Summary | Real Analysis

Introduction

— | | $\wedge$ | and | | $\vee$ | or | | $\rightarrow$ | then | | $\implies$ | implies | | $\Leftarrow$ | implied by | | $\iff$ | if and only if | | $\forall$ | for all | | $\exists$ | there exists | | $\sim$ | not |

Let's take $a \rightarrow b$.

1. Contrapositive or transposition: $\sim b \rightarrow \sim a$. This is equivalent to the original.
2. Inverse: $\sim a \rightarrow \sim b$. Does not depend on the original.
3. Converse: $b \rightarrow a$. Does not depend on the original.

$$a \rightarrow b \equiv \sim a \vee b \equiv \sim b \rightarrow \sim a$$

Required proofs

- $\sim \forall x\, P(x) \equiv \exists x \sim P(x)$
- $\sim \exists x\, P(x) \equiv \forall x \sim P(x)$
- $\exists x\, \exists y P(x, y) \equiv \exists y\, \exists x P(x, y)$
- $\forall x\, \forall y P(x, y) \equiv \forall y \forall x P(x, y)$
- $\exists x\, \forall y P(x, y) \implies \forall y\, \exists x P(x, y)$
- $(A \rightarrow C) \wedge (B \rightarrow C) \equiv (A \vee B) \rightarrow C$

Methods of proofs

1. Just proof what should be proven

2. Prove the contrapositive
3. Proof by contradiction
4. Proof by induction

Proof by contradiction

Suppose $a \implies b$ has to be proven. If $a \land \sim b$ is proven to be false, then, by proof by contradiction, $a \implies b$ can be trivially proven.

Logic behind proof by contradiction

$$a \land \sim b = F$$

$$\sim (a \land \sim b) = \sim F$$

$$\sim a \lor b = T$$

$$a \to b = T$$

$$a \implies b$$

Set theory

Zermelo-Fraenkel set theory with axiom of choice (ZFC) — [9 axioms all together] — is being used in this module.

Definitions

- $x \in A^{\mathrm{c}} \iff x \notin A$
- $x \in A \cup B \iff x \in A \vee x \in B$
- $x \in A \cap B \iff x \in A \wedge x \in B$
- $A \subset B = \forall x(x \in A \implies x \in B)$
- $A - B = A \cap B^{\mathrm{c}}$

Required proofs

- $(A \cap B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B^{\mathrm{c}}$
- $(A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}}$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \subset A \cup B$
- $A \cap B \subset A$

The axioms

These are mentioned here for reference and they are not exact and formal definitions. A formal definition can be found on [ZFC set theory - Wikipedia]().

Axiom of extensionality

Two sets are equal (are the same set) if they have the same elements.

$$A = B \iff ((\forall z \in A \implies z \in B) \wedge (\forall z \in B \implies z \in A))$$

Axiom of regularity

A set cannot be an element of itself.

Axiom of specification

Subsets that are constructed using set builder notation, always exists.

Axiom of pairing

If x and y are sets, then there exists a set which contains both x and y as elements.

$$\forall x \forall y \exists z((x \in z) \land (y \in z))$$

Axiom of union

The union of the elements of a set exists.

Axiom schema of replacement

The image of a set under a definable function will also be a set.

Axiom of infinity

There exists a set having infinitely many elements.

Axiom of power set

For any set x, there exists a set y that contains every subset of x:

$$\forall x \exists y \forall z(z \subset x \implies z \in y)$$

Axiom of well-ordering (choice)

I don't understand this axiom. If you do, let me know.

Set of Numbers

Sets of numbers

- Positive integers: $\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$.
- Natural integers: $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$.
- Negative integers: $\mathbb{Z}^- = \{-1, -2, -3, -4, \dots\}$.
- Integers: $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$.
- Rational numbers: $\mathbb{Q} = \left\{ \frac{p}{q} \middle| q \neq 0 \wedge p, q \in \mathbb{Z} \right\}$.
- Irrational numbers: limits of sequences of rational numbers (which are not rational numbers)
- Real numbers: $\mathbb{R} = \mathbb{Q}^c \cup \mathbb{Q}$.

Complex numbers are taught in a separate set of lectures, and not included under real analysis lectures.

Axiomatic definiton of real numbers

Set of real numbers is a set satisfying all these axioms:

- Field axioms
- Order axioms
- Completeness axiom

Archimedean property

$$\forall y \in \mathbb{R}^{+}\ \exists k \in \mathbb{Z}^{+} \text{ s.t. } \frac{1}{k} < y$$

Continued Fraction Expansion

The process

- Separate the integer part
- Find the inverse of the remaining part. Result will be greated than 1.
- Repeat the process for the remaining part.

Finite expansion

Take $\frac{420}{69}$ for example.

$$\frac{420}{69} = 6 + \frac{6}{69}$$

$$\frac{420}{69} = 6 + \frac{1}{\frac{69}{6}}$$

$$\frac{420}{69} = 6 + \cfrac{1}{11 + \cfrac{3}{6}}$$

$$\frac{420}{69} = 6 + \cfrac{1}{11 + \cfrac{1}{2}}$$

As $\frac{420}{69}$ is finite, its continued fraction expansion is also finite. And it can be written as $\frac{420}{69} = [6; 11, 2]$.

Infinite expansion

For irrational numbers, the expansion will be infinite.

For example π:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}$$

Conintued fraction expansion of π is $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots]$.

Field Axioms

Field Axioms of $\mathbb{R}$

$\mathbb{R} \neq \emptyset$ with two binary operations $+$ and $\cdot$ satisfying the following properties

1. Closed under addition: $\forall a, b \in \mathbb{R}; a + b \in \mathbb{R}$
2. Commutative: $\forall a, b \in \mathbb{R}; a + b = b + a$
3. Associative: $\forall a, b, c \in \mathbb{R}; (a + b) + c = a + (b + c)$
4. Additive identity: $\exists 0 \in \mathbb{R}\, \forall a \in \mathbb{R}; a + 0 = 0 + a = a$
5. Additive inverse: $\forall a \in \mathbb{R}\, \exists(-a); a + (-a) = (-a) + a = 0$
6. Closed under multiplication: $\forall a, b \in \mathbb{R}; a \cdot b \in \mathbb{R}$
7. Commutative: $\forall a, b \in \mathbb{R}; a \cdot b = b \cdot a$

8. Associative: $\forall a, b, c \in \mathbb{R}; (a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. Multiplicative identity: $\exists 1 \in \mathbb{R} \, \forall a \in \mathbb{R}; a \cdot 1 = 1 \cdot a = a$
10. Multiplicative inverse: $\forall a \in \mathbb{R} - \{0\} \, \exists a^{-}; a \cdot a^{-} = a^{-} \cdot a = 1$
11. Multiplication is distributive over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$

Required proofs

The below mentioned propositions can and should be proven using the above-mentioned axioms. $a, b, c \in \mathbb{R}$.

