Math 104A Practice Final Problems

1. Find the natural (free) cubic spline that interpolates the function values $f(0) = 1$, $f(1) = 2$, and $f(2) = 9$ in the interval $[0, 2]$.

2. Let $f(0.1) = 1.10517$, $f(0.2) = 1.22140$, and $f(0.3) = 1.34986$. a) Find a first order approximation to $f'(0.2)$. b) Find a second order approximation to $f'(0.2)$.

3. Suppose that $N(h)$ is an approximation to M for every $h > 0$ and that
$$M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$
for some constants K_1, K_2, K_3, ... Use the values $N(h)$, $N(\frac{h}{3})$, and $N(\frac{h}{9})$ to produce and $O(h^3)$ approximation to M.

4. The (simple) trapezoidal rule applied to $\int_0^4 f(x)dx$ gives the value 2 and Simpson's rule gives the value 1. What is $f(2)$?

5. Derive a quadrature formula for approximating
$$\int_0^3 f(x)dx$$
in terms of $f(0)$, $f(1)$, $f(2)$ and $f(3)$.

6. a) Define the degree of accuracy or precision of a quadrature formula. b) What's the degree of accuracy for the Composite Trapezoidal rule? c) What's the degree of accuracy for the Composite Simpson's rule?

7. Consider the integral
$$\int_0^1 e^x x^3 dx$$
Approximate this integral using a) the Composite Trapezoidal rule with $n = 4$ and b) the Composite Simpson rule with $n = 4$. c) Which approximation do you expect to be more accurate? Explain.

8. Consider the integral
$$\int_0^1 x^2 dx$$
If the Composite Trapezoidal rule and the Composite Simpson rule quadratures are used to approximate this integral, what is the error produced by these approximations in this particular case? Be precise.

9. Consider a smooth function f on $[a, b]$ (i.e.with as many continuous derivatives as you need) and divide this interval into N subintervals of equal length. The composite trapezoidal rule is then given by
$$T(h) = h\left(\frac{1}{2}f(a) + \sum_{k=1}^{N-1} f(x_k) + \frac{1}{2}f(b)\right),$$

where $h = (b-a)/N$ and $x_k = a + kh$. Knowing that $T(h) = \int_a^b f(x)dx + c_2h^2 + c_4h^4 + \dots$ apply Richardson extrapolation to obtain the fourth order (Simpson) rule:

$$S(h) = \frac{h}{3}\left(f(a) + 4f(x_1) + 2\sum_{k=1}^{N-1}[f(x_{2k}) + 2f(x_{2k+1})] + f(b)\right).$$

10. a) Define the local truncation error $\tau_{i+1}(h)$ for a difference method of the form

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(t_i, w_i) \quad i = 0, 1, \dots, N-1 \end{aligned}$$

b) show that $\tau_{i+1}(h) = O(h^2)$ for the midpoint method.

11. Consider the initial value problem (IVP):

$$y' = \frac{2t}{y^2} \qquad 0 \le t \le 1 \qquad y(0) = 1. \tag{1}$$

a) Determine the exact solution of this IVP.
b) Use Euler Method to find an approximation to the solution of this IVP with $h = 0.5$.
c) Reduce h by half ($h = 0.25$) and compute the corresponding approximation.
d) Which answer is more accurate (at corresponding points)? Explain.

12. a) What is the order of accuracy (order of the local truncation error) of the Runge-Kutta Heun's method? b) Use the Heun's method to find an approximation (1), problem 11, using $h = 0.5$.

13. a) What would be the main reason for using an Adams-Bashforth method instead of a Runge-Kutta method, if both are of the same order of accuracy?
b) Give one advantage and one disadvantage of using an Adams-Moulton method instead of a Adams-Bashforth method.

14. Consider again the IVP (1) of question 11. Use the two-step Adams-Bashforth method to find an approximation for the IVP using $h = 0.25$. To initialize the two-step method you can either use a second order RK or the exact solution.

15. Consider the initial value problem (IVP):

$$y' = -y + \sin 2\pi t \qquad 0 \le t \le 1 \qquad y(0) = 1. \tag{2}$$

Use the two-step Adams-Bashforth method to find an approximation for the IVP using $h = 0.25$. To initialize the two-step method use midpoint RK method.

16. Consider the function values: $f(0) = 0$, $f(0.1) = 0.3090$, $f(0.2) = 0.5878$, $f(0.3) = 0.8090$, and $f(0.4) = 0.9511$
a) Compute two approximations to $f'(0.2)$ using the (2nd order) centered difference formula.

For one approximation use the values $f(0)$ and $f(0.4)$ and for the other use $f(0.1)$ and $f(0.3)$.
b) Which approximation in a) is more accurate? Explain in terms of the error.
c) Knowing that the error in the centered difference formula has the form:

$$K_2h^2 + K_4h^4 + K_6h^6 \ldots,$$

use Richardson extrapolation and the two approximations in a) to obtain a new approximation that is fourth order accurate.

17. Describe four of the main problems that we studied in this course.