Introduction to Differential Equations

Equations which are composed of an unknown function and its derivatives.

Ordinary Differential Equations

When a differential equation involves one independent variable, and one or more dependent variables.

An example:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x)$$

Partial Differential Equations

When a differential equation involves more than one independent variables, and more than one dependent variables.

$$\frac{\partial y}{\partial x} = \cos(x)$$

Linear

A linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function (dependant variable) and its derivatives, that is an equation of the form:

$$P_0(x)y + P_1(x)y' + \ldots + P_n(x)y^{(n)} + Q(x) = 0$$

Where

- All (differentiable) functions of x (depends only on x, not on y).
- y and its successive derivatives of the unknown function y of the independent variable x.

Nonlinear

Nonlinear differential equations are any equations that cannot be written in the above form. In particular, these include all equations that include:

- y and/or its successive derivatives raised to any power (obv. other than 1)
- nonlinear functions of y or any derivative
- any product or function of these

Order

Highest order derivative.

Degree

Power of highest order derivative.

Picard’s Existence and Uniqueness Theorem

Consider the below IVP.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \ ; \ y(x_0) = y_0$$

Suppose: D is an open neighbourhood in $\mathbb{R}^2$ containing the point $\left(x_0, y_0\right)$.

If f and $\frac{\partial f}{\partial y}$ are continuous functions in D, **then** the IVP has a unique solution in some closed interval containing x_0.

Solving First Order Ordinary Differential Equations

Separable equation

Separable if x and y functions can be separated into separate one-variable functions (as shown below).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

$$\int \frac{1}{g(y)}\mathrm{d}y = \int f(x)\mathrm{d}x$$

Homogenous equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

Here the function $f(x, y)$ is homogenous when $f(x, y) = f(\lambda x, \lambda y)$.

To solve:

- Use $y = vx$ substitution, where v is a function of x and y.
- By differentiating both sides: $\mathrm{d}y = v + v\mathrm{d}x$
- Applying both of these into the equation, simplies it to be separable.

Reduction to homogenous type

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax+by+c}{Ax+By+C}$$

This type of equation can be reduced to homogenous form.

If $a : b = A : B$, use the substitution: $u = ax + by$.

In other cases:

- Find h and k such that $ah + bk + c = 0$ and $Ah + Bk + C = 0$.
- Use substitutions:
  - $X = x + h$
  - $Y = y + k$

The reduced equation would be:

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{aX+bY}{AX+BY}$$

Linear equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

The above form is called **the standard form**.

When $Q(x) = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = 0$, the equation would be separable.

Otherwise:

- Identify P(x) from the standard form
- Calculate integrating factor: $I = e^{\int P(x)\mathrm{d}x}$. Integrate P(x). Put it as the power of e.
- Multiply both sides by I. L.H.S becomes $\frac{d}{dx}(yI)$. We can solve by integrating both sides.

Bernoulli's equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

The above equation is Bernoulli's equations when $n \in \mathbb{R}$.

When $n = 0$ or $n = 1$, the equation would be linear.

Otherwise we can use $v = y^{1-n}$ to convert it to linear form.

None of the above

The equation must be converted to one of the above by using a substitution.

Higher Order Ordinary Differential Equations

Linear Differential Equations

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + p_n(x)y = q(x)$$

Based on $q(x)$, the above equation is categorized into 2 types

- **Homogenous** if $q(x) = 0$
- **Non-homogenous** if $q(x) \neq 0$

> **Note**
>
> For 1st semester, only higher order, linear, ordinary differential equations with constant coefficients are focused on. They can be written as:
>
> $$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_n y = q(x)$$

Solution

The general solution of the equation is $y = y_p + y_c$.

Here

- y_p - **particular solution**
- y_c - **complementary solution**

Particular solution

Doesn't exist for homogenous equations. For non-homogenous equations check steps section of 2nd order ODE.

Complementary solution

Solutions assuming $LHS = 0$ (as in a homogenous equations).

$$y_c = \sum_{i=1}^{n} c_i y_i$$

Here

- c_i - constant coefficients
- y_i - a linearly-independent solution

Linearly dependent & independent

n-th order linear differential equations have n linearly independent solutions.

Two solutions of a differential equation u, v are said to be **linearly dependent**, if there exists constants $c_1, c_2\ (\neq 0)$ such that $c_1 u(x) + c_2 v(x) = 0$.

Otherwise, the solutions are said to be **linearly independent**, which means:

$$\sum_{i=1}^{n} c_i y_i = 0 \to \forall c_i = 0$$

Linear differential operators with constant coefficients

Differential operator

Defined as:

$$\mathrm{D}^i = \frac{\mathrm{d}^i}{\mathrm{d}x^i}\ ;\ n \in \mathbb{Z}^+$$

We can write the above equation using the differential operator:

$$\mathrm{D}^n y + a_1 \mathrm{D}^{n-1} y + \ \ldots\ + a_n y = q(x)$$

Here if we factor out y (***how tf?***), we get:

$$(\mathrm{D}^n + a_1 \mathrm{D}^{n-1} + \ \ldots\ + a_n) y = P(D) y = q(x)$$

where $P(D) = (\mathrm{D}^n + a_1 \mathrm{D}^{n-1} + \ \ldots\ + a_n)$.