- $a \cdot 0 = 0$ Hint: Start with $a(1 + 0)$
- $1 \neq 0$
- Additive identity (0) is unique
- Multiplicative identity (1) is unique
- Additive inverse ($-a$) is unique for a given a
- Multiplicative inverse (a^{-1}) is unique for a given a
- $a + b = 0 \implies b = -a$
- $a + c = b + c \implies a = b$
- $-(a + b) = (-a) + (-b)$
- $-(-a) = a$
- $ac = bc \implies a = b$
- $ab = 0 \implies a = 0 \vee b = 0$
- $-(ab) = (-a)b = a(-b)$
- $(-a)(-b) = ab$
- $a \neq 0 \implies (a^{-1})^{-1} = a$
- $a, b \neq 0 \implies ab^{-1} = a^{-1}b^{-1}$

Field

Any set satisfying the above axioms with two binary operations (commonly $+$ and $\cdot$) is called a **field**. Written as:

$$(\mathbb{R}, +, \cdot) \text{ is a Field}$$

$$(\mathbb{R}, \cdot, +) \text{ is not a field}$$

Field or Not?

	Is field?	Reason (if not)
$(\mathbb{R}, +, \cdot)$	True	
$(\mathbb{R}, \cdot, +)$	False	Axiom 11 is invalid
$(\mathbb{Z}, +, \cdot)$	False	Multiplicative inverse doesn't exist
$(\mathbb{Q}, +, \cdot)$	True	
$(\mathbb{Q}^c, +, \cdot)$	False	$\sqrt{2} \cdot \sqrt{2} \notin \mathbb{Q}^c$
Boolean algebra	False	Additive inverse doesn't exist
$(\{0, 1\}, + \text{ mod } 2, \cdot \text{ mod } 2)$	True	
$(\{0, 1, 2\}, + \text{ mod } 3, \cdot \text{ mod } 3)$	True	
$(\{0, 1, 2, 3\}, + \text{ mod } 4, \cdot \text{ mod } 4)$	False	Multiplicative inverse doesn't exist

Completeness Axiom

Let A be a non empty subset of $\mathbb{R}$.

- u is the upper bound of A if: $\forall a \in A; a \leq u$
- A is bounded above if A has an upper bound
- Maximum element of A: $\max A = u$ if $u \in A$ and u is an upper bound of A
- Supremum of A $\sup A$, is the smallest upper bound of A
- Maximum is a supremum. Supremum is not necessarily a maximum.
- l is the lower bound of A if: $\forall a \in A; a \geq l$
- A is bounded below if A has a lower bound
- Minimum element of A: $\min A = l$ if $l \in A$ and l is a lower bound of A
- Infimum of A $\inf A$, is the largest lower bound of A
- Minimum is a infimum. Infimum is not necessarily a minimum.

Theorems

Let A be a non empty subset of $\mathbb{R}$.

- Say u is an upper bound of A. Then $u = \sup A$ **iff**: $\forall \epsilon > 0\ \exists a \in A;\ a + \epsilon > u$
- Say l is a lower bound of A. Then $l = \inf A$ **iff**: $\forall \epsilon > 0\ \exists a \in A;\ a - \epsilon < l$

Proof Hint

Prove the contrapositive. Use $\epsilon = \frac{1}{2}(L - sup(A))$ for supremum proof.

Required proofs

- $sup(a, b) = b$
- $inf(a, b) = a$

Completeness property

A set A is said to have the completeness property **iff** every non-empty subset of A:

- Which is bounded below has a infimum in A
- Which is bounded above has a supremum in A

Both $\mathbb{R}, \mathbb{Z}$ have the completeness property. $\mathbb{Q}$ doesn't.

In addition to that:

- Every non empty subset of $\mathbb{Z}$ which is bounded above has a maximum
- Every non empty subset of $\mathbb{Z}$ which is bounded below has a minimum

Order Axioms

- **Trichotomy**: $\forall a, b \in \mathbb{R}$ exactly one of these holds: $a > b$, $a = b$, $a < b$
- **Transitivity**: $\forall a, b, c \in \mathbb{R}; a < b \land b < c \implies a < c$
- **Operation with addition**: $\forall a, b \in \mathbb{R}; a < b \implies a + c < b + c$
- **Operation with mutliplication**: $\forall a, b, c \in \mathbb{R}; a < b \land 0 < c \implies ac < bc$

Definitions

- $a < b \equiv b > a$
- $a \leq b \equiv a < b \lor a = b$
- $a \neq b \equiv a < b \lor a > b$
- $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$

Triangular inequalities

$$|a| - |b| \leq |a + b| \leq |a| + |b|$$

$$\Big||a| - |b|\Big| \leq |a + b|$$

> **Proof Hint**
>
> For first:
>
> - Use $-|a| \leq a \leq |a|$
>
> For second:
>
> - Use the below substitutions in first conclusion
>   - $a = a - b \quad \wedge \quad b = b$
>   - $a = b - a \quad \wedge \quad b = a$

Required proofs

- $\forall a, b, c \in \mathbb{R}; a < b \wedge c < 0 \implies ac > bc$
- $1 > 0$
- $-|a| \leq a \leq |a|$
- Triangular inequalities

Theorems

- $\exists a\ \forall\epsilon > 0,\ a < \epsilon \implies a \leq 0$
- $\exists a\ \forall\epsilon > 0,\ 0 \leq a < \epsilon \implies a = 0$
- $\forall\epsilon > 0\ \exists a, a < \epsilon \not\Longrightarrow a \leq 0$

> **Caution**
>
> $\forall\epsilon > 0\ \exists a,\ a < \epsilon \implies a \leq 0$ is **not** valid.

Relations

Definitions

- Cartesian Product of sets A, B $\ A \times B = \{(a, b)|a \in A, b \in B\}$
- Ordered pair $(a, b) = \Big\{ \{a\}, \{a, b\} \Big\}$

Relation

Let $A, B \neq \emptyset$. A relation $R : A \to B$ is a non-empty subset of $A \times B$.

