

Growth of Functions

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- **Asymptotic notation**
 - O -notation
 - Ω -notation
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Analogy

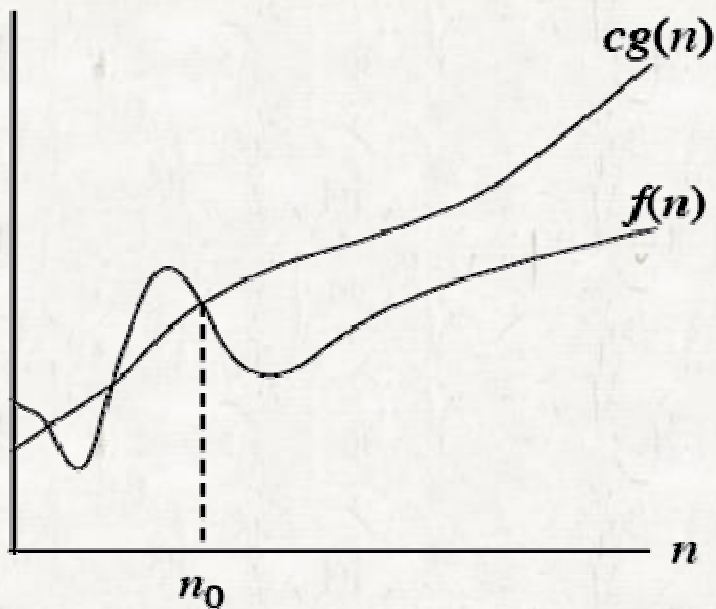
● Analogy

- $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.
- $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.
- $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.
- Examples

O-notation

● *O*-notation

$O(g(n)) = \{f(n): \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}.$

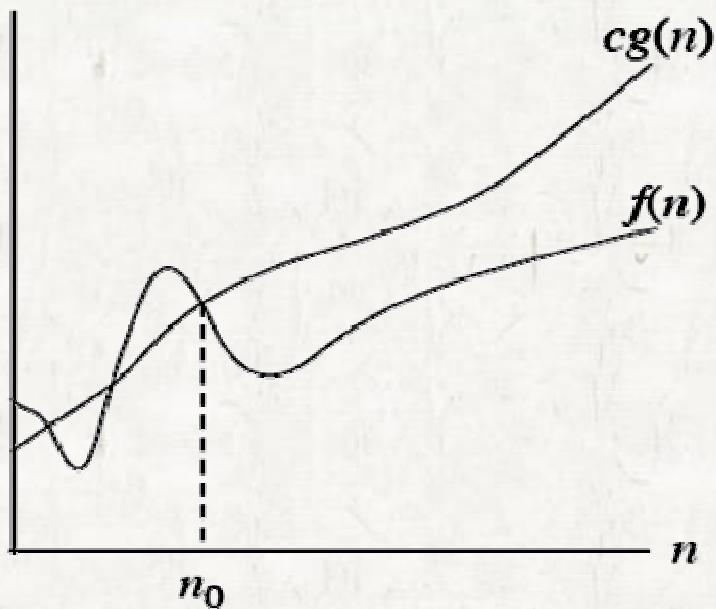


- For all values n to the right of n_0 , the value of the function $f(n)$ is on or below $cg(n)$.

O-notation

• *O*-notation

$O(g(n)) = \{f(n): \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}.$



- $g(n)$ is called an *asymptotic upper bound* of $f(n)$.
- $f(n) = O(g(n))$ denotes $f(n) \in O(g(n))$.

