Getting Started

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Contents

- Sorting problem
- 2 sorting algorithms
  - Insertion sort
  - Merge sort

Sorting problem

keys

- **Input**
  - A sequence of n number $\langle a_1, a_2, \ldots, a_n \rangle$.
- **Output**
  - A permutation (reordering) $\langle a'_1, a'_2, \ldots, a'_n \rangle$ of the input sequence such that $a'_1 \leq a'_2 \leq \ldots \leq a'_n$.
- Ex>
  - Input: < 5, 2, 4, 6, 1, 3>
  - Output: < 1, 2, 3, 4, 5, 6>

Insertion sort

- **Insertion sort**
  - Description
  - Correctness
  - Performance

Description

- What is insertion sort?
  - A sorting algorithm using insertion.

- What is insertion?
  - Given a key and a sorted list of keys, insert the key into the sorted list preserving the sorted order.
  - ex> Insert 3 into <1, 2, 4, 5, 6>

Description

- Insertion sort uses insertion incrementally.
  - Let $A[1..n]$ denote the array storing keys.
  - Insert $A[2]$ into $A[1]$.
  - Insert $A[3]$ into $A[1..2]$.
  - Insert $A[4]$ into $A[1..3]$.

    .

    .

    .
  - Insert $A[n]$ into $A[1..n\text{-}1]$.

Description: example

- 5 2 4 6 1 3
- 5 2 4 6 1 3
- 2 5 4 6 1 3
- 2 4 5 6 1 3
- 2 4 5 6 1 3
- 1 2 4 5 6 3
- 1 2 3 4 5 6

Description: pseudo code

INSERTION-SORT(*A*)

Pseudocode conventions are given in p. 19 - 20 of the textbook.

```
for j = 2 to A.length
    key = A[j]
    i = j - 1
    while i > 0 and A[i] > key
        A[i + 1] = A[i]
        i = i - 1
    A[i + 1] = key
```

n-1 iterations of insertion.

Insert *A*[*j*] into *A*[1..*j* - 1].

Find a place to put *A*[*j*].

Put *A*[*j*].

Insertion sort

- Insertion sort
  - Description
  - Correctness
  - Performance
    - Running time
    - Space consumption

Running time

- How to analyze the running time of an algorithm?
  - Consider running the algorithm on a specific machine and measure the running time.
    - We cannot compare the running time of an algorithm on a machine with the running time of another algorithm on another machine.
    - So, we have to measure the running time of every algorithm on a specific machine, which is impossible.
  - Hence, we count the number of instructions used by the algorithm.

Instructions

- Arithmetic
  - Add, Subtract, Multiply, Divide, remainder, floor, ceiling
- Data movement
  - Load, store, copy
- Control
  - Conditional branch
  - Unconditional branch
  - Subroutine call and return

Running time

- The running time of an algorithm grows with the input size, which is the number of items in the input.
- For example, sorting 10 keys is faster than sorting 100 keys.
- So the running time of an algorithm is described as a function of input size n, for example, $T(n)$.

Running time of insertion sort

	cost	*times*
INSERTION-SORT(*A*)		
for $j = 2$ **to** *A.length*	c_1	n
$key = A[j]$	c_2	$n - 1$
$i = j - 1$	c_4	$n - 1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i + 1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i - 1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i + 1] = key$	c_8	$n - 1$

- ***T*(*n*)**: The sum of product of *cost* and *times* of each line.

Running time of insertion sort

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\sum_{j=2}^{n} t_j + c_6\sum_{j=2}^{n}(t_j-1) + c_7\sum_{j=2}^{n}(t_j-1) + c_8(n-1)$$

cost	*times*
c_1	n
c_2	n - 1
c_4	n - 1
c_5	$\sum_{j=2}^{n} t_j$
c_6	$\sum_{j=2}^{n} (t_j - 1)$
c_7	$\sum_{j=2}^{n} (t_j - 1)$
c_8	n - 1

- ***T*(*n*)**: The sum of product of *cost* and *times* of each line.