- $a\,R\,b \equiv (a, b) \in R$
- Domain of R: $dom(R) = A$
- Codomain of R: $codom(R) = B$
- Range of R: $ran(R) = \{y|(x, y) \in R\}$
- $ran(R) \subseteq B$
- Pre-range of R: $preran(R) = \{x\,|\,(x, y) \in R\}$
- $preran(R) \subseteq A$
- $R(a) = \{b\,|\,(a, b) \in R\}$

Everywhere defined

R is everywhere defined $\iff A = dom(R) = preran(R)$
$\iff \forall a \in A,\ \exists b \in B;\ (a, b) \in R.$

Onto

R is onto $\iff B = codom(R) = ran(R) \iff \forall b \in B\, \exists a \in A\, (a, b) \in R$

Aka. **surjection**.

Inverse

Inverse of a relation R:

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

Types of relation

one-many

$\iff \exists a \in A,\ \exists b_1, b_2 \in B\ ((a, b_1), (a, b_2) \in R \wedge b_1 \neq b_2)$

Not one-many

$\iff \forall a \in A,\ \forall b_1, b_2 \in B\ ((a, b_1), (a, b_2) \in R \implies b_1 = b_2)$

many-one

$\iff \exists a_1, a_2 \in A,\ \exists b \in B\ ((a_1, b), (a_2, b) \in R \wedge a_1 \neq a_2)$

Not many-one

$\iff \forall a_1, a_2 \in A,\ \forall b \in B\ ((a_1, b), (a_2, b) \in R \implies a_1 = a_2)$

many-many

iff R is **one-many** and **many-one**.

one-one

iff R is **not one-many** and **not many-one**. Aka. **injection**.

Bijection

When a relation is **onto** and **one-one**.

Functions

A function $f: A \to B$ is a relation $f: A \to B$ which is everywhere defined and not one-many.

- $dom(f) = A = preran(f)$

Inverse

For a function $f: A \to B$ to have its inverse relation $f^{-1}: B \to A$ be also a function, we need:

- f is onto
- f is not many-one (in other words, f must be one-one)

The above statement is true for all unrestricted function f that has an inverse f^{-1}:

$$f(f^{-1}(x)) = x = f^{-1}(f(x)) = x$$

Real-valued functions

When both domain and codomains of a function are subsets of $\mathbb{R}$, the function is said to be a real-valued function.

Composition

Composition of relations

Let $R : A \to B$ and $S : B \to C$ are 2 relations. Composition can be defined when $\operatorname{ran}(R) = \operatorname{preran}(S)$.

Say $\operatorname{ran}(R) = \operatorname{preran}(S) = D$. Composition of the 2 relations is written as:

$$S \circ R = \{(a, c) \,|\, (a, b) \in R,\ (b, c) \in S,\ b \in D\}$$

Identity relation

From the properties of the inverse relation, $R \circ R^{-1}, R^{-1} \circ R$ are both defined always. This relation is called the identity relation and denoted by I.

Composition of functions

Let $f : A \to B$ and $g : B \to C$ be 2 functions where f is onto.

$$g \circ f = \{(x, z) \,|\, (x, y) \in f,\ (y, z) \in g,\ y \in B\} = g(f(x))$$

The notation $g \circ f$ can be written as $g(f(x))$.

Countability

A set A is countable **iff** $\exists f : A \to Z^+$, where f is a one-one function.

Examples

- Countable: Any finite set, $\mathbb{Z}$, $\mathbb{Q}$
- Uncountable: $\mathbb{R}$, Any open/closed intervals in $\mathbb{R}$.

Transitive property

Say $B \subset A$.

$$A \text{ is countable} \implies B \text{ is countable}$$

$$B \text{ is not countable} \implies A \text{ is not countable}$$

Limits

$\lim_{x \to a} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

Defining δ in terms of a given ϵ is enough to prove a limit.

One sided limits

In x-limit

$$\lim_{x \to a} f(x) = L \iff \left(\lim_{x \to a^-} f(x) = L \land \lim_{x \to a^-} f(x) = L \right)$$

Right limit

$\lim_{x \to a^-} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (-\delta < x - a < 0 \implies |f(x) - L| < \epsilon)$$

Left limit

$\lim_{x \to a^+} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < x - a < \delta \implies |f(x) - L| < \epsilon)$$

In the answer

$$\lim_{x \to a} f(x) = L \iff \left(\lim_{x \to a} f(x) = L^+ \lor \lim_{x \to a} f(x) = L^- \right)$$

Top limit

$\lim_{x \to a} f(x) = L^+$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies 0 \leq f(x) - L < \epsilon)$$

Bottom limit

$\lim_{x \to a} f(x) = L^-$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies -\epsilon < f(x) - L \leq 0)$$

Limits including infinite

In x-limit

Positive infinity

$\lim_{x \to \infty} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \ \exists N > 0 \ \forall x \ (x > N \implies |f(x) - L| < \epsilon)$$

Negative infinity

$\lim_{x \to -\infty} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \ \exists N > 0 \ \forall x \ (x < -N \implies |f(x) - L| < \epsilon)$$

In the answer

Positive infinity

$\lim_{x \to a} f(x) = \infty$ **iff**:

$$\forall M > 0 \ \exists \delta > 0 \ \forall x \ (0 < |x - a| < \delta \implies f(x) > M)$$

Negative infinity

$\lim_{x \to a} f(x) = -\infty$ **iff**:

$$\forall M > 0 \ \exists \delta > 0 \ \forall x \ (0 < |x - a| < \delta \implies f(x) < -M)$$

Known Limits

Well-known limits

Existing limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Proof hint

Squeeze theorem with $\sin\theta\cos\theta < \theta < \tan\theta$.

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

Limits that DNE

$$\lim_{x \to \infty} \sin x$$

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

Indeterminate forms

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $\infty \cdot 0$
- $\infty - \infty$
- ∞^0
- 0^0
- 1^∞

Continuity

A function f is continuous at a **iff**:

$$\lim_{x \to a} f(x) = f(a)$$

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (|x - a| < \delta \implies |f(x) - f(a)| < \epsilon)$$

One-side continuous

Continuous from left

A function f is continuous from left at a **iff**:

$$\lim_{x \to a^-} f(x) = f(a)$$

Continuous from right

A function f is continuous from right at a **iff**:

$$\lim_{x \to a^+} f(x) = f(a)$$

On interval

Open interval

A function f is continuous in (a, b) **iff** f is continuous on every $c \in (a, b)$.

