UNIVERSITY OF COPENHAGEN

Seminar: Monte Carlo Methods in Econometrics and Finance

Variance Reduction Techniques Applications in Monte Carlo Valuation of European Call Options

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Contents

1 Introduction

A financial derivative is a contract which provides to the holder a future payoff that depends on the performance of an underlying entity such as an asset, stocks or interest rate. Options are one of the financial derivatives, serve as the function of speculating and hedging. There are different kinds of options. A European option is a not path-dependent option which gives its holder the right to buy or sell stock for a predetermined price at a specified maturity, in contrast to an American option, where allows holders to exercise the option rights at any time before and including the day of expiration. An Asian option is a path-dependent option where the payoff depends on the average price of the underlying asset over a certain period of time. There are two basic kinds of options: call option and put option. A call option gives the buyer the right to buy the underlying asset while a put gives the buyer the right to sell.

In option pricing, Black and Scholes (1973) apply the partial differential equation to describe the evolution of a derivative's price over time. In Flemming Dose[1] comment, derivative pricing is a problem of solving the differential equation. One way of solving the differential equation is to compute the present value of the expected payoff under a risk-neutral probability measure. Owing to the complexity of the payoff structure and the stochastic process dominating the evolution of the underlying asset, the valuation problem can be solved by either using analytic formulas or numerical methods. The closed form solutions are available for pricing the European options. For the options like American options and arithmetic Asian options, we price them by Monte Carlo method or other numerical methods because there are no closed solutions for pricing them. In this paper, we price the European call option by using the analytical formula and Monte Carlo method. Even though there is no need for us to use Monte Carlo method to price a European call option because it has closed form solution, it still worthwhile for us to simulate it from learning perspective. Furthermore, we mainly focus on the variance reduction techniques, getting intuition about how variance reduction techniques reduce the standard error in the Monte Carlo simulations. We also want to the discuss the connection between control variates and other variance reduction techniques, and then illustrate with simulations. Taking Monte Carlo method to price a European call option therefore may be a good idea from my perspective.

When we use Monte Carlo method to price a European call option, its simulated result always associated with a variance which limits the precision of the simulation results. In order to make a simulation statically efficient, variance reduction techniques can be used. In this paper, we introduce control variates, antithetic variates, moment matching and stratified sampling to discuss their variance reduction efficiency in Monte Carlo valuation of a European call option. These are based on (Glasserman (2004),p185-276) and Boyle et al. (1997) work. Further, we also discuss the relationships between control variates and antithetic variates, moment matching, and stratified sampling. These are based on Glynn and Szechtman (2002), Grant (1983) and Boyle et al. (1997) work.

This essay is organized as follows. Section 2 describes the problem of computing the Black-Scholes price of a European call option and Monte Carlo valuation of a European call option. Variance reduction techniques are presented in Section 3. In Section 4, control variates relation to other variance reduction techniques are discussed. Section 5 concludes the essay.

2 Option Pricing under BSM model and Monte Carlo Methods

European call option gives the buyer the right, but not the obligation, to buy the underlying asset at a certain price at a specified maturity. Black and Scholes (1973) and Merton (1975) laid the foundation for option pricing model, which is Black-Scholes-Merton(BSM) model. According to Zhang (2009) classification,

[1] the example of good paper

there are other three kinds of option pricing methods, which are binomial methods, finite difference methods and Monte Carlo methods . In the essay, we focus on BSM model and Monte Carlo methods.

2.1 Black-Scholes-Merton (BSM) Model

We define S_t as the price of the stock at time t and K as the strike price of the option. r is risk free rate and σ is the volatility of the stock. T is the expire time. For simplification, we assume $t = 0$ in the following cases, then the time to maturity is T. We define $\Phi(\cdot)$ as the cumulative distribution function of the standard normal distribution and $W(T)$ as a standard Brownian motion. The stock price dynamic under the risk-neutral probability measure Q is:

$$\frac{dS_T}{S_T} = rdT + \sigma dW(T) \tag{2.1}$$

We assume $W(T) \sim N(0,T)$, $Z \sim N(0,1)$, then $W(T)$ is also the distribution of $\sqrt{T}Z$. The solution of equation 2.1 is:

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z} \tag{2.2}$$

A European call option C price at time T is:

$$C_T = e^{-rT} max(S_T - K, 0) \tag{2.3}$$

According to Black and Scholes (1973) and Merton (1975) method, a European call option price $C(S_0, K, T, \sigma, r)$ in BSM model is:

$$\begin{aligned} C(S_0, K, T, \sigma, r) &= E\left[e^{-rT} max(S_T - K, 0)\right] \\ &= S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) \end{aligned} \tag{2.4}$$

$$d_{1,2} = \frac{ln(S_0/K) + (r \pm \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

2.2 Monte Carlo Methods

Monte Carlo methods are simulation algorithms, which base on repeated computation and random sampling to estimate numerical results[2]. In Monte Carlo valuation of a European call option, assume we can draw a sequence$Z_1, Z_2, \ldots, Z_i$ of i.i.d. $N(0,1)$ variables, then we can estimate the value of a European call option $E\left[e^{-rT} max(S_T - K, 0)\right]$ by using the following algorithm:

[2]Wikipedia definition

Algorithm 1 Monte Carlo valuation of European call option

Input: S_0, K, r, σ, T, n

1. **for** $i = 1$ to n
2. **generate** $Z_i \sim N(0,1)$
3. **set** $C_i = e^{-rT} max(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma Z_i\sqrt{T}} - K, 0)$
4. **end for**
5. **set** $\tilde{C_n} = \frac{\sum_{i=1}^{n} C_i}{n}$, $\tilde{\sigma_n^2} = \frac{\sum_{i=1}^{n}(C_i-\tilde{C_n})^2}{n-1}$

For $n \geq 1$, the expecation of $\tilde{C_n}$ is equal to the true value C, $\tilde{C_n}$ therefore is unbiased.

$$E\left[\tilde{C_n}\right] = C = E\left[e^{-rT} max(S_T - K, 0)\right] \tag{2.5}$$

The estimator is also strongly consistent, then we have:

$$\tilde{C_n} \to C \text{ with probability 1 as } n \to \infty$$

For $C_1, C_2 \ldots C_i$ i.i.d. with $E[C_i] = C < \infty$ and $Var(C_i) = \sigma^2 < \infty$,for $i = 1, \ldots, n$, then by Central Limit Theorem (CLT) (Chopra (2007)) ,we have:

$$\frac{\tilde{C_n} - C}{\sigma/\sqrt{n}} \Rightarrow N(0,1) \tag{2.6}$$

The Monte Carlo estimator $\tilde{C_n}$ approximates a normal distribution, the accuracy of this approximation increases as n increases. As the true value of σ is unknown to us, we replace σ with sample standard deviation $\tilde{\sigma}$ as an estimator:

$$\tilde{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(C_i - \tilde{C_n})^2}{n-1}} \tag{2.7}$$

