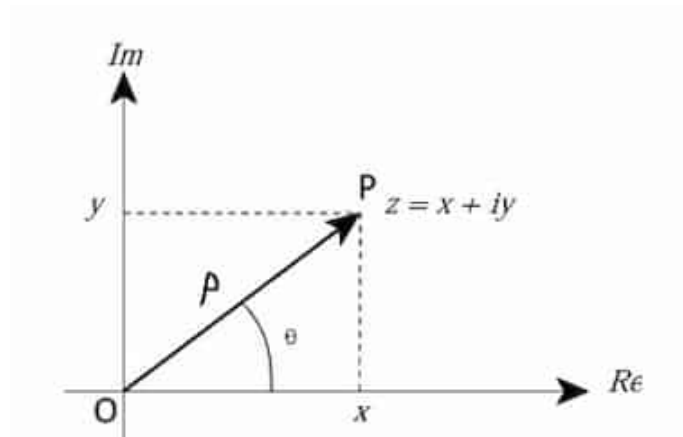# Summary | Complex Numbers

Introduction

Representation methods

The methods are:

- Cartesian representation: $z = x + iy$
- Polar representation: $z = pe^{i\theta}$

Here:

- $x = p \cos \theta$ - real part
- $y = p \sin \theta$ - imaginary part
- $p = \sqrt{x^2 + y^2}$ - modulus
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ - arg angle

Euler's Formula

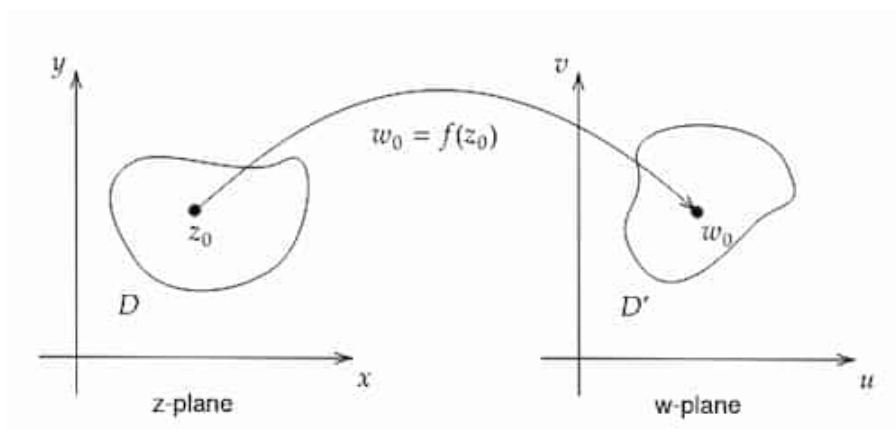For $x \in \mathbb{R}$:

$$e^{ix} = \cos x + i \sin x$$

Euler's Identity

One of the most beautiful equations in mathematics.

$$e^{i\pi} + 1 = 0$$

Complex Functions

Suppose $w = f(z)$ where $z, w \in \mathbb{C}$. Input and output points are denoted in 2 separate complex planes.

Here:

- D - domain of f
- D' - codomain of f

Image

Image of f is the set:

$$\{f(z) \mid z \in D\}$$

Cartesian form

$$f(z) = u(x, y) + iv(x, y)$$

Here

$$u, v$$

are real functions.

Limit of Complex Functions

$\lim\limits_{z \to z_0} f(z) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall z \; (0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon)$$

Complex limit properties are similar to real limits.

Difference from real functions

For real functions, when considering the limit at a point, we could approach the point either from the left or from the right.

For complex functions, the point can be approached along any path in the complex plane. The distance $|z - z_0|$ decreases to 0.

Real and imaginary limits

Suppose $f(z) = u(x, y) + iv(x, y)$, $\lim\limits_{(x,y)\to(x_0,y_0)} u(x, y) = L_1$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x, y) = L_2$, where $z_0 = x_0 + iy_0$ and $z = x + iy$. Then $\lim\limits_{z\to z_0} f(z) = L_1 + iL_2$.

Continuity

$f(z)$ is continuous at z_0 **iff**:

$$\lim_{z\to z_0} f(z) = f(z_0)$$

$$\iff \forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon)$$

Differentiability

A complex function f is differentiable at z_0 **iff**:

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L = f'(z_0)$$

$f'(z_0)$ is called the derivative of f at z_0. The rules for differentiation in real functions can also be applied to complex functions. So, go through [Differentiability — Real Analysis](#).

Singular point

If $f(z)$ is not differentiable at z_0 then z_0 is called a singular point of $f(z)$.

Neighbourhood

Suppose $z_0 \in \mathbb{C}$. A neighborhood of z_0 is the region contained in the circle $|z - z_0| = r > 0$.

Analytic

A function f is said to be analytic at z_0 **iff** it is differentiable throughout a neighbourhood of z_0.

Analytic implies differentiable

$$f \text{ is analytic at } z_0 \implies f \text{ is differentiable at } z_0$$