

# Summary | Riemann Integration

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## Introduction

### Interval

Let  $I = [a, b]$ . Length of the interval  $|I| = b - a$ .

### Disjoint interval

When 2 intervals don't share any common numbers.

### Almost disjoint interval

When 2 intervals are disjoint or intersect only at a common endpoint.

## Riemann Integral

Let  $f : [a, b] \rightarrow \mathbb{R}$  is a bounded (not necessarily continuous) function on a closed, bounded (compact) interval.

Riemann integral of  $f$  is:  $\int_a^b f$

### Definite integral

When  $a, b$  are constants.

### Indefinite integral

When  $a$  is a constant but  $b$  is replaced with  $x$ .

## Partition

Let  $I$  be a non-empty, compact interval (closed and bounded). A partition of  $I$  is a finite collection  $\{I_1, I_2, \dots, I_n\}$  of almost disjoint, non-empty, compact sub-intervals whose union is  $I$ .

A partition is determined by the endpoints of all sub-intervals:  $a = x_0 < x_1 < \dots < x_n = b$ .

A partition can be denoted by:

- its intervals -  $P = \{I_1, I_2, \dots, I_n\}$
- the endpoints of its intervals -  $P = \{x_0, x_1, \dots, x_n\}$

## Riemann Sum

Let

- $f : [a, b] \rightarrow \mathbb{R}$  is a bounded function on the compact interval  $I = [a, b]$  with  $M = \sup_I f$  and  $m = \inf_I f$ .
- $P = \{I_1, I_2, \dots, I_n\}$
- $M_k = \sup_{I_k} f = \sup \{f(x) : x \in [x_{k-1}, x_k]\}$
- $m_k = \inf_{I_k} f = \inf \{f(x) : x \in [x_{k-1}, x_k]\}$

### Upper riemann sum

$$U(f; P) = \sum_{k=1}^n M_k |I_k|$$

### Lower riemann sum

$$L(f; P) = \sum_{k=1}^n m_k |I_k|$$

$$m_k < M_k \implies L(f; P) \leq U(f; P)$$

When  $P_1, P_2$  are any 2 partitions of  $I$ :  $L(f; P_1) \leq U(f; P_2)$

## Refinements

$Q$  is called a refinement of  $P \iff$  if  $P$  and  $Q$  are partitions of  $[a, b]$  and  $P \subseteq Q$ .

When  $Q$  is a refinement of  $P$ :

$$L(f; P) \leq L(f; Q) \leq U(f; Q) \leq U(f; P)$$

**Note**

If  $P_1$  and  $P_2$  are partitions of  $[a, b]$ , then  $Q = P_1 \cup P_2$  is a refinement of both  $P_1$  and  $P_2$ . In that case:

$$L(f; P_1) \leq L(f; Q) \leq U(f; Q) \leq U(f; P_2)$$

## Upper & Lower integral

Let  $\mathbb{P}$  be the collection of all possible partitions of the interval  $[a, b]$ .

### Upper Integral

$$U(f) = \inf \{U(f; P); P \in \mathbb{P}\} = \overline{\int_a^b f}$$

### Lower Integral

$$L(f) = \sup \{L(f; P); P \in \mathbb{P}\} = \underline{\int_a^b f}$$

For a bounded function  $f$ , always  $L(f) \leq U(f)$

## Riemann Integrable

A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable on  $[a, b]$  **iff**  $U(f) = L(f)$ . In that case, the Riemann integral of  $f$  on  $[a, b]$  is denoted by  $\int_a^b f(x) \, dx$ .

## Reimann Integrable or not

Function	Yes or No?	Proof hint
Unbounded	No	By definition
Constant	Yes	$\forall P$ (any partition) $L(f; P) = U(f; P)$
Monotonically increasing/decreasing	Yes	Take a partition such that $\Delta x < \delta = \frac{\epsilon}{f(b)-f(a)}$
Continuous	Yes	Take a partition such that $\Delta x < \delta = \frac{\epsilon}{2(b-a)}$

### Note

If the set of points of discontinuity of a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is finite, then  $f$  is Riemann integrable on  $[a, b]$ .

### Note

If the set of points of discontinuity of a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is finite number of limit points, then  $f$  is integrable on  $[a, b]$ .

A function may have infinitely many discontinuous points, but if the set of all discontinuous points have finite number of limit points, then  $f$  is integrable on  $[a, b]$ .