Closed interval

A function f is continuous in $[a, b]$ **iff** f is:

- continuous on every $c \in (a, b)$
- right-continuous at a
- left-continuous at b

Uniformly continuous

Suppose a function f is continuous on (a, b). f is uniformly continuous on (a, b) **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

If a function f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$.

> ⚠ **Todo**
>
> Is this section correct? I am not 100% sure.

Continuity Theorems

Extreme Value Theorem

If f is continuous on $[a, b]$, f has a maximum and a minimum in $[a, b]$.

Proof Hint

Proof is quite hard.

Intermediate Value Theorem

Let f is continuous on $[a, b]$. If $\exists u$ such that $f(a) > u > f(b)$ or $f(a) < u < f(b)$: $\exists c \in (a, b)$ such that $f(c) = u$.

Proof Hint

- Define $g(x) = f(x) - u$
- Define $A = \{x \in [a, b) \mid g(x) > 0\}$
- Show that $\sup A$ ($= c$) exists. Assume and contradict these cases:
  - $c = a$ (use $2\epsilon = g(a)$)
  - $c = b$ (use $2\epsilon = -g(b)$)
  - $c \in (a, b)$ then contradict:
    - $g(c) > 0$ (similar to $c = a$ case)
    - $g(c) < 0$ (similar to $c = b$ case)

Sandwich (or Squeeze) Theorem

Let:

- For some $\delta > 0 : \forall x(0 < |x-a| < \delta \implies f(x) \leq g(x) \leq h(x))$
- $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \in \mathbb{R}$

Then $\lim_{x\to a} g(x) = L$.

> **Note**
>
> Works for any kind of x limits.

"No sudden changes"

Positive

Let f be continuous on a and $f(a) > 0$

$$\implies \exists\delta > 0; \forall x \left(|x-a| < \delta \implies f(x) > 0\right)$$

> **Proof Hint**
>
> Take $\epsilon = \frac{f(a)}{2}$

Negative

Let f be continuous on a and $f(a) < 0$

$$\implies \exists\delta > 0; \forall x \left(|x-a| < \delta \implies f(x) < 0\right)$$

> **Proof Hint**
>
> Take $\epsilon = -\frac{f(a)}{2}$

Differentiability

A function f is differentiable at a **iff**:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'(a)$$

When it is differentiable, $f'(a)$ is called the derivative of f at a.

Critical point

$c \in [a, b]$ is called a critical point **iff**:

$$f \text{ is not differentiable at } c \quad \vee \quad f'(c) = 0$$

One-side differentiable

f is differentiable at a **iff** f is left differentiable at a **and** f is right differentiable at a.

Left differentiable

A function f is left-differentiable at a **iff**:

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_-(a)$$

Right differentiable

A function f is right-differentiable at a **iff**:

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_+(a)$$

On intervals

Open interval

A function f is differentiable in (a, b) **iff** f is differentiable on every $c \in (a, b)$.

Closed interval

A function f is differentiable in $[a, b]$ **iff** f is:

- differentiable on every $c \in (a, b)$
- right-differentiable at a
- left-differentiable at b

Continuously differentiable functions

A function f is said to be continosly differentiable at a **iff** :

- f is differentiable at a **and**
- f' is continous at a

Differentiability implies continuity

$$f \text{ is differentiable at } a \implies f \text{ is continuous at } a$$

Likewise, one-sided differentiability implies corresponding one-sided continuity.

Proof Hint

Use $\delta = min(\delta_1, \frac{\epsilon}{1+|f'(a)|})$.

Note

Suppose f is differentiable at a. Define g:

$$g(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

g is continuous at a.

Properties of differentiation

Addition

$$\frac{\mathrm{d}}{\mathrm{d}x}(f \pm g) = f' \pm g'$$

Multiplication

$$\frac{\mathrm{d}}{\mathrm{d}x}(fg) = fg' + fg'$$

Division

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{g}\right)=\frac{gf'-fg'}{g^2}$$

Composition

$$\frac{\mathrm{d}}{\mathrm{d}x}f(g(x))=f'(g(x))\,g'(x)$$

Power

$$\frac{\mathrm{d}}{\mathrm{d}x}f^n=nf^{n-1}(x)f'(x)$$

Darboux's Theorem

Let f be differentiable on $[a,b]$, $f'(a)\neq f'(b)$ and u is strictly between $f'(a)$ and $f'(b)$:

$$\exists c\in(a,b)\text{ s.t. }f'(c)=u$$

> **Proof Hint**
>
> Use $g(x)=ux-f(x)$ and follow the proof pattern of IVT proof.

Extremums

Suppose $f:[a,b]\to\mathbb{R}$, and $F=f([a,b])=\Big\{\,f(x)\mid x\in[a,b]\,\Big\}$. Both minimum and maximum values are called the extremums.

Maximum

Maximum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\forall x \in [a, b],\ f(c) \geq f(x)$$

aka. **Global Maximum**. Maximum doesn't exist always.

Local Maximum

A Local maximum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\exists \delta\ \forall x\, (0 < |x - c| < \delta \implies f(c) \geq f(x))$$

Global maximum is obviously a local maximum.

The above statement can be simplified when $c = a$ or $c = b$.

When $c = a$:

$$\exists \delta\ \forall x\, (0 < x - c < \delta \implies f(c) \geq f(x))$$

When $c = b$:

$$\exists \delta\ \forall x\, (-\delta < x - c < 0 \implies f(c) \geq f(x))$$

Minimum

Minimum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\forall x \in [a, b],\ f(c) \leq f(x)$$

aka. **Global Minimum**. Minimum doesn't exist always.

Local Minimum

$$\exists \delta\ \ \forall x\,(0 < |x - c| < \delta \implies f(c) \leq f(x))$$

Global minimum is obviously a local maximum.

The above statement can be simplified when $c = a$ or $c = b$.

