数计院 2011 级 1 班《数学分析》期中考试卷

出卷人　戴道清

2011 年 11 月 18 日*

1. 用 ε-N 或 ε-δ 语言证明下列极限. $(2 \times 5$ 分)

   (1) $\lim\limits_{n\to\infty}\left(\sin\sqrt{n+1}-\sin\sqrt{n}\right)=0$

   (2) $\lim\limits_{x\to 1}\sqrt{\dfrac{7}{16x^2-9}}=1$

2. 求下列极限. $(5 \times 5$ 分)

   (1) $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2+n+1}+\dfrac{2}{n^2+n+2}+\dfrac{3}{n^2+n+3}+\cdots+\dfrac{n}{n^2+n+n}\right)$.

   (2) $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3}\right)^n$ $(a,\,b,\,c>0)$.

   (3) $\lim\limits_{n\to\infty}\sin^2\left(\pi\sqrt{n^2+n}\right)$.

   (4) $\lim\limits_{x\to 0}x\left[\dfrac{1}{x}\right]$ ($[x]$ 表示不超过 x 的最大整数).

   (5) $\lim\limits_{x\to 0}\dfrac{x(1-\cos x)}{(1-e^x)\sin x^2}$.

3. 计算下列各题. $(4 \times 6$ 分)

   (1) $y=\dfrac{1+x^2}{1-x^2}$, 求 y'.

   (2) $y=x^3\sin(2x)$, 求 $y^{(10)}$.

   (3) $y=(1+x)^{\frac{1}{x}}$, 求 $\mathrm{d}y$.

   (4) $y=e^{-x}\cos x$, 求 d^2y.

4. (8 分) 证明曲线 $\begin{cases} x=a(\cos t+t\sin t),\\ y=a(\sin t-t\cos t),\end{cases}$ 上任一点的法线到原点的距离恒等于 a.

5. (8 分) 定义序列 $\{a_n\}_{n=1}^{\infty}$ 为: $a_1=a>0$, $a_{n+1}=\dfrac{1}{2}\left(a_n+\dfrac{2}{a_n}\right)$, $n=1,2,\cdots$. 讨论此序列的敛散性. 若其收敛, 则求出其极限值.

6. (10 分) 证明: 若函数 $f:[a,b]\to[a,b]$ 连续, 则方程 $f(x)=x$ 在 $[a,b]$ 上至少有一个不动点, 即存在 $x_0\in[a,b]$, 使得 $f(x_0)=x_0$.

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7. $(8+7$ 分$)$

(1) 证明: 若序列 $\{x_n\}$ 收敛, 则对任意给定的 $\varepsilon > 0$, 存在 $N \in \mathbf{N}^+$, 使得当 $m, n > N$ 时, 都有 $|x_m - x_n| < \varepsilon$. 并试以此证明序列 $\{\sin n\}$ 发散.

(2) 对 Dirichlet 函数

$$D(x) = \begin{cases} 1, & x \text{ 为有理数}, \\ 0, & x \text{ 为无理数}, \end{cases}$$

证明: $D(x)$ 在 $\mathbf{R}$ 上处处不连续. 进一步, 是否存在 $\mathbf{R}$ 上的实函数 $f(x)$, 使得其仅在 $x=0$ 处连续而在其余点均间断? 若存在, 请举列说明; 若不存在, 请证明之.

数计院 2011 级 2 班《数学分析》期中考试卷

出卷人　黄煜

2011 年 11 月 19 日*

1. (10 分) 已知 $f'(x_0)$ 存在, 求 $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+4\Delta x)-f(x_0-2\Delta x)}{\Delta x}$.

2. (10 分) 设 $\begin{cases} x=\varphi(t), \\ y=\phi(t), \end{cases}$ 求 $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

3. 求下列数列极限. (2×5 分)

   (1) $\lim\limits_{n\to\infty}\dfrac{n^{\frac{2}{3}}\sin n^2}{n+1}$.

   (2) $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}\right)$

4. 证明数列 $a_n=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{n^2}$ 收敛.

5. 求下列函数极限. (3×5 分)

   (1) $\lim\limits_{x\to\frac{\pi}{2}}(1+\cos x)^{\frac{3}{\cos x}}$.