By applying this, we can compute the confidence interval:

$$\tilde{C_n} \pm z_{\delta/2}\frac{\tilde{\sigma}}{\sqrt{n}} \tag{2.8}$$

κ_δ is the $1-\delta$ quantile of the standard normal distribution and $\frac{\tilde{\sigma}}{\sqrt{n}}$ is the standard error ($s.e.$). The probability that the confidence interval covers C approaches $1-\delta$ as $n \to \infty$. From the form of the standard error we know that if we want to reduce the length of the confidence interval, we can either reduce $\tilde{\sigma}$ or increase simulation time n. In the flowing simulations, we take n simulation times to estimate the European call option price. Standard errors are estimated by re-simulation. We take m re-simulation trials and each one based on n replications of the estimator. The sample standard deviation of the m simulation estimates gives an estimate of the standard error of a single simulation estimate. Table 2.1 reports a European call option price implied by the BSM model and Monte Carlo method[3] .

[3]All the simulations in this paper, we have controled random number generation.

K	n	BS-European Call	MC-European Call	$s.e.$	Computation time t	$s.e./\sqrt{t}$
80	100	24.5888	24.6215	1.9278	0.0203	0.1424
	1000		24.5863	0.6080	0.2003	0.4475
	10000		24.5896	0.1921	1.8887	1.3743
100	100	10.4506	10.4647	1.4796	0.0217	0.1473
	1000		10.4458	0.4678	0.1942	0.4407
	10000		10.4513	0.1471	2.0371	1.4273
120	100	3.2475	3.2557	0.8710	0.0235	0.1534
	1000		3.2434	0.2750	0.1983	0.4453
	10000		3.2477	0.0861	1.9652	1.4018

Table 2.1: Monte Carlo Valuation of European Call Option. European call option with $S_0 = 100, r = 0.05, \sigma = 0.2, T = 1, m = 10000$.

In 2.1, the European call option price under Monte Carlo method will converge to the BSM model price as simulation time n increases. The standard error $s.e.$ will decrease when n increases. However, the more simulation times we want, the more computation time we need. Hence, there is a trade off between simulation times and computation time.

3 Variance Reduction Techniques

Variance reduction techniques are used to increase the efficiency of Monte Carlo simulations. The variance reduction techniques introduced in (Glasserman (2004),p185-276) reduce variance of simulation estimates from two perspectives: one is to reduce the uncertainty in simulation inputs, the other one is to apply tractable features of a model to modify or correct simulation outputs. In this paper, we will introduce control variates, antithetic variates, moment matching and stratified sampling to discuss their variance reduction efficiency in Monte Carlo simulations .

3.1 Antithetic Variates

In (Glasserman (2004),p205-208) definition, antithetic variates attempts to reduce variance by using negative correlation between pairs of variables. Assuming we can draw a sequence$Z_1, Z_2, \ldots, Z_i$ of i.i.d. $N(0,1)$ variables, then we can also get its pair sequence $-Z_1, -Z_2, \ldots, -Z_i$ of i.i.d $N(0,1)$ variables. A large value of Z_i can be balanced by its small value of antithetic $-Z_i$, resulting in variance reduction. In Monte Carlo valuation of European call option, we can price European call option C_i and its antithetic value C_i^{AV} by using the following formulas:

$$C_i = e^{-rT} max(S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma Z_i \sqrt{T}} - K, 0) \tag{3.1}$$

$$C_i^{AV} = e^{-rT} max(S_0 e^{(r-\frac{\sigma^2}{2})T - \sigma Z_i \sqrt{T}} - K, 0) \tag{3.2}$$

Each pair (C_i, C_i^{AV}) is i.i.d. For each i, C_i and C_i^{AV}have the same distribution but not independent. The antithetic variates estimator $\bar{C}_{AV}$ is the mean of $2n$ observations.

$$\bar{C}_{AV} = \frac{1}{2n}(\sum_{i=1}^{n} C_i + \sum_{i=1}^{n} C_i^{AV}) = \frac{1}{n}\sum_{i=1}^{n} \frac{C_i + C_i^{AV}}{2} \tag{3.3}$$

It is meaningful to compare the variances of $\bar{C}_{AV}$ with the variance of Monte Carlo estimator $\tilde{C}_n$ if they have the same mean. 2.5 ensures both estimators $\bar{C}_{AV}$ and $\tilde{C}_n$ have the same mean. Antithetic variates reduces variance if

$$Var(C_i + C_i^{AV}) < 2Var(C_i) \tag{3.4}$$

The left hand side of 3.4 can be written as $Var(C_i^{AV}) + Var(C_i) + 2Cov(C_i^{AV}, C_i)$. C_i and C_i^{AV} have the same variance $Var(C_i^{AV}) = Var(C_i))$ because they have the same distribution. Finally, antithetic variates reduce variance if

$$Cov(C_i^{AV}, C_i) < 0 \tag{3.5}$$

The algorithm of antithetic variates application in Monte Carlo valuation of European call option shows as follow:

Algorithm 2 Antithetic Variates Application in Monte Carlo valuation of European call option

Input: S_0, K, r, σ, T, n
1. **for** $i = 1$ to n
2. **generate** $Z_i \sim N(0,1)$
3. **generate** C_i and C_i^{AV} according to 3.1 and 3.2 respectively
4. **set** $C_{AV} = \frac{C_i + C_i^{AV}}{2}$
5. **end for**
6. **set** $\tilde{C}_n = \frac{\sum_{i=1}^{n} C_{AV}}{n}$, $\tilde{\sigma_n^2} = \frac{\sum_{i=1}^{n}(C_{AV} - \tilde{C_n})^2}{n-1}$

3.2 Moment Matching

Let sequence$Z_1, Z_2, \ldots, Z_i, i = 1, \ldots, n$ are i.i.d. $N(0,1)$ variables. The sample moments of the n Z's are not fully match those of the standard normal. However, we can convert the Z's to match a finite number of the underlying population moments. This is meaningful because it ensures that the underlying assets can be priced accurately within finite-sample. We apply moment matching to reduce the variance in Monte Carlo valuation of European call option based on Boyle et al. (1997) work. We can use the following formula to match the first moment of the standard normal.

