

# Summary | Matrices

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## Introduction

Revise Matrices unit from G.C.E. (A/L) Combined Mathematics and G.C.E. (O/L) Mathematics.

## Types of matrices

### Square matrix

Number of columns equal to number of rows.

#### Main diagonals of a square matrix

Formed by elements having equal subscripts.

### Diagonal matrix

A square matrix whose only non-zero elements are main-diagonal elements. Denoted by  $D$ . Subset of triangular matrices.

### Identity matrix or Unit matrix

A diagonal matrix whose diagonal elements are all equal to 1. Denoted by  $I$ . Subset of diagonal matrices.

### Zero matrix / Null matrix

All elements are 0.

### Column matrix (column vector)

Only 1 column.

### Row matrix (row vector)

Only 1 row.

## Triangular matrix

Upper triangular matrix or lower triangular matrix.

### Upper triangular matrix

All elements below the main diagonal are 0. Subset of square matrices.

### Lower triangular matrix

All elements above the main diagonal are 0. Subset of square matrices.

## Matrix operations

### Addition and subtraction

Order of the 2 matrices must be same. Matrix obtained by adding or subtracting corresponding elements.

### Scalar multiplication

Matrix obtained by multiplying all elements by the scalar.

### Matrix multiplication

Explained on a separate page: [Matrix multiplication](#)

#### Note

Other operations are also defined in separate pages.

## Matrix Multiplication

Defined only when the number of columns of the first matrix is equal to the number of rows of the second matrix.

Suppose  $A = (a_{ij})_{m \times p}$  and  $B = (b_{ij})_{p \times n}$ .

$$A \times B = C = (c_{ij})_{m \times n} \text{ where } c_{ij} = \sum_{k=1}^p a_{ik} \times b_{kj}$$

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### Note

- Generally  $A \times B \neq B \times A$
- $A \times B = 0 \not\Rightarrow A = 0 \vee B = 0$
- $A \neq 0 \wedge B \neq 0 \not\Rightarrow A \times B \neq 0$

## Properties of matrix multiplication

$A, B, C, I$  (Identity) matrices must be chosen so that below-mentioned product matrices are defined.

1. Associative:  $A(BC) = (AB)C$
2. Right distributive over addition:  $(A + B)C = AC + BC$
3. Left distributive over addition:  $C(A + B) = CA + CB$
4.  $AI = IA = A$

## Transpose

Matrix obtained from a given matrix by interchanging its rows and columns. Denoted by a superscript T, like  $A^T$ .

## Properties

1.  $(A^T)^T = A$
2. Distributive over addition:  $(A + B)^T = A^T + B^T$
3.  $(kA)^T = kA^T$
4.  $(A \times B)^T = B^T \times A^T$

## More types of matrices

### Symmetric matrix

If  $A = A^T$ . Subset of square matrices.

## Skew-symmetric matrix

If  $A = -A^T$ . Subset of square matrices. All elements in main diagonal are 0.

### Note

Any square matrix can be expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

## Determinant

Defined only for square matrices. Denoted by  $|A|$ .

### For 2x2

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

### For higher order

#### Minor of an element

Suppose  $A = (a_{ij})$ .

Minor of an element  $a_{ij}$ , is the matrix obtained by deleting  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $A$ . Denoted by  $M_{ij}$ .

#### Co-factor of an element

Suppose  $A = (a_{ij})$ .

Co-factor of an element  $a_{ij}$ , is defined as (commonly denoted as  $A_{ij}$ ):

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

### Definition

If  $A = (a_{ij})_{n \times n}$  then the **determinant** of  $A$  is denoted by  $|A|$  and is defined by:

$$|A| = \sum_{j=1}^n a_{ij} A_{ij}$$

where  $1 \leq j \leq n$ .

### Properties of determinants

- $|A^T| = |A|$
- Every element of a row or column of a matrix is 0 then the value of its determinant is 0.
- If 2 columns or 2 rows of a matrix are identical then its determinant is 0.
- If A and B are two square matrices then  $|AB| = |A||B|$ .
- The value of the determinant of a matrix remains unchanged if a scalar multiple of a row or column is added to any other row or column.
- If a matrix  $B$  is obtained from a square matrix  $A$  by an interchange of two columns or rows:  
 $|B| = -|A|$ .
- If every entry in any row or column is multiplied by  $k$ , then the whole determinant is multiplied by  $k$ .

