Team 2
I pledge my Honor that I have abided by the Stevens Honor System.
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Query 2

1. $\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \widehat{y_i})^2 = \sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_i)]^2 =$

Knowing that β_0 and β_1 represent the constants of the regression model we can take derivation in regards to them.

Derivation for β_0

$\frac{\partial S}{\partial \beta_0}[\sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_i)]^2] = 0 \rightarrow$ chain rule $y_i - (\beta_0 + \beta_1 x_i) = u \rightarrow$

$\frac{\partial S}{\partial u}(2u) \rightarrow \frac{\partial u}{\partial \beta_0}(2(y_i - \beta_0 - \beta_1 x_i)) \rightarrow 2(\frac{\partial u}{\partial \beta_0} y_i - \frac{\partial u}{\partial \beta_0}\beta_0 - \frac{\partial u}{\partial \beta_0}\beta_1 x_i))$

$=2(0 - 1 - 0) =- 2$

$\frac{\partial S}{\partial \beta_0}[\sum_{i=1}^{n} - 2[y_i - \beta_0 - \beta_1 x_i)]=0 \rightarrow - 2\sum_{i=1}^{n} y_i + 2\sum_{i=1}^{n} \beta_0 + 2\sum_{i=1}^{n} \beta_1 x_i = 0 \rightarrow$

$-\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \beta_0 + \sum_{i=1}^{n} \beta_1 x_i = 0 \rightarrow \sum_{i=1}^{n} \beta_0 + \sum_{i=1}^{n} \beta_1 x_i = \sum_{i=1}^{n} y_i \rightarrow$

$n\beta_0 + \sum_{i=1}^{n} \beta_1 x_i = \sum_{i=1}^{n} y_i \rightarrow$ $\beta_0 = \bar{y} - \beta_1 \bar{x}$

Derivation for β_1

$\frac{\partial S}{\partial \beta_1}[\sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_i)]^2] = 0 \rightarrow$ chain rule $y_i - (\beta_0 + \beta_1 x_i) = u \rightarrow$

$\frac{\partial S}{\partial u}(2u) \rightarrow \frac{\partial u}{\partial \beta_1}(2(y_i - \beta_0 - \beta_1 x_i)) \rightarrow 2(\frac{\partial u}{\partial \beta_1} y_i - \frac{\partial u}{\partial \beta_1}\beta_0 - \frac{\partial u}{\partial \beta_1}\beta_1 x_i))$

$=2(0 - 0 - x_i) =- 2x_i$

$\frac{\partial S}{\partial \beta_1}[\sum_{i=1}^{n} - 2[y_i - \beta_0 - \beta_1 x_i)(x_i)]=0 \rightarrow - 2\sum_{i=1}^{n} y_i x_i + 2\sum_{i=1}^{n} \beta_0 x_i + 2\sum_{i=1}^{n} \beta_1 x_i^2 = 0$

$\beta_0 \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i x_i$ One possible answer is $\beta_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i x_i - \beta_0 \sum_{i=1}^{n} x_i$

Alternatively substitute the equation found for β_0

$$(\overline{y} - \beta_1\overline{x})\, n\overline{x} + \beta_1 \sum_{i=1}^{n} x_i^{\,2} = \sum_{i=1}^{n} y_i x_i \rightarrow \text{Simplifies to } \boxed{\beta_1 = \frac{\sum_{i=1}^{n} y_i x_i - n\overline{y}\,(\overline{x})}{\sum_{i=1}^{n} x_i^{\,2} - n(\overline{x})^2}}$$

2. Our derivatives in number one were set to zero, hence we can see our equations come out to be:

$$\beta_0 = \overline{y} - \beta_1\overline{x}$$

$$\beta_1\, n\,(\overline{x})^2 = n(\overline{y})(\overline{x}) - \beta_0 n\,(\overline{x}) \rightarrow \beta_1\,(\overline{x})^2 + \beta_0(\overline{x}) = (\overline{y})(\overline{x}) \rightarrow$$
$$(\overline{y})(\overline{x}) - \beta_1\,(\overline{x})^2 = \beta_0(\overline{x}) \rightarrow (\overline{y}) - \beta_1(\overline{x}) = \beta_0$$

3. $$\beta_1 = \frac{\begin{vmatrix} n & \Sigma y_i \\ \Sigma x_i & \Sigma x_i y_i \end{vmatrix}}{\begin{vmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{vmatrix}} = \frac{n\Sigma x_i y_i - \Sigma x_i y_i}{n\Sigma x_i^2 - \Sigma x_i^2} = \frac{\Sigma x_i y_i (n-1)}{\Sigma x_i^2 (n-1)} = \frac{\Sigma x_i y_i}{\Sigma x_i^2} = \frac{\Sigma y_i}{\Sigma x_i}$$