   (2) $\lim\limits_{x\to 0^+}\dfrac{x}{\sqrt{1-\cos x}}$.

   (3) $\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x}{\cos x-1}$.

6. (10 分) 已知 $f(x)=\begin{cases} \dfrac{\ln(1+x)}{x}, & x>0, \\ 1, & x=0, \\ \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}, & x<0, \end{cases}$ 证明 $f(x)$ 在 $x=0$ 处连续.

7. (10 分) 若 $f(x)$ 在 $[a,b]$ 连续, 恒正, 证明 $\dfrac{1}{f(x)}$ 在 $[a,b]$ 连续.

8. (5 分) 若 $\forall\, x\in\mathbf{R}$, 有 $f'(x)=kf(x)$, 求 $f(x)$.

9. (10 分) f 在 $[a,b]$ 可导, 证明: $\exists\,\varepsilon\in(a,b)$ 使得 $3\,\varepsilon^2\big(f(a)-f(b)\big)=\big(b^3-a^3\big)\,f'(\varepsilon)$.

10. (10 分) $\lim\limits_{x\to a^-}f(x)=A$ 的充要条件是: 对于任意的严格单调上升且以 a 为极限的数列 $\{x_n\}$ 均有 $\lim\limits_{n\to\infty}f(x_n)=A$.

*指录入时间

数计院 2011 级 3, 4 班《数学分析》期中考试卷

出卷人　何伟弘

2011 年 11 月 15 日*

1. (10 分)

   (1) 设有数列 $\{x_n\}$ 和常数 A, 如果 A 的任意邻域 $(A-\varepsilon, A+\varepsilon)$ 内都包含有 $\{x_n\}$ 的无穷多项, 问 $\lim\limits_{n\to\infty} x_n = A$ 吗? 肯定则给出证明, 否定则举出反例说明.

   (2) 若 $\lim\limits_{x\to+\infty} f(x)g(x)=0$, 且 $g(x)$ 有界, 则 $\lim\limits_{x\to\infty} f(x)=0$ 吗? 肯定则给出证明, 否定则举出反例说明.

2. (10 分) 设 $f(x)=\begin{cases}1, & 0\leqslant x\leqslant 1,\\ 2, & 1<x\leqslant 2,\end{cases}$ $g(x)=f(2x)+f(x+1)$, 写出 $g(x)$ 的定义式, 并作出 $g(x)$ 的图形.

3. (10 分) 求 a, 使 $y=ax^2$ 与 $y=\ln x$ 相切, 并求出切点处的切线方程.

4. (10 分) 设 $C>0$, $x_1=\sqrt{C}$, $x_{n+1}=\sqrt{C+x_n}$, 证明 $\lim\limits_{n\to\infty} x_n$ 存在并求其最大值.

5. (10 分) 讨论函数 $y=\dfrac{1}{1-e^{\frac{x}{x-1}}}$ 的间断点 (类型).

6. (30 分) 计算

   (1) 设 $y=\tan(x+y)$, 求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ 和 $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

   (2) 设 $\begin{cases}y=\ln\tan\frac{t}{2}+\cos t,\\ x=\sin t,\end{cases}$ 求 $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{1}{2}}$ 和 $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

   (3) 已知 $\lim\limits_{x\to+\infty}\left((x^5+7x^4+2)^c-x\right)$ 存在且不为 0, 确定 c 并求出极限值.

   (4) 设 $y=\dfrac{4x^2-1}{x^2-1}$, 求 $y^{(n)}$.

   (5) $\lim\limits_{x\to+\infty}\ln(1+2^x)\ln\left(1+\dfrac{3}{x}\right)$.

7. (10 分) 讨论函数 $f(x)=\lim\limits_{n\to+\infty}\sqrt[n]{2+(2x)^n+x^{2n}}$ 在 $[0,+\infty)$ 上的连续性及其导函数.

8. (10 分) 证明 Darboux 定理: 若 $f(x)$ 在 $[a,b]$ 可导, $f'(a)\neq f'(b)$, c 为介于 $f'(a)$ 与 $f'(b)$ 之间的实数, 则存在 $\xi\in(a,b)$ 使得 $f'(\xi)=c$.