Running time of insertion sort

- t_i: The number of times the **while** loop test is executed for *j*.

- Note that **for**, **while** loop test is executed one time more than the loop body.

Running time of insertion sort

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1)$$
$$+ c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

- Although the size of the input is the same, we have
  - best case
  - average case, and
  - worst case.

Running time of insertion sort

- Best case
  - If $A[1..n]$ is already sorted, $t_j = 1$ for $j = 2, 3, \ldots, n$.

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + \cancel{c_6 \sum_{j=2}^{n} (t_j - 1)}$$

$$+ \cancel{c_7 \sum_{j=2}^{n} (t_j - 1)} + c_8(n-1)$$

$$= c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1)$$

$$= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$$

- This running time can be expressed as $an+b$ for *constants a* and *b*; it is thus a ***linear function*** of *n*.

Running time of insertion sort

- Worst case
  - If $A[1..n]$ is sorted in reverse order, $t_j = j$ for $j = 2, 3,\ldots, n$.

$$\sum_{j=2}^{n} j = \frac{n(n+1)}{2} - 1 \quad \text{and} \quad \sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$$

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right)$$

$$+ c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1)$$

$$= \left(\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2}\right)n^2 + \left(c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8\right)n - (c_2 + c_4 + c_5 + c_8)$$

- This running time can be expressed as $an^2 + bn + c$ for constants a, b, and c; it is thus a ***quadratic function*** of n.

Running time of insertion sort

- **Only the degree of leading term is important.**
  - Because we are only interested in the ***rate of growth*** or ***order of growth.***
  - For example, a quadratic function grows faster than any linear function.

- The degree of leading term is expressed as Θ–notation.
  - The worst-case running time of insertion sort is $\Theta(n^2)$.

Space consumption of insertion sort

- $\Theta(n)$ space.

- Moreover, the input numbers are ***sorted in place*.**
  - $n + c$ space for some constant c.

Self-study on Insertion Sort

- **Exercise 2.1-1**
- **Exercise 2.1-2**

Content

- Sorting problem
- Sorting algorithms
  - Insertion sort - $\Theta(n^2)$.
  - Merge sort - $\Theta(n \lg n)$.

Merge

- What is merge sort?
  - A sorting algorithm using merge.

- What is merge?
  - Given two sorted lists of keys, generate a sorted list of the keys in the given sorted lists.
  - <1, 5, 6, 8> < 2, 4, 7, 9> → < 1, 2, 4, 5, 6, 7, 8, 9>

Merge

- Merging example
  - <1, 5, 6, 8> <2, 4, 7, 9> → < 1 >
  - < 5, 6, 8> <2, 4, 7, 9> → < 1, 2 >
  - < 5, 6, 8> < 4, 7, 9> → < 1, 2, 4 >
  - < 5, 6, 8> < 7, 9> → < 1, 2, 4, 5 >
  - < 6, 8> < 7, 9> → < 1, 2, 4, 5, 6 >
  - < 8> < 7, 9> → < 1, 2, 4, 5, 6, 7 >
  - < 8> < 9> → < 1, 2, 4, 5, 6, 7, 8 >
  - < > < 9> → < 1, 2, 4, 5, 6, 7, 8, 9>

