UNIVERSITY OF BATANGAS

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Properties of Determinants

Research Paper

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Index:

Abstract:
Determinants are mathematical objects that are very useful in the analysis and solution of system of linear equations. These applications are only defined for square matrices. In this study, the readers will find a brief note on determinants and its properties based on online articles and class lectures.

Introduction:
The determinant, in Linear Algebra, is a value that can be computed from the elements of a square matrix. The determinant of a matrix A is denoted by $d(A), \det(A)$, or $|A|$. Geometrically, it can be viewed as the volume scaling factor of the linear transformation of the matrix.

This is also the signed volume of the n-dimensional parallelepiped spanned by the column or row vectors of a matrix. The determinant is positive or negative according to whether the linear mapping preserves or reverses the orientation of n-space.

In the case of a 2 x 2 matrix, the determinant may be defined as:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Similarly, for 3 x 3 matrix,

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} x & x & x \\ x & e & f \\ x & h & i \end{vmatrix} - b \begin{vmatrix} x & x & x \\ d & x & f \\ g & x & i \end{vmatrix} + c \begin{vmatrix} x & x & x \\ d & e & x \\ g & h & x \end{vmatrix}$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei - ahf - bdi + bgf + cdh - cge$$

Each determinant of a 2 x 2 matrix in this equation is called a ‘minor’ of the matrix ‘A’. This procedure can be extended to give a recursive definition for the determinant of an n x n matrix, the minor expansion formula.

Definitions, related to determinants:

(1) Square matrix:

In mathematics, a square matrix is a matrix with the same number of rows and columns. An n-by-n matrix is known as a square matrix of order n. any two square matrices of the same order can be added and multiplied.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Fig-1: Square Matrix

(2) Linear transformations:

A linear map which is also known as a linear mapping or linear transformation or linear function, is a mapping $V \to W$ between the two modules that preserves the operation of addition and scalar multiplication.

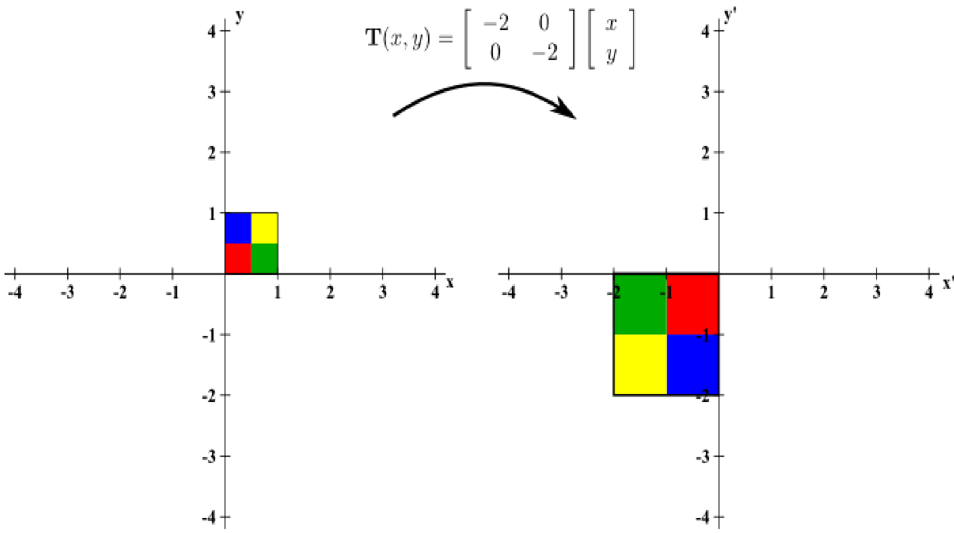

Fig-2: Determinants and Linear Transformation

(3) Minor:

A minor of a matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows and columns. Minors obtained by removing just one row and one column from square matrices are required for calculating matrix cofactors which in turn are useful for computing both the determinant and inverse of square matrices.