9. (10 分) (选做) 设 $f(x)$ 在 $x=0$ 点连续, $f(0)=0$, 且 $\lim\limits_{x\to 0}\dfrac{f(2x)-f(x)}{x}=A$, 求证 $f'(0)=A$.

*指录入时间

数计院 2011 级 5 班《数学分析》期中考试卷

出卷人　张海樟

2011 年 11 月 21 日*

1. (8 分) 用 ε-δ 语言求极限 $\lim\limits_{x\to 1}\dfrac{1+x^2}{x+1}$.

2. (20 分) 求下列极限:

   (1) $\lim\limits_{n\to\infty}\dfrac{4-n+n^2-2n^3}{n^2+n^3}$

   (2) $\lim\limits_{n\to\infty}\sqrt[n]{1+3^n+5^n}$

   (3) $\lim\limits_{x\to\pi}\dfrac{\sin x}{\sqrt{x}-\sqrt{\pi}}$

   (4) $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-1}{x}$

   (5) $\lim\limits_{x\to\infty}\left(\dfrac{2x+3}{2x-1}\right)^{3x-1}$

   (6) $\lim\limits_{x\to 0}\dfrac{\cos 4x-\cos 2x}{x^2}$

   (7) $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x-a}$ $(a>0)$

3. (10 分) 选择做下列两小题中的一题.

   (1) 设 $x_1=1$, $x_{n+1}=\dfrac{x_n+2}{3+x_n}$. 求证: x_n 收敛并求其极限.

   (2) 设 c 为一正常数, $x_1=1$, $x_{n+1}=\dfrac{c}{n}\cdot x_n$. 求证: x_n 收敛并求其极限.

4. (8 分) (函数极限的保序性) 设 $\lim\limits_{x\to x_0}f(x)=a$, $\lim\limits_{x\to x_0}g(x)=b$ 且存在 $\delta>0$ 使得

$$f(x)\leqslant g(x),\quad \forall\, 0<|x-x_0|<\delta.$$

   求证: $a\leqslant b$.

5. (18 分) 求下列函数的高阶导数 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$:

   (1) $y=e^x\tan x$

   (2) $y=\dfrac{(1+x^3)(x^4+2-\sin x)}{x^2+e^x}$

   (3) $y=\dfrac{x^2-x+1}{e^x+1}$

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(4) $y=x^{\ln(\pi+\arcsin x)}$

(5) $x=e^{3t}\cos^2 t$, $y=e^{3t}\sin^2 t$

(6) $\arctan\dfrac{y}{x}=\ln\sqrt[3]{x^2+y^2}$

6. (16 分) 求下列函数的高阶导数:

(1) $y=\dfrac{1}{x^2-2x-3}$, 求 $y^{(n)}$.

(2) $y=\dfrac{x}{\sqrt{1-x}}$, 求 $y^{(n)}$.

(3) $y=x^2\sin 2x$, 求 $y^{(30)}$.

(4) $e^{xy}-xy+3=0$, 求 $y^{(2)}$.

7. (10 分) 设 $\lim\limits_{n\to\infty}x_n=a$, $\lim\limits_{n\to\infty}y_n=b$, 其中 a, b 为有限实数. 记

$$u_n=\frac{x_1y_n+x_2y_{n-1}+\cdots+x_{n-1}y_2+x_ny_1}{n}.$$

求证: $\lim\limits_{n\to\infty}u_n=ab$.

8. (10 分) 设 f 为从闭区间 $[0,1]$ 到 $[0,1]$ 的连续函数 (即 f 的值域包含于 $[0,1]$ 中). 求证: f 在 $[0,1]$ 有不动点, 即证明存在 $\xi\in[0,1]$ 使得 $f(\xi)=\xi$.

9. (附加题, 3 分) 分析 Riemann 函数

$$f(x)=\begin{cases}\dfrac{1}{p}, & \text{若 } x=\dfrac{p}{q},\ p,\ q \text{ 为互质正整数},\\ 0, & \text{若 } x \text{ 为无理数}\end{cases}$$

在 $(0,1)$ 的连续性.

Midterm Exam for Mathematical Analysis, Fall 2011, Sun Yat-sen University

2:30-5:00pm, Thursday, November 10th, 2011

1. (8 pts) Find the limit $\lim_{x\to 1} \frac{1+x^2}{x+1}$ by the $\varepsilon - \delta$ language.

