一、(每小题6分,共12分)用定义证明下列极限:

1. 用$\varepsilon-N$语言证明$\lim\limits_{n\to\infty}\dfrac{\sin n}{n}=0$;

证明:$\forall\varepsilon>0$,取$N=[\dfrac{1}{\varepsilon}]+1$,则当$n>N$时,有

$$\left|\frac{\sin n}{n}-0\right|\le\frac{1}{n}<\varepsilon,$$

故$\lim\limits_{n\to\infty}\dfrac{\sin n}{n}=0.$

2. 用$\varepsilon-\delta$语言证明$\lim\limits_{x\to1}\dfrac{x^3-1}{x-1}=3.$

证明:$x\ne1$时,$\left|\dfrac{x^3-1}{x-1}-3\right|=\left|x^2+x-2\right|=\left|x+2\right|\cdot\left|x-1\right|$,

不妨设$0<\left|x-1\right|<1\Rightarrow\left|x+2\right|=\left|x-1+3\right|\le\left|x-1\right|+3<4\Rightarrow\left|\dfrac{x^3-1}{x-1}-3\right|<4\left|x-1\right|$,

$\forall\varepsilon>0$,取$\delta=\min(1,\dfrac{\varepsilon}{4})$, 则当$0<\left|x+1\right|<\delta$时,有

$$\left|\frac{x^3-1}{x-1}-3\right|<\varepsilon,$$

故$\lim\limits_{x\to1}\dfrac{x^3-1}{x-1}=3.$

二、(每小题6分,共30分)求极限:

1. $\lim\limits_{n\to\infty}(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}})$;

解:$\dfrac{n}{\sqrt{n^2+n}}\le\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}\le\dfrac{n}{\sqrt{n^2+1}}$,

又$\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+1}}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\dfrac{1}{n^2}}}=1$, $\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+n}}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\dfrac{1}{n}}}=1$,

故$\lim\limits_{n\to\infty}(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}})=1.$

2. $\lim\limits_{n\to\infty}\left(\dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\cdots+\dfrac{1}{n\cdot(n+1)}\right)$

解:原式$=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{3}+\cdots+\dfrac{1}{n}-\dfrac{1}{n+1}\right)$

$$=\lim_{n\to\infty}(1-\frac{1}{n+1})=1.$$

3. $\lim\limits_{x\to+\infty}\dfrac{x\sin x}{x^2-4}$;

解:(方法一) $\lim\limits_{x\to+\infty}\dfrac{x}{x^2-4}=0$，$|\sin x|\le 1$，则$\sin x$有界，故$\lim\limits_{x\to+\infty}\dfrac{x\sin x}{x^2-4}=0.$

(方法二) $0\le\left|\dfrac{x\sin x}{x^2-4}\right|\le\dfrac{|x|}{x^2-4}$，而$\lim\limits_{x\to+\infty}\dfrac{|x|}{x^2-4}=0$，

则$\lim\limits_{x\to+\infty}\left|\dfrac{x\sin x}{x^2-4}\right|=0$，故$\lim\limits_{x\to+\infty}\dfrac{x\sin x}{x^2-4}=0.$

4. $\lim\limits_{x\to0}\dfrac{\sin 4x}{\sqrt{x+1}-1}$;

解：原式$=\lim\limits_{x\to0}\dfrac{\sin 4x\cdot 4x\cdot(\sqrt{x+1}+1)}{4x(\sqrt{x+1}-1)(\sqrt{x+1}+1)}=\lim\limits_{x\to0}\left(\dfrac{\sin 4x}{4x}4(\sqrt{x+1}+1)\right)=\lim\limits_{x\to0}4(\sqrt{x+1}+1)=8.$

5. $\lim\limits_{x\to0}(\dfrac{2-x}{2})^{\frac{1}{x}}$.