O-notation

- Example

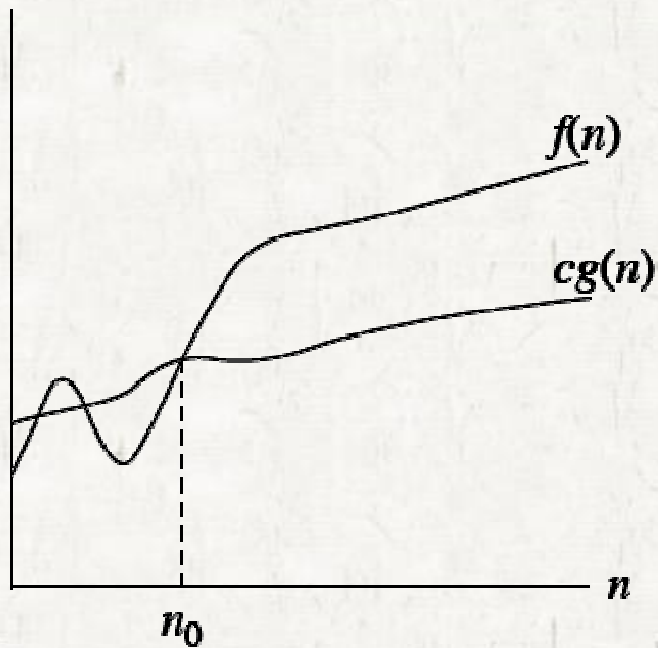
$$3n + 1 = O(n^2)$$

- Show there are c and n_0 such that $3n + 1 \leq cn^2$ for all $n \geq n_0$.
- Dividing by n^2 yields $\frac{3}{n} + \frac{1}{n^2} \leq c$.
- The inequality holds for any $n \geq 1$ and $c \geq 4$.

Ω -notation

Ω -notation

$\Omega(g(n)) = \{f(n): \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0\}.$



- For all values n to the right of n_0 , the value of $f(n)$ is **on or above** $cg(n)$.
- $g(n)$ is called an **asymptotic lower bound** of $f(n)$.

Ω -notation

- Example

$$3n^2 - 4n + 1 = \Omega(n)$$

- Show there are c and n_0 such that $3n^2 - 4n + 1 \geq cn$ for all $n \geq n_0$.

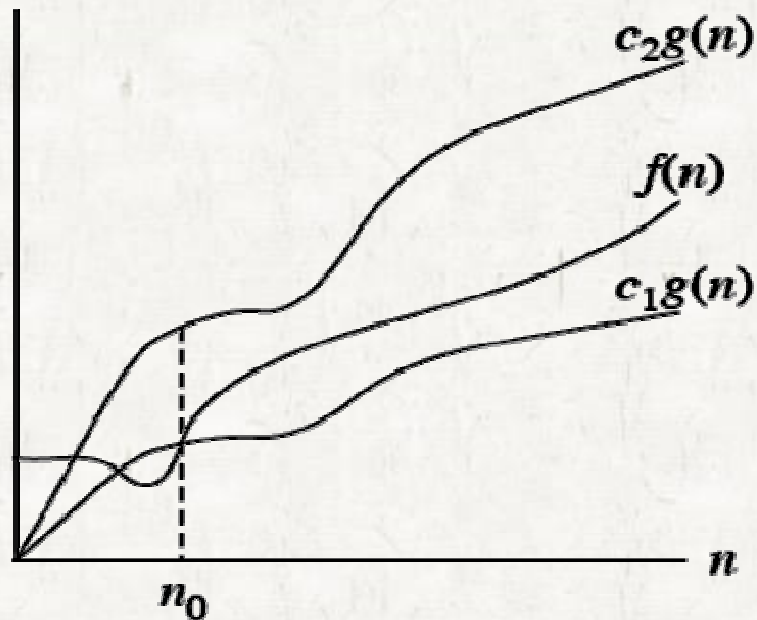
- Dividing by n yields $3n - 4 + \frac{1}{n} \geq c$.

- The inequality holds for any $n \geq 2$ and $c = 2$.

Θ -notation

Θ -notation

$\Theta(g(n)) = \{f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \leq c_1g(n) \leq f(n) \leq c_2g(n) \text{ for all } n \geq n_0\}.$



- For all values of n to the right of n_0 , the value of $f(n)$ lies on or above $c_1g(n)$ and on or below $c_2g(n)$.
- $g(n)$ is called an *asymptotically tight bound* for $f(n)$.

Θ -notation

• Example

$$\frac{1}{2}n^2 - 3n = \Theta(n^2)$$

To show there exist positive constants c_1 , c_2 and n_0 such that

$$c_1 n^2 \leq \frac{1}{2}n^2 - 3n \leq c_2 n^2 \text{ for all } n \geq n_0.$$

Dividing by n^2 yields $c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2.$

Θ -notation

- Example

$$c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2.$$

- The right-hand inequality holds for $n \geq 1$ by choosing $c_2 \geq 1/2$.
- The left-hand inequality holds for $n \geq 7$ by choosing $c_1 \leq 1/14$.
- Thus, by choosing $c_1 = 1/14$, $c_2 = 1/2$, and $n_0 = 7$,

we can verify that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

Θ -notation

● Example

- Consider any quadratic function $f(n) = an^2 + bn + c$, where a , b , and c are constants and $a > 0$.
- Throwing away the lower-order terms and ignoring the constant yields $f(n) = \Theta(n^2)$.
- The reader may verify that $0 \leq c_1n^2 \leq an^2 + bn + c \leq c_2n^2$ for all $n \geq n_0$. (Self-study)
- In general, for any polynomial $p(n) = \sum_{i=0}^d a_i n^i$ where the a_i are constants and $a_d > 0$, we have $p(n) = \Theta(n^d)$.