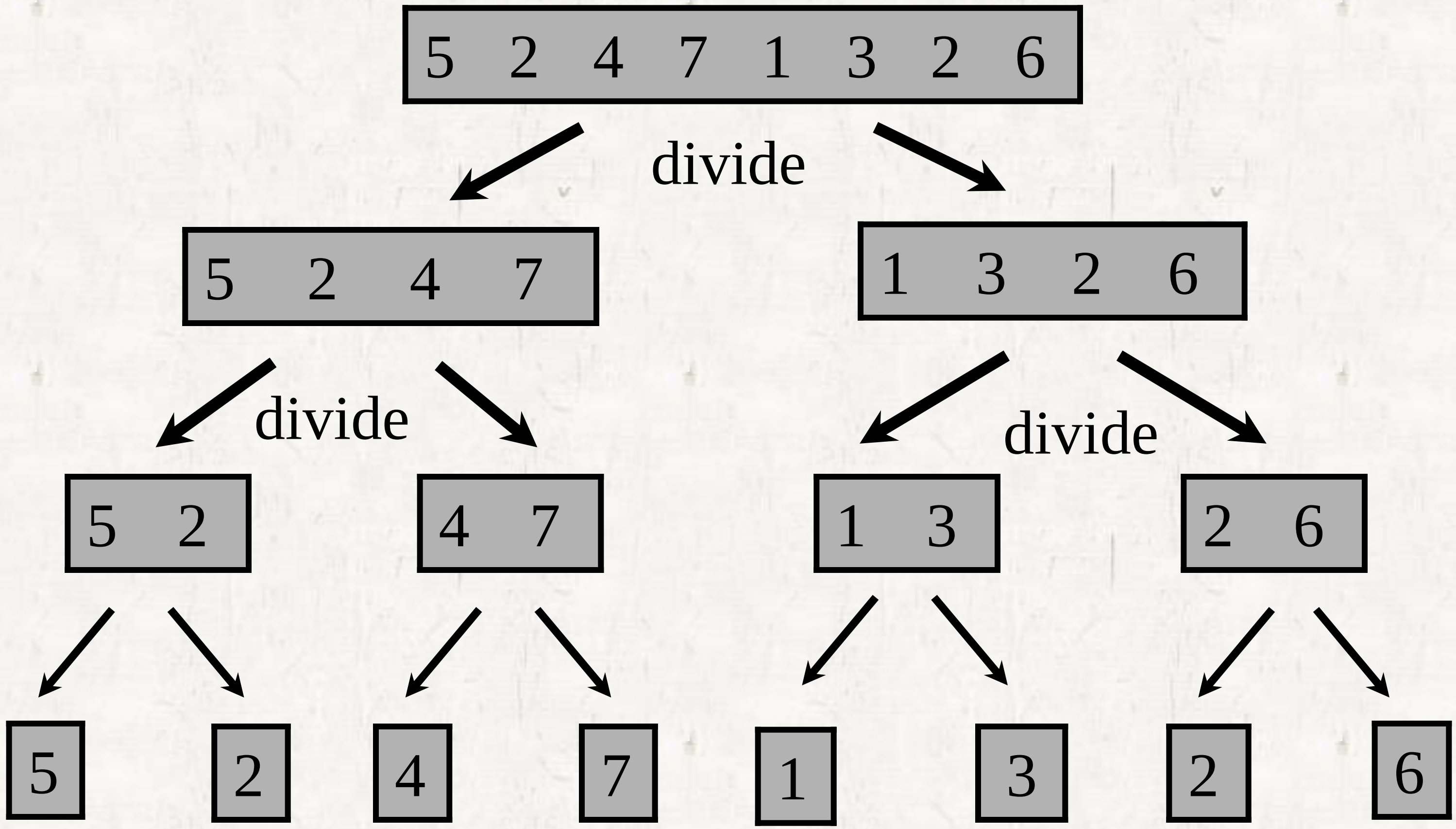
Merge

- Running time of merge
  - Let n_1 and n_2 denote the lengths of two sorted lists.
  - $\Theta(n_1 + n_2)$ time.
    - Main operations: compare and move
    - #comparison $\leq$ #movement
    - Obviously, #movement $= n_1 + n_2$
    - So, #comparison $\leq n_1 + n_2$
    - Hence, #comparison + #movement $\leq 2(n_1 + n_2)$
    - which means $\Theta(n_1 + n_2)$.