## Cauchy Criterion

### Theorem

A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable **iff** for every  $\epsilon > 0$  there exists a partition  $P_\epsilon$  of  $[a, b]$ , which may depend on  $\epsilon$ , such that:

$$U(f, P_\epsilon) - L(f, P_\epsilon) \leq \epsilon$$

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### Proof Hint

- To prove  $\implies$  : consider  $L(f) - \frac{\epsilon}{2} < L(f; P)$  and  $U(f; P) < U(f) + \frac{\epsilon}{2}$
- To prove  $\impliedby$  : consider  $L(f; P) < L(f)$  and  $U(f) < U(f; P)$

### Note

$f : [a, b] \rightarrow \mathbb{R}$  is integrable on  $[a, b]$  when:

- The set of points of discontinuity of a bounded function  $f$  is finite.
- The set of points of discontinuity of a bounded function  $f$  is finite number of limit points. (may have infinite number of discontinuities)

## Theorems on Integrability

### Theorem 1

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded, and integrable on  $[c, b]$  for all  $c \in (a, b)$ . Then  $f$  is integrable on  $[a, b]$ . Also valid for the other end.

### Proof Hint

- Isolate a partition on the required end.
- Choose  $x_1$  or  $x_{n-1}$  such that  $\Delta x < \frac{\epsilon}{4M}$  where  $M$  is an upper or lower bound.

### Theorem 2

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded, and continuous on  $[c, b]$  for all  $c \in (a, b)$ . Then  $f$  is integrable on  $[a, b]$ . Also valid for the other end.

### TODO: Proof Hint

# Properties of Integrals

## Notation

If  $a < b$  and  $f$  is integrable on  $[a, b]$ , then:

$$\int_a^b f = - \int_b^a f$$

## Properties

Suppose  $f$  and  $g$  are integrable on  $[a, b]$ .

### Addition

$f + g$  will be integrable on  $[a, b]$ .

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$$

#### Proof Hint

- Prove  $f + g$  is integrable using:
  - $\sup(f + g) \leq \sup(f) + \sup(g)$
  - $\inf(f + g) \geq \inf(f) + \inf(g)$
- Start with  $U(f + g)$  and show  $U(f + g) \leq U(f) + U(g)$
- Start with  $L(f + g)$  and show  $L(f + g) \geq L(f) + L(g)$

### Constant multiplication

Suppose  $k \in \mathbb{R}$ .  $kf$  will be integrable  $[a, b]$ .

$$\int_a^b kf = k \int_a^b f$$

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**Proof Hint**

- Prove for  $k \geq 0$ . Use  $U - L < \frac{\epsilon}{k}$
- Prove for  $k = -1$
- Using the above results, proof for  $k < 0$  is apparent

## Bounds

If  $m \leq f(x) \leq M$  on  $[a, b]$ :

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

If  $f(x) \leq g(x)$  on  $[a, b]$ :

$$\int_a^b f \leq \int_a^b g$$

## Modulus

$|f|$  will be integrable on  $[a, b]$ .

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

**Proof Hint**

Start with  $-|f| \leq f \leq |f|$ . And integrate both sides.

## Multiple

$fg$  will be integrable on  $[a, b]$ .

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### Proof Hint

- Suppose  $f$  is bounded by  $k$
- Prove  $f^2$  is integrable (Use  $\frac{\epsilon}{2k}$ )
- $fg$  is integrable because:

$$fg = \frac{1}{2} [(f+g)^2 - f^2 - g^2]$$

### Max, Min

$\max(f, g)$  and  $\min(f, g)$  are integrable.

Where  $\max$  and  $\min$  functions are defined as:

$$\max(f, g) = \frac{1}{2} (|f - g| + f + g)$$

$$\min(f, g) = \frac{1}{2} (-|f - g| + f + g)$$

### Additivity

$\iff f$  is Riemann integrable on  $[a, c]$  and  $[c, b]$  where  $c \in (a, b)$ .

### Proof Hint

- $\implies$  : Use Cauchy criterion after defining these:
  - $P' = \{c\} \cup P$
  - $Q = P' \cap [a, c]$
  - $R = P' \cap [c, b]$
- $\impliedby$  : Use Cauchy criterion on  $[a, c]$ ,  $[c, b]$  separately and then combine using a union partition

After the integrability is proven,



$$\int_a^b f = \int_a^c f + \int_c^b f$$

**Proof Hint**

1. Let  $Q$  be a partition on  $[a, c]$  and  $R$  be a partition on  $[c, b]$ . And  $P = Q \cup R$ .
2. Prove the below using Cauchy criteria:

$$\int_a^b f < L(f; P) + \epsilon \implies \int_a^b f \leq \int_a^c f + \int_c^b f$$

3. Prove the below using Cauchy criteria (by considering RHS):

$$\int_a^c f + \int_c^b f \leq \int_a^b f$$

## Sequential Characterization of Integrability

A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable **iff**  $\exists \{P_n\}$  a sequence of partitions, such that:

$$\lim_{n \rightarrow \infty} [U(f; P_n) - L(f; P_n)] = 0$$

In that case:

$$\int_a^b f = \lim_{n \rightarrow \infty} U(f; P_n) = \lim_{n \rightarrow \infty} L(f; P_n)$$

**Proof Hint**

Cauchy criteria and squeeze theorem is used for both side proof.