We call $P(D)$ a polynomial differential operator with constant coefficients.

Solving Second Order Ordinary Differential Equations

Homogenous

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a\frac{\mathrm{d}y}{\mathrm{d}x} + +by = 0\,;\;a, b\,\text{are constants}$$

Consider the function $y = e^{mx}$. Here m is a constant to be found.

By applying the function to the above equation, we get:

$$m^2 + am + b = 0$$

The above equation is called the **Auxiliary equation** or **Characteristic equation**.

Case 1: Distinct real roots

$$y = Ae^{m_1x} + Be^{m_2x}$$

Case 2: Equal real roots

$$y = (Ax + B)e^{mx}$$

Case 3: Complex conjugate roots

$$y = Ae^{(p+iq)x} + Be^{(p-iq)x} = e^{px}(C\cos qx + D\sin qx)$$

Non-homogenous

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a\frac{\mathrm{d}y}{\mathrm{d}x} + +by = q(x)\,;\;a, b\,\text{are constants}$$

Method of undetermined coefficients

We find y_p by guessing and substitution which depends on the nature of $q(x)$.

If $q(x)$ is:

- a constant, y_p is a constant

- kx , $y_p = ax + b$
- kx^2 , $y_p = ax^2 + bx + c$
- $k \sin x$ or $k \cos x$, $y_p = a \sin x + b \cos x$
- e^{kx} , $y_p = ce^{kx}$ (Only works if k is **not** a root of auxiliary equation)

Steps

- Solve for y_c
- Based on the form of $q(x)$, make an initial guess for y_p .
- Check if any term in the guess for y_p is a solution to the complementary equation.
- If so, multiply the guess by x . Repeat this step until there are no terms in y_p that solve the complementary equation.
- Substitute y_p into the differential equation and equate like terms to find values for the unknown coefficients in y_p .
- If coefficients were unable to be found (they cancelled out or something like that), multiply the guess by x and start again.
- $y = y_p + y_c$

Wronskian

Consider the equation, where P, Q are functions of x alone, and which has 2 fundamental solutions $u(x), v(x)$:

$$y'' + Py' + Qy = 0$$

The Wronskian $w(x)$ of two solutions $u(x), v(x)$ of differential equation, is defined to be:

$$w(x) = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix}$$

Theorem 1

The Wronskian of two solutions of the above differential equation is **identically zero or never zero**.

> **Note**
>
> Identically zero means the function is always zero.

Proof

Consider the equation, where P, Q are functions of x alone.

$$y'' + Py' + Qy = 0$$

Let $u(x), v(x)$ be 2 fundamental solutions of the equation:

$$u'' + Pu' + Qu = 0 \quad \wedge \quad v'' + Pv' + Qv = 0$$

$$w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$$

$$w' = uv'' - vu'' = -P[uv' - vu'] = -Pw$$

By solving the above relation:

$$w = ce^{-\int P \,\mathrm{d}x}$$

Suppose there exists x_0 such that $w(x_0) = 0$. That implies $c = 0$. That implies w is always 0
.

Theorem 2

The solutions of the above differential equation are linearly dependent **iff** their Wronskian vanish identically.

Variation of parameters

Consider the equation, where P, Q are functions of x alone, and which has 2 fundamental solutions y_1, y_2:

$$y'' + Py' + Qy = f(x)$$

The general solution of the equation is:

$$y_g = c_1 y_1 + c_2 y_2$$

Now replace c_1, c_2 with $u(x), v(x)$ and we get $y_p = uy_1 + vy_2$ which can be found using the method of variation of parameters.

$$u = -\int \frac{y_2 f}{W(x)} \, \mathrm{d}x \ \land \ v = \int \frac{y_1 f}{W(x)} \, \mathrm{d}x$$

Proof

$$y_p = uy_1 + vy_2$$

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$$

Set $u'y_1 + v'y_2 = 0 \ \ (1)$ to simplify further equations. That implies $y_p' = uy_1' + vy_2'$.

$$y_p'' = uy_1'' + u'y_1' + vy_2'' + v'y_2$$

Substituting y_p'', y_p', y_p to the differential equation:

$$y_p'' + Py_p' + Qy_p = u'y_1' + v'y_2'$$

This implies $u'y_1' + v'y_2' = f(x) \ \ (2)$.

From equations (1) and (2), where $W(x)$ is the wronskian of y_1, y_2:

$$u' = -\frac{y_2 f}{W(x)} \ \land \ v' = \frac{y_1 f}{W(x)}$$

$$u = -\int \frac{y_2 f}{W(x)}\,\mathrm{d}x \;\wedge\; v = \int \frac{y_1 f}{W(x)}\,\mathrm{d}x$$

y_p can be found now using u, v, y_1, y_2