解：原式$=\lim\limits_{x\to0}(1-\dfrac{x}{2})^{(-\frac{2}{x})(-\frac{x}{2})\frac{1}{x}}=\lim\limits_{x\to0}\left[(1-\dfrac{x}{2})^{(-\frac{2}{x})}\right]^{-\frac{1}{2}}=e^{-\frac{1}{2}}.$

或原式$\overset{y=-\frac{x}{2}}{=}\lim\limits_{y\to0}(1+y)^{\frac{1}{-2y}}=\lim\limits_{y\to0}\left[(1+y)^{\frac{1}{y}}\right]^{-\frac{1}{2}}=e^{-\frac{1}{2}}.$

三、(每小题5分,共20分)求下列函数的导数$\dfrac{dy}{dx}$:

1. $y=\dfrac{x\cos x-\ln x}{x+1}$;

解：$y'=\dfrac{(1\cdot\cos x-x\sin x-\frac{1}{x})(x+1)-(x\cos x-\ln x)\cdot 1}{(x+1)^2}$

$$=\frac{-x^2\sin x-x\sin x+\cos x+\ln x-1-\frac{1}{x}}{(x+1)^2}=\frac{-x^3\sin x-x^2\sin x+x\cos x+x\ln x-x-1}{x(x+1)^2}.$$

2. $y=x^{\ln x},x>0$;

解:(方法一) $\ln y=\ln(x^{\ln x})=\ln^2 x$,在两边同对$x$求导可得

$$\frac{y'}{y}=2\ln x\cdot\frac{1}{x},$$

故$y'=y(\dfrac{2}{x}\ln x)=x^{\ln x}(\dfrac{2}{x}\ln x)=2x^{\ln x-1}\ln x.$

(方法二) $y=x^{\ln x}=e^{\ln(x^{\ln x})}=e^{\ln^2 x}$,

$$y'=e^{\ln^2 x}\cdot 2\ln x\cdot\frac{1}{x}=2x^{\ln x-1}\ln x.$$

3. $y = x + \arctan y$;

解：(方法一)在方程两边同对x求导得

$$y' = 1 + \frac{1}{1+y^2}y', \qquad 则\ y' = \frac{1+y^2}{y^2}.$$

(方法二) 在方程两边同求微分得

$$dy = dx + \frac{1}{1+y^2}dy, \qquad 解得 y' = \frac{dy}{dx} = \frac{1+y^2}{y^2}.$$

4. $\begin{cases} x = e^{2t}\cos^2 t \\ y = e^{2t}\sin^2 t \end{cases}.$

解：$\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{2e^{2t}\sin^2 t + 2e^{2t}\cos t\sin t}{2e^{2t}\cos^2 t - 2e^{2t}\cos t\sin t} = \tan t\dfrac{\sin t + \cos t}{\cos t - \sin t}.$

四、(每小题5分,共10分)讨论下列函数的间断点并说明其类型:

1. $f(x) = \dfrac{x}{(1+x)^2}$;

解：因$\lim\limits_{x\to -1} f(x) = -\infty$, 故$x = -1$是函数的第二类间断点，此时也称无穷间断点.

2. $f(x) = \dfrac{\sin x}{|x|}$.

解：$x = 0$是函数的间断点.

因$\lim\limits_{x\to 0^+} f(x) = 1, \lim\limits_{x\to 0^-} f(x) = -1$, 故$x = 0$是函数的第一类间断点(跳跃间断点).

五、(每小题6分,共12分)按要求完成下列各题:

1. 已知$y = e^{3u}$, $u = \dfrac{1}{2}\ln t$, $t = x^3 - 2x + 5$, 求微分dy;

解：(方法一) $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dt}\dfrac{dt}{dx} = 3e^{3u}\cdot\dfrac{1}{2}\dfrac{1}{t}\cdot(3x^2-2) = \dfrac{3e^{\frac{3}{2}\ln(x^3-2x+5)}(3x^2-2)}{2(x^3-2x+5)}$,

$$故 dy = \frac{3e^{\frac{3}{2}\ln(x^3-2x+5)}(3x^2-2)}{2(x^3-2x+5)}dx = \frac{3\sqrt{(x^3-2x+5)}(3x^2-2)}{2}dx.$$

(方法二) $dy = 3e^{3u}du = 3e^{3u}\cdot\dfrac{1}{2}\dfrac{1}{t}dt = 3e^{3u}\cdot\dfrac{1}{2}\dfrac{1}{t}\cdot(3x^2-2)dx = \dfrac{3\sqrt{(x^3-2x+5)}(3x^2-2)}{2}dx.$

2. 设$y=\dfrac{1}{x(x-1)}$，求$y^{(n)}$.