When $c = a$:

$$\exists \delta\ \ \forall x\,(0 < x - c < \delta \implies f(c) \leq f(x))$$

When $c = b$:

$$\exists \delta\ \ \forall x\,(-\delta < x - c < 0 \implies f(c) \leq f(x))$$

Special cases

f is continuous

Then by [Extreme Value Theorem](), we know f has a minimum and maximum in $[a, b]$.

f is differentiable

- If $f(a)$ is a local maximum: $f'_+(a) \leq 0$
- If $f(b)$ is a local maximum: $f'_-(b) \geq 0$
- $c \in (a, b)$ and If $f(c)$ is a local maximum: $f'(c) = 0$
- If $f(a)$ is a local minimum: $f'_+(a) \geq 0$
- If $f(b)$ is a local minimum: $f'_-(b) \leq 0$
- $c \in (a, b)$ and If $f(c)$ is a local minimum: $f'(c) = 0$

Other Theorems

Rolle's Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). And $f(a) = f(b)$. Then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$

Proof Hint

By Extreme Value Theorem, maximum and minimum exists for f.

Consider 2 cases:

1. Both minimum and maximum exist at a and b.
2. One of minimum or maximum occurs in (a, b).

Mean Value Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then:

$$\exists c \in (a,b) \text{ s.t. } f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof Hint

- Define $g(x) = f(x) - \left(\frac{f(a)-f(b)}{a-b}\right)x$
- $g(a)$ will be equal to $g(b)$
- Use Rolle's Theorem for g

Cauchy's Mean Value Theorem

Let f and g be continuous on $[a,b]$ and differentiable on (a,b), and $\forall x \in (a,b)\ g'(x) \neq 0$
Then:

$$\exists c \in (a,b) \text{ s.t. } \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof Hint

- Define $h(x) = f(x) - \left(\frac{f(a)-f(b)}{g(a)-g(b)}\right)g(x)$
- $h(a)$ will be equal to $h(b)$
- Use Rolle's Theorem for h

Mean value theorem can be obtained from this when $g(x) = x$.

Generalized MVT for Riemann Integrals

Let f, g be continuous on $[a, b]$ ($\implies$ f, g are integrable), and g does not change sign on (a, b). Then $\exists \zeta \in (a, b)$ such that:

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\zeta)\int_a^b g(x)\mathrm{d}x$$

Proof Hint

- Use Extreme value theorem for f
- Multiply by $g(x)$. Then integrate. Then divide by $\int_a^b g(x)$.
- Use intermediate value theorem to find $f(\zeta)$

L'Hopital's Rule

Note

Be careful with the pronunciation.

- It's not "Hospital's Rule", there are no "s"
- It's not "Hopital's Rule" either, there is a "L'"

Learn the correct pronounciation from this video on YouTube.

L'Hopital's Rule can be used when all of these conditions are met. (here δ is some positive number). Select the appropriate x range (as in the limit definition), say I.

1. Either of these conditions must be satisfied

- $f(a) = g(a) = 0$
- $\lim f(x) = \lim g(x) = 0$
- $\lim f(x) = \lim g(x) = \infty$

2. f, g are continuous on $x \in I$ (closed interval)
3. f, g are differentiable on $x \in I$ (open interval)
4. $g'(x) \neq 0$ on $x \in I$ (open interval)
5. $\lim\limits_{x \to a^+} \frac{f'(x)}{g'(x)} = L$

Then: $\lim\limits_{x \to a^+} \frac{f(x)}{g(x)} = L$

Here, L can be either a real number or $\pm\infty$. And it is valid for all types of “x limits”.

> ⓘ **Proof Hint**
>
> L’Hopital’s rule can be proven using Cauchy’s Mean Value Theorem.

Higher Order Derivatives

Suppose f is a function defined on (a, b). f is n times differentiable or n-th differentiable **iff**:

$$\lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a} = L \in \mathbb{R} = f^{(n)}(a)$$

Here $f^{(n)}$ denotes n-th derivative of f. And $f^{(0)}$ means the function itself.

$f^{(n)}(a)$ is the n-th derivative of f at a.

Note

$$f \text{ is } n\text{-th differentiable at } a \implies f^{(n-1)} \text{ is continuous at } a$$

Second derivative test

Suppose $f'(x) = 0$ and $f''(x)$ is continuous at c:

- $f''(c) > 0 \implies$ a local minimum is at c. Converse is **not** true.
- $f''(c) < 0 \implies$ a local maximum is at c. Converse is **not** true.

The above conclusion is from Taylor's theorem when $n = 1$:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(\zeta)}{2!}(x-c)^2$$

$$f(x) - \text{Tangent line} = \frac{f''(\zeta)}{2!}(x-c)^2$$

Taylor's Theorem

Let f is $n+1$ differentiable on (a, b). Let $c, x \in (a, b)$. Then $\exists \zeta \in (c, x) \text{ s.t. }:$

$$f(x) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}$$

Mean value theorem can be derived from taylor's theorem when $n = 0$.

Proof Hint

$$F(t) = f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

$$G(t) = (x-t)^{n+1}$$

- Define F, G as mentioned above
- Consider the interval $[c, x]$
- Use Cauchy's mean value theorem for F, G after making sure the conditions are met.

The above equation can be written like:

$$f(x) = T_n(x,c) + R_n(x,c)$$

Taylor Polynomial

This part of the above equation is called the Taylor polynomial. Denoted by $T_n(x,c)$.

$$T_n(x,c) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Remainder

Denoted by $R_n(x,c)$.

$$R_n(x,c) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}$$

Integral form of the remainder

$$R_n(x,c) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n \mathrm{d}t$$

> ⓘ **Proof Hint**
>
> - Method 1: Use integration by parts and mathematical induction.
> - Method 2: Use Generalized MVT for Riemann Integrals where:
>   - $F = f^{(n+1)}$
>   - $G = (x-t)^n$

Sequence

A sequence on a set A is a function $u : \mathbb{Z}^+ \to A$.

Image of the n is written as u_n. A sequence is indicated by one of these ways:

$$\left\{u_n\right\}_{n=1}^{\infty} \text{ or } \left\{u_n\right\} \text{ or } \left(u_n\right)_{n=1}^{\infty}$$

Increasing or Decreasing

A sequence $\left(u_n\right)$ is

- Increasing **iff** $u_n \geq u_m$ for $n > m$
- Decreasing **iff** $u_n \leq u_m$ for $n > m$
- Monotone **iff** either increasing or decreasing
- Strictly increasing **iff** $u_n > u_m$ for $n > m$
- Strictly decreasing **iff** $u_n < u_m$ for $n > m$

Convergence

Converging

A sequence $\left(u_n\right)_{n=1}^{\infty}$ is converging (to $L \in \mathbb{R}$) **iff**: $\lim_{n \to \infty} u_n = L$

$$\forall \epsilon > 0\ \exists N \in \mathbb{Z}^+\ \forall n\ (n > N \implies |u_n - L| < \epsilon)$$

> **Note**
>
> $$\forall x \in \mathbb{R} \quad \lim_{n \to \infty} \frac{x^n}{n!} = 0$$

Diverging

A sequence is diverging **iff** it is not converging.

$$\lim_{n \to \infty} u_n = \begin{cases} \infty \\ -\infty \\ \text{undefined}, & \text{when } u_n \text{ is osciallating} \end{cases}$$

Convergence test

All converging sequences are bounded.