Since $\beta_0 = \overline{y} - \beta_1(\overline{x})$ and $\overline{x} = \frac{1}{n}\Sigma\, x_i$, $\overline{y} = \frac{1}{n}\Sigma\, y_i$:

$$\beta_0 = \overline{y} - \frac{\Sigma y_i}{\Sigma x_i}(\overline{x}) = \overline{y} - \frac{\Sigma y_i}{\Sigma x_i}\left(\tfrac{1}{n}\Sigma\, x_i\right) = \overline{y} - \Sigma\, y_i\left(\tfrac{1}{n}\right) = \overline{y} - \overline{y} = 0$$

$$(\beta_0, \beta_1) = \left(0, \frac{\Sigma y_i}{\Sigma x_i}\right)$$

4.

$$\begin{vmatrix} \frac{\partial s^2}{\partial \beta^2_{\ 0}} SSE & \frac{\partial s^2}{\partial \beta_0 \partial \beta_1} SSE \\ \frac{\partial s^2}{\partial \beta_0 \partial \beta_1} SSE & \frac{\partial s^2}{\partial \beta^2_{\ 1}} SSE \end{vmatrix} = \begin{vmatrix} n & 0 \\ 0 & \sum_{i=1}^{n} x_i^2 \end{vmatrix}$$

$$\frac{\partial S}{\partial \beta_0^{\ 2}} (n\, \beta_0 + n\beta_1 \overline{x} \ - n\overline{y}\,)$$

$$(\frac{\partial S}{\partial \beta_0} n\, \beta_0 + \frac{\partial S}{\partial \beta_0} n\beta_1 \overline{x} - \frac{\partial S}{\partial \beta_0} n\overline{y}\,)$$

$$(n\, 1\, +\, 0\, -\, 0\,)\, =\, n$$

$$\frac{\partial S}{\partial \beta_0\, \partial \beta_1} (\overline{y} - \beta_1 \overline{x})\, n\overline{x} \ + \beta_1 \sum_{i=1}^{n} x_i^{\ 2} - \sum_{i=1}^{n} y_i x_i$$

$$\frac{\partial S}{\partial \beta_0\, \partial \beta_1} [(\overline{y} - \beta_1 \overline{x})\, n\overline{x} \ + n\beta_1 (\overline{x})^2 - n(\overline{x} * \overline{y})]$$

$$[((\overline{y} - \beta_1 \overline{x})\, n\overline{x}) \frac{\partial S}{\partial \beta_0} \ + (n\beta_1 (\overline{x})^2) \frac{\partial S}{\partial \beta_0} - (n(\overline{x} * \overline{y})) \frac{\partial S}{\partial \beta_0}] \ = \ [0\, +\, 0\, -\, 0] \,=\, 0$$

$$\frac{\partial S}{\partial \beta_1^{\ 2}} (\overline{y} - \beta_1 \overline{x})\, n\overline{x} \ + \beta_1 \sum_{i=1}^{n} x_i^{\ 2} - \sum_{i=1}^{n} y_i x_i$$

$$\frac{\partial S}{\partial \beta_1} (\overline{y} - \beta_1 \overline{x})\, n\overline{x} \ + (\frac{\partial S}{\partial \beta_1}) \beta_1 \sum_{i=1}^{n} x_i^{\ 2} - \frac{\partial S}{\partial \beta_1} \sum_{i=1}^{n} y_i x_i$$

$$\frac{\partial S}{\partial \beta_1} (\overline{y} - \beta_1 \overline{x})\, n\overline{x} \ + (\frac{\partial S}{\partial \beta_1}) \beta_1 \sum_{i=1}^{n} x_i^{\ 2} - \frac{\partial S}{\partial \beta_1} \sum_{i=1}^{n} y_i x_i = (0\, +\, \sum_{i=1}^{n} x_i^{\ 2} - 0) \,=\, \sum_{i=1}^{n} x_i^{\ 2}$$

5. $\begin{vmatrix} n & 0 \\ 0 & \sum_{i=1}^{n} x_i^2 \end{vmatrix} \rightarrow$ critical point - $(\beta_0 = 0, \beta_1 = \frac{\Sigma y_i}{\Sigma x_i})$

   The Hessian matrix is symmetric. Since n represents the number of values in a given dataset as seen with $\sum_{i=1}^{n} x_i^2$ n is a non-zero, positive value. The given Hessian matrix is in its reduced form. Since the pivots are positive, we can conclude that this is a positive definite matrix.

6. According to the multivariable second derivative test, critical point is a local minimum if:

$$\frac{\partial S}{\partial \beta_0^{\ 2}} (\beta_0, \beta_1) > \ 0 \text{ and } \frac{\partial S}{\partial \beta_0^{\ 2}} (\beta_0, \beta_1) \cdot \frac{\partial S}{\partial \beta_1^{\ 2}} (\beta_0, \beta_1) - \frac{\partial S}{\partial \beta_0\, \partial \beta_1} (\beta_0, \beta_1) \ > \ 0$$

$n > 0$ and $n \cdot \sum_{i=1}^{n} x_i^2 - 0 > 0$

Both of these are true, due to how n is a positive value since it represents the number of values in a dataset. Moreover $(n \cdot \sum_{i=1}^{n} x_i^2 - 0)$ represents a positive value since the $\sum_{i=1}^{n} x_i^2$ summation will always result in a positive value since x_i is squared.

Also, according to the slides, upon proving that the Hassian matrix is positive definite, the critical point is proven to be the unique minimizer of SSE.