Proof: We shall show that the limit is 1. To this end, observe that

$$\left|\frac{1+x^2}{x+1} - 1\right| = \frac{|x|}{|1+x|}|x-1|.$$

When $0 < |x-1| < 1$, we have $0 < x$. In this case,

$$\frac{|x|}{|1+x|} = \frac{x}{1+x} < 1,$$

which follows that

$$\left|\frac{1+x^2}{x+1} - 1\right| \le |x-1|.$$

Therefore, for any $\varepsilon > 0$, it suffices to choose $\delta = \min\{1, \varepsilon\}$. $\square$

2. (20 pts) Find the following limits. (Show your work!):

$$(2.1)\quad \lim_{n\to\infty} \frac{4-n+n^2-2n^3}{n^2+n^3} \qquad (2.2)\quad \lim_{n\to\infty} \sqrt[n]{1+3^n+5^n}$$

$$(2.3)\quad \lim_{x\to\pi} \frac{\sin x}{\sqrt{x}-\sqrt{\pi}} \qquad (2.4)\quad \lim_{x\to 0} \frac{\sqrt[3]{1+x}-1}{x}$$

$$(2.5)\quad \lim_{x\to\infty} \left(\frac{2x+3}{2x-1}\right)^{3x-1} \qquad (2.6)\quad \lim_{x\to 0} \frac{\cos 4x - \cos 2x}{x^2}$$

$$(2.7)\quad \lim_{x\to a} \frac{a^x - x^a}{x-a} \ (a > 0).$$

- (2.1)

$$\lim_{n\to\infty} \frac{4-n+n^2-2n^3}{n^2+n^3} \lim_{n\to\infty} \frac{\frac{4}{n^3} - \frac{1}{n^2} + \frac{1}{n} - 2}{\frac{1}{n}+1} = -2.$$

- (2.2) As $5^n < 1+3^n+5^n < 3\times 5^n$,

$$5 < \sqrt[n]{1+3^n+5^n} \le 5\sqrt[n]{3}.$$

By $\lim_{n\to\infty} \sqrt[n]{3} = 1$ and the squeeze theorem, $\lim_{n\to\infty} \sqrt[n]{1+3^n+5^n} = 5$.

- (2.3)

$$\lim_{x\to\pi} \frac{\sin x}{\sqrt{x}-\sqrt{\pi}} = \lim_{x\to\pi} \frac{\sin x - \sin\pi}{x-\pi}(\sqrt{x}+\sqrt{\pi}) = (\sin x)'\big|_{x=\pi} (2\sqrt{\pi}) = -2\sqrt{\pi}.$$

- (2.4)

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x}-1}{x} = \lim_{x\to 0} \frac{\sqrt[3]{1+x}-1}{x-0} = \left(\sqrt[3]{1+x}\right)'\Big|_{x=0} = \frac{1}{3}.$$

- (2.5)

$$\lim_{x\to\infty} \left(\frac{2x+3}{2x-1}\right)^{3x-1} = \lim_{x\to\infty} \left(1+\frac{4}{2x-1}\right)^{\frac{2x-1}{4}\frac{4(3x-1)}{2x-1}} = e^6.$$

- (2.6)
$$\lim_{x\to 0}\frac{\cos 4x-\cos 2x}{x^2}=\lim_{x\to 0}\frac{-2\sin(3x)\sin(x)}{x^2}=\lim_{x\to 0}-6\frac{\sin(3x)}{3x}\frac{\sin x}{x}=-6.$$
- (2.7)
$$\lim_{x\to a}\frac{a^x-x^a}{x-a}=(a^x-x^a)'\big|_{x=a}=a^a\ln a-aa^{a-1}=a^a(\ln a-1).$$

3. (10 pts) Do one and only one of the following two questions:

(3.1) Let $x_1=1$ and $x_{n+1}=\dfrac{x_n+2}{3+x_n}$. Prove that x_n converges and find the limit.

(3.2) Let c be a positive constant, $x_1=1$ and $x_{n+1}=\dfrac{c}{n}x_n$. Show that x_n converges and find the limit.