Increasing and bounded above

Let (u_n) be increasing and bounded above. Then (u_n) is converging (to $\sup \{u_n\}$).

Proof Hint

- $\{u_n\}$ has a $\sup u_n (= s)$
- Prove: $\lim_{n \to \infty} u_n = s^-$

Decreasing and bounded below

Let (u_n) be decreasing and bounded below. Then (u_n) is converging (to $\inf \{u_n\}$).

Proof Hint

- $\{u_n\}$ has a $\inf u_n (= l)$
- Prove: $\lim_{n \to \infty} u_n = l^+$

Newton's method of finding roots

Suppose f is a function. To find its roots:

- Select a point x_0
- Draw a tangent at x_0
- Choose x_1 which is where the tangent meets $y = 0$
- Continue this process repeatedly

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Subsequence

Suppose $u : \mathbb{Z}^+ \to \mathbb{R}$ be a sequence and $v : \mathbb{Z}^+ \to \mathbb{Z}^+$ be an increasing sequence. Then $u \circ v : \mathbb{Z}^+ \to \mathbb{R}$ is a subsequence of u.

Existence of subsequence

Every sequence has a monotonic subsequence.

Proof Hint

- Let $n \in \mathbb{Z}^+$ be called "good" **iff** $\forall m > n,\ u_n > u_m$.
- Suppose u_n has infinitely many "good" points. That implies u_n has a decreasing subsequence.
- Suppose u_n has finitely many "good" points. Let N is the maximum of those. $\forall n_1 > N,\ n_1 \text{ is not "good"}$ That implies u_n has a increasing subsequence.

Bolzano-Weierstrass

Every bounded sequence on $\mathbb{R}$ has a converging subsequence.

Proof Hint

From the above theorem, there is a monotonic subsequence u_{n_k} which is also bounded. Bounded monotone sequences converge.

Note

For a set A, all 3 statements are equivalent:

- A has the completeness property
- A is complete
- Bolzano-Weierstrass theorem on A

Theorem 1

Suppose u_n is a sequence converging to L, and u_{n_k} is a subsequence of u_n. Then u_{n_k} is converging to L.

$$\lim_{n\to\infty} u_k = L \implies \lim_{n_k\to\infty} u_{n_k} = L$$

Proof Hint

Note that $n_k \geq k$.

Theorem 2

Suppose u_n is a sequence diverging to ∞, and u_{n_k} is a subsequence of u_n. Then u_{n_k} is diverging to ∞.

$$\lim_{n\to\infty} u_k = \infty \implies \lim_{n_k\to\infty} u_{n_k} = \infty$$

> **Proof Hint**
>
> Note that $n_k \geq k$.

Subsequence of a cauchy sequence

If u_n is Cauchy and u_{n_k} is a subsequence converging to L, then u_n converges to L.

Cauchy Sequence

A sequence $u : \mathbb{Z}^+ \to A$ is Cauchy **iff**:

$$\forall \epsilon > 0 \, \exists N \in \mathbb{Z}^+ \, \forall m, n; m, n > N \implies |u_n - u_m| < \epsilon$$

Bounded

All Cauchy sequences are bounded. (has an upper bound).

> **Proof Hint**
>
> - Consider the Cauchy definition
> - Take $n > m = N + 1 > N$

Convergence & Cauchy

A sequence is converging **iff** it is Cauchy.

Proof Hint

To prove $implies$:

- Consider the limit definition of converging sequences
- Introduce the converging value (say L) into the inequality and split into 2 parts

To prove $impliedby$:

- Consider the definition of Cauchy sequences
- Show that the sequence is bounded

Complete

A set A is complete **iff**:

$$\forall u : \mathbb{Z}^+ \to A;\ u \text{ converges to } L \in A$$

IMPORTANT: Q is **not** complete because:

$$\sum_{k=1}^{\infty} \frac{1}{k!} = e - 1 \notin \mathbb{Q}$$

IMPORTANT: $\mathbb{R}$ is complete.

Proof Hint

Proof is quite hard.

Series

Let (u_n) be a sequence, and a series (a new sequence) can be defined from it such that:

$$s_n = \sum_{k=1}^{n} u_k$$

Convergence

If (s_n) is converging:

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \sum_{k=1}^{n} u_k = \sum_{k=1}^{\infty} u_k = S \in \mathbb{R}$$

Absolutely Converging

$\sum_{k=1}^{n} u_k$ is absolutely converging **iff** $\sum_{k=1}^{n} |u_k| \text{ is converging.}$

$$\sum_{k=1}^{n} |u_k| \text{ is converging } \implies \sum_{k=1}^{n} u_k \text{ is converging}$$

> **Proof Hint**
>
> Use this inequality:
>
> $$0 \leq \ |u_k| - u_k \ \leq 2|u_k|$$

Theorem

A series s_n is absolutely converging to s **iff** rearranged series of s_n converges to s.

Conditionally Converging

$\sum_{k=1}^{n} u_k$ is condtionally converging **iff**:

$$\sum_{k=1}^{n} |u_k| \text{ is diverging} \quad \text{and} \sum_{k=1}^{n} u_k \text{ is converging}$$

Theorem

Suppose s_n is a conditionally converging series. Then:

1. Sum of all the positive terms limits to ∞
2. Sum of all the negative terms limits to $-\infty$
3. s_n can be rearranged to have the sum:
    - Any real number x
    - ∞
    - $-\infty$
    - Does not exist

Divergence test

$$\sum_{k=1}^{n} u_k \text{ is converging} \implies \lim_{k \to \infty} u_k = 0$$

The converse is more useful:

$$\lim_{k\to\infty} u_k \neq 0 \implies \sum_{k=1}^{n} u_k \text{ is diverging}$$

A secret note

For any $p > 0$, as n tends to ∞, the below inequality holds:

$$\ln n < n^p < n!$$

The above inequality can be used to

Convergence Tests

Known series

Direct Comparison Test

Let $0 < u_k < v_k$.

$$\sum_{k=1}^{\infty} v_k \text{ is converges} \implies \sum_{k=1}^{\infty} u_k \text{ is converges}$$

Proof Hint

- Note that $\sum_{k=1}^{n} u_k$ and $\sum_{k=1}^{n} v_k$ are increasing
- Show that $\sum_{k=1}^{\infty} v_k$ converges to its supremum v which is an upper bound of $\sum_{k=1}^{n} u_k$

Example

Proving the convergence of $\sum_{k=1}^{\infty} \frac{1}{k!}$, by using $k! \geq 2^{k-1}$ for all $k \geq 0$.

