

AI1103

Challenging Problem 7

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Download LaTex file from below link :

<https://github.com/KRISHNASAI1105/demo/blob/main/Challenging%20problem%207/LaTex/Challenging%20problem%207.tex>

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Consider a Markov chain with state space $\{0,1,2,3,4\}$ and transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $\lim_{n \rightarrow \infty} p_{23}^{(n)}$ equals

- 1) $\frac{1}{3}$
- 2) $\frac{1}{2}$
- 3) 0
- 4) 1

Solution

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{4}{9} & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & 0 \\ \frac{1}{9} & \frac{2}{9} & \frac{1}{9} & \frac{2}{9} & \frac{1}{9} \\ 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} & \frac{4}{9} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{14}{27} & \frac{4}{27} & \frac{5}{27} & \frac{1}{9} & \frac{1}{27} \\ \frac{5}{27} & \frac{5}{27} & \frac{7}{27} & \frac{5}{27} & \frac{5}{27} \\ \frac{2}{27} & \frac{1}{9} & \frac{5}{27} & \frac{4}{27} & \frac{14}{27} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{46}{81} & \frac{1}{9} & \frac{4}{27} & \frac{8}{27} & \frac{2}{27} \\ \frac{81}{20} & \frac{9}{4} & \frac{27}{17} & \frac{81}{4} & \frac{27}{20} \\ \frac{81}{2} & \frac{27}{8} & \frac{81}{4} & \frac{27}{1} & \frac{81}{46} \\ \frac{27}{2} & \frac{81}{8} & \frac{27}{1} & \frac{9}{1} & \frac{81}{8} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \dots & \dots & \frac{\frac{1}{30}n^5 - \frac{1}{3}n^4 + \frac{3}{2}n^3 - \frac{8}{3}n^2 + \frac{37}{15}n}{3^n} & \dots & \dots \\ \dots & \dots & \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{3^n} & \dots & \dots \\ \dots & \dots & \frac{\frac{1}{30}n^5 - \frac{1}{3}n^4 + \frac{3}{2}n^3 - \frac{8}{3}n^2 + \frac{37}{15}n}{3^n} & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

As we only require 2nd row 3rd column element in the p^n matrix, so no need to generalise remaining terms.

$$p_{23}^{(n)} = \frac{\frac{1}{30}n^5 - \frac{1}{3}n^4 + \frac{3}{2}n^3 - \frac{8}{3}n^2 + \frac{37}{15}n}{3^n} \quad (0.0.1)$$

So,

$$\lim_{n \rightarrow \infty} p_{23}^{(n)} = \lim_{n \rightarrow \infty} \frac{\frac{1}{30}n^5 - \frac{1}{3}n^4 + \frac{3}{2}n^3 - \frac{8}{3}n^2 + \frac{37}{15}n}{3^n} \quad (0.0.2)$$

By L'Hospital's Rule,

$$\lim_{n \rightarrow \infty} p_{23}^{(n)} = \lim_{n \rightarrow \infty} \frac{\frac{1}{6}n^4 - \frac{4}{3}n^3 + \frac{9}{2}n^2 - \frac{16}{3}n + \frac{37}{15}}{3^n \log_{10} 3} \quad (0.0.3)$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{2}{3}n^3 - 4n^2 + 9n - \frac{16}{3}}{3^n (\log_{10} 3)^2} \quad (0.0.4)$$

$$= \lim_{n \rightarrow \infty} \frac{2n^2 - 8n + 9}{3^n (\log_{10} 3)^3} \quad (0.0.5)$$

$$= \lim_{n \rightarrow \infty} \frac{4n - 8}{3^n (\log_{10} 3)^4} \quad (0.0.6)$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3^n (\log_{10} 3)^5} \quad (0.0.7)$$

$$= 0. \quad (\because \text{As } n \rightarrow \infty, \frac{1}{3^n} \rightarrow 0.) \quad (0.0.8)$$