- (3.1) Note that $x_2=\frac{3}{4}<1=x_1$. We notice that for $n\geq 2$
$$x_{n+1}-x_n=\frac{x_n+2}{3+x_n}-\frac{x_{n-1}+2}{3+x_{n-1}}=\frac{x_n-x_{n-1}}{(3+x_n)(3+x_{n-1})}.$$
It can then be shown by induction that x_n is decreasing. Clearly, $0<x_n<1$. By the bounded monotone convergence theorem, x_n converges. Denote by a the limit. Then
$$a=\frac{2+a}{3+a},$$
which together with $0\leq a\leq 1$ yields that $a=\sqrt{3}-1$.
- (3.2) Clearly, $x_n>0$ for all $n\in\mathbb{N}_+$. For $n>[c]$, $c/n<1$. Thus, $x_{n+1}=\frac{c}{n}x_n<x_n$. It implies that x_n is decreasing after excluding some finite terms. By the bounded monotone convergence theorem, x_n converges. Denote by a the limit. Then
$$a=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\frac{c}{n}x_n=\lim_{n\to\infty}\frac{c}{n}\lim_{n\to\infty}x_n=0\cdot a=0.$$

4. (8 pts) Suppose that $\lim_{x\to x_0}f(x)=a$, $\lim_{x\to x_0}g(x)=b$, and there exists $\delta>0$ such that
$$f(x)\leq g(x),\ \forall\ 0<|x-x_0|<\delta.$$
Show that $a\leq b$.

Proof: Assume to the contrary that $a>b$. Then for $\varepsilon_0=\frac{a-b}{2}$, there exists some $\delta>0$ such that for $0<|x-x_0|<\delta$
$$|f(x)-a|<\varepsilon_0=\frac{a-b}{2}\Rightarrow f(x)>a-\frac{a-b}{2}=\frac{a+b}{2}$$
and
$$|g(x)-b|<\varepsilon_0=\frac{a-b}{2}\Rightarrow g(x)<b+\frac{a-b}{2}=\frac{a+b}{2}.$$
The above two equation contradicts the assumption that $f(x)\leq g(x)$ when x is sufficiently close to x_0. □

5. (18 pts) Find first order derivatives $\frac{dy}{dx}$ of the following functions:

(5.1) $y = e^x \tan x$

(5.2) $y = \dfrac{(1+x^3)(x^4+2-\sin x)}{x^2+e^x}$

(5.3) $y = \dfrac{x^2-x+1}{e^x+1}$

(5.4) $y = x^{\ln(\pi+\arcsin x)}$

(5.5) $x = e^{3t}\cos^2 t,\ y = e^{3t}\sin^2 t$

(5.6) $\arctan \frac{y}{x} = \ln \sqrt[3]{x^2+y^2}$.

- (5.1) $y' = e^x \tan x + \frac{e^x}{\cos^2 x}$.
- (5.2)

$$\ln|y| = \ln|1+x^3| + \ln(x^4+2-\sin x) - \ln(x^2+e^x).$$

Taking the derivative on both sides above yields

$$\frac{y'}{y} = \frac{3x^2}{1+x^3} + \frac{4x^3-\cos x}{x^4+2-\sin x} - \frac{2x+e^x}{x^2+e^x}.$$

Thus,

$$y' = \frac{(1+x^3)(x^4+2-\sin x)}{x^2+e^x}\left(\frac{3x^2}{1+x^3} + \frac{4x^3-\cos x}{x^4+2-\sin x} - \frac{2x+e^x}{x^2+e^x}\right).$$

- (5.3)

$$y' = \frac{(2x-1)(e^x+1) - e^x(x^2-x+1)}{(e^x+1)^2}.$$

- (5.4)

$$\ln y = \ln(\pi+\arcsin x)\cdot \ln x.$$

Taking the derivative on both sides above gives

$$\frac{y'}{y} = \frac{1}{\sqrt{1-x^2}(\pi+\arcsin x)}\ln x + \frac{\ln(\pi+\arcsin x)}{x}.$$

Therefore,

$$y' = x^{\ln(\pi+\arcsin x)}\left(\frac{\ln x}{(\pi+\arcsin x)\sqrt{1-x^2}} + \frac{\ln(\pi+\arcsin x)}{x}\right).$$

- (5.5)