$$\tilde{Z}_i = Z_i - \tilde{Z} \tag{3.6}$$

where $\tilde{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$. The underlying stock prices now can be written as :

$$\tilde{S}_T(i) = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\tilde{Z}_i} \tag{3.7}$$

If the Z_i's are normally distributed, then $\tilde{Z}_i$'s are also normally distributed. However, $\tilde{Z}_i$'s are no longer independent, moment-matched estimators $\tilde{C}_i^{MM1}$ are therefore not independent. We can also match two moments of a standard normal distribution by using the following formula.

$$\tilde{Z}_i = \frac{(Z_i - \tilde{Z})}{s_z} \tag{3.8}$$

where s_z is the sample standard deviation of the $\tilde{Z}_i$'s. Unlike the $\tilde{Z}_i$'s in 3.6, $\tilde{Z}_i$'s in 3.8 are not normally distributed even though Z_i's are normal. The estimators $\tilde{C}_i^{MM2}$ therefore are biased estimators of the true option value C. However, European call option is not a path-dependent option, its value only depends on the underlying stock price at the expire time T, and the mean and variance of the terminal stock price S_T is also known. Hence, the moment matching could be applied to terminal stock price $S_T(i)$. In the first moment, $\tilde{S}_T(i)$ can be written as

$$\tilde{S}_T(i) = S_T(i) - \tilde{S}_T + \mu_{S_T} \tag{3.9}$$

where $\mu_{S_T} = S_0 e^{rT}$ and $\tilde{S}_T = \frac{1}{n}\sum_{i=1}^{n} S_T(i)$. We can also get the first two moments by using the formula:

$$\tilde{S}_T(i) = (S_T(i) - \tilde{S}_T)\frac{\sigma_{S_T}}{s_{S_T}} + \mu_{S_T} \tag{3.10}$$

where $\sigma_{S_T} = S_0\sqrt{e^{2rT}(e^{\sigma^{2T-1}} - 1)}$ and s_{S_T} is the standard deviation of $S_T(i)$'s. The algorithm of moment matching in Monte Carlo valuation of European call option similars to 1 which we simply replace the orginal underlying stock price process with different underlying stock price process. The numerical results of various moment matching are given in Table 3.1

K	σ	$s.e.$ without moment matching	$s.e.$ in 3.6	$s.e.$ in 3.8	$s.e.$ in 3.9	$s.e.$ in 3.10[4]
80	0.2	1.9278	0.4686	0.1221	0.2564	0.6938
	0.3	2.7866	1.0321	0.1688	0.6194	1.0110
	0.4	3.6882	1.6892	0.2923	1.0683	1.4099
100	0.2	1.4796	0.7296	0.1962	0.6015	0.6125
	0.3	2.2609	1.2362	0.3406	0.9943	0.9068
	0.4	3.1387	1.8461	0.5597	1.4738	1.2891
120	0.2	0.8710	0.6459	0.3241	0.6046	0.5021
	0.3	1.6606	1.1734	0.5306	1.0627	0.7344
	0.4	2.5550	1.7927	0.7910	1.5929	1.0667

Table 3.1: Moment Matching Application in Monte Carlo Valuation of European Call Option. European call option with $S_0 = 100, r = 0.05, T = 1, n = 100, m = 10000$

The results in Table 3.1 show that both matching the first moment and the first two moments can reduce the standard error. Matching the first two moments by transforming the standard normal is the most effective way to reduce standard error, which can reduce error by a factor ranging from 2 to 16. However, if we match the first two moments by transforming the terminal stock prices, it can only reduce a factor ranging from 1 to 3. Matching the first moment in both two transformed methods can reduce error by a factor ranging from 1 to 8. Further, the estimates in matching the first two moments by transforming the terminal stock prices are biased[5]. In moment matching, we can match higher order moments as well. Further, moments can be also used as control variates rather than for moment matching. Boyle et al. (1997) argue that matching the first two moments is asymptotically equivalent to use the first two moments as control variates. More discussions will be showed in 4.3.

[4]In Boyle et al. (1997) paper, they conclude that matching two moments by using both two methods dominates matching one moments. However, matching two moments by using 3.10 is not always working better than matching one moments in our case. If we replace $\frac{s_{S_T}}{\sigma_{S_T}}$ with $\frac{\sigma_{S_T}}{s_{S_T}}$ in 3.10, matching two moments dominates matching one moments, but the numerical result still biased.

[5]See the appendix 6.1.1

3.3 Stratified Sampling

In (Glasserman (2004),p209-232) and Ridder et al. (2013) definition on stratified sampling, stratified sampling is a sampling mechanism that limits the fraction of observations taken from specific strata of the sample space. The following part is largely based on the discussion of stratified sampling in (Glasserman (2004),p209-232) work.

We denote X as stratification variable which can take values in an arbitrary set. Assume C is an option function of X, our goal is to estimate $E[C]$. In stratified sampling, instead of estimating $E[C]$ over the domain U, stratified sampling split up the domain U of X into separate strata, estimate the conditional expectation X over each strata separately, then average estimate for each strata. Assume $A_1, \ldots, A_K$ are disjoint strata which $\sum_{i=1}^{K} P(X \in A_i) = 1$ and then we have

$$E[C] = \sum_{i=1}^{K} P(X \in A_i) E[C|X \in A_i] = \sum_{i=1}^{K} p_i E[C|X \in A_i] \tag{3.11}$$

where $p_i = P(X \in A_i)$. $X_1, \ldots, X_K$ are independent, which also have the same distribution as X. The fraction of these samples falling in A_i will not equal to theoretical probabilityp_i, though it will converge to the p_i as $n \to \infty$. In stratified sampling, we decide what fraction of the samples would be drawn from the each stratum A_i. Each observation drawn from A_i is subjected to have the distribution of Xconditional on $X \in A_i$.

We use proportional sampling in Monte Carlo valuation of call option, which means the fraction of the samples would be drawn from A_i matches the theoretical probability $p_i = P(X \in A_i)$. We geneartae $n_i = np_i$ samples from A_i if total sample size is n. Denote $C_{ij}, i = 1, \ldots, K, j = 1, \ldots, n_j$ be independent draws from the distribution of Xconditional on $X \in A_i$. An unbiased estimator of $E[C]$ is given as:

$$\bar{C} = \sum_{i=1}^{K} p_i \cdot \frac{1}{n_i} \sum_{j=1}^{n_i} C_{ij} = \frac{1}{n} \sum_{i=1}^{K} \sum_{j=1}^{n_i} C_{ij} \tag{3.12}$$

The variance of $\bar{C}$ is given as:

$$Var[\bar{C}] = \sum_{i=1}^{K} p_i^2 \cdot Var[\frac{1}{n_i} \sum_{j=1}^{n_i} C_{ij}] = \frac{1}{n} \sum_{i=1}^{K} p_i Var(C_{ij}) \tag{3.13}$$

Where $Var(C_{ij}) = Var[C|X \in A_i]$. We need to consider two issues when we apply stratified sampling. One is choosing the stratification variable X, the strata $A_1, \ldots, A_K$ and the stratified sample size $n_1, \ldots, n_K$. For the first issue, we find that stratified sampling is most effective when the variability of C within each stratum is small. The other one is generating samples from the distribution of (X, C) conditional on $X \in A_i$. We can solve it by using stratifying uniforms.