Merge sort

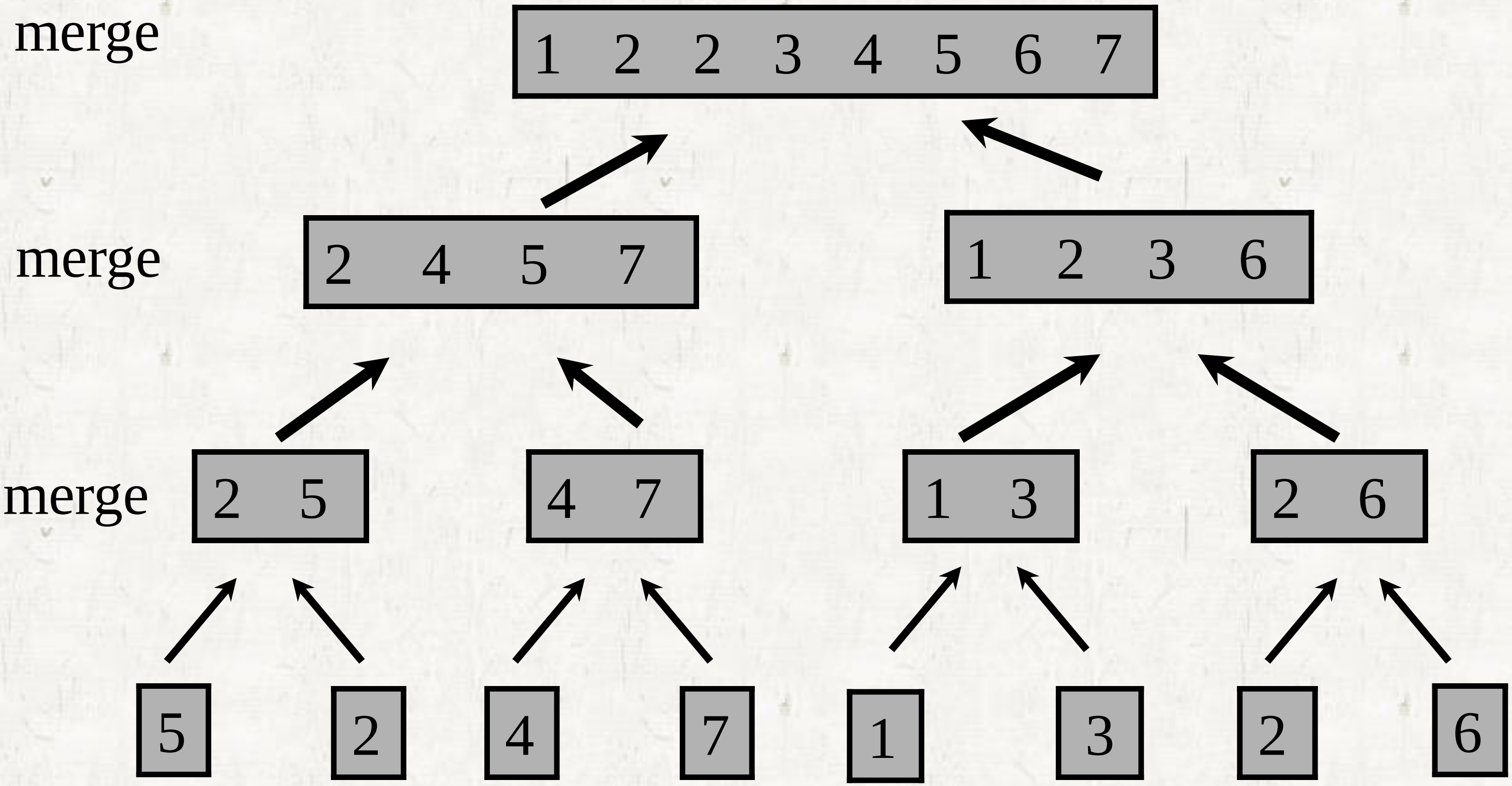
- A divide-and-conquer approach
  - **Divide:** Divide the n keys into two lists of $n/2$ keys.
  - **Conquer:** Sort the two lists recursively using merge sort.
  - **Combine:** Merge the two sorted lists.

