AI1103
Challenging Problem 12

Nagubandi Krishna Sai
MS20BTECH11014

Download LaTex file from below link :

https://github.com/KRISHNASAI1105/demo/blob/main/Challenging%20problem%2012/LaTex/Challenging%20problem%2012.tex

UGC NET JUNE 2019 *Q*.51

Consider a Markov chain with state space {0,1,2,3,4} and transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $lim_{n\to\infty} p_{23}^{(n)}$ equals

1) $\frac{1}{3}$
2) $\frac{1}{2}$
3) 0
4) 1

Solution

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For eigenvalue calculation,

$$|A - \lambda I| = 0 \quad A = P^\intercal$$

$$A = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} - \frac{1}{3}\begin{vmatrix} 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} \tag{0.0.1}$$

$$= (1-\lambda)\left(\frac{1}{3}-\lambda\right)\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} - \frac{1}{3}(1-\lambda)\begin{vmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3}-\lambda & 0 \\ 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} + \frac{1}{9}\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & 0 \\ 0 & \frac{1}{3} & 1-\lambda \end{vmatrix} \tag{0.0.2}$$

$$= (1-\lambda)^2\left(\frac{1}{3}-\lambda\right)^3 - \frac{2}{9}(1-\lambda)^2\left(\frac{1}{3}-\lambda\right) + \frac{1}{9}(1-\lambda)\left(\frac{1}{3}-\lambda\right)^2 - \frac{1}{81}(1-\lambda) \tag{0.0.3}$$

$$= (1-\lambda)^2\left(\frac{1}{3}-\lambda\right)\left[\left(\frac{1}{3}-\lambda\right)^2 - \frac{2}{9}\right] + \frac{1}{9}(1-\lambda)\left[\left(\frac{1}{3}-\lambda\right)^2 - \frac{1}{9}\right] \tag{0.0.4}$$

$$= \frac{-27\lambda^5 + 81\lambda^4 - 84\lambda^3 + 32\lambda^2 - \lambda - 1}{27} \tag{0.0.5}$$

Eigenvalues are,

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = \frac{1}{3}$$

$$\lambda_4 = \frac{1-\sqrt{2}}{3}$$
$$\lambda_5 = \frac{1+\sqrt{2}}{3}$$

Corresponding Eigenvectors are:
Eigenvector V_1 for $\lambda_1 = 1$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvector V_2 for $\lambda_2 = 1$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvector V_3 for $\lambda_3 = \frac{1}{3}$:

$$\begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Eigenvector V_4 for $\lambda_4 = \frac{1-\sqrt{2}}{3}$:

$$\begin{bmatrix} 1 \\ -2-\sqrt{2} \\ 2+2\sqrt{2} \\ -2-\sqrt{2} \\ 1 \end{bmatrix}$$

Eigenvector V_5 for $\lambda_5 = \frac{1+\sqrt{2}}{3}$:

$$\begin{bmatrix} 1 \\ -2+\sqrt{2} \\ 2-2\sqrt{2} \\ -2+\sqrt{2} \\ 1 \end{bmatrix}$$

$$V = [V_1 V_2 V_3 V_4 V_5]$$

$$V = \begin{bmatrix} 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & -2-\sqrt{2} & -2+\sqrt{2} \\ 0 & 0 & 0 & 2+2\sqrt{2} & 2-2\sqrt{2} \\ 0 & 0 & -2 & -2-\sqrt{2} & -2+\sqrt{2} \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$V^{-1} = \frac{1}{8}\begin{bmatrix} 8 & 6 & 4 & 2 & 0 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & -2+\sqrt{2} & -2+2\sqrt{2} & -2+\sqrt{2} & 0 \\ 0 & -2-\sqrt{2} & -2-2\sqrt{2} & -2-\sqrt{2} & 0 \end{bmatrix}$$

$$V^{-1}AV = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\sqrt{2}}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1+\sqrt{2}}{3} \end{bmatrix}$$

$$V^{-1}AV \times \mathrm{V}^{-1}AV \times \text{......(n times)}\mathrm{V}^{-1}AV = V^{-1}A^nV$$

As the 2^{nd} row 3^{rd} column element in the above matrix is Zero.

$\therefore$ Hence, $\lim_{n\to\infty} p_{23}^{(n)} = 0$.

$\therefore$ $\boxed{\textbf{Option 3} \text{ is } \textbf{correct} \text{ answer.}}$