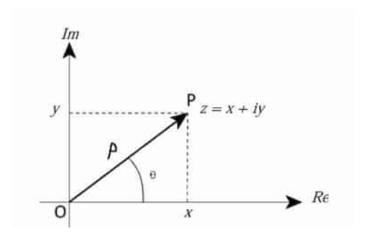
Summary | Complex Numbers

Introduction

Representation methods



The methods are:

ullet Cartesian representation: z=x+iy

• Polar representation: $z=pe^{i\theta}$

Here:

 $ullet \ x = p\cos heta$ - real part

 $ullet \ y = p\sin heta$ - imaginary part

 $ullet p = \sqrt{x^2 + y^2}$ - modulus

 $oldsymbol{ heta} = an^{-1}\left(rac{y}{x}
ight)$ - arg angle

Euler's Formula

For $x \in \mathbb{R}$:

$$e^{ix} = \cos x + i \sin x$$

(i) Proof Hint

Use power series for e^x , $\cos x$, $\sin x$.

$$e^x = \sum_{n=0}^{\infty} rac{x^n}{n!} = 1 + rac{x}{1!} + rac{x^2}{2!} + rac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n rac{x^{2n+1}}{(2n+1)!} = x - rac{x^3}{3!} + rac{x^5}{5!} - rac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n rac{x^{2n}}{(2n)!} = 1 - rac{x^2}{2!} + rac{x^4}{4!} - rac{x^6}{6!} + \cdots$$

Euler's Identity

One of the most beautiful equations in mathematics.

$$e^{i\pi}+1=0$$

Roots of Unity

n-th roots of unity (1) are the complex numbers that satisfy the equation, $z^n=1$. There are n distinct solutions.

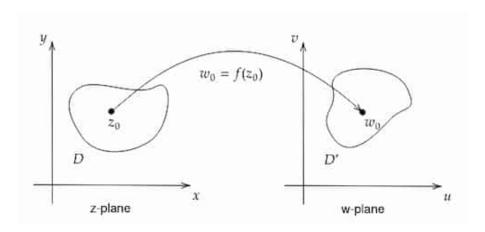
$$z = \exp\left(i\Big(rac{2m\pi}{n}\Big)
ight) \ ext{ where } \ m \in \mathbb{Z} \cup [0,n)$$

The solution can be written as $1, w, w^2, w^3, \dots, w^{n-1}$.

 ${f 1}$ is called the trivial solution. Other solutions are primitive ${\it n}$ -th roots.

Complex Functions

Suppose w=f(z) where $z,w\in\mathbb{C}$. Input and output points are denoted in 2 separate complex planes.



Here:

- ullet D domain of f
- ullet D' codomain of f

Image

Image of \boldsymbol{f} is the set:

$$\big\{f(z)\mid z\in D\big\}$$

Cartesian form

$$f(z)=u(x,y)+iv(x,y)$$

Here

u, v

are real functions.

Limit of Complex Functions

$$\lim_{z o z_0}f(z)=L$$
 iff:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall z \; (0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon)$$

Complex limit properties are similar to real limits.

Difference from real functions

For real functions, when considering the limit at a point, we could approach the point either from the left or from the right.

For complex functions, the point can be approached along any path in the complex plane. The distance $|z-z_0|$ decreases to 0.

Real and imaginary limits

Suppose
$$f(z)=u(x,y)+iv(x,y)$$
, $\lim_{(x,y) o(x_0,y_0)}u(x,y)=L_1$ and $\lim_{(x,y) o(x_0,y_0)}v(x,y)=L_2$, where $z_0=x_0+iy_0$ and $z=x+iy$. Then $\lim_{z o z_0}f(z)=L_1+iL_2$.

Continuity

f(z) is continuous at z_0 iff:

$$\lim_{z o z_0}f(z)=f(z_0)$$

$$\iff orall \epsilon > 0 \; \exists \delta > 0 \; orall x \; (|z-z_0| < \delta \; \Longrightarrow \; |f(z)-f(z_0)| < \epsilon)$$

Differentiability

A complex function f is differentiable at z_0 iff:

$$\lim_{z o z_0}rac{f(z)-f(z_0)}{z-z_0}=L=f'(z_0)$$

 $f'(z_0)$ is called the derivative of f at z_0 . The rules for differentiation in real functions can also be applied to complex functions. So, go through <u>Differentiability</u> — <u>Real Analysis</u>.

Singular point

If f(z) is not differentiable at z_0 then z_0 is called a singular point of f(z).

Neighbourhood

Suppose $z_0 \in \mathbb{C}$. A neighborhood of z_0 is the region contained in the circle $|z-z_0|=r>0$.

Analytic

A function f is said to be analytic at z_0 iff it is differentiable throughout a neighbourhood of z_0 .

Analytic implies differentiable

$$f$$
 is analytic at $z_0 \implies f$ is differentiable at z_0

Cauchy Riemann Equations

The set of equations mentioned below are the Cauchy Riemann Equations, where u,v are functions of x,y.

$$rac{\partial u}{\partial x}=u_x=rac{\partial v}{\partial y}=v_y \quad \wedge \quad rac{\partial u}{\partial y}=u_y=-rac{\partial v}{\partial x}=-v_x$$

Theorem 1

Suppose f(z)=u(x,y)+iv(x,y), and f is differentiable at $z_0.$ Then

- All partial derivatives u_x, u_y, v_x, v_y exist
- They satisfy the Cauchy Riemann equations

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

(i) Note

Contrapositive is useful when proving f is **not** differentiable at z_0 .

Theorem 2

Suppose f(z)=u(x,y)+iv(x,y). All partial derivatives exist, and they are all continuous at z_0 . Then f is differentiable at z_0 . And:

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$$

Theorem 3

If f is analytic at z_0 , then its first-order partial derivatives are continuous in a neighbourhood of z_0 .

Entire Functions

A complex function that is differentiable everywhere. Entire functions are analytic everywhere.

Examples:

- polynomial functions
- e^z

Counter examples:

• Rational functions are not entire functions

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