Merge sort



5 2 4 7 1 3 2 6
divide
5 2 4 7
1 3 2 6
divide
divide
5 2
4 7
1 3
2 6
5
2
4
7
1
3
2
6

Merge sort



Pseudo code

```
MERGE-SORT(A, p, r)
if p < r
    q = ⌊(p + r)/2⌋
    MERGE-SORT(A, p, q)
    MERGE-SORT(A, q + 1, r)
    MERGE(A, p, q, r)
```

Running time

- **Divide:** $\Theta(1)$
  - The divide step just computes the middle of the subarray, which takes constant time.
- **Conquer:** $2T(n/2)$
  - We recursively solve two subproblems, each of size $n/2$.
- **Combine:** $\Theta(n)$
  - We already showed that merging two sorted lists of size $n/2$ takes $\Theta(n)$ time.

Running time

- $T(n)$ can be represented as a recurrence.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1, \\ 2T(n/2) + \Theta(n) & \text{if } n>1 \end{cases}$$

Running time

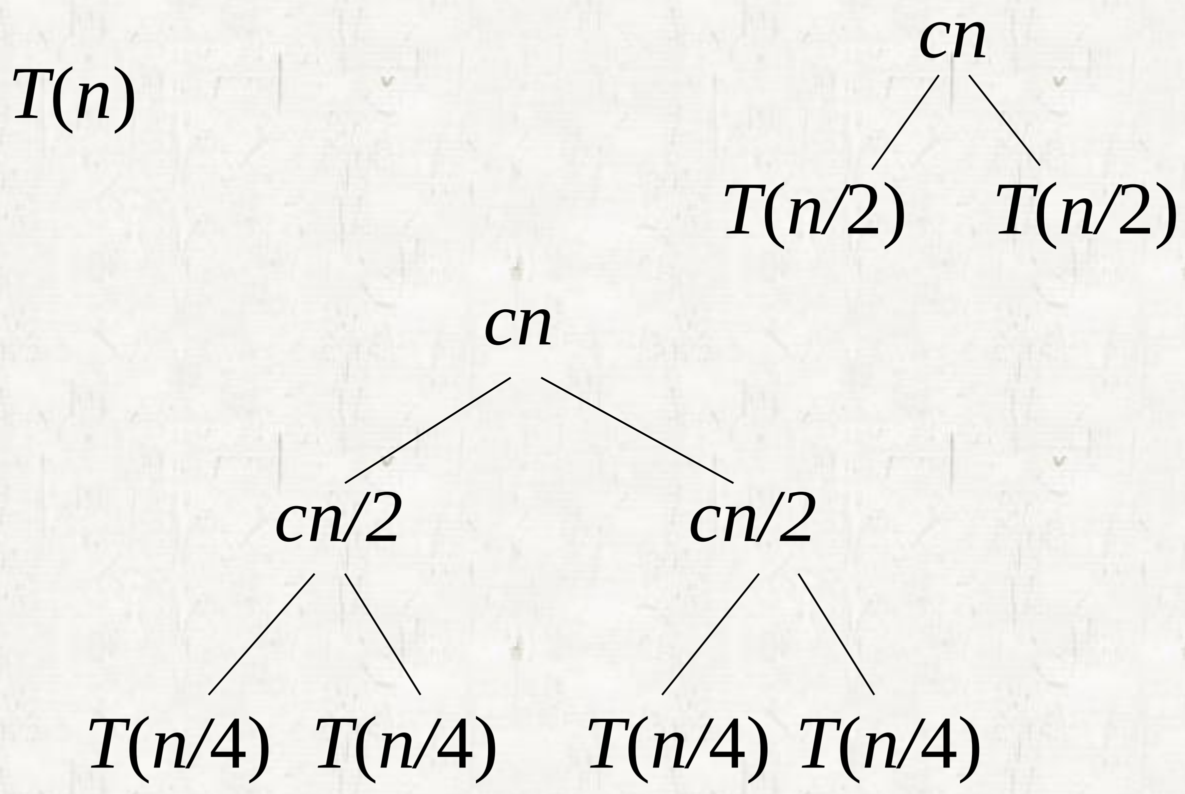
- where the constant c represents the time required to solve problems of size 1 as well as the time per array element of the divide and combine steps.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

