AI1103
Challenging Problem 12

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Download LaTex file from below link :

https://github.com/KRISHNASAI1105/demo/blob/main/Challenging%20problem%2012/LaTex/Challenging%20problem%2012.tex

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Consider a Markov chain with state space {0,1,2,3,4} and transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $lim_{n\to\infty} p_{23}^{(n)}$ equals

1) $\frac{1}{3}$
2) $\frac{1}{2}$
3) 0
4) 1

Solution

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{4}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{2}{9} & \frac{1}{3} & \frac{2}{9} & \frac{1}{9} \\ 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{14}{27} & \frac{4}{27} & \frac{5}{27} & \frac{1}{9} & \frac{1}{27} \\ \frac{5}{27} & \frac{5}{27} & \frac{7}{27} & \frac{5}{27} & \frac{5}{27} \\ \frac{1}{27} & \frac{1}{9} & \frac{5}{27} & \frac{4}{27} & \frac{14}{27} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{46}{81} & \frac{1}{9} & \frac{4}{27} & \frac{8}{81} & \frac{2}{27} \\ \frac{20}{81} & \frac{4}{27} & \frac{17}{81} & \frac{4}{27} & \frac{20}{81} \\ \frac{2}{27} & \frac{8}{81} & \frac{4}{27} & \frac{1}{9} & \frac{46}{81} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \dots & \dots & \frac{\frac{1}{30}n^5 - \frac{1}{3}n^4 + \frac{3}{2}n^3 - \frac{8}{3}n^2 + \frac{37}{15}n}{3^n} & \dots & \dots \\ \dots & \dots & \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{3^n} & \dots & \dots \\ \dots & \dots & \frac{\frac{1}{30}n^5 - \frac{1}{3}n^4 + \frac{3}{2}n^3 - \frac{8}{3}n^2 + \frac{37}{15}n}{3^n} & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

As we only require 2^{nd} row 3^{rd} column element in the p^n matrix, so no need to generalise remaining terms.

$$p_{23}^{(n)} = \frac{\frac{1}{30}n^5 - \frac{1}{3}n^4 + \frac{3}{2}n^3 - \frac{8}{3}n^2 + \frac{37}{15}n}{3^n} \tag{0.0.1}$$

So,

$$\lim_{n\to\infty} p_{23}^{(n)} = \lim_{n\to\infty} \frac{\frac{1}{30}n^5 - \frac{1}{3}n^4 + \frac{3}{2}n^3 - \frac{8}{3}n^2 + \frac{37}{15}n}{3^n} \tag{0.0.2}$$

By L'Hospital's Rule,

$$\lim_{n\to\infty} p_{23}^{(n)} = \lim_{n\to\infty} \frac{\frac{1}{6}n^4 - \frac{4}{3}n^3 + \frac{9}{2}n^2 - \frac{16}{3}n + \frac{37}{15}}{3^n \log_{10} 3} \tag{0.0.3}$$

$$= \lim_{n\to\infty} \frac{\frac{2}{3}n^3 - 4n^2 + 9n - \frac{16}{3}}{3^n (\log_{10} 3)^2} \tag{0.0.4}$$

$$= \lim_{n\to\infty} \frac{2n^2 - 8n + 9}{3^n (\log_{10} 3)^3} \tag{0.0.5}$$

$$= \lim_{n\to\infty} \frac{4n - 8}{3^n (\log_{10} 3)^4} \tag{0.0.6}$$

$$= \lim_{n\to\infty} \frac{4}{3^n (\log_{10} 3)^5} \tag{0.0.7}$$

$$= 0. \qquad (\because \text{As } n \to \infty, \frac{1}{3^n} \to 0) \tag{0.0.8}$$

$\therefore$ **Option 3** is **correct** answer.