### In relation with eigenvalues

For a  $n \times n$  matrix A with  $n$  number of [eigenvalues](#):

$$|A| = \prod_{i=1}^n \lambda_i$$

## Adjoint

Suppose  $A = (a_{ij})_{n \times n}$ .

$$\text{adj}A = (A_{ij})_{n \times n}^T$$

Where  $A_{ij}$  is the [co-factor of](#)  $a_{ij}$ .

## Properties

Suppose  $A$  is a  $n \times n$  matrix.

- $\text{adj}(I) = I$
- $\text{adj}(cA) = c^{n-1}\text{adj}(A)$
- $\text{adj}(A^T) = (\text{adj}(A))^T$
- $\text{adj}(A) A = A \text{adj}(A) = |A|I$

## Inverse

Suppose  $A$  and  $B$  are square matrices of the same order. If  $AB = BA = I$  then  $B$  is called the inverse of  $A$  and is denoted by  $A^{-1}$ .

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

## Singular or Non-singular

A square matrix is singular **iff**  $|A| = 0$ . Otherwise its non-singular or invertible.

## Properties of Inverse

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $A(\text{adj}A) = (\text{adj}A)A = |A|I$

## Elementary Transformations

- Interchange of any columns or rows
- Addition of multiple of any row or column to any other row or column
- Multiplication of each element of a column or a row by a non-zero constant

When a matrix  $B$  is obtained by applying elementary transformations to a matrix  $A$ , then  $A$  **is equivalent to**  $B$ . Denoted by  $A \approx B$ .

## Theorem

The elementary row operations that reduce a given matrix  $A$  to the identity matrix, also transform the identity matrix to the inverse of  $A$ .

## Augmented Matrix

Two matrices are written as a single matrix with a vertical line in-between. Denoted by  $(A|B)$ .

Example:

$$\left[ \begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right]$$

## Inverse using elementary row transformations

Let  $A$  be a square matrix with order  $n \times n$ .

- Start with  $(A_{n \times n} | I_n)$
- Repeatedly add **row** transformations (not column) to both of the matrices until the  $LHS$  becomes an identity matrix.
  - Convert all elements outside the main diagonal to  $0$ .
  - Convert elements on the main diagonal to  $1$  by multiplying by a constant.
- When  $LHS$  is an identity matrix,  $RHS$  is  $A^{-1}$ .

⚠ TODO

What about singular matrices?

## Echelon Form

A matrix is in row echelon form (or just “row echelon” form) **iff**:

- All rows having only zero entries are at the bottom.
- For all row that does not contain entirely zeros, the first non-zero entry is 1.
- For 2 successive non-zero rows, the leading 1 in the higher row is further left than the leading 1 in the lower row.

The process of reducing the augmented matrix to row Echelon form is known as **Gaussian elimination**.

## Column echelon form

A matrix  $A$  is in column echelon form if  $A^T$  is in row echelon form.

## System of Linear Equations

Any system of linear equations can be represented in matrix notation as shown below.

- $a_{11}x + a_{12}y + a_{13}z = b_1$
- $a_{21}x + a_{22}y + a_{23}z = b_2$
- $a_{31}x + a_{32}y + a_{33}z = b_3$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \implies AX = B$$

2 types based on  $B$ :

- $= 0$  : [Homogeneous system](#)
- $\neq 0$  : [Non-homogeneous system](#)

## Consistent

When the system of equations has at least 1 solution. Otherwise inconsistent.



# Rank

Number of non-zero rows of row echelon form of a matrix  $A$ . Denoted by  $\text{Rank } A$ .

## Note

$\text{Rank } A \leq \text{Rank } (A|B)$  is always true.

## Relation with non-homogenous system of solutions

Consider the system:  $A_{n \times n} X_{n \times 1} = B_{n \times 1}$ .

- $|A| \neq 0 \iff \text{Rank } A = \text{Rank } (A|B) = n \iff$  unique solution exists
- $|A| = 0 \implies$  no solution  $\vee$  infinitely many solutions
- $\text{Rank } A < \text{Rank } (A|B) \implies$  no solutions
- $\text{Rank } A = \text{Rank } (A|B) < n \implies$  infinitely many solutions

## Solutions of Homogenous Systems

Consider the system:

$$A_{m \times n} X_{n \times 1} = O_{m \times 1}$$

Any homogenous system is consistent, because  $X = O$  is always a solution.

