Rational Fractal Spline

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Introduction to Fractal Interpolation

- Fractal interpolation is a modern technique in approximation theory used to fit and analyze scientific data.
- The graph of these FIF can be used to approximate image component such as the profiles of mountains ranges, the tops of clouds and horizons over forests.
- This method utilizes an iterated function system (IFS) with rational functions of the form $\frac{p_i(x)}{q_i(x)}$.
- Here, $p_i(x)$ and $q_i(x)$ are cubic polynomials that include two shape parameters.

Advantages of Rational Cubic Fractal IFS

- The rational cubic IFS provides additional flexibility over classical rational cubic interpolants.
- This flexibility arises from the inclusion of scaling factors and shape parameters.
- Classical rational cubic functions are a special case of the proposed fractal interpolants.

Shape Preservation

- The fractal interpolation scheme is computationally efficient and adaptable (local, moderately local, or global) based on scaling factors and shape parameters.
- Sufficient conditions on scaling factors and shape parameters ensure that the interpolation function preserves shape:
  - Monotonicity
  - Positivity
  - Convexity

Given Data and IFS

- Consider a data set $\{(x_i, f_i), i = 1, 2, \ldots, n\}$, where $x_1 < x_2 < \cdots < x_n$.
- Define the IFS $I^* = \{\mathbb{R}^2; w_i(x, f) = (L_i(x), F_i(x, f)), i = 1, 2, \ldots, n-1\}$.
- Where:

$$L_i(x) = a_i x + b_i$$

- $L_i(x)$ satisfies the condition

$$L_i(x_1) = x_i, L_i(x_n) = x_i + 1.$$

- $F_i(x, f) = \alpha_i f + r_i(x)$, where:

$$r_i(x) = \frac{p_i(x)}{q_i(x)}$$

- $p_i(x)$ and $q_i(x)$ are cubic polynomials, and $q_i(x) \neq 0 \quad \forall x \in [x_1, x_n]$.
- $$p_i(x) \equiv P_i(\theta) = U_i(1-\theta)^3 + M_i\theta(1-\theta)^2 + N_i\theta^2(1-\theta) + Z_i\theta^3,$$
  $$q_i(x) \equiv Q_i(\theta) = (1-\theta)^3 + v_i\theta(1-\theta)^2 + w_i\theta^2(1-\theta) + \theta^3$$
- Where U_i, M_i, N_i, and Z_i, for $i = 1, 2, \ldots, n-1$, are real parameters as follows:
  $$U_i = f_i - \alpha_i f_1,$$
  $$Z_i = f_{i+1} - \alpha_i f_n,$$
  $$M_i = v_i f_i + h_i d_i - \alpha_i \left[(x_n - x_1)d_1 + f_1 v_i\right],$$
  $$N_i = w_i f_{i+1} - h_i d_{i+1} + \alpha_i \left[(x_n - x_1)d_n - f_n w_i\right].$$
- $|\alpha_i| < a_i$, for $i = 1, 2, \ldots, n-1$.

Derivative Conditions

- Let $F_i'(x, f) = \frac{\alpha_i f + r_i'(x)}{a_i}$, where $r_i'(x)$ is the derivative of $r_i(x)$ with respect to x.
- Boundary conditions:

$$F_i(x_1, f_1) = f_i, \quad F_i(x_n, f_n) = f_{i+1}$$

$$F_i'(x_1, f_1) = d_i, \quad F_i'(x_n, f_n) = d_{i+1}$$

- for $i = 1, 2, \ldots, n-1$.
- $h_i = x_{i+1} - x_i \;\; ; i = 1 \; to \; n$
- d_i $(i = 1, 2, \ldots, n)$ be the derivative values at the knots.
- The value of ends points is given by:

$$d_1 = \begin{cases} 0, & \text{if } \Delta_1 = 0, \\ \Delta_1 + \frac{(\Delta_1 - \Delta_2)h_1}{h_1 + h_2}, & \text{otherwise.} \end{cases}$$

- For d_n:

$$d_n = \begin{cases} 0, & \text{if } \Delta_{n-1} = 0, \\ \Delta_{n-1} + \frac{(\Delta_{n-1} - \Delta_{n-2})h_{n-1}}{h_{n-1} + h_{n-2}}, & \text{otherwise.} \end{cases}$$

- The value of d_i for $i = 2, 3, \ldots, n-1$ is given by:

$$d_i = \begin{cases} 0, & \text{if } \Delta_{i-1} = 0 \text{ or } \Delta_i = 0, \\ \frac{\Delta_i h_{i-1} + \Delta_{i-1} h_i}{h_i + h_{i-1}}, & \text{otherwise.} \end{cases}$$

- The variable θ is defined as:

$$\theta = \frac{x - x_1}{x_n - x_1}, \quad x \in [x_1, x_n].$$

- Then the attractor of IFS is the graph of rational cubic FIF.

