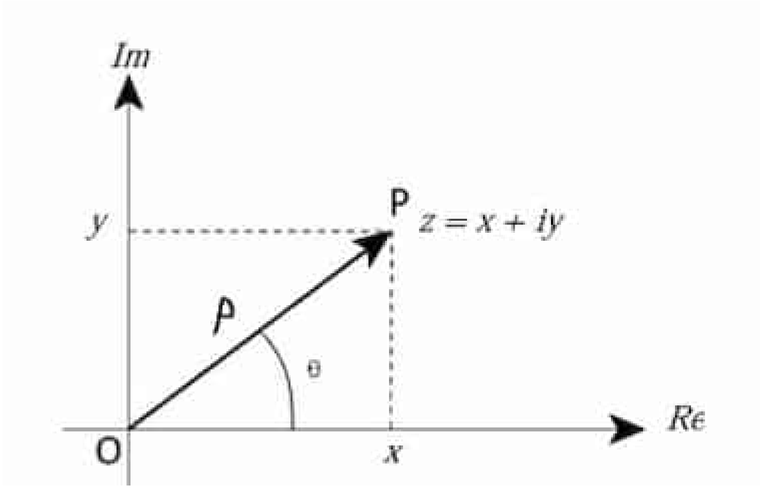
Summary | Complex Numbers

Introduction

Representation methods



The methods are:

- Cartesian representation: $z = x + iy$
- Polar representation: $z = pe^{i\theta}$

Here:

- $x = p\cos\theta$ - real part
- $y = p\sin\theta$ - imaginary part
- $p = \sqrt{x^2 + y^2}$ - modulus
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ - arg angle

Euler's Formula

For $x \in \mathbb{R}$:

$$e^{ix} = \cos x + i\sin x$$

Proof Hint

Use power series for $e^x, \cos x, \sin x$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

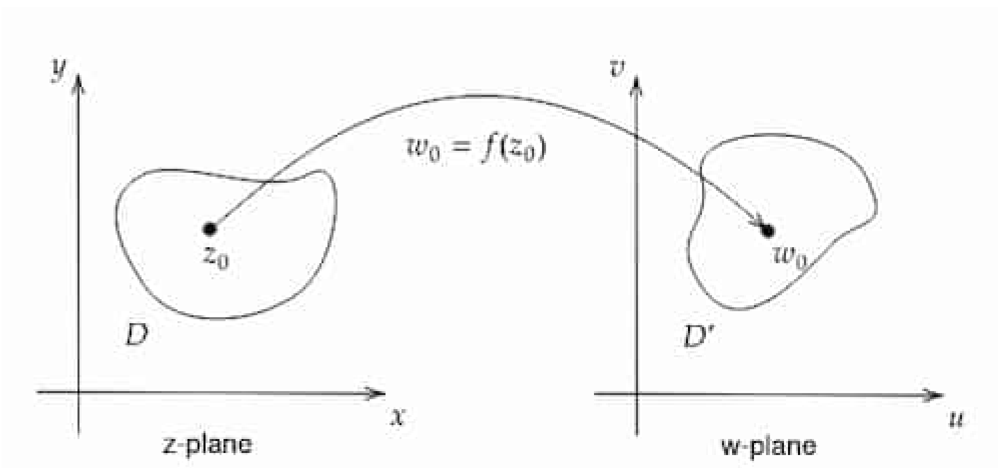
Euler’s Identity

One of the most beautiful equations in mathematics.

$$e^{i\pi} + 1 = 0$$

Complex Functions

Suppose $w = f(z)$ where $z, w \in \mathbb{C}$. Input and output points are denoted in 2 separate complex planes.



Here:

- D - domain of f
- D' - codomain of f

Image

Image of f is the set:

$$\{f(z) \mid z \in D\}$$

Cartesian form

$$f(z) = u(x, y) + iv(x, y)$$

Here

$$u, v$$

are real functions.

Limit of Complex Functions

$\lim_{z \to z_0} f(z) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall z \; (0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon)$$

Complex limit properties are similar to real limits.

Difference from real functions

For real functions, when considering the limit at a point, we could approach the point either from the left or from the right.

For complex functions, the point can be approached along any path in the complex plane. The distance $|z - z_0|$ decreases to 0.

Real and imaginary limits

Suppose $f(z) = u(x, y) + iv(x, y)$, $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = L_1$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = L_2$, where $z_0 = x_0 + iy_0$ and $z = x + iy$. Then $\lim_{z\to z_0} f(z) = L_1 + iL_2$.

Continuity

$f(z)$ is continuous at z_0 **iff**:

$$\lim_{z\to z_0} f(z) = f(z_0)$$

$$\iff \forall\epsilon > 0 \ \exists\delta > 0 \ \forall x \ (|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon)$$