- $\text{Rank } A = \text{Rank } (A|B) = n \iff$  unique solution exists
- $\text{Rank } A = \text{Rank } (A|B) < n \implies$  infinitely many solutions

## Solution of Non-homogenous Systems

### Method 1: Direct approach

Used when coefficient matrix  $A$  is invertible. It means the system has a unique set of solutions.

$$AX = B \implies X = A^{-1}B$$

## Method 2: Cramer's Rule

Let  $AX = B$ , where  $A$  is the coefficient matrix and  $X = (x_i)_{n \times 1}$ .

$$x_i = \frac{|A_i|}{|A|}$$

Where  $A_i$  is the matrix obtained by replacing  $i$ th column in matrix  $A$  by  $B$ .

## Method 3: Reducing to Echelon Form

Start with  $(A|B)$ . Convert the LHS to echelon form using elementary row transformations. The solution can be found now. If a contradiction is encountered while solving the equation, that means the system has no solutions.

# Eigenvalues & Eigenvectors

## Definitions

### Characteristic Polynomial

Let  $A$  be a  $n \times n$  matrix.

$$p(\lambda) = |A - \lambda I|$$

## Eigenvalues

Roots of the equation  $p(\lambda) = 0$  are the eigenvalues of  $A$ .

### Note

- [Determinant of a matrix](#) can be written in terms of all of its eigenvalues.
- If  $\lambda$  is an eigenvalue of  $A$ , then  $\lambda^2$  is an eigenvalue of  $A^2$

## Eigenvectors

The column vectors satisfying the equation  $(A - \lambda_i I)X_i = 0$ .

## Normalized eigenvectors

An eigenvector with the magnitude (norm) of 1. Normalizing factor  $k$  of any eigenvector is:

$$\frac{1}{k} = \sqrt{\sum_{i=1}^n X_i^2}$$

## Norm

Norm of a column or row matrix  $W_{n \times n}$  is denoted by  $||W||$  and defined as:

$$||W|| = \sqrt{\sum_{i=1}^n w_i^2}$$

## Algebraic Multiplicity

If the characteristic polynomial consists of a factor of the form  $(\lambda - \lambda_i)^r$  and  $(\lambda - \lambda_i)^{r+1}$  is not a factor of the characteristic polynomial then  $r$  is the algebraic multiplicity of the eigenvalue  $\lambda$ .

## Spectrum

Set of all eigenvalues.

## Spectral Radius

$$R = \max \left\{ |\lambda_i| \text{ where } \lambda_i \in \text{Spectrum} \right\}$$

## Linear Independence of Eigenvectors

Suppose  $X_1, X_2, X_3, \dots, X_n$  is a set of eigenvectors.  $k_1, k_2, k_3, \dots, k_n$  is a set of scalars.

All those eigenvectors are independent **iff**:

$$k_1 X_1 + k_2 X_2 + k_3 X_3 + \dots + k_n X_n = 0 \implies k_1 = k_2 = k_3 = \dots = k_n = 0$$

## For special matrices

### Real symmetric matrix

Suppose  $A$  is a symmetric matrix with all real entries. Then:

- The eigenvalues of  $A$  are all real:  $\forall \lambda \in S_A, (\lambda_i \in \mathbb{R})$
- The eigenvectors of  $A$  (corresponding to distinct values of  $\lambda$ ) are mutually orthogonal

### Upper triangular matrix

The eigenvalues are the diagonal entries.

## Orthogonal & Orthonormal Vectors

Consider 2 column matrices  $v_1$  and  $v_2$ :

$$v_1 = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \wedge v_2 = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

### Product

The product of  $v_1$  and  $v_2$  is defined as:

$$v_1 \cdot v_2 = \sum_{k=1}^n a_k b_k = v_2 \cdot v_1 = v_1^T v_2$$

## Orthogonal

### Orthogonal matrix

For a square matrix  $A$  with real entries, it is orthogonal **iff**  $A^{-1} = A^T$

### Orthogonal vectors

$v_1$  and  $v_2$  are orthogonal **iff**  $v_1 \cdot v_2 = 0$ .

For a set of  $n$  column vectors, they are orthogonal **iff** they are pairwise orthogonal. That is:

$$\forall i, j \in \{1, \dots, n\} \wedge i \neq j, (v_i \cdot v_j = 0)$$

### **Note**

$v_1, v_2$  are orthogonal  $\implies v_1, v_2$  are linearly independent.

Converse is **not** true.

