of the Sun, Moon, and Planets," in preparation for IAU publication, http://iau-comm4.jpl.nasa.gov/XSChap8.pdf, Guidance, Navigation, and Control, Solar System Dynamics Group, Jet Propulsion Laboratory, Pasadena, CA, 2007.

[Vallado2006] Vallado, D. A., Crawford, P., Hujsak, R., and Kelso, T. S., Revisiting Spacetrack Report #3, American Institute for Aeronautics and Astronautics, Inc., Report 2006-6753, Reston, VA, 2006.

[Yeomans2006] Standish, E. Myles, "Keplerian Elements for Approximate Positions of the Major Planets," Guidance, Navigation, and Control, Solar System Dynamics Group, http://ssd.jpl.nasa.gov/?planet_pos, Jet Propulsion Laboratory, Pasadena, CA., Feb. 2006.