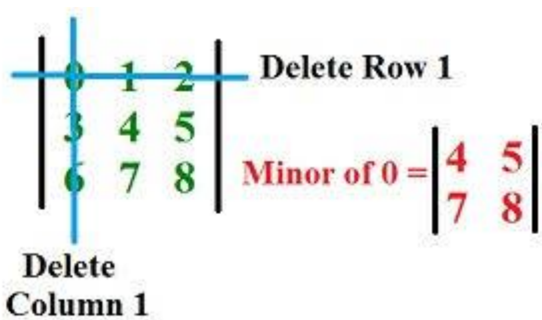

Fig-3: Minor of a Matrix

(4) <u>Cofactors:</u>

The cofactor of a square matrix A of order (n x n) is equal to $(-1)^{i+j}M_{ij}$, where M_{ij} represents the minor of the elements a_{ij} of the matrix A. The notation to represent the cofactor is C_{ij}. To find the cofactor of a matrix, we need to follow the steps:

a. Find the minor M_{ij} of the given matrix
b. Then put the values of i, j and put this value as power to -1.
c. Multiply this obtained value which would either be

The value obtained is the cofactor of the element:

$$C_{ij} = (-1)^{i+j}M_{ij}$$

(5) <u>System of linear equations:</u>

A system of linear equations or linear system is a collection of two or more linear equations involving the same set of variables. For examples,

$3x + 2y - z = 1$
$2x - 2y + 4z = -2$
$-x + \frac{1}{2}y - z = 0$

is a system of three equations in the three variables x, y, z. A solution to a linear system is an assignments of values to the variables such that all the equations are simultaneously satisfied. A solution to system above is given by-

$$x = 1, y = -2, and\ z = -2$$

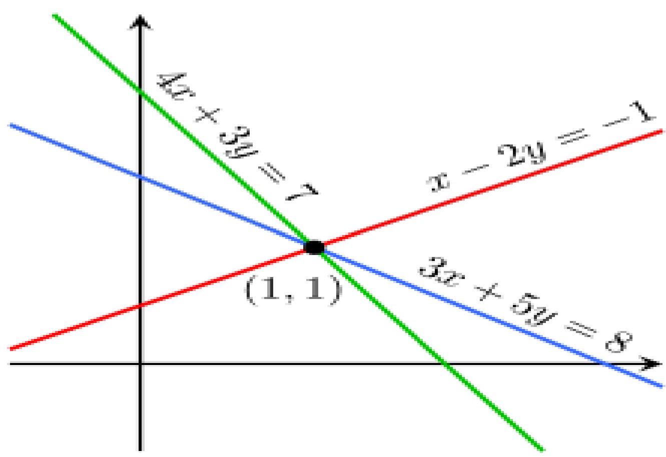

Fig-4: System of Linear Equations

(6) <u>Invertible matrix:</u>

An n-by-n square matrix is called invertible, if there exists an n-by-n matrix B such that

$$AB = BA = I_n$$

Where, I_n denotes the n-by-n identity matrix and the multiplication used is ordinary matrix multiplication.

(7) Characteristic polynomial:

The characteristic polynomial of a matrix is a polynomial which is invariant under matrix similarity and has the eigenvalues as roots. It has the determinants and the trace of the matrix coefficient. The characteristics equation is the equation obtained by equating to zero the characteristics polynomial.

Let, we consider an n x n matrix A, The characteristics polynomial of A denoted by $p_A(t)$ is the polynomial defined by,

$$p_A(t) = \det(tI - A)$$

Where, I denotes n-by-n identity matrix. Some authors define the characteristics polynomial to be $\det(A - tI)$. The polynomial differs from the one defined here by a sign $(-1)^n$, so it makes no difference for properties like having as roots the eigenvalues of A.

(8) Eigenvalues:

Eigenvalues are the special set of scalars associated with a linear system of equations that are sometimes also known as characteristic roots or characteristic values, proper values or latent roots. The determination of the eigenvalues and eigenvectors of a system is extremely important in physics and engineering, where it is equivalent to matrix diagonalization and arises in such common applications as stability analysis, the physics of rotating bodies, and small oscillating of vibrating systems, to name only a few. Each eigenvalue is paired with a corresponding so called eigenvector.

If T is a linear transformation from a vector space V over a field F into itself and v is a vector in V that is not the zero vector, then v is an eigenvector of T, if $T(v)$ is scalar multiple of v. This condition can be written as the equation

$$T(v) = \lambda v$$

Where, λ is a scalar in the field F, known as the eigenvalue or characteristic value or characteristic root associated with the eigenvector v.