解：$y=\dfrac{1}{x-1}-\dfrac{1}{x}$,

$$y^{(n)}=(\frac{1}{x-1})^{(n)}-(\frac{1}{x})^{(n)}=(-1)^n n!\frac{1}{(x-1)^{n+1}}-(-1)^n n!\frac{1}{x^{n+1}}$$

$$=(-1)^n n!\left(\frac{1}{(x-1)^{n+1}}-\frac{1}{x^{n+1}}\right).$$

六、(10分)证明：若$a_n>0$, 且$\lim\limits_{n\to\infty}\dfrac{a_n}{a_{n+1}}=l>1$, 则$\lim\limits_{n\to\infty}a_n=0$.

证明：（方法一） $\lim\limits_{n\to\infty}\dfrac{a_n}{a_{n+1}}=l>1$，显然$\exists q$,满足$l>q>1$, 由数列极限保序性，

则$\exists N$, 当$n>N$时，有$\dfrac{a_n}{a_{n+1}}>q>1$,　此时，

$$0<a_n=\frac{a_n}{a_{n-1}}\cdots\frac{a_{N+2}}{a_{N+1}}a_{N+1}<q^{-1}\cdots q^{-1}a_{N+1}=\left(q^{-1}\right)^{n-N-1}a_{N+1}=\left(q^{-1}\right)^n\left(q^{-1}\right)^{-N-1}a_{N+1},$$

又$\lim\limits_{n\to\infty}\left(q^{-1}\right)^n\left(q^{-1}\right)^{-N-1}a_{N+1}=0$，故$\lim\limits_{n\to\infty}a_n=0$.

（方法二）$\lim\limits_{n\to\infty}\dfrac{a_n}{a_{n+1}}=l>1$，由数列极限保序性，则$\exists N$, 当$n>N$时，有$\dfrac{a_n}{a_{n+1}}>1$, 而$a_n>0$，

故当$n>N$时，$\{a_n\}$单调下降，

由$a_n>0$，可知$\{a_n\}$有下界，由单调有界数列极限定理可知$\{a_n\}$有极限，

设$\lim\limits_{n\to\infty}a_n=a$, 现证$a=0$.

反证法：若$a\neq0$, 则$\lim\limits_{n\to\infty}\dfrac{a_n}{a_{n+1}}=\dfrac{a}{a}=1$,与题目$l>1$矛盾，故$\lim\limits_{n\to\infty}a_n=0$.

七、(6分)设$f(x)=\begin{cases}\dfrac{1}{x}\sin x^2 & x>0\\ x+1 & x\le0\end{cases}$，讨论$f(x)$在$x=0$处的可导性；如果可导，求$f'(x)$.

解：(方法一） $\lim\limits_{x\to0^-}f(x)=\lim\limits_{x\to0^-}(x+1)=1$,

$$\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{1}{x}\sin x^2=\lim_{x\to0^+}x\frac{\sin x^2}{x^2}=0\times1=0,$$

故$f(x)$在$x=0$处不连续，因此$f'(0)$不存在.

（方法二） $f_-'(0)=\lim\limits_{x\to0^-}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to0^-}\dfrac{x+1-1}{x}=1$,

$$f_+'(0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x}=\lim_{x\to0^+}\frac{\frac{1}{x}\sin x^2-1}{x}=\lim_{x\to0^+}\left(\frac{\sin x^2}{x^2}-\frac{1}{x}\right)\text{不存在},$$

故$f(x)$在$x=0$处不可导.

当$x<0$时，$f'(x)=(x+1)'=1$,

当$x>0$时，$f'(x)=(\dfrac{1}{x}\sin x^2)'=-\dfrac{1}{x^2}\sin(x^2)+\dfrac{1}{x}\cos(x^2)\cdot2x=2\cos(x^2)-\dfrac{1}{x^2}\sin(x^2)$.