For  $\Leftarrow$  :

- Consider the limit definition.
- Prove  $f$  is Riemann integrable on  $P_n$  by Cauchy criteria.
- Use squeeze theorem for  $U(f; P_n) - U(f) \leq U(f; P_n) - L(f; P_n)$  to prove limit of upper sum
- Prove limit of lower sum using the limit of upper sum

For  $\Rightarrow$  : Consider the below, where  $n \in \mathbb{N}$ .

$$0 \leq U(f; P_n) - L(f; L_n) \leq \frac{1}{n}$$

## Theorem

Suppose  $f$  is Riemann integrable on  $[a, b]$  and  $\epsilon > 0$ . Then  $\exists \delta > 0 \forall P$ :

$$|P| < \delta \implies \left| \int_a^b f - \sum_{j=1}^n f(\zeta_j) I_j \right| < \epsilon$$

where  $\zeta_j \in [x_{j-1}, x_j], j = 1, 2, \dots, n$ .

**Proof Hint**

$$\underline{\int_a^b f - \epsilon} < L(f; P) \leq \sum_{j=1}^n f(\zeta_j) I_j \leq U(f; P) < \overline{\int_a^b f + \epsilon}$$

## Intermediate Value Theorem for Integrals

Suppose  $f$  is a continuous function on  $[a, b]$ . Then  $\exists x \in (a, b)$ :

$$f(x) = \frac{1}{b-a} \int_a^b f$$

### Proof Hint

Suppose  $f_{\max} = M = f(x_0)$  and  $f_{\min} = m = f(y_0)$ .

When  $M = m$ :  $f$  is a constant function. Proof is trivial.

Otherwise:

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

Then there exists  $x \in (x_0, y_0)$ .

## Generalized IVT

Suppose  $f, g$  are continuous functions on  $[a, b]$  and  $g \geq 0$ . Then  $\exists x \in (a, b)$ :

$$f(x) \int_a^b g = \int_a^b fg$$

### Proof Hint

Consider this and proof is similar to IVT.

$$mg \leq fg \leq Mg$$

# Convergence Functions

Convergence of functions is introduced in [Sequence of Functions | Real Analysis](#).

## Uniform Convergence Theorem

Let  $f_n$  be a sequence of Riemann integrable functions on  $[a, b]$ . Suppose  $f_n$  converges to  $f$  uniformly. Then  $f$  is Riemann integrable on  $[a, b]$  and  $\forall x \in [a, b]$ :

$$\int_a^x f_n(x) \, dx \text{ converges to } \int_a^x f(x) \, dx \text{ uniformly}$$

and:

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx$$

### Proof Hint

- Consider  $\frac{\epsilon}{2(b-a)}$  in place of  $\epsilon$ .
- Consider Cauchy criteria for  $f_N$ .
- Prove  $f - f_N$  is Riemann integrable using Cauchy criteria.
- $f$  is Riemann integrable as  $f = f_N + (f - f_N)$ .

When  $f_n$  converges to  $f$  pointwise, we cannot be sure if  $f$  is Riemann integrable or not. An example where  $f$  is not Riemann integrable:

$$\lim_{n \rightarrow \infty} u_n = \begin{cases} 1 & x = q_k \text{ where } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Here  $q_k$  is the enumeration of rational numbers in  $[0, 1]$ .

## Dominated Convergence Theorem

Let  $f_n$  be a sequence of Riemann integrable functions on  $[a, b]$ . Suppose  $f_n$  converges to  $f$  pointwise where  $f$  is Riemann integrable on  $[a, b]$ . If  $\exists M > 0 \forall n \forall x \in [a, b]$  s.t.  $|f_n(x)| \leq M$ :

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx$$

## Monotone Convergence Theorem

Let  $f_n$  be a sequence of Riemann integrable functions on  $[a, b]$ , and they are monotone (all increasing or decreasing, like  $f_1 \leq f_2 \leq \dots \leq f_n$ ). Suppose  $f_n$  converges to  $f$  pointwise where  $f$  is Riemann integrable on  $[a, b]$ . If  $\exists M > 0 \forall n \forall x \in [a, b]$  s.t.  $|f_n(x)| \leq M$ :

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx$$

Can be proven from the dominated convergence theorem.