Denote $U_1, \ldots, U_n$ be independent and uniformly distributed between 0 and 1 and let

$$V_i = \frac{i-1}{n} + \frac{U_i}{n} \tag{3.14}$$

Where $V_i, i = 1, \ldots, n$ is uniformly distributed between $\frac{i-1}{n}$ and $\frac{i}{n}$. It has the distribution of Uconditional on $U \in A_i$ for $U \sim Unif[0, 1]$. $V_1, \ldots, V_n$ therefore construct a strantified sample from the uniform distribution. Further, European call option can only be exercised at time T and the payoff is not a path-dependent. Thus, we use an inverse transformation method to stratify the last value $W(T)$. The inverse functions $\Phi^{-1}(V_i), i = 1, \ldots, n$, form a stratified sample as $\Phi^{-1}(V_i) \sim N(0, 1)$. The algorithm of stratified sampling in Monte Carlo valuation of European call option therefore shows as follow[6]:

[6] k is number of strata; nk is sample size in each strata

Algorithm 3 Stratified Sampling Application in Monte Carlo valuation of European call option

Input: S_0, K, r, σ, T, n, k, nk
for $i = 1$ to k
generate $v \sim rand(nk, 1)$
genearte $u = \frac{i-1}{k} + \frac{v}{k}$
set $z = norminv(u)$
generate C_i and S_T^i according to equation 3.1 and 2.2
set $C_i^{SS} = \frac{\sum_{i=1}^{nk} C_i}{nk}$, $\tilde{\sigma_i^2} = \frac{\sum_{i=1}^{nk}(C_i - C_i^{SS})^2}{nk-1}$
end for
set $C^{SSF} = \frac{\sum_{i=1}^{k} C_i^{SS}}{k}$, $\tilde{\sigma_F^2} = \frac{\sum_{i=1}^{k}(C_i^{SS} - C^{SSF})^2}{k-1}$

3.4 Control Variates

In (Glasserman (2004),p185) definition, control variates tends to reduce the error in an estimate of an unknown quantity by using information about the errors in estimates of known quantities. Assume $C_1, C_2, \ldots, C_i, i = 1, \ldots, n$ are i.i.d. realizations of call option value at time T, which can be obtained from n simulated paths. Assume now that along with each realization C_i, we calculate another output X_i, with known expectation $E[X]$. We also assume that each pair (C_i, X_i) is i.i.d. Then, for any constant β, we can compute $C_i(\beta)$ for each simulated path:

$$C_i(\beta) = C_i - \beta(X_i - E[X]) \tag{3.15}$$

Then we compute the sample mean, we will have:

$$\bar{C}(\beta) = \bar{C} - \beta(\bar{X} - E[X]) \tag{3.16}$$

where $\bar{C} = \frac{1}{n}\sum_{i=1}^{n} C_i$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. The estimators show above also follow the same properties as the Monte Carlo estimator $\tilde{C_n}$. The estimator $\bar{C}(\beta)$ is unbiased becasue $E[\bar{C}(\beta)] = E[C]$ and it is also consistent becasue $\bar{C}(\beta)$ converges to the true value of $E[C]$ with probability 1 as $n \to \infty$. For each $C_i(\beta)$, it has variance

$$\begin{aligned} Var[C_i(\beta)] &= Var[C_i - \beta(X_i - E[X])] \\ &= Var(C) + \beta^2 Var(X) - 2\beta Cov(C, X) \end{aligned} \tag{3.17}$$

where we use the fact that $Var(E[X]) = 0$. The control variate estimator's effectiveness depends on the coefficient β. The optimal coefficient $\beta^*$is chosen to reduce the variance in 3.15. We can obtain β^* by minimizing 3.17 and is given by

$$\beta^* = \frac{Cov(C, X)}{Var(X)} \tag{3.18}$$

In practice, $Cov(C, X)$ is unknown to us if $E[C]$ is unknown. We therefore can not obtain β^*. However, we can replace β^* with the sample estimator $\hat{\beta}$:

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(C_i - \bar{C})(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \tag{3.19}$$

Dividing numerator and denominator by n and applying Law of Large Numbers (LLN)(T. Kotiah (1994)) ensures that $\hat{\beta} \to \beta^*$ with probability 1. However, replace β^* with $\hat{\beta}$ will lead to some bias. By using the fact that $E[\hat{\beta}] = \beta^*$,$E[\bar{C}] = E[C]$ and $Cov(\hat{\beta}, \bar{X}) = E[\hat{\beta}\bar{X}] - E[\hat{\beta}]E[\bar{X}]$, then we replace β with $\hat{\beta}$ and take expecation in both two sides in 3.16, we can obtain the bias :

$$\begin{aligned} Bias(\bar{C}(\hat{\beta})) &= E[\bar{C}(\hat{\beta})] - E[C] \\ &= -E[\hat{\beta}(\bar{X} - E[X])] \\ &= -E[\hat{\beta}\bar{X}] + \beta^* E[X] \\ &= -Cov(\hat{\beta}, \bar{X}) - \beta^* E[X] + \beta^* E[X] \\ &= -Cov(\hat{\beta}, \bar{X}) \end{aligned} \tag{3.20}$$

The bias can turn to be zero if $\hat{\beta}$ and $\bar{X}$ are independent. We can eliminate bias by using a different sequence of random numbers, then we can ensure independence. In practice, the bias generated by using 3.19 is pretty small and vanishes asymptotically. We therefore can ignore the bias and let $E[\bar{C}(\hat{\beta})] = E[C]$.