## Orthonormal

For a set of  $n$  column vectors, they are orthonormal **iff**:

- They are pairwise orthogonal **AND**
  - For all  $n$  column vectors their norm is 1
- $$\forall i \in \{1, \dots, n\}, \|v_i\| = 1$$

## Properties of orthogonal matrices

- Product of 2 orthogonal matrices of the same order is also an orthogonal matrix
- The columns or rows of an orthogonal matrix form an orthogonal set of vectors

## Trace

Suppose  $A = (a_{ij})_{n \times n}$  is a square matrix. Trace of  $A$  is the sum of the diagonal entries.

$$\text{trace}(A) = \text{Tr}(A) = \sum_{i=1}^n a_{ii}$$

Trace is also equal to the sum of eigenvalues.

$$\text{trace}(A) = \sum \lambda_i \text{ where } \lambda_i \in \text{spectrum of } A$$

# Diagonalization

## Similar matrices

2 square matrices  $A$  and  $B$  of the same order, are similar **iff** there exists an invertible matrix  $P$  such that:

$$B = P^{-1}AP$$

Similarity of 2 matrices is commutative.

Similar matrices have the set of eigenvalues.

### Note

If  $A$  and  $B$  are similar, then  $A^2$  and  $B^2$  are similar.

## Definition

A matrix  $A$  is **diagonalizable** if it is similar to a [diagonal matrix](#).

$$\exists D, P \text{ s.t. } D = P^{-1}AP$$

Here:

- $D$  is a diagonal matrix
- $P$  is an invertible matrix

## Steps

- Find eigenvalues of  $A_{n \times n}$ :  $\lambda_1, \lambda_2, \dots, \lambda_n$
- Find corresponding eigenvectors:  $X_1, X_2, \dots, X_n$
- Construct  $P$  by joining the eigenvectors as columns

$$P = (X_1 X_2 \dots X_n)_{n \times n} \wedge D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

### ⓘ Note

Order of those eigenvectors is **not** a problem. Here the matrix  $P$  differs based on the order, and hence is not unique.

### ⓘ Real symmetric matrix

Suppose  $A_{n \times n}$  is a **real symmetric matrix**. If it has **distinct** eigenvalues then it has  $n$  **mutually orthogonal linearly-independent** eigenvectors.

Hence the diagonalizing matrix  $P$  (formed by using the normalized eigenvectors) is an **orthogonal matrix**.

## Uses

### Finding integer powers

Suppose  $A_{n \times n}$  is diagonalizable, and  $k \in \mathbb{R}$ .

$$A = P^{-1}DP \implies A^k = P^{-1}D^kP$$

## Cayley-Hamilton Theorem

If  $p(\lambda)$  is the characteristic polynomial of the matrix  $A_{n \times n}$ , then  $p(A) = O$

## Uses

- Easily compute the inverse of a matrix
- Easily express higher powers of a matrix in terms of its lower powers

# Matrix Norms

Let  $A_{n \times n}$ . A norm of  $A$  is denoted by  $\|A\|$ .

## Definitions

Suppose  $A = (a_{ij})_{m \times n}$  for all the definitions below.

### 1-norm

Maximum of the absolute column sums.

$$\|A\|_1 = \max \left\{ \sum_{i=1}^m |a_{ij}| ; j \in [1, n] \right\}$$

### 2-norm

Square root of the sum of all elements squared. Also called as Euclidean norm.

$$\left(\|A\|_2\right)^2 = \left(\|A\|_E\right)^2 = \sum_{i=1}^m \sum_{j=1}^n (a_{ij})^2$$

### Infinity norm

Maximum of the row sums.

$$\|A\|_\infty = \max \left\{ \sum_{j=1}^n |a_{ij}| ; i \in [1, m] \right\}$$

#### Note

For any matrix  $X \in \mathbb{R}^n$ :

$$\|X\|_\infty \leq \|X\|_2 \leq \|X\|_1$$



## Vector norm

Norm defined for column vectors.

## Properties of Norms

Works for all types of norms.

Suppose  $A, B$  are  $m \times n$  ordered.

1.  $\|A\| \geq 0$
2.  $\|A\| = 0 \iff A = 0$
3.  $\|kA\| = |k| \times \|A\|$
4.  $\|A + B\| \leq \|A\| + \|B\|$  (triangle inequality)
5.  $\|AB\| \leq \|A\| \times \|B\|$

## Unit Ball

A unit ball in  $\mathbb{R}^n$  with respect to a norm  $\|\cdot\|$ .

$$\{X \mid X \in \mathbb{R}^n, \|X\| \leq 1\}$$

## Unit disc

When  $n = 2$ , unit ball is also called the unit disc.