## Fundamental Theorem of Calculus

### Theorem I

If  $g$  is continuous on  $[a, b]$  that is differentiable on  $(a, b)$  and if  $g'$  is integrable on  $[a, b]$  then

$$\int_a^b g' = g(b) - g(a)$$

#### Proof Hint

Consider a general partition and use [Mean Value Theorem](#) on each partition.

## Integration by parts

Suppose  $u, v$  are continuous functions on  $[a, b]$  that are differentiable on  $(a, b)$ . If  $u'$  and  $v'$  are Riemann integrable on  $[a, b]$ :

$$\int_a^b u(x)v'(x) \, dx + \int_a^b u'(x)v(x) \, dx = u(b)v(b) - u(a)v(a)$$

### Proof Hint

Consider  $g = uv$  and use [FTC I](#).

## Theorem II

Suppose  $f$  is an Riemann integrable function on  $[a, b]$ . For  $x \in (a, b)$ .

$$F(x) = \int_a^x f(t) \, dt$$

- $F(x)$  is uniformly continuous on  $[a, b]$
- $f$  is continuous at  $x_0 \in (a, b) \implies F$  is differentiable and  $F'(x_0) = f(x_0)$

### Proof Hint

For the first point:

- Consider 2 points in the interval  $x, y (> x)$  such that  $|x - y| < \delta = \frac{\epsilon}{M}$
- Show  $|F(y) - F(x)| \leq \epsilon$

For the second point: Consider the continuity definition of  $f$  and prove is quite trivial.

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \epsilon$$

## Theorem

Suppose  $f$  is Riemann integrable on an open interval  $I$  containing the values of differentiable functions  $a, b$ . Then:

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt = f(b(x))b'(x) - f(a(x))a'(x)$$

### Proof Hint

Can be done using FTC I and II. Proof is quite trivial.

## Theorem - Change of Variable

Suppose  $u$  is a differentiable function on an open interval  $J$  such that  $u'$  is continuous. Let  $I$  be an open interval such that  $\forall x \in J, u(x) \in I$ .

If  $f$  is continuous on  $I$ , then  $f \circ u$  is continuous on  $J$  and:

$$\int_a^b (f \circ u)(x) u'(x) dx = \int_{u(a)}^{u(b)} f(u) du$$

## Improper Riemann Integrals

Initially Riemann integrals are defined only for **bounded** functions defined on a set of **compact** intervals.

### Type 1

A function that is **not** integrable at one endpoint of the interval.

Suppose  $f : (a, b] \rightarrow \mathbb{R}$  is integrable on  $[c, b] \forall c \in (a, b)$ .

$$\int_a^b f = \lim_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b f$$

Can be similarly defined on the other endpoint. The above integral converges **iff** the limit exists and finite. Otherwise diverges.

### Examples

$$\int_0^1 \frac{1}{x^p} dx = \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^1 \frac{1}{x^p} dx$$

The above integral converges (to  $\frac{1}{p-1}$ ) **iff**  $0 < p < 1$ . When  $p \geq 1$ , it diverges to  $\infty$ .

### Type 2

A function defined on unbounded interval (including  $\infty$ ).

Suppose  $f : [a, \infty) \rightarrow \mathbb{R}$  is integrable on  $[a, r] \forall r > a$ .

$$\int_a^{\infty} f = \lim_{r \rightarrow \infty} \int_a^r f$$

Can be similarly defined on the other endpoint. The above integral converges **iff** the limit exists and finite. Otherwise diverges.

### Examples

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{r \rightarrow \infty} \int_1^r \frac{1}{x^p} dx$$

The above integral converges (to  $\frac{1}{p-1}$ ) **iff**  $p > 1$ . When  $0 < p \leq 1$ , it diverges to  $\infty$ .

### Type 3

A function that is undefined at finite number of points. The integral can be split into multiple integrals of type 1. Similarly integrals from  $-\infty$  to  $\infty$  can be defined.



### Note

The integral can be split into multiple integrals only when all those integrals exist.

Convergence of improper integrals is similar to the convergence of [series](#).

## Absolute convergence test

$$\int_a^b |f| \text{ converges} \implies \int_a^b f \text{ converges}$$

## Gamma function

Defined as below for  $n > 0$ :

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

Aka. Eulerian integral of the second kind.