Shape Preservation in Fractal Interpolation

Problem:
For an arbitrary selection of scaling factors and shape parameters, the rational cubic fractal interpolation function (FIF) may fail to maintain desired shape properties:

- **Monotonicity**: The curve may oscillate, even if the data is monotonic.
- **Positivity**: The interpolation could dip below zero, even if the data is positive.
- **Convexity (Concavity)**: The curvature may not reflect the original data's convex or concave shape.

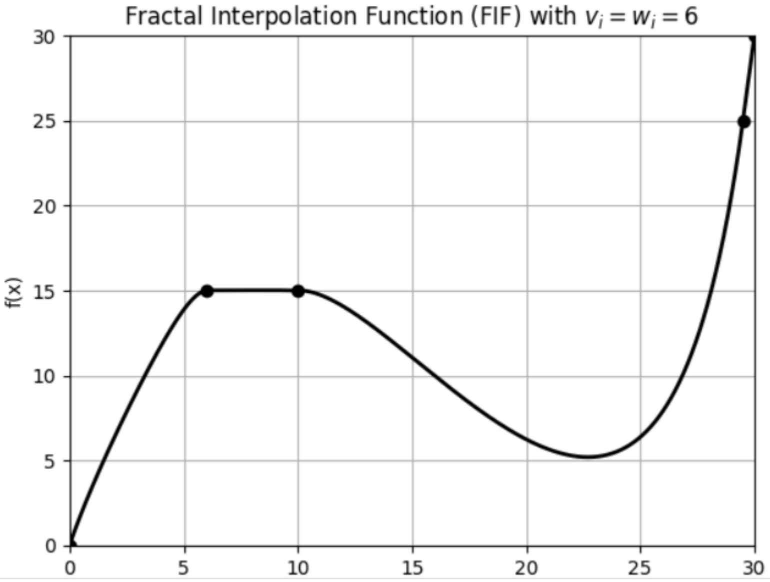
This is very similar to the ordinary spline schemes that do not provide the desired shape features of a data.Thus some mathematical treatment is required to achieve a monotonicity, positivity, and convexity (concavity) preserving rational cubic spline FIF for a given data

Approach to solve the problem

There are two approach to address this:

- Trail and Error= Manually adjusting the scalling factor and shape parameter but it can work sometime usually not very efficient and precise.
- Automated Method= Develop automated process to compute the appropriate scalling factor and shape parameter that preserve desired shape properties

Figure



Fractal Interpolation Function (FIF) with $v_i = w_i = 6$
f(x)
0
5
10
15
20
25
30
0
5
10
15
20
25
30

Sufficient conditions for monotonicity

Theorem:
Let $\{(x_i, f_i, d_i), i = 1, 2, \ldots, n\}$ be a given monotonic data set. The derivative values must satisfy the necessary conditions for monotonicity:

- $d_i = d_{i+1} = 0, \quad \mathit{for} \quad \Delta_i = 0;$
- $\mathrm{sgn}(d_i) = \mathrm{sgn}(d_{i+1}) = \mathrm{sgn}(\Delta_i) \quad \mathit{for} \;\; \Delta_i \neq 0.$

Now, define the scaling factors α_i for $i = 1, 2, \ldots, n-1$ as:

$$\alpha_i \in \begin{cases} [0, \mu_i] & \text{if } \mu_i < a_i \\ [0, a_i) & \text{if } \mu_i \geq a_i \end{cases}$$

where $\mu_i = \min \left\{ \frac{a_i d_i}{d_1}, \frac{a_i d_{i+1}}{d_n}, \frac{f_{i+1} - f_i}{f_n - f_1} \right\}$.
Additionally, the shape parameters v_i and w_i for $i = 1, 2, \ldots, n-1$ are selected as:

- **Option 1:**

$$v_i = l_i \cdot d_i^* \cdot i^*, \quad w_i = k_i \cdot d_{i+1}^* \cdot i^*, \quad l_i, k_i \in \mathbb{R}^+ \text{ such that } \frac{1}{l_i} + \frac{1}{k_i} \leq 1$$

- **Option 2:**

$$v_i = \eta_i \cdot d_i^* \cdot d_{i+1}^* \cdot i^*, \quad w_i = \nu_i \cdot d_i^* \cdot d_{i+1}^* \cdot i^*$$

where $\eta_i, \nu_i \geq 1$, $d_i^* = d_i - \alpha_i \cdot \frac{d_1}{a_i}$, $d_{i+1}^* = d_{i+1} - \alpha_i \cdot \frac{d_n}{a_i}$, and $i^* = i - \alpha_i \cdot \frac{f_n - f_1}{h_i}$.

Conclusion: Form this theorem we find a way to choose scaling factors α_i, shape parameters v_i and w_i, such monotonicity-preserving C^1-rational cubic FIF, whose graph is the attractor of the rational cubic IFS.

Thank you for your attention!