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3e^{3t}\cos^2 t - 2e^{3t}\cos t\sin t}{3e^{3t}\sin^2 t + 2e^{3t}\sin t\cos t} = \frac{3\cos^2 t - 2\cos t\sin t}{3\sin^2 t + 2\sin t\cos t} = \frac{3-2\tan t}{3\tan^2 t + 2\tan t}.$$

- (5.6) Taking derivatives about x on both sides, we have

$$\frac{1}{1+\frac{y^2}{x^2}}\frac{y'x-y}{x^2} = \frac{1}{3}\frac{1}{x^2+y^2}(2x+2yy').$$

By simplifying the above equation, we get

$$xy'-y = \frac{1}{3}(2x+2yy').$$

It follows that $y' = (2x+3y)/(3x-2y)$.

6. (16 pts) Find higher order derivatives of the following functions:

(6.1) $y = \frac{1}{x^2-2x-3}$, find $y^{(n)}$ (6.2) $y = x^2 \sin 2x$, find $y^{(30)}$

(6.3) $y = \frac{x}{\sqrt{1-x}}$, find $y^{(n)}$ (6.4) $e^{xy} - xy + 3 = 0$, find $y^{(2)}$.

- (6.1) Note that $y = \frac{1}{4}(\frac{1}{x-3} - \frac{1}{x+1})$. Thus,

$$y^{(n)} = \frac{(-1)^n n!}{4}\left(\frac{1}{(x-3)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right).$$

- (6.2) By Leibniz's formula,

$$\begin{aligned}(x^2\sin 2x)^{(30)} &= \binom{30}{0}x^2(\sin 2x)^{(30)} + \binom{30}{1}(x^2)'(\sin 2x)^{(29)} + \binom{30}{2}(x^2)''(\sin 2x)^{(28)} \\ &= x^2 2^{30}\sin(2x+\frac{30\pi}{2}) + 60x2^{29}\sin(2x+\frac{29}{2}\pi) + 870\cdot 2^{28}\sin(2x+\frac{28}{2}\pi) \\ &= -2^{30}x^2\sin(2x) + 60\cdot 2^{29}x\cos(2x) + 870\cdot 2^{28}\sin(2x).\end{aligned}$$

- (6.3) By Leibniz's formula,

$$\begin{aligned}\left(\frac{x}{\sqrt{1-x}}\right)^{(n)} &= x\left((1-x)^{-\frac{1}{2}}\right)^{(n)} + nx\left((1-x)^{-\frac{1}{2}}\right)^{(n-1)} \\ &= x(1-x)^{-\frac{2n+1}{2}}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n} + nx(1-x)^{-\frac{2n-1}{2}}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^{n-1}}.\end{aligned}$$

- (6.4) Taking the derivatives with respect to x on both sides, we get

$$e^{xy}(y+xy') - (y+xy') = 0,$$

which implies that

$$y' = -\frac{y(e^{xy}-1)}{x(e^{xy}-1)} = -\frac{y}{x}.$$

Thus,

$$y'' = -\frac{y'x-y}{x^2} = -\frac{-\frac{y}{x}x-y}{x^2} = \frac{2y}{x^2}.$$

7. (10 pts) Suppose that $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$, where a, b are finite real numbers. Set

$$u_n = \frac{x_1y_n + x_2y_{n-1} + \cdots + x_{n-1}y_2 + x_ny_1}{n}.$$

Show that $\lim_{n\to\infty} u_n = ab$.

Proof: Since x_n and y_n are convergent, they are both bounded. Thus, there exists some constant $M > 0$ such that

$$|x_n| \le M, \quad |y_n| \le M \text{ for all } n.$$

Then $|a|, |b| \le M$. Let $\varepsilon > 0$ be fixed. There exists some integer $m > 0$ such that for $n > m$

$$|x_n - a| \le \frac{\varepsilon}{4M}, \quad |y_n - b| \le \frac{\varepsilon}{4M}.$$

We then observe that for $n > m$

$$\begin{aligned}|u_n - ab| &= \frac{|(x_1y_n - ab) + (x_2y_{n-1} - ab) + \cdots + (x_ny_1 - ab)|}{n} \\ &\le \frac{|x_1y_n - ab| + \cdots + |x_my_{n+1-m} - ab|}{n} \\ &\quad + \frac{|x_{m+1}y_{n-m} - ab| + \cdots + |x_{n-m}y_{m+1} - ab|}{n} \\ &\quad + \frac{|x_{n+1-m}y_m - ab| + \cdots + |x_ny_1 - ab|}{n}.\end{aligned}$$