Limit Comparison Test

Let $0 < u_k, v_k$ and $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = R$.

$$R > 0 \implies \left(\sum_{n=1}^{\infty} u_n \text{ is converging} \iff \sum_{n=1}^{\infty} v_n \text{ is converging}\right)$$

$$R = 0 \implies \left(\sum_{n=1}^{\infty} v_n \text{ is converging} \implies \sum_{n=1}^{\infty} u_n \text{ is converging}\right)$$

$$R = \infty \implies \left(\sum_{n=1}^{\infty} v_n \text{ is diverging} \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}\right)$$

Proof Hint

Only possibilities are $R = 0, R > 0, R = \infty$.

For $R > 0$:

- Consider limit definition with $\epsilon = \frac{L}{2}$
- Direct comparison test can be used for the 2 set of inequalities

For $R = 0$:

- Consider limit definition with $\epsilon = 1$
- Direct comparison test can be used now

For $R = \infty$:

- Consider limit definition with $M = 1$
- Direct comparison test can be used now

Integral Test

Let $u(x) > 0$, decreasing and integrable on $[1, M]$ for all $M > 1$. Then:

$$\sum_{n=1}^{\infty} u_n \text{ is converging} \iff \int_1^{\infty} u(x)\,\mathrm{d}x \text{ is converging}$$

Proof Hint

As $u(x)$ is decreasing, it is apparent that it is integrable.

Make use of this inequality:

$$s_n - u_1 \leq \int_1^n u(x)\,\mathrm{d}x \leq s_n - u_n$$

For $\Longleftarrow$:

- Note that s_n is increasing
- Show that s_n is bounded above by $\int_1^\infty u(x)\,\mathrm{d}x + u_1$

For $\Longrightarrow$:

- Define $F(n) = \int_1^n u(x)\,\mathrm{d}x$
- Note that $F(n)$ is increasing
- Note that $\lim\limits_{n\to\infty} u_n = 0$
- Show that $F(n)$ is bounded above by $\lim\limits_{n\to\infty} s_n$

> **Note**
>
> $$\sum_{n=1}^{\infty} u_n \text{ is converging} \implies \lim_{k\to\infty} u_k = 0$$
>
> $$\int_1^{\infty} u(x)\,\mathrm{d}x \text{ is converging} \implies \lim_{k\to\infty} u(k) = 0$$

Ratio Test

Let $u(x) > 0$ and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$.

$$L < 1 \implies \sum_{n=1}^{\infty} u_n \text{ is converging}$$

$$L > 1 \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}$$

Proof Hint

- Consider the limit definition with:
  - For the $L < 1$ case: $\epsilon = \frac{1}{2}(1 - L)$
  - For the $L > 1$ case: $\epsilon = \frac{1}{2}(L - 1)$
- Show that: $\frac{1}{2}(3L - 1) < \frac{u_{k+1}}{u_k} < \frac{1}{2}(1 + L)$
- Recursively simplify the inequality to reach u_{N+1} which is a constant
- Use $\sum_{k=1}^{\infty} r^k$ is converging **iff** $r < 1$

Root Test

Let $u(x) > 0$ and $\lim\limits_{n \to \infty} u_n^{1/n} = L$.

$$L < 1 \implies \sum_{n=1}^{\infty} u_n \text{ is converging}$$

$$(L > 1 \lor L = \infty) \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}$$

Proof Hint

Consider the limit definition with:

- For $L < 1: \epsilon = \frac{1}{2}(1 - L)$
- For $L > 1: \epsilon = \frac{1}{2}(L - 1)$
- For $L = \infty: M > 1$

Known Series

These series are helpful when using the direct comparison test or limit comparison test.

Convergent

When $s > 1$:

$$\sum_{k=1}^{\infty} \frac{1}{k^s}$$

The above series is known as p-series (not power series) and occurs in the definition of Riemann zeta function.

When $|r| < 1$:

$$\sum_{k=1}^{\infty} r^k$$

Divergent

When $s \leq 1$:

$$\sum_{k=1}^{\infty} \frac{1}{k^s}$$

When $|r| \geq 1$:

$$\sum_{k=1}^{\infty} r^k$$

Alternating Series

Suppose $u_k > 0$. An alternating series is:

$$\sum_{k=1}^{n} (-1)^{k-1} u_k = u_1 - u_2 + u_3 - u_4 + \cdots$$

Convergence test

If $\forall k \; u_k > 0$, decreasing and $\lim_{n \to \infty} u_n = 0$, **then**:

$$\sum_{k=1}^{n} (-1)^{k-1} u_k \text{ is converging}$$

Proof

For odd-indexed elements:

$$s_{2m+3} \leq s_{2m+1} \leq s_1 = u_1$$

For even-indexed elements:

$$s_{2m+2} \geq s_{2m} \geq s_2 = u_1 - u_2$$

Combining these 2:

$$0 \leq u_1 - u_2 \leq s_2 \leq s_{2m} \leq s_{2m+1} \leq s_1 = u_1$$

s_{2m} is bounded above by u_1 and increasing. s_{2m+1} is bounded below by 0 and decreasing. So both converges.

$$\lim_{m \to \infty} (s_{2m+1} - s_{2m}) = \lim_{m \to \infty} u_{2m+1} = 0$$

$$\implies \lim_{m \to \infty} s_{2m+1} = \lim_{m \to \infty} s_{2m} = s$$

Both converges to the same number.

Power Series

A series of the form:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

Here:

- x - a variable
- c - a constant

Convergence of a power series can be checked using ratio test or root test.