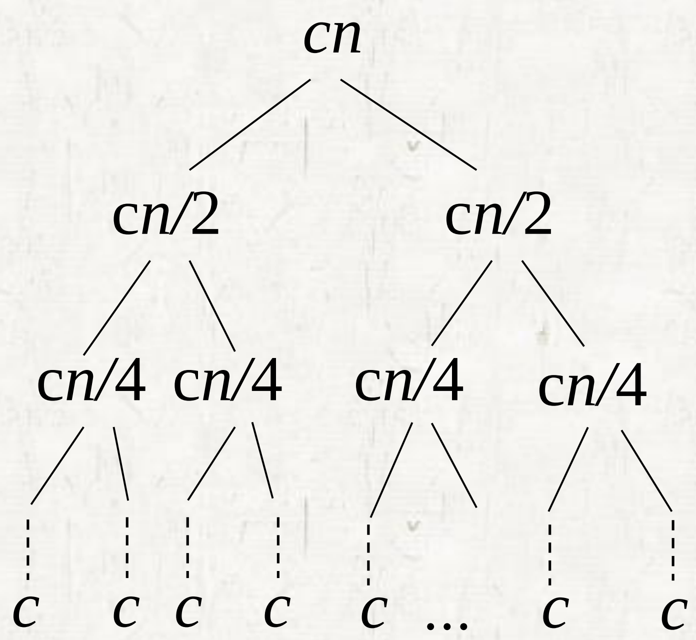
$$\Downarrow$$

$$T(n) = \begin{cases} c & \text{if } n=1, \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

