AI1103
Challenging Problem 12

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Download LaTex file from below link :

https://github.com/KRISHNASAI1105/demo/blob/main/Challenging%20problem%2012/LaTex/Challenging%20problem%2012.tex

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Consider a Markov chain with state space {0,1,2,3,4} and transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $lim_{n\to\infty} p_{23}^{(n)}$ equals

1) $\frac{1}{3}$
2) $\frac{1}{2}$
3) 0
4) 1

Solution

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For eigenvalue calculation,

$$|A - \lambda I| = 0 \quad A = P$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} \tag{0.0.1}$$

$$= (1-\lambda)\left(\frac{1}{3}-\lambda\right)\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & 0 & 1-\lambda \end{vmatrix}$$
$$- \frac{1}{3}(1-\lambda)\begin{vmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3}-\lambda & \frac{1}{3} \\ 0 & 0 & 1-\lambda \end{vmatrix} \tag{0.0.2}$$

$$= (1-\lambda)^2\left(\frac{1}{3}-\lambda\right)^3 - \frac{2}{9}(1-\lambda)^2\left(\frac{1}{3}-\lambda\right) \tag{0.0.3}$$

$$= (1-\lambda)^2\left(\frac{1}{3}-\lambda\right)\left[\left(\frac{1}{3}-\lambda\right)^2 - \frac{2}{9}\right] \tag{0.0.4}$$

$$= (1-\lambda)^2\left(\frac{1}{3}-\lambda\right)\left(\lambda^2 - \frac{2\lambda}{3} - \frac{1}{9}\right) \tag{0.0.5}$$

For eigenvalue calculation,

$$|A - \lambda I| = 0$$

Eigenvalues are,

1) $\lambda_1 = 1$
2) $\lambda_2 = 1$
3) $\lambda_3 = \frac{1}{3}$
4) $\lambda_4 = \frac{1-\sqrt{2}}{3}$
5) $\lambda_5 = \frac{1+\sqrt{2}}{3}$

Definition 1. ***Eigendecomposition*** *is the factorization of a matrix into a canonical form, whereby the matrix is represented in terms of its* ***eigenvalues*** *and* ***eigenvectors*****.**

Corresponding Eigenvectors are:

1) Eigenvector V_1,V_2 for λ_1=1(multiplicity =2) :
   So, for $\lambda = 1$ we get 2 eigenvectors.

(A-λ_1).$V_1 = 0$

$$\begin{bmatrix} 1-\lambda_1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda_1 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda_1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 1-\lambda_1 \end{bmatrix}.V_1 = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ e_1 \end{bmatrix} = 0$$

$$\text{Null space}=d_1\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + e_1\begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So, eigenvectors are,

$$V_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

2) Eigenvector V_3 for λ_3= $\frac{1}{3}(multiplicity = 1)$:
   $(A-\lambda_3).V_3 = 0$

$$\begin{bmatrix} 1-\lambda_3 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda_3 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda_3 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda_3 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1-\lambda_3 \end{bmatrix}.V_3 = 0$$

$$\begin{bmatrix} \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2}{3} \end{bmatrix}.\begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \\ e_3 \end{bmatrix} = 0$$

$$V_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

3) Eigenvector V_4 for λ_4= $\frac{1-\sqrt{2}}{3}$
   (multiplicity =1) :
   $(A-\lambda_4).V_4 = 0$

$$\begin{bmatrix} 1-\lambda_4 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda_4 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda_4 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda_4 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1-\lambda_4 \end{bmatrix}.V_4 = 0$$

$$\begin{bmatrix} \frac{2+\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2+\sqrt{2}}{3} \end{bmatrix}.\begin{bmatrix} a_4 \\ b_4 \\ c_4 \\ d_4 \\ e_4 \end{bmatrix} = 0$$

$$V_4 = \begin{bmatrix} a_4 \\ b_4 \\ c_4 \\ d_4 \\ e_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

4) Eigenvector V_5 for λ_5= $\frac{1+\sqrt{2}}{3}$
   (multiplicity =1) :
   $(A-\lambda_5).V_5 = 0$

$$\begin{bmatrix} 1-\lambda_5 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}-\lambda_5 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3}-\lambda_5 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3}-\lambda_5 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1-\lambda_5 \end{bmatrix}.V_5 = 0$$

$$\begin{bmatrix} \frac{2-\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2-\sqrt{2}}{3} \end{bmatrix}.\begin{bmatrix} a_5 \\ b_5 \\ c_5 \\ d_5 \\ e_5 \end{bmatrix} = 0$$

$$V_5 = \begin{bmatrix} a_5 \\ b_5 \\ c_5 \\ d_5 \\ e_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

$$V = [V_1 V_2 V_3 V_4 V_5]$$

$$V = \begin{bmatrix} 4 & -3 & 0 & 0 & 0 \\ 3 & -2 & -1 & 1 & 1 \\ 2 & -1 & 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$V^{-1} =$

$$\frac{1}{4\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} \\ 0 & 0 & 0 & 0 & 4\sqrt{2} \\ \sqrt{2} & -2\sqrt{2} & 0 & 2\sqrt{2} & -\sqrt{2} \\ -\sqrt{2}+1 & \sqrt{2} & -2 & \sqrt{2} & -\sqrt{2}+1 \\ -\sqrt{2}-1 & \sqrt{2} & 2 & \sqrt{2} & -\sqrt{2}-1 \end{bmatrix}$$

$V^{-1}AV =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ -1+\frac{1}{\sqrt{2}} & \frac{2-\sqrt{2}}{4} & 0 & \frac{1-\sqrt{2}}{3} & 0 \\ -1-\frac{1}{\sqrt{2}} & \frac{2+\sqrt{2}}{4} & 0 & 0 & \frac{1+\sqrt{2}}{3} \end{bmatrix}$$

Definition 2. *Let $X_0, X_1, \ldots, X_n$ be a Markov chain with state space $E = \{1, \ldots, l\}$, initial distribution $\boldsymbol{\alpha}$ and one-step transition matrix $\mathbf{P}$. Then the vector $\boldsymbol{\alpha}_n = (\alpha_{n1}, \ldots, \alpha_{nl})^\top$ of the probabilities $\alpha_{ni} = P(X_n = i)$ is given by the equation,*

$$\alpha_n^\top = \alpha^\top P^n \tag{0.0.6}$$

Sir, the above definition is not completely related to this question.

The above matrix is a diagonal matrix, with eigenvalues has diagonal elements.

$$V^{-1}AV \times \mathrm{V}^{-1}AV \times \text{......(n times)} \mathrm{V}^{-1}AV = V^{-1}A^nV$$

$$V^{-1}A^nV = \mathrm{B}$$

Let, B be some matrix after the whole multiplication.

$$A^n = VBV^{-1}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{70} & \frac{9}{70} & 0 & 0 & 0 \\ -\frac{153}{35} & \frac{153}{35} & 0 & 0 & 0 \end{bmatrix}$$

$$A^n = V^{-1}BV = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{3}{8} & 0 & 0 & 0 & \frac{11}{8} \\ -1 & 0 & 0 & 0 & 2 \\ -\frac{7}{8} & 0 & 0 & 0 & \frac{15}{8} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

As the 2^{nd} row 3^{rd} column element in the above matrix is Zero.

$\therefore$ Hence, $\lim_{n\to\infty} p_{23}^{(n)} = 0$.

$\therefore$ $\boxed{\textbf{Option 3} \text{ is } \textbf{correct} \text{ answer.}}$