

AI1103

Challenging Problem 12

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Download LaTex file from below link :

<https://github.com/KRISHNASAI1105/demo/blob/main/Challenging%20problem%202012/LaTex/Challenging%20problem%202012.tex>

$$|A - \lambda I| = (1 - \lambda) \begin{vmatrix} \frac{1}{3} - \lambda & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} - \lambda & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} - \lambda & \frac{1}{3} \\ 0 & 0 & 0 & 1 - \lambda \end{vmatrix} \quad (0.0.1)$$

UGC NET JUNE 2019 Q.51

Consider a Markov chain with state space $\{0,1,2,3,4\}$ and transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $\lim_{n \rightarrow \infty} p_{23}^{(n)}$ equals

- 1) $\frac{1}{3}$
- 2) $\frac{1}{2}$
- 3) 0
- 4) 1

Solution

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For eigenvalue calculation,

$$|A - \lambda I| = 0 \quad A = P$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} - \lambda & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} - \lambda & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} - \lambda & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 - \lambda \end{bmatrix}$$

$$= (1 - \lambda) \left(\frac{1}{3} - \lambda \right) \begin{vmatrix} \frac{1}{3} - \lambda & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} - \lambda & \frac{1}{3} & 0 \\ 0 & 0 & 1 - \lambda & \frac{1}{3} \end{vmatrix} \quad (0.0.2)$$

$$- \frac{1}{3} (1 - \lambda) \begin{vmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} - \lambda & \frac{1}{3} \\ 0 & 0 & 1 - \lambda \end{vmatrix} \quad (0.0.2)$$

$$= (1 - \lambda)^2 \left(\frac{1}{3} - \lambda \right)^3 - \frac{2}{9} (1 - \lambda)^2 \left(\frac{1}{3} - \lambda \right) \quad (0.0.3)$$

$$= (1 - \lambda)^2 \left(\frac{1}{3} - \lambda \right) \left[\left(\frac{1}{3} - \lambda \right)^2 - \frac{2}{9} \right] \quad (0.0.4)$$

$$= (1 - \lambda)^2 \left(\frac{1}{3} - \lambda \right) \left(\lambda^2 - \frac{2\lambda}{3} - \frac{1}{9} \right) \quad (0.0.5)$$

For eigenvalue calculation,

$$|A - \lambda I| = 0$$

Eigenvalues are,

- 1) $\lambda_1 = 1$
- 2) $\lambda_2 = 1$
- 3) $\lambda_3 = \frac{1}{3}$
- 4) $\lambda_4 = \frac{1 - \sqrt{2}}{3}$
- 5) $\lambda_5 = \frac{1 + \sqrt{2}}{3}$

Corresponding Eigenvectors are:

- 1) Eigenvector V_1, V_2 for $\lambda_1=1$ (multiplicity =2) :
So, for $\lambda = 1$ we get 2 eigenvectors.

$$(A - \lambda_1) \cdot V_1 = 0$$

$$\begin{bmatrix} 1 - \lambda_1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} - \lambda_1 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} - \lambda_1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} - \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 1 - \lambda_1 \end{bmatrix} \cdot V_1 = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ e_1 \end{bmatrix} = 0$$

$$\text{Null space} = d_1 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + e_1 \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So, eigenvectors are,

$$V_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

2) Eigenvector V_3 for $\lambda_3 = \frac{1}{3}$ (multiplicity = 1) :

$$(A - \lambda_3).V_3 = 0$$

$$\begin{bmatrix} 1 - \lambda_3 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} - \lambda_3 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} - \lambda_3 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} - \lambda_3 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 - \lambda_3 \end{bmatrix} \cdot V_3 = 0$$

$$\begin{bmatrix} \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \\ e_3 \end{bmatrix} = 0$$

$$V_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

3) Eigenvector V_4 for $\lambda_4 = \frac{1 - \sqrt{2}}{3}$

$$(A - \lambda_4).V_4 = 0$$

$$\begin{bmatrix} 1 - \lambda_4 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} - \lambda_4 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} - \lambda_4 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} - \lambda_4 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 - \lambda_4 \end{bmatrix} \cdot V_4 = 0$$

$$\begin{bmatrix} \frac{2 + \sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2 + \sqrt{2}}{3} \end{bmatrix} \cdot \begin{bmatrix} a_4 \\ b_4 \\ c_4 \\ d_4 \\ e_4 \end{bmatrix} = 0$$

$$V_4 = \begin{bmatrix} a_4 \\ b_4 \\ c_4 \\ d_4 \\ e_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

4) Eigenvector V_5 for $\lambda_5 = \frac{1 + \sqrt{2}}{3}$ (multiplicity = 1) :

$$(A - \lambda_5).V_5 = 0$$

$$\begin{bmatrix} 1 - \lambda_5 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} - \lambda_5 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} - \lambda_5 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} - \lambda_5 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 - \lambda_5 \end{bmatrix} \cdot V_5 = 0$$

$$\begin{bmatrix} \frac{2 - \sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-\sqrt{2}}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{2 - \sqrt{2}}{3} \end{bmatrix} \cdot \begin{bmatrix} a_5 \\ b_5 \\ c_5 \\ d_5 \\ e_5 \end{bmatrix} = 0$$

$$V_5 = \begin{bmatrix} a_5 \\ b_5 \\ c_5 \\ d_5 \\ e_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

$$V = [V_1 V_2 V_3 V_4 V_5]$$

$$V = \begin{bmatrix} 4 & -3 & 0 & 0 & 0 \\ 3 & -2 & -1 & 1 & 1 \\ 2 & -1 & 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$V^{-1} = \frac{1}{4\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} \\ 0 & 0 & 0 & 0 & 4\sqrt{2} \\ \sqrt{2} & -2\sqrt{2} & 0 & 2\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} + 1 & \sqrt{2} & -2 & \sqrt{2} & -\sqrt{2} + 1 \\ -\sqrt{2} - 1 & \sqrt{2} & 2 & \sqrt{2} & -\sqrt{2} - 1 \end{bmatrix}$$

$$V^{-1}AV = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ -1 + \frac{1}{\sqrt{2}} & \frac{2-\sqrt{2}}{4} & 0 & \frac{1-\sqrt{2}}{3} & 0 \\ -1 - \frac{1}{\sqrt{2}} & \frac{2+\sqrt{2}}{4} & 0 & 0 & \frac{1+\sqrt{2}}{3} \end{bmatrix}$$

The above matrix is a diagonal matrix, with eigenvalues as diagonal elements.

$$V^{-1}AV \times V^{-1}AV \times \dots \times (n \text{ times}) V^{-1}AV = V^{-1}A^n V$$

$$V^{-1}A^n V = B$$

Let, B be some matrix after the whole multiplication.

$$A^n = VBV^{-1}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{70} & \frac{9}{70} & 0 & 0 & 0 \\ -\frac{153}{35} & \frac{153}{35} & 0 & 0 & 0 \end{bmatrix}$$

$$A^n = V^{-1}BV = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{3}{8} & 0 & 0 & 0 & \frac{11}{8} \\ -1 & 0 & 0 & 0 & 2 \\ -\frac{7}{8} & 0 & 0 & 0 & \frac{15}{8} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

As the 2nd row 3rd column element in the above matrix is Zero.

$$\therefore \text{Hence, } \lim_{n \rightarrow \infty} p_{23}^{(n)} = 0.$$

\therefore **Option 3 is correct answer.**