The effectiveness of control variates does not only depend on the choice of coefficient β, but also on the choice of the control variate X_i. Using β^* and then inserting it into 3.17, the variance of the control estimator can be written as:

$$Var[C_i(\beta)] = Var[C_i](1 - \rho^2(C, X)) \tag{3.21}$$

where $\rho(C, X) = \frac{Cov(C,X)}{\sqrt{Var(C)Var(X)}}$ is the correlation between C and X. High correlation between the payoff of the call option and its control variate will greatly reduce the variance.We can also apply multiple controls in the method of control variates. We will discuss the multiple control variates when we discuss control variates relation to other variance reduction techniques.

When we apply control variates in Monte Carlo valuation of European call option, we take the underlying stock price as a control variate. In risk-neutral probability measure Q, $e^{-rT}S_T$ is a martingale. We therefore have $E[e^{-rT}S_T] = S_0$. Generating each independent path $S^i, i = 1, \ldots, n$, over the time period $[0, T]$, we can obtain the control variate estimator $\bar{C}(\beta^*)$

$$\bar{C}(\beta^*) = \frac{1}{n}\sum_{i=1}^{n}(C_i - \beta^*[S_T^i - e^{rT}S_0]) \tag{3.22}$$

The algorithm of control variates in Monte Carlo valuation of European call option is as follow:

Algorithm 4 Control Variates Application in Monte Carlo valuation of European call option

Input: S_0, K, r, σ, T, n

1. **for** $i = 1$ to n
2. **generate** $Z_i \sim N(0, 1)$
3. **generate** C_i, S_T^i and β^* according to equation 3.1, 2.2 and 3.18
4. **set** $C_i(\beta^*) = C_i - \beta^*(S_T^i - e^{rT}S_0)$
5. **end for**
6. **set** $\bar{C}(\beta^*) = \frac{\sum_{i=1}^{n} C_i}{n}$, $\tilde{\sigma_n^2} = \frac{\sum_{i=1}^{n}(C_i(\beta^*) - \bar{C}(\beta^*))^2}{n-1}$

We take pricing European call option as an example again to compare different variance reduction techniques. In moment matching, we match two moments of a standard normal distribution because it is the most effective methods to reduce standard error in compared other three moment matching methods in this paper. In control variates, the underlying stock price is used as a control variate. In stratified sampling, we take 100 stratas. Results appear in Table 3.2.

K	σ	$s.e.$ without Variance Reduction	$s.e.$ in AV	$s.e.$ in MM	$s.e.$ in SS	$s.e.$ in CV
80	0.2	1.9278	0.4591	0.1221	0.1190	0.2406
	0.3	2.7866	1.0272	0.1688	0.2317	0.5166
	0.4	3.6882	1.6819	0.2923	0.3996	0.7775
100	0.2	1.4796	0.7296	0.1962	0.1186	0.5552
	0.3	2.2609	1.2252	0.3406	0.2315	0.8402
	0.4	3.1387	1.7964	0.5597	0.3995	1.1185
120	0.2	0.8710	0.5677	0.3241	0.1182	0.5721
	0.3	1.6606	1.0595	0.5306	0.2312	0.9326
	0.4	2.5550	1.6247	0.7910	0.3994	1.2727

Table 3.2: Different Variance Reduction Techniques Applications in Monte Carlo Valuation of European Call Option. European call option with $S_0 = 100, r = 0.05, T = 1, n = 100, m = 10000$

In Table 3.2, using the antithetic variates reduces error by a factor ranging from 1 to 4. Taking underlying stock price as a control variate reduces error by a factor ranging from 1 to 8. Matching the first two moments by transforming the orginal standard normal can reduce error by a factor ranging from 2 to 16. Stratified sampling is the most effective methods, which can reduce error by a factor ranging from 6 to 16. As a matter of fact, if we divide the sample space into more stratums, it will reduce error much more effective[7].According to Boyle et al. (1997) comment on different variance reduction techniques, antithetic variates is easy to implement, but the variance reduction is often very moderate. Moment matching is also easy to implement and it will reduce the error, but its bias is a potential problem. Though stratified sampling technique can lead to very huge error reductions, conditional sampling may not easy to obtain at some cases like radial stratification. We will introduce poststratification to solve this problem when we discuss the connection between control variates and stratified sampling. The control variate technique can lead to very substantial error reduction in principle, but its effectiveness depends on the choice of the control variate.

4 Control variates relation to other variance reduction techniques

After showing how variance reduction techniques improve the preciseness of the Monte Carlo method, now we turn to discuss how control variates relates to antithetic variates, moment matching, and stratified sampling. Glynn and Szechtman (2002) establish the connections between the method of control variates and antithetics variates,and stratified sampling. Boyle et al. (1997) showed that moments can alternatively be used as control variates. They argued that moment matching is asymptotically equivalent to a control variate technique with suboptimal coefficients, and is therefore dominated by the optimal use of moments as controls. The following part outlines these relations and illustrate it in the Monte Carlo Valuation of European Call Option. The following discussions are largely based on Boyle et al. (1997), Grant (1983) and Glynn and Szechtman (2002) work.

4.1 Control Variates and Antithetic Variates

Glynn and Szechtman (2002) and Grant (1983) take $C_i - C_i^{AV}, i = 1, \ldots, n$ as control variates to estiblish the relationship between control variates and antithetic variates. A control variate $C_i - C_i^{AV}$ follows the properties stated in 3.1, $C_i - C_i^{AV}$ therefore has mean zero. Hence, for any constant β, we can compute:

[7]See appendix 6.1.2

$$\begin{aligned} C_i(\beta) &= C_i - \beta(C_i - C_i^{AV}) \\ &= \beta C_i^{AV} + (1-\beta)C_i \end{aligned} \tag{4.1}$$

For antithetic variates, C_i and C_i^{AV} are identically distributed. For each $C_i(\beta)$, it has variance:

$$Var[C_i(\beta)] = [(1-\beta)^2 + \beta^2 + 2\beta(1-\beta)\rho(C_i, C_i^{AV})]Var(C_i) \tag{4.2}$$

where we use the fact that $Var(C_i) = Var(C_i^{AV})$ and $Cov(C_i, C_i^{AV}) = \rho(C_i, C_i^{AV})Var(C_i)$. We can obtain β^* by minimizing 4.2 and is given by

$$2(\beta^* - 1) + 2\beta^* + (2 - 4\beta^*)\rho(C_i, C_i^{AV}) = 0 \tag{4.3}$$

which has solution $\beta^* = 1/2$. Thus no matter how the sign of the covariance, $\beta^* = 1/2$ is the optimal coefficient. The equation 4.2 now becomes:

$$Var[C_i(\beta)] = [\frac{1}{2} + \frac{1}{2}\rho(C_i, C_i^{AV})]Var(C_i) \tag{4.4}$$

Inserting $\beta^* = 1/2$ into 4.1. we have $C_i(\beta) = \frac{C_i^{AV}+C_i}{2}$. In other words, we put the same weight on C_i and C_i^{AV}. Hence, taking $C_i - C_i^{AV}$ as control variates is the same as we merely apply antithetic variates to reduce variance.