For $1 \le j \le m$,

$$|x_jy_{n+1-j} - ab| \le M^2 + |ab| \le 2M^2 \quad \text{and} \quad |x_{n+1-j}y_j - ab| \le M^2 + |ab| \le 2M^2.$$

For $m + 1 \le j \le n - m$,

$$|x_jy_{n+1-j} - ab| \le |x_jy_{n+1-j} - ay_{n+1-j}| + |ay_{n+1-j} - ab| \le M\frac{\varepsilon}{4M} + |a|\frac{\varepsilon}{4M} \le \frac{\varepsilon}{2}.$$

By the above two equations,

$$\frac{|x_1y_n - ab| + \cdots + |x_my_{n+1-m} - ab|}{n} \le \frac{2mM^2}{n},$$

$$\frac{|x_{m+1}y_{n-m} - ab| + \cdots + |x_{n-m}y_{m+1} - ab|}{n} \le \frac{n - 2m}{n}\frac{\varepsilon}{2},$$

and

$$\frac{|x_{n+1-m}y_m - ab| + \cdots + |x_ny_1 - ab|}{n} \le \frac{2mM^2}{n}.$$

Combining the above three estimates, we have for $n > m$

$$|u_n - ab| \le \frac{4mM^2}{n} + \frac{\varepsilon}{2}.$$

As $\frac{4mM^2}{n} \to 0$ as $n \to \infty$, there exists $m' > 0$ such that for $n > m'$

$$\frac{4mM^2}{n} < \frac{\varepsilon}{2}.$$

Thus, for $n > \max\{m, m'\}$, $|u_n - ab| < \varepsilon$, which completes the proof. □

8. (10 pts) Let f be a continuous function from $[0, 1]$ to $[0, 1]$, that is, the range of f is contained in $[0, 1]$. Show that f has a fixed point in $[0, 1]$, namely, show that there exists $\xi \in [0, 1]$ such that $f(\xi) = \xi$.

   Proof: Set $g(x) = f(x) - x$. Then g is continuous on $[0, 1]$. The question reduces to show the existence of a zero of g on $[0, 1]$. Since the range of f is contained in $[0, 1]$,

$$0 \le f(x) \le 1, \text{ for all } x \in [0, 1].$$

   As a consequence, $f(0) \ge 0$ and $f(1) \le 1$. It follows that $g(0) \ge 0$ and $g(1) \le 0$. By the intermediate value theorem for continuous functions, there exists some $\xi \in [0, 1]$ such that $g(\xi) = 0$, or equivalently, $f(\xi) = \xi$. □

9. (Bonus question, 3 pts) Analyze the continuity of the following function f on $(0,1)$

$$f(x) = \begin{cases} \frac{1}{p}, & \text{if } x = \frac{p}{q},\ p, q \text{ are relatively prime positive integers,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Proof: We first show that for all $x_0 \in (0,1)$

$$\lim_{x \to x_0} f(x) = 0.$$

Let $\varepsilon > 0$ be fixed. We see that $|f(x) - 0| \geq \varepsilon$ only when $x = \frac{p}{q}$ and

$$\frac{1}{p} \geq \varepsilon,$$

which is equivalent to $p \leq 1/\varepsilon$. When $x = \frac{p}{q}$ and $|x - x_0| < \frac{x_0}{2}$, we must have

$$\frac{x_0}{2} < \frac{p}{q},$$

which implies that

$$q < \frac{2p}{x_0}.$$

The conclusion is that when $|x - x_0| \leq \frac{x_0}{2}$ for $|f(x) - 0| \geq \varepsilon$, we must have $x = \frac{p}{q}$, and

$$p \leq 1/\varepsilon,\ q < \frac{2p}{x_0} \leq \frac{2}{x_0 \varepsilon}.$$

Such rational points are finitely many. We denote them by $x_1, x_2, \ldots, x_m$ (the point x_0 is not counted here). Let

$$\delta = \min\{\frac{x_0}{2}, |x_1 - x_0|, \ldots, |x_m - x_0|\}.$$

For $0 < |x - x_0| < \delta$, x is not in the set $\{x_1, \ldots, x_m\}$. As a result, $|f(x) - 0| < \varepsilon$. We have hence proved that

$$\lim_{x \to x_0} f(x) = 0.$$

Therefore, rational points are removable discontinuity points of f while f is continuous on irrational points. $\square$