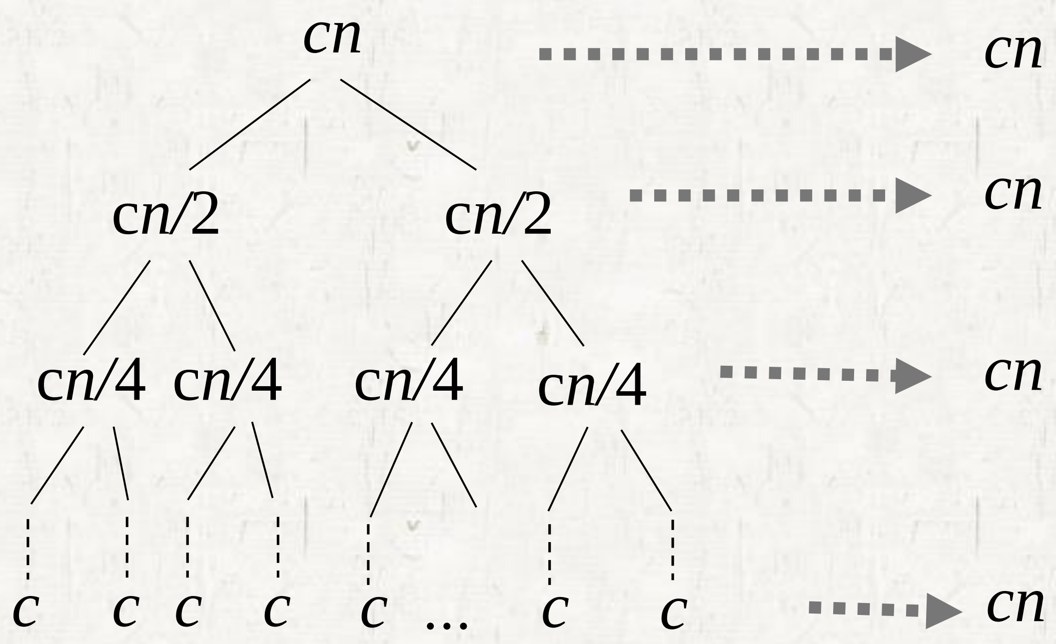
Recursion tree



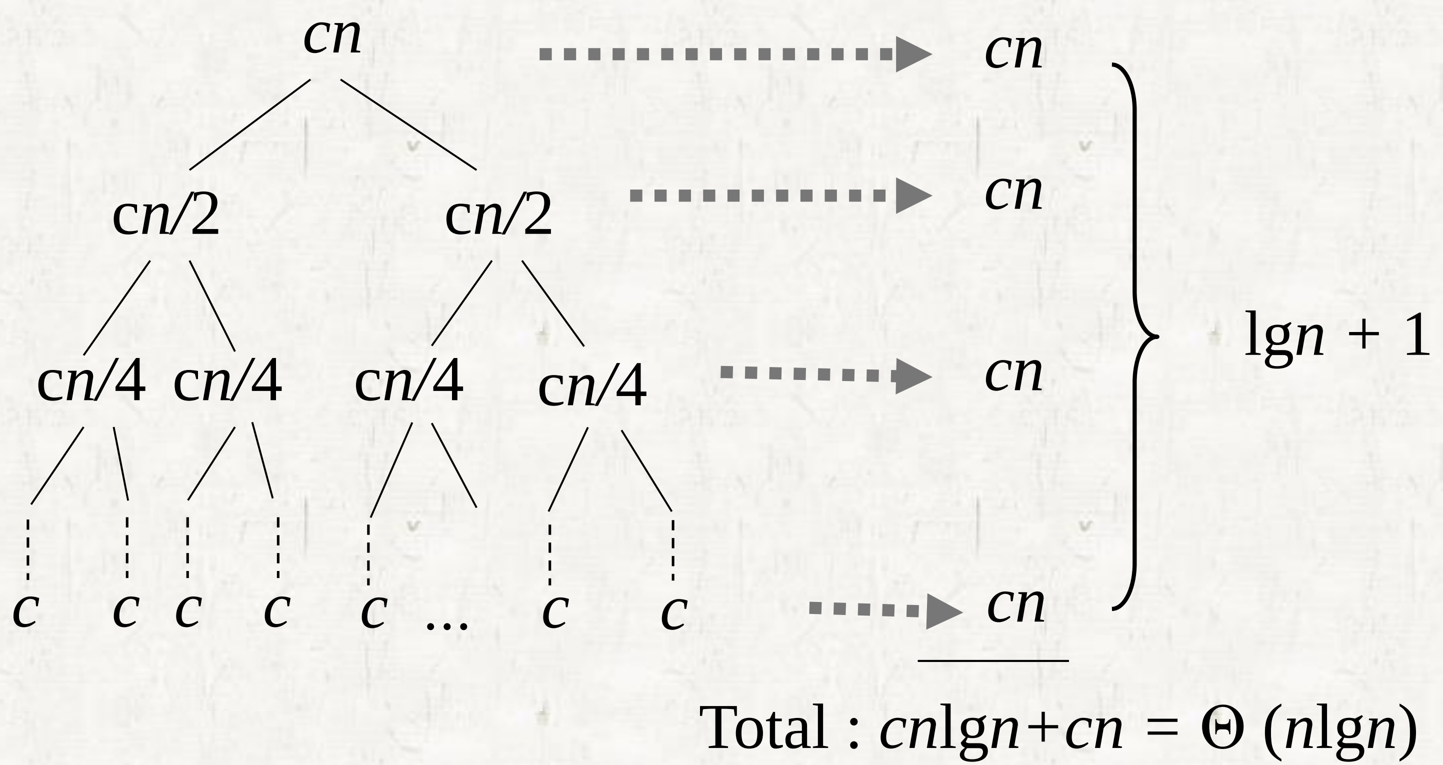
Recursion tree



Recursion tree

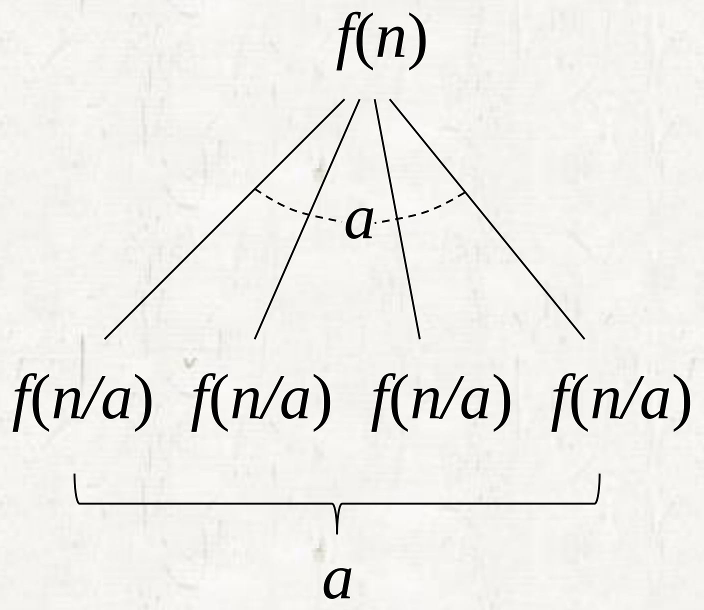


Recursion tree



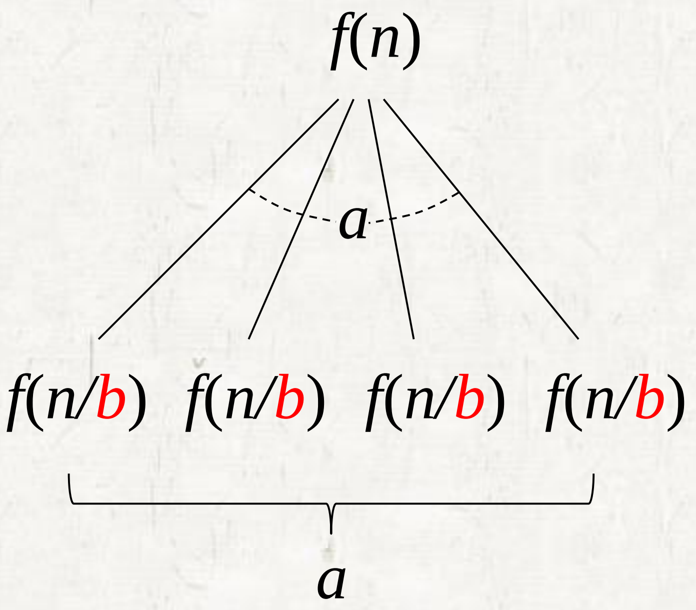
Divide and conquer

- Divide and conquer with *a* subproblems and each of which is 1/*a* the size of the original.



Divide and conquer

- Divide and conquer with a subproblems and each of which is $1/b$ the size of the original.



Divide and conquer

- Suppose that our division of the problem yields *a* subproblems, each of which is $1/b$ the size of the original.
- Let $D(n)$ denote time to divide the problem into subproblems.
- Let $C(n)$ denote time to combine the solutions to the subproblems into the solution to the original problem.
- We get the recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le c, \\ aT(n/b) + D(n) + C(n) & \text{otherwise.} \end{cases}$$

Divide and conquer

- For merge sort,
  - $a = b = 2$.
  - $D(n) = \Theta(1)$.
  - $C(n) = \Theta(n)$.

- The worst-case running time $T(n)$ of merge sort:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1, \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Self-study

- **Merge sort**
  - Exercise 2.3-1
  - Exercise 2.3-2

- **Horner’s rule**
  - Problem 2-3 (a) (b)

More (sorting) algorithms

- **Binary Search**
  - Exercise 2.3-5

- **Selection sort**
  - Exercise 2.2-2

Programming assignment

- **Program sorting algorithms in non-increasing order.**

  1. **Insertion sort**

  2. **Merge sort**

  3. **Insertion sort on small arrays in merge sort**
     - Problem 2-1

- **Due on**