4.2 Control Variates and Stratified Sampling

In (Glasserman (2004),p232-234) comment, stratified sampling requires us to know the stratum probabilities and a mechanism for conditional sampling from strata. However, conditional sampling may not easy to obtain at some cases like radial stratification. We therefore use poststratification when conditional sampling is costly. The following part relates to poststratification is largely based on (Glasserman (2004),p232-234) work.

Poststratification combines satratum probabilities with indepedent sampling to reduce variance, at least asymptotically. Assume again we want to estimate $E[C]$, we also have a mechanism for obtaining independent replications $(C_1, X_1), \ldots, (C_n, X_n)$ of the pair (C, X). The theoretical probability $p_i = P(X \in A_i)$ for each stratum $A_i, i = 1, \ldots, K$ are known and $\sum_{i=1}^{K} p_i = 1$. Denote N_i is the number of samples that fall in stratum i, then we have

$$N_i = \sum_{j=1}^{n} 1\{X_j \in A_i\} \tag{4.5}$$

Denote S_i is the sum of C_j for which X_i fall in stratum i, then we have

$$S_i = \sum_{i=1}^{n} 1\{X_j \in A_i\} C_j \tag{4.6}$$

Now the sample mean $\bar{C} = \frac{C_1+\cdots+C_n}{n}$ can be written as

$$\bar{C} = \frac{S_1 + \cdots + S_K}{n} = \sum_{i=1}^{K} \frac{N_i}{n} \cdot \frac{S_i}{N_i} \tag{4.7}$$

By LLN, $\frac{N_i}{n}$ will converge to the theoretical probability p_i with probability 1 and $\frac{S_i}{N_i}$ will also converge to the stratum mean $E[C|X \in A_i]$ with probability 1. Poststratification replaces $\frac{N_i}{n}$ with its theoretical probability p_i to produce the estimator $\hat{C}$

$$\hat{C} = \sum_{i=1}^{K} p_i \frac{S_i}{N_i} \tag{4.8}$$

The sample mean $\bar{C}$ assigns weight $\frac{1}{n}$ for every obervations, the poststratified estimator $\hat{C}$ assigns weight $\frac{p_i}{N_i}$ for the values falling in stratum i. In other words, values from undersampled strata ($N_i < np_i$) get more weights, and vice versa. Poststratification is as efficient as stratified sampling in reducing variance as the sample size grows.We combine control variates with poststratification based on Glynn and Szechtman (2002) work. Denotep_i^o equals to $\frac{N_i}{n}$. Assume the theoretical probability p_i are known, then we have

$$G = (p_1^o - p_1, p_2^o - p_2, \cdots, p_K^o - p_K) \tag{4.9}$$

where G is a K-dimensional control variate (having mean 0). The standard estimator C^{SS} can be obtained by using 1 and the control variates estimator is C_{cv}^{SS} . Then we have

$$C_{cv}^{SS} = C^{SS} - \beta^T G \tag{4.10}$$

For each $C_{cv_i}^{SS}, i = 1, \ldots, K$, it has variance

$$Var(C_i^{SS} - \beta^T G_i) = Var(C_i^{SS}) - 2\beta^T \Sigma_{C^{SS}G} + \beta^T \Sigma_{C^{SS}C^{SS}} \beta \tag{4.11}$$

Where $\Sigma_{C^{SS}C^{SS}}$ and $\Sigma_{C^{SS}G}$ corresponds to the covariance matrix of C^{SS} and the vector of covariances between (C^{SS}, G) respectively. We can obtain β^* by minimizing 4.11 and is given by

$$\beta^* = \Sigma_{C^{SS}}^{-1} \Sigma_{C^{SS}G} \tag{4.12}$$

when we take G as control variates, the optimal $\beta_i^* = E[C|X \in A_i]$ for $1 \le i \le K$ according to Glynn and Szechtman (2002) proof.

In the Monte Carlo simulations, we simply equally divide the sample space into two stratums out of my limited programming knowledge. The theoretical probability for each stratum should be $\frac{1}{2}$. G now is a 2-dimensional control variate. The corresponding numerical results described in Table 4.1.

K	σ	$s.e.$ without Variance Reduction	$s.e.$ in SS	$s.e.$ in CVSS
80	0.2	1.9278	1.1654	1.1589
	0.3	2.7866	1.7938	1.7856
	0.4	3.6882	2.5753	2.5610
100	0.2	1.4796	1.0504	1.0446
	0.3	2.2609	1.7391	1.7299
	0.4	3.1387	2.5489	2.5345
120	0.2	0.8710	0.8034	0.7944
	0.3	1.6606	1.5023	1.4858
	0.4	2.5550	2.3051	2.2805

Table 4.1: Taking G as Control Variates ($CVSS$) and Stratified Sampling (SS)in Monte Carlo Valuation of European Call Option. European call option with $S_0 = 100, r = 0.05, T = 1, n = 100, m = 10000$

Numerical results appear in Table 4.1. Both stratified sampling and taking G as control variates reduce error by a factor ranging from 1 to 2. To be specific, taking G as control variates slightly increases the standard error reduction efficiency, but the increment is very small. We therefore roughly suggest it is undifferentiated for us to take G as control variates or apply stratified sampling to reduce the standard error in the Monte Carlo valuation of a European call option.

4.3 Control Variates and Moment Matching

Boyle et al. (1997) takes the first two moments as control variates in order to estiblish the relationship between control variates and moment matching. Assume $Z_1, Z_2, \ldots, Z_i, i = 1, \ldots, n$ are i.i.d. $N(0,1)$ variables and we want to estimate $E[f(Z)]$ for European call option function f. The standard estimator is $C^{MM} = \frac{1}{n}\sum_{i=1}^{n} f(Z_i)$ and the moment matching estimator is $C_{cv}^{MM} = \frac{1}{n}\sum_{i=1}^{n} f(\tilde{Z}_i)$ where $\tilde{Z}_i$ defined in 3.8. Denote s_z is the sample standard deviation of $Z_1, Z_2, \ldots, Z_i, i = 1, \ldots, n$ and $\tilde{Z}$ is their sample mean. By CLT, the scaled difference $\sqrt{n}(\tilde{Z}_i - Z_i)$ coverges in distrubution for $\tilde{Z}$ and s_z. We take first order condition for f w.r.t $\tilde{Z}_i$.Then we have[8]

$$f(\tilde{Z}_i) = f(Z_i) + f\text{'}(Z_i)[\tilde{Z}_i - Z_i] + o_p(\frac{1}{\sqrt{n}}) \tag{4.13}$$