Uses of determinants:

Determinants occur throughout mathematics. For example, a matrix is often used to represent the co-efficient in a system of linear equations; the determinant can be used to solve those equations, although other methods of solution are much more computationally efficient. In linear algebra, a matrix, with entries in a field, is invertible if and only if its determinant is non-zero, and correspondingly the matrix is singular if and only if the determinant is zero. This leads to the use of determinants in defining the characteristics polynomial of a matrix, whose roots are the eigenvalues. In analytic geometry, determinants express the signed n-dimensional volumes of n-dimensional parallelepipeds. This leads to the use of determinants in calculus, the Jacobean determinant in the change of variables rule for integrals of function of several variables. Determinants appear frequently in algebraic identities such as the Vandermonde identity.

Properties of determinants:

Determinants possess many algebraic properties, including the determinant of a product of matrices. Special types of matrices have special determinants. Determinants have a number of useful properties in the form of theorems.

<u>Theorem-1: Multiplying a row or column by a constant</u>

If each element of any row (or column) of a determinant is multiplied by a constant k, the new determinant is k times the original.

Partial proof:

Let C_{ij} is the cofactor of a_{ij} then expanding by first row, we have

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13})$$

$$= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Theorem-1 also states that a factor common to all elements of a row (or column) can be taken out as a factor of the determinant.

Example: Taking out a common factor of a column

$$\begin{vmatrix} 6 & 1 & 3 \\ -2 & 7 & -2 \\ 4 & 5 & 0 \end{vmatrix}$$

$$2\begin{vmatrix} 3 & 1 & 3 \\ -1 & 7 & -2 \\ 2 & 5 & 0 \end{vmatrix}$$

Where, 2 is a common factor of the first column.

Theorem-2: Row or column of zeros

If every element in a row (or a column) is zero, the value of the determinant is zero.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} = 0$$

Theorem-2 is immediate consequence of theorem-1, and it is illustrated in the following example:

$$\begin{vmatrix} 3 & -2 & 5 \\ 0 & 0 & 0 \\ -1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 9 \\ -2 & 0 & 4 \\ 3 & 0 & -1 \end{vmatrix} = 0$$

Theorem-3: Interchanging row or columns

If two rows (or two columns) of a determinant are interchanged, the new determinant is the negative of the original.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix}$$

A proof of theorem-3 even for a determinant of order 3 is notationally involved. It is easy to prove the theorem-3 partially by direct expansion of the determinants before and after the interchange of two rows or two columns. The theorem is illustrated by the following example:

$$\begin{vmatrix} 1 & 0 & 9 \\ -2 & 1 & 5 \\ 3 & 0 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 9 & 0 \\ -2 & 5 & 1 \\ 3 & 7 & 0 \end{vmatrix}$$

Theorem-4: Equal rows or columns

If the corresponding elements are equal in two rows (or columns), the value of determinant is zero.

Proof:

The general proof of Theorem-4 follows directly Theorem-3. Let, a determinant $D,$ has two rows (or columns) equal and interchanging the equal rows (or columns), the new determinant will be the same as the original. By the Theorem-3,

$$D = -D$$
$$2D = 0$$
$$D = 0$$

Example:

$$D = \begin{vmatrix} a & a & d \\ b & b & e \\ c & c & f \end{vmatrix} = 0$$

Where, the first and second column are equal, then $D = 0$

Theorem-5: Addition of rows or columns

If a multiple of any row (or column) of a determinant is added to any other row (or column), the value of the determinant is not changed.

Partial proof:

In a general third-order determinant, adding a k multiple of the second column to the first and then expand by the first column, then obtain (where C_{ij} is the cofactor of a_{ij} in the original determinant),

$$\begin{vmatrix} a_{11}+ka_{12} & a_{12} & a_{13} \\ a_{21}+ka_{22} & a_{22} & a_{23} \\ a_{31}+ka_{32} & a_{32} & a_{33} \end{vmatrix} = (a_{11}+ka_{12})C_{11} + (a_{21}+ka_{22})C_{21} + (a_{31}+ka_{32})C_{31}$$

$$= (a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}) + k(a_{12}C_{11} + a_{22}C_{21} + a_{32}C_{31})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The determinant following k is 0, because the first and second columns are equal.

References:

01. Fundamental of Linear Algebra and Optimization – Jean Gallier and Jocelyn Quaintance
02. Linear Algebra, Theory and Applications – Kenneth Kuttler
03. Classnotes/msm/graduate school/University of Batangas
04. Determinant - Wikipedia
05. Mathworld.wolf.com/eigenvalue.html
06. Byjus.com/math/minors-cofactors-determinant/
07. Sosmath.com/matrix/determ0/determ0.html