## Convergence

$\Gamma(n)$  is convergent **iff**  $n > 0$ .

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### Proof Hint

Direct comparison test is used. Proved in 3 cases:

- Case 1: for positive integer  $n$ 
  - Consider the [lemma 2](#)'s limit definition
  - Take  $\epsilon = 1$
  - Use the convergence of  $\int_0^\infty e^{-x/2} dx$
- Case 2: for  $n > 1$  non-integers
  - Use  $\lfloor n \rfloor < n < \lfloor n \rfloor + 1$
  - Use  $x^{y-1}e^{-x} \leq x^{\lfloor n \rfloor}e^{-x}$
  - $\Gamma(\lfloor n \rfloor + 1)$  is converging from case 1
- Case 3: for  $0 < n < 1$ .
  - Proof is similar to case 1
  - But  $\int_0^N e^{-x}x^{n-1} dx$  is an improper
  - Prove that it is also converging

## Properties

Proofs are required for each property mentioned below.

- $\Gamma(1) = 1$
- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma(n+1) = n!$

## Extension of gamma function

$\Gamma(n)$  function can be extended for negative non-integers using:

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

This cannot be used to define  $\Gamma(0)$  because of the denominator. And through induction,  $\Gamma$  function cannot be defined for negative integers.

## Lemmas

### Lemma 1

$$\forall s > 0 \int_0^{\infty} e^{-sx} dx \text{ converges}$$

### Lemma 2

$$\forall n \in \mathbb{Z}^+ \lim_{x \rightarrow \infty} \frac{x^{n-1}}{e^{x/2}} = 0$$

## Transformations

Alternate forms of  $\Gamma(n)$ .

### Form 1

$\forall n > 0$ :

$$\Gamma(n) = \frac{1}{n} \int_0^{\infty} e^{-x^{1/n}} dx$$

**Proof Hint**

Use  $x^n = t$ .

**ⓘ Note**

One of the most frequently used integrals in mathematics:

$$\int_0^\infty e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

**Form 2**

$$\int_0^\infty e^{-kx} x^{n-1} \mathrm{d}x = \frac{\Gamma(n)}{k^n}$$

**ⓘ Proof Hint**

Use  $x = kt$ .

**Form 3**

$$\Gamma(n) = \int_0^1 \ln\left(\frac{1}{x}\right)^{n-1} \mathrm{d}x$$

**ⓘ Proof Hint**

Use  $e^{-x} = t$ .

**Form 4**

$$\Gamma(n) = 2 \int_0^\infty e^{-x^2} x^{2n-1} \mathrm{d}x$$

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**Proof Hint**

Use  $x = t^2$ .

## Beta function

Beta function is defined as below, for  $m, n > 0$ :

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Aka. Eulerian integral of the first kind.

**Note**

For  $m, n \leq 0$ , the beta function is divergent.

## Properties

### Symmetrical

From the definition:

$$B(m, n) = B(n, m)$$

**Proof Hint**

Use  $t = 1 - x$ .

## Transformations

### Form 1

$$B(m, n) = \int_0^\infty \frac{x^{n-1}}{(x+1)^{m+n}} dx = \int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}} dx$$

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**ⓘ Proof Hint**

Use  $x = \frac{1}{1+t}$  in the definition.

**Form 2**

$$B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

**ⓘ Proof Hint**

Use  $x = \frac{1}{t}$  in Form 1.

**Form 3**

$$\int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx = \frac{B(m, n)}{a^m b^n}$$

**ⓘ Proof Hint**

Use  $x = \frac{a}{b}t$  in Form 1.

**Form 4**

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}(\theta) \cos^{2n-1}(\theta)}{(a \sin^2 \theta + b \cos^2 \theta)^{m+n}} d\theta = \frac{B(m, n)}{2a^m b^n}$$

**ⓘ Proof Hint**

Use  $x = \tan \theta$  in Form 3.

### Form 5

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}} dx = \frac{B(m,n)}{a^n(1+a)^m}$$

#### Proof Hint

Use the substitution in the definition.

$$x = \frac{t(1+a)}{t+a}$$

### Form 6

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1} dx = (b-a)^{m+n-1} B(m,n)$$

#### Proof Hint

Use  $x = at + b(1-t)$  in the definition.

### Form 7

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+(b-a)x)^{m+n}} dx = \frac{B(m,n)}{a^n b^m}$$

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}} dx = \frac{B(m,n)}{(b+c)^n b^m}$$

#### Proof Hint

I don't know.

## Relation with gamma function

$\forall m, n > 0.$

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

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