AI1103
Challenging Problem 12

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Download LaTex file from below link :

https://github.com/KRISHNASAI1105/demo/blob/main/Challenging%20problem%2012/LaTex/Challenging%20problem%2012.tex

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Consider a Markov chain with state space {0,1,2,3,4} and transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $lim_{n\to\infty} p_{23}^{(n)}$ equals

1) $\frac{1}{3}$
2) $\frac{1}{2}$
3) 0
4) 1

Solution

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For eigenvalue calculation,

$$|A - \lambda I| = 0 \quad A = P^{\top}$$

$$A = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} - \frac{1}{3}\begin{vmatrix} 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} \tag{0.0.1}$$

$$= (1-\lambda)\left(\frac{1}{3}-\lambda\right)\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} - \frac{1}{3}(1-\lambda)\begin{vmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3}-\lambda & 0 \\ 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} + \frac{1}{9}\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} \tag{0.0.2}$$

$$= (1-\lambda)^2\left(\frac{1}{3}-\lambda\right)^3 - \frac{2}{9}(1-\lambda)^2\left(\frac{1}{3}-\lambda\right) + \frac{1}{9}(1-\lambda)\left(\frac{1}{3}-\lambda\right)^2 - \frac{1}{81}(1-\lambda) \tag{0.0.3}$$

$$= (1-\lambda)^2\left(\frac{1}{3}-\lambda\right)\left[\left(\frac{1}{3}-\lambda\right)^2 - \frac{2}{9}\right] + \frac{1}{9}(1-\lambda)\left[\left(\frac{1}{3}-\lambda\right)^2 - \frac{1}{9}\right] \tag{0.0.4}$$

$$= \frac{-27\lambda^5 + 81\lambda^4 - 84\lambda^3 + 32\lambda^2 - \lambda - 1}{27} \tag{0.0.5}$$

$$= (1-\lambda)^2\left(\frac{1}{3}-\lambda\right)\left(\lambda^2 - \frac{2\lambda}{3} - \frac{1}{9}\right) \tag{0.0.6}$$

For eigenvalue calculation,

$$|A - \lambda I| = 0$$

Eigenvalues are,

1) $\lambda_1 = 1$
2) $\lambda_2 = 1$
3) $\lambda_3 = \frac{1}{3}$
4) $\lambda_4 = \frac{1-\sqrt{2}}{3}$
5) $\lambda_5 = \frac{1+\sqrt{2}}{3}$

Corresponding Eigenvectors are:

1) Eigenvector V_1,V_2 for λ_1=1(multiplicity =2) : So, for $\lambda = 1$ we get 2 eigenvectors.

(A-λ_1).$V_1 = 0$

$$\begin{bmatrix} 1-\lambda_1 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3}-\lambda_1 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda_1 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda_1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1-\lambda_1 \end{bmatrix} .V_1 = 0$$

$$\begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{-2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \end{bmatrix} . \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ e_1 \end{bmatrix} = 0$$

Null space=a$_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So, eigenvectors are,

$$\text{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

2) Eigenvector V_3 for λ_3= $\frac{1}{3}(multiplicity = 1)$: $(A - \lambda_3).V_3 = 0$

$$\begin{bmatrix} 1-\lambda_3 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3}-\lambda_3 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda_3 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda_3 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1-\lambda_3 \end{bmatrix} .V_3 = 0$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix} . \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \\ e_3 \end{bmatrix} = 0$$

$$\text{V}_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

3) Eigenvector V_4 for λ_4= $\frac{1-\sqrt{2}}{3}(multiplicity = 1)$: $(A - \lambda_4).V_4 = 0$

$$\begin{bmatrix} 1-\lambda_4 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3}-\lambda_4 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda_4 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda_4 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1-\lambda_4 \end{bmatrix} .V_4 = 0$$

$$\begin{bmatrix} \frac{2+\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2+\sqrt{2}}{3} \end{bmatrix} . \begin{bmatrix} a_4 \\ b_4 \\ c_4 \\ d_4 \\ e_4 \end{bmatrix} = 0$$

$$\text{V}_4 = \begin{bmatrix} a_4 \\ b_4 \\ c_4 \\ d_4 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2-\sqrt{2} \\ 2+2\sqrt{2} \\ -2-\sqrt{2} \\ 1 \end{bmatrix}$$

4) Eigenvector V_5 for λ_5= $\frac{1+\sqrt{2}}{3}(multiplicity = 1)$: $(A - \lambda_5).V_5 = 0$

$$\begin{bmatrix} 1-\lambda_5 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3}-\lambda_5 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda_5 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda_5 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1-\lambda_5 \end{bmatrix} .V_5 = 0$$

$$\begin{bmatrix} \frac{2-\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{-\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-\sqrt{2}}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2-\sqrt{2}}{3} \end{bmatrix} . \begin{bmatrix} a_5 \\ b_5 \\ c_5 \\ d_5 \\ e_5 \end{bmatrix} = 0$$

$$\text{V}_5 = \begin{bmatrix} a_5 \\ b_5 \\ c_5 \\ d_5 \\ e_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2+\sqrt{2} \\ 2-2\sqrt{2} \\ -2+\sqrt{2} \\ 1 \end{bmatrix}$$

$$V = [V_1 V_2 V_3 V_4 V_5]$$

$$V = \begin{bmatrix} 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & -2-\sqrt{2} & -2+\sqrt{2} \\ 0 & 0 & 0 & 2+2\sqrt{2} & 2-2\sqrt{2} \\ 0 & 0 & -2 & -2-\sqrt{2} & -2+\sqrt{2} \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$V^{-1} = \frac{1}{8}\begin{bmatrix} 8 & 6 & 4 & 2 & 0 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & -2+\sqrt{2} & -2+2\sqrt{2} & -2+\sqrt{2} & 0 \\ 0 & -2-\sqrt{2} & -2-2\sqrt{2} & -2-\sqrt{2} & 0 \end{bmatrix}$$

$$V^{-1}AV = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\sqrt{2}}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1+\sqrt{2}}{3} \end{bmatrix}$$

The above matrix is a diagonal matrix, with eigenvalues has diagonal elements.

$$V^{-1}AV \times V^{-1}AV \times ......(\text{n times}) V^{-1}AV = V^{-1}A^nV$$

As the 2^{nd} row 3^{rd} column element in the above matrix is Zero.

$$V^{-1}A^nV = B$$

Let, B be some matrix after the whole multiplication.

$$A^n = VBV^{-1}$$

V,V^{-1} are left and right eigenvectors. Even if we don't know B matrix, A^n resembles P^n.
As $\lim_{n\to\infty} {Pr_n}^n$ approaches a matrix which has structure that all rows of matrix are identical.
$\therefore$ Hence, $\lim_{n\to\infty} p_{23}^{(n)} = 0$.

$\therefore$ $\boxed{\textbf{Option 3} \text{ is } \textbf{correct} \text{ answer.}}$