Θ -notation

• Theorem 3.1

For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Notation

• Notation in equations and inequalities

- $n = O(n^2)$ (set)
- $T(n) = 2T(n/2) + \Theta(n)$ (element)
- $2n^2 + \Theta(n) = \Theta(n^2)$

Analogy

● Analogy

- $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.
- $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.
- $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.
- $f(n) = o(g(n)) \approx f(n) < g(n)$ in degree.
- $f(n) = \omega(g(n)) \approx f(n) > g(n)$ in degree.

o-notation

- The asymptotic upper bound provided by *O*-notation may or may not be asymptotically tight.
- The bound $2n^2 = O(n^2)$ is asymptotically tight, but the bound $2n = O(n^2)$ is not.
- We use *o*-notation to denote an upper bound that is not asymptotically tight.

o-notation

• *o*-notation

$o(g(n)) = \{f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0\}.$

- $f(n)$ is called an *asymptotically smaller* than $g(n)$.
- For example, $2n = o(n^2)$, but $2n^2 \neq o(n^2)$.

o-notation

- The main difference between O and o .
 - $f(n) = O(g(n))$, the bound $0 \leq f(n) \leq cg(n)$ holds for *some* constant $c > 0$
 - $f(n) = o(g(n))$, the bound $0 \leq f(n) < cg(n)$ holds for *all* constants $c > 0$.
- Intuitively, in the o -notation, the function $f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity; that is,

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$$

ω -notation

- ω -notation is used to denote a lower bound that is not asymptotically tight.

- **ω -notation**

$\omega(g(n)) = \{f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \leq cg(n) < f(n) \text{ for all } n \geq n_0\}$

- $f(n)$ is called an *asymptotically larger* than $g(n)$.
- So, $f(n) \in \omega(g(n))$ if and only if $g(n) \in o(f(n))$.

ω -notation

• Example

- $n^2/2 = \omega(n)$, but $n^2/2 \neq \omega(n^2)$.
- The relation $f(n) = \omega(g(n))$ implies that

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$$

if the limit exists.

- That is, $f(n)$ becomes arbitrarily large relative to $g(n)$ as n approaches infinity.

Comparison of functions

• Comparison of functions

- Transitivity
- Reflexivity
- Symmetry
- Transpose symmetry

Comparison of functions

• Comparison of functions

- Transitivity ($=, \leq, \geq, <, >$)
- Reflexivity ($=, \leq, \geq$)
- Symmetry ($=$)
- Transpose symmetry ($\leq \leftrightarrow \geq, < \leftrightarrow >$)

Transitivity

• Transitivity ($=, \leq, \geq, <, >$)

- $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$,
- $f(n) = O(g(n))$ and $g(n) = O(h(n))$ imply $f(n) = O(h(n))$,
- $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$,
- $f(n) = o(g(n))$ and $g(n) = o(h(n))$ imply $f(n) = o(h(n))$,
- $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$ imply $f(n) = \omega(h(n))$.

Reflexivity

• Reflexivity ($=, \leq, \geq$)

- $f(n) = \Theta(f(n))$
- $f(n) = O(f(n))$
- $f(n) = \Omega(f(n))$

Symmetry and transpose symmetry

• Symmetry (=)

- $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

• Transpose symmetry ($\leq \leftrightarrow \geq$, $< \leftrightarrow >$)

- $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$,
- $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

Comparison of functions

• Trichotomy

- For any two real numbers a and b , exactly one of the following must hold: $a < b$, $a = b$, $a > b$.
- That is, any two numbers are comparable.
- Are any two functions asymptotically comparable?
 - Is it possible $f(n) \neq O(g(n))$ and $f(n) \neq \Omega(g(n))$?
 - n and $n^{1+\sin n}$

Self-study

• Exercise 3.1-1

- Show $\max(f(n), g(n)) = \Theta(f(n) + g(n))$

• Exercise 3.1-4

- Is $2^{n+1} = O(2^n)$?
- Is $2^{2n} = O(2^n)$?

• Problem 3-2 for O , Θ , and Ω .

- Use $\lg(n!) = \Theta(n \lg n)$