Assuming that up to terms $o_p(\frac{1}{\sqrt{n}})$ the moment matching estimator and standard estimator are related via

$$\begin{aligned} C_{cv}^{MM} \approx& \frac{1}{n}\sum_{i=1}^{n} f(Z_i) + \frac{1}{n}\sum_{i=1}^{n} f'(Z_i)[\tilde{Z}_i - Z_i] \\ =& \frac{1}{n}\sum_{i=1}^{n} f(Z_i) + \frac{1}{n}\sum_{i=1}^{n} f'(Z_i)[(\frac{1}{s_z} - 1)Z_i - \frac{1}{s_z}\tilde{Z}] \\ =& \frac{1}{n}\sum_{i=1}^{n} f(Z_i) + (\frac{1}{n}\sum_{i=1}^{n} f'(Z_i)Z_i)(\frac{1}{s_z} - 1) + \frac{1}{n}\sum_{i=1}^{n} f'(Z_i)(-\frac{\tilde{Z}}{s_z}) \\ =& \frac{1}{n}\sum_{i=1}^{n} f(Z_i) + \hat{\beta}_1(\frac{1}{s_z} - 1) - \hat{\beta}_2\frac{\tilde{Z}}{s_z} \end{aligned} \tag{4.14}$$

$\hat{\beta}_1$ and $\hat{\beta}_2$ converge to the coefficients β_1 and β_2 as $n \to \infty$, then we have $\beta_1 = E[f'(Z)Z]$ and $\beta_2 = E[f'(Z)]$. We replace $\hat{\beta}_1$ and $\hat{\beta}_2$ with β_1 and β_2 in 4.14. However, β_1 and β_2 are not coincide with the optimal coefficients β_1^* and β_2^* in Boyle et al. (1997) comment, they will also introduce bias as monment matching itself.

In Monte Carlo valuation of European call option, we denote European call option function as $f(Z_i) = e^{-rT}(S_T - K)H(S_T - K)$, where $S_T = S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma Z_i\sqrt{T}}$ and $H(S_T - K)$ is a Heavyside function. Heavyside function $H(S_T - K)$ is defined as

$$H(S_T - K) = \begin{cases} 0, & S_T - K \leq 0 \\ 1, & S_T - K > 0 \end{cases} \tag{4.15}$$

The Dirac delta function $\delta(S_T - K)$ is the derivative of $H(S_T - K)$, it is given as

$$\delta(S_T - K) = \frac{dH(S_T - K)}{d(S_T - K)} = \begin{cases} \infty, & S_T - K = 0 \\ 0, & otherwise \end{cases} \tag{4.16}$$

When we take first order condition for f w.r.t Z_i, we utilize one of the Dirac delta function features, $(S_T - K)\delta(S_T - K) = 0$. Then $(f(Z_i))'$ is given as:

$$(f(Z_i))' = \sigma\sqrt{T}S_0 e^{-\frac{\sigma^2}{2}T + \sigma Z_i\sqrt{T}} H(S_T - K) \tag{4.17}$$

[8] $o_p(\frac{1}{\sqrt{n}})$ is a strict upper bound

The numerical results of taking the first two moments as control variates and matching the first two moments in Monte Carlo valuation of European Call Option are shown in Table 4.2

K	σ	$s.e.$ without Variance Reduction	$s.e.$ in MM(3.8)	$s.e.$ in CVMM
80	0.2	0.6080	0.0386	0.0393
	0.3	0.8797	0.0544	0.0560
	0.4	1.1664	0.0960	0.0982
100	0.2	0.4678	0.0622	0.0630
	0.3	0.7154	0.1093	0.1104
	0.4	0.9941	0.1810	0.1825
120	0.2	0.2750	0.1029	0.1035
	0.3	0.5248	0.1693	0.1702
	0.4	0.8082	0.2535	0.2547

Table 4.2: Taking the First Two Moments as Control Variates ($CVMM$) and Moment Matching (MM) in Monte Carlo Valuation of European Call Option. European call option with $S_0 = 100, r = 0.05, T = 1, n = 1000, m = 10000$

From the numerical results we know that both taking the first two moments as control variates and matching the first two moments can reduce error by a factor ranging from 2 to 16. To be specific, taking the first two moments as control variates slightly decreases the standard error reduction efficiency, but the decrement is very small. Hence, we argue that matching the first two moments are asymptocially equivalent to use $(\frac{1}{s_z} - 1)$ and $\frac{\bar{Z}}{s_z}$ as controls, which is consistent with Boyle et al. (1997) argument.

5 Concluding Remarks

In this paper we discuss the variance reduction techniques applications in the Monte Carlo valuation of European call option. Even though there is no need for us to use Monte Carlo method to price a European call option because it has closed form solution, it still worthwhile for us to simulate it from learning perspective, getting intuition about how variance reduction techniques work, laying the foundation for my future research work like applying variance reduction techinuqes in the Monte Carlo valuation of path-dependent options. We also discuss how control variates relates to antithetic variates, moment matching,and stratified sampling.

We firstly argue that it is feasible for us to do the Monte Carlo valuation of European call options , then we take antithetic variates, moment matching, stratified sampling and control variates to reduce the standard error made by the Monte Carlo simulation. From the numerical results we know that stratified sampling will work much more effective if we can divide the sample space into more stratums. The effectiveness of the control variates depends on the optimal choice of the control variate. Antithetic method is easy to implement, but at the same time the variance reduction is often very moderate. As for moment matching, matching the first moments and the second moments reduce standard error much more effective than antithetic variates. However, the bias induce by the moment matching is a potential problem. Finally, more variance reduction techniques like importance sampling and Latin hypercube sampling could be contained in my future research work.

In the discussion of relationships between control variates and other variance reduction techniques, we show how control variates relates to antithetic variates through mathematical derivation. In establishing the connection between control variates and stratified sampling, and moment matching, we illustrated it through Monte Carlo valuation of a European call option. We introduce the poststratification in establishing the relationship between control variates and stratified sampling and the numercial results support the Glynn and Szechtman (2002) argument. Further, we take the first two moments as control variates to discuss the connection between control variates and moment matching. The numerical results of Monte carlo simulations support the Boyle et al. (1997) argument. Matching the first two moments is asymptotically equivalent to use the first two moments as control variates. In Glynn and Szechtman (2002) paper, they also establish the connection between control variates and numerical integration, and rotation sampling. This give us an inspiration to illustrate them with simulations in our future research. We can use Monte Carlo methods to price other options like an Asian option to illusrtae those connection in our furture research.

