Computational Methods in Economics

Nonlinear Equations

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Nonlinear Equations: Definitions

Consider

$$f(x) = 0 \tag{1}$$

- A large class of problems
- Integral equations/ System of Differential Equations.

Nonlinear Equations (1)

Consider the basic neoclassical growth model

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_t U(c_t)$$
$$c_t + k_{t+1} = (1-\delta)k_t + e^{z_t} k_t^{\alpha}, \qquad t = 0, 1, 2, \ldots..,$$
$$z_t = \rho z_{t-1} + \epsilon_t \text{ with } \epsilon_t \sim \mathbb{N}(0,1).$$

The FOC:

$$u'(c_t) = \beta \mathbb{E}_t \{u'(c_{t+1})(1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} - \delta)\} \tag{2}$$

Nonlinear Equations (2)

- It is not possible to determine the zero of this function.
- This is true for most utility functions.

Linearization

- Linear utility.
- Linear approximation of nonlinear relationship

$$Ax = b \tag{3}$$

- Basic Matrix Algebra (Gauss Elimination)

Nonlinear Equations (5)

- Since it is not possible to explicitly recover the zero. Approximation is usually the good approach.
- Approximation is done through an iterative process. From an initial starting value x_0, successive approximates x_i to the zero are computed from an iteration function.

Nonlinear Equations (6)

- What is a good iteration function? How do you find a suitable iteration function.
- Under what conditions will the sequence x_i converge?
- How quickly will the sequence converge.

Nonlinear Equations (3)

- Fixed point theorems..
- Brouwer fixed point: "Every continuous function from a disk to itself has a fixed point".
- Harder for problems of the kind $f(x, g(x)) = 0$. Banach Fixed point.

Nonlinear Equations (4)

Proposition

Let $f : R^n \longrightarrow R^n$ be an iteration function with fixed point xi. Suppose that there are a neighborhood $U(\xi)$ of ξ, a number p, and a constant such that $\forall x \in U(\xi)$

$$||f(x) - \xi|| \leq c||x - \xi||^P$$

Then there is a neighborhood $V(\xi) \subseteq U(\xi)$ of ξ so that for all starting values $x_0 \in V(\xi)$ the iteration method defined by f generates iterates x_i with $x_i \in V(\xi)$ for all i that converge to ξ at least with order p.

Methods

- Bisection
- Newton

Bisection

- Given $f()$, set lower and upper bound denoted by a and b such that $f(b) * f(a) < 0$
- Set c = (a+b)/2:
  - If $f(c)f(a) < 0$ then b=c,
  - If $f(c)f(b) < 0$ then a=c,
- Iterate until $f(c) \approx 0$.

Newton

- Given $f()$, set an initial guess x_0
- Update $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$