数计院 2011 级 6 班《数学分析》期中考试卷

出卷人　冼军

2011 年 11 月 25 日*

一. 填空题. (4×4 分)

1. 函数 $f(x) = x - [x]$, $x \in (-\infty, +\infty)$ 的最小正周期为______.

2. 已知函数满足 $af(x) + bf(\frac{1}{x}) = \frac{c}{x}$, a, b, c 均为常数, $|a| \neq |b|$, 则函数 $f(x) =$______.

3. $f(x)$, $x \in (-\infty, +\infty)$ 如何表示成为两个非负函数之差______.

4. 已知 $\lim\limits_{x\to\infty} \left(\sqrt{ax^2+bx+c} - \alpha x - \beta\right)$ (a, b, c 为常数, $a > 0$), 则 $\alpha =$______.

二. 计算和证明题. (84 分)

1. 用定义证明: (13 分)

(1) $\lim\limits_{n\to\infty} \dfrac{1}{n^\alpha} = 0$, 其中 $\alpha > 0$

(2) $\lim\limits_{n\to\infty} \dfrac{1+2+\cdots+n}{n^3} = 0$

(3) $\lim\limits_{n\to\infty} a_n = 1$, 其中 $a_n = \begin{cases} \frac{n-1}{n}, & n \text{ 为偶数}, \\ \frac{\sqrt{n^2+n}}{n}, & n \text{ 为奇数}. \end{cases}$

2. 确定函数 $x_1 > 0$, $x_{n+1} = \frac{3(1+x_n)}{3+x_n}$, $n = 1, 2, 3, \cdots$, 证明此数列存在极限并求极限. (7 分)

3. 计算下列函数极限. (5×4 分)

(1) $\lim\limits_{n\to\infty} \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{2n-1}{2n}\right)$

(2) $\lim\limits_{n\to\infty} \dfrac{\sum_{p=1}^{n} p!}{n!}$

(3) 求函数 $f(x) = \dfrac{1}{x} - \left[\dfrac{1}{x}\right]$ 在点 $x = \dfrac{1}{n}$ (n 为自然数) 的左、右极限.

(4) $\lim\limits_{x\to 0} (1+ax)^{\frac{1}{\sin x}}$

*指录入时间

(5) $\lim\limits_{n\to\infty}(1+x)\left(1+x^2\right)\cdots\left(1+x^{2n}\right)$, 其中 $|x|<1$

4. 讨论函数 $f(x)=\begin{cases}\cos\frac{\pi x}{2}, & |x|\leqslant 1,\\ |x-1|, & |x|>1\end{cases}$ 的连续区间与间断点. (8 分)

5. 求下列函数的导数或微分. (4×5 分)

(1) $\arcsin\frac{1+2\cos x}{2+\cos x}$

(2) $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$

(3) $\ln|x-1|$

(4) $x^y-y^x=1$, 求 $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

(5) $y=y(x)$ 是方程 $\begin{cases}3t^2+2t+3=x,\\ e^y\sin t-y+1=0\end{cases}$ 所确定的隐函数, 求 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

6. 证明 $\lim\limits_{x\to x_0^+}f(x)=\infty$ 的充要条件是对任何数列 $\{x_n\}$ 且 $x_n>x_0$, $x_n\to x_0(n\to\infty)$, 都有 $\lim\limits_{n\to\infty}f(x_n)=\infty$. (8 分)

7. 设 $f(x)$ 在 $[a,b]$ 上连续, 且当 $f(a)=f(b)$, 证明一定存在长度为 $\frac{b-a}{2}$ 的区间 $[\alpha,\beta]\subset[a,b]$, 使得 $f(\alpha)=f(\beta)$, 即在区间 $[a,\frac{b+a}{2}]$ 上一定存在 α, 使得 $f(\alpha)=f(\alpha+\frac{b-a}{2})$.