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6 Appendix

6.1 Table

6.1.1 Moment Matching Application in Monte Carlo Valuation of European Call Option

As we mentioned above, the estimates in matching the first two moments by transforming the terminal stock prices are biased. The numerical results show in table 6.1

K	σ	BS-European Call	Option price without moment matching	Option price in 3.6	Option price in 3.8	Option price in 3.9	Option price in 3.10
80	0.2	24.5888	24.6215	24.5471	24.5519	24.5779	32.0372
	0.3	26.4621	26.5000	26.3694	26.3974	26.4380	35.5762
	0.4	28.9764	29.0181	28.8243	28.8740	28.9422	38.6275
100	0.2	10.4506	10.4647	10.3845	10.4162	10.4060	20.8747
	0.3	14.2313	14.2517	14.1224	14.1707	14.1692	25.2298
	0.4	18.0230	18.0512	17.8665	17.9279	17.9485	28.9227
120	0.2	3.2475	3.2557	3.2029	3.2174	3.2093	12.8383
	0.3	6.9040	6.9197	6.8180	6.8491	6.8410	17.4171
	0.4	10.8060	10.8305	10.6730	10.7163	10.7233	21.3878

Table 6.1: Moment Matching Application in Monte Carlo Valuation of European Call Option. European call option with $S_0 = 100, r = 0.05, T = 1, n = 100, m = 10000$

6.1.2 Stratified Sampling Application in Monte Carlo Valuation of European Call Option

If we divide the sample space into more stratums in stratified sampling, it will reduce error much more effective. Table 6.2 illustrates this argument.

K	Stratum	BS-European Call	MC-European Call	*s.e.*	MC-European Call in SS	*s.e.* in SS
80	100	24.5888	24.5840	0.0607	24.5889	0.0036
	1000				24.5878	0.0011
	10000				24.5885	0.0002
100	100	10.4506	10.5160	0.0467	10.4462	0.0035
	1000				10.4519	0.0012
	10000				10.4508	0.0003
120	100	3.2475	3.2757	0.0278	3.2469	0.0034
	1000				3.2466	0.0012
	10000				3.2473	0.0002

Table 6.2: Stratified Sampling Application in Monte Carlo Valuation of European Call Option. European call option with $S_0 = 100, r = 0.05, T = 1, \sigma = 0.2, n = 10000$.

6.2 MATLAB Code

Here are the MATLAB codes about stratified sampling and the codes about the relationships between control variates and moment matching, and stratified sampling. The codes I use in the stratified sampling is largely based on Cheng Feng's work. All the simulations in this paper, we have controled random number generation.

6.2.1 Stratified Sampling

```
% Stratified Sampling in Monte Carlo valuation of European call option
% The underlying stock price follows a geometric Brownian motion.
function [v,se] = option_euro_stratification(s0, r, sigma, T, K, n,stra)
% s0 - initial stock price; r - risk-free interest rate;
% sigma - volatility; T - terminal time; K - strike price;
% stra -number of strata; n-simulation times; nsim-resimulation times
sz=n/stra; % sample size in each strata
for i=1:stra
   v=rand(sz,1);  % samples from uniform distribution on [0,1]
   u= (i-1)/stra+v/stra;
   z= norminv(u);  % inverse transformation on the i-th stratum
   x=s0 *exp(-sigma^2*T/2 +sigma*sqrt(T)*z); % discounted terminal stock
      price
   optionpayoff=max(x-exp(-r*T)*K, 0); % discounted option payoff
   vstra(i) = sum(optionpayoff)/sz;  % monte carlo simulation for  i-th
      stratum
   sestra(i) = sqrt(sum((optionpayoff - vstra(i)).^2)/(sz-1)); %the
      standard error for i-th stratum
end
v=sum(vstra)/stra;
se=sqrt(sum(sestra.^2/sz))/stra;
end
```

6.2.2 Combined Control Variates with Moment Matching

```
function [value, se] = CVMM(s0, r, sigma, T, K, n,nsim)
% connection between control variate and moment matching
rng(1000)
z = randn(n,nsim); % n samples from N(0,1)
sez=std(z);
y = s0*exp((r-0.5*sigma^2)*T + sigma*sqrt(T)*z); % stock price at T
x=exp(-r*T)*max(y-K,0); % payoff of the option at T
v = sum(x)/n;  % MC estimate of the value of the option
% getting bata 1 and 2
beta1=sigma*s0*exp((-0.5*sigma^2)*T + sigma*sqrt(T)*z).*z.*heaviside(y-K);
beta2=sigma*s0*exp((-0.5*sigma^2)*T + sigma*sqrt(T)*z).*heaviside(y-K);
% optimal beta 1 and 2
optimalbeta1=sum(beta1)/n;
optimalbeta2=sum(beta2)/n;
% control variates 1 and 2
control1=(1./sez-1);
```

```
control2=(1./sez).*(sum(z)./n);
control=x+optimalbeta1.*control1-optimalbeta2.*control2;
v = sum(control)/n;
value=sum(v)/nsim
se= std( v )
end
```

6.2.3 Combined Control Variates with Stratified Sampling

```
function [v,se] = cvss(s0, r, sigma, T, K, n)
% connection between control variate and stratified sampling
% for simplication, here we just divide into 2 stratum. Therefore, the
% theortical probability should be 1/2 for each stratum
z=randn(n,1)
k1=z>0
p1=sum(k1)/numel(k1)
k2=z<0
p2=sum(k2)/numel(k2)
a1=find(z>0)
z1=z(a1)
a2=find(z<0)
z2=z(a2) % getting the actuall probability for stratum 1 and 2
 % discounted terminal stock price and option payoff for stratum 1 and 2
s1 = s0*exp((r-0.5*sigma^2)*T + sigma*sqrt(T)*z1);
x1=exp(-r*T)*max(s1-K,0);
s2 = s0*exp((r-0.5*sigma^2)*T + sigma*sqrt(T)*z2);
x2=exp(-r*T)*max(s2-K,0);
y = s0*exp((r-0.5*sigma^2)*T + sigma*sqrt(T)*z);
x=exp(-r*T)*max(y-K,0);
control=x-mean(x1)*(p1-0.5)-mean(x2)*(p2-0.5) % mean(x1) and mean(x2) are
   the optimal beta respectively
v=sum(control)/n
se= std( control ) / sqrt( length( control ))
end
```