Radius of convergence

Maximum radius of x in where the series converges.

$$R = \sup \left\{ r \mid \text{series converges for } |x-c| < r \right\}$$

The below equation can be used to find R:

$$\lim_{k \to \infty} |a_k|^{\frac{1}{k}} = \frac{1}{R}$$

The series may converge or diverge for $|x-c| = R$.

Range of convergence

$(c-R, c+R)$ is the range of convergence. Aka. interval of convergence. The series may converge or diverge at the endpoints. Endpoints must be checked separately to find out if they must be included in the range of convergence.

Theorem 1

If $R \in (0, \infty)$ and $|x - a| \leq p$ for $p < R$, then $s_n(x)$ is uniformly (and absolutely) converging.

> ⓘ **Proof Hint**
>
> - Note the relation between R and a_k
> - Prove $(\frac{p+R}{2pR})^k$ is an upperbound to $|a_k|^{\frac{1}{k}}$, using it's infinity limit
> - Define $M_k = (\frac{p+R}{2R})^k = r^k$
> - Prove M_k is a bound to u_k
> - Prove $\sum_{k=1}^{n} r^k$ is converging as $0 < r < 1$

Taylor Series

Let f be infinitely many times differentiable on (a, b) and $c, x \in (a, b)$.

If $\lim_{n \to \infty} R_n(x) = 0$ for $x \in (c - R, c + R) \subset (a, b)$, then Taylor series of f at c is given by:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n$$

> ⓘ **Note**
>
> Usually Taylor series expansion is done with $c = 0$. This is a special case of Taylor series, and called the **Maclaurin series**.

Examples

e^x

Range of convergence is $\mathbb{R}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

ln (1+x)

Range of convergence is $(-1, 1]$.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

sin x

Range of convergence is $\mathbb{R}$.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

cos x

Range of convergence is $\mathbb{R}$.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Sequence of Functions

Types of Convergence

Pointwise convergence

$$\forall \epsilon > 0 \; \forall x \in [a, b] \; \exists N \in \mathbb{Z}^+ \; \forall n > N \; ; \; \left| f_n(x) - f(x) \right| < \epsilon$$

Here N depends on ϵ, x.

Examples:

- x^n on $[0, 1]$

Uniformly convergence

$$\forall \epsilon > 0 \; \exists N \in \mathbb{Z}^+ \; \forall x \in [a, b] \; \forall n > N \; ; \; \left| f_n(x) - f(x) \right| < \epsilon$$

Here N depends on ϵ only. Implies pointwise convergence.

Examples:

- $\frac{x^2}{n}$ on $[0, 1]$

Uniform convergence tests

Supremum test

A sequence of functions $u_n(x)$ converges to $u(x)$ uniformly **iff**:

$$\lim_{n \to \infty} \sup_x |u_n(x) - u(x)| = 0$$

ⓘ **Proof Hint**

Let $l_n = |u_n(x) - u(x)|$.

To prove $\implies$:

- Consider the epsilon-delta definition of uniform convergence
- $\frac{\epsilon}{2}$ is an upperbound of l_n
- $\sup_x l_n \leq \frac{\epsilon}{2} < \epsilon$

To prove $\impliedby$:

- Consider the epsilon-delta definition of the above limit
- $l_n < \sup_x l_n < \epsilon$

Properties of uniform convergence

Continuity is preserved

If $u_n(x)$ is continuous and converging to $u(x)$, then $u(x)$ is also continuous.

ⓘ **Proof Hint**

Consider the limit definitions of:

1. $u_n(x)$ converges to $u(x)$
2. $u_n(x)$ is continuous at a

Consider $|u(x) - u(a)|$. Introduce $u_n(x)$ and $u_n(a)$ in there. Split into 3 absolute values. Show that the sum is lesser than 3ϵ.

Limit and integral can be switched

Explained in [Converging Functions | Riemann Integration].

Differentiation is complicated

Uniform convergence-differentiation pair doesn't go as smooth like integration was.

Suppose $u_n(x)$ is a sequence of differentiable functions, and they uniformly converges to $u(x)$. Then we can't say, for sure, $u(x)$ is differentiable. An example is:

$$u_n(x) = \sqrt{x^2 + \frac{1}{n}}$$

Theorem

If (all conditions must be met):

1. $u_n(x)$ is differentiable on $[a, b]$
2. $u_n(x_0)$ converges (pointwise) for some $x_0 \in [a, b]$
3. $u'_n(x)$ converges to $f(x)$ uniformly on $[a, b]$

Then:

1. $u_n(x)$ converges to $u(x)$ uniformly on $[a, b]$
2. $u(x)$ is differentiable on $[a, b]$
3. $u'(x) = f(x)$ **OR** in other words $u'_n(x)$ converges to $u'(x)$ uniformly

Uniformly Cauchy

$u_n(x)$ in $x \in A$ is said to be uniformly cauchy **iff**:

$$\forall \epsilon > 0 \exists N \in \mathbb{Z}^+ \forall m, n > N \forall x \in A; |u_n(x) - u_m(x)| < \epsilon$$

If $u_n(x)$ is a sequence of functions on $\mathbb{R}$, then:

$$u_n(x) \text{ converges uniformly} \iff u_n(x) \text{ is uniformly Cauchy}$$

Proof Hint

To prove $\implies$:

- Consider $|u_n(x) - u_m(x)|$
- Introduce $u(x)$ in the inequality
- Split the inequality and and use the definition of uniform convergence

To prove $\impliedby$:

- Consider the definition of uniformly Cauchy
- Let m go to ∞

Series of Functions

Let $u_k(x)$ is a sequence of integrable functions. And series of those functions is defined as:

$$s_n(x) = \sum_{k=1}^{n} u_k(x)$$

Convergence

$s_n(x)$ converges to $s(x)$ uniformly.

⚠ TODO

Include the Proof Hint.

Convergence tests

Weierstrass M-test

To test if a series of functions converges uniformly and absolutely.

Let f_n be a sequence functions on a set A. And both these conditions are met:

- $\forall n \geq 1 \, \exists M_n \geq 0 \, \forall x \in A \, ; |f_n(x)| \leq M_n$
- $\sum_{n=1}^{\infty} M_n$ converges

Then:

$$\sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly \& absolutely}$$

Differentiation

Theorem

If (all conditions must be met):

1. $u_n(x)$ is differentiable ($\implies$ $s_n(x)$ is differentiable) on $[a, b]$
2. $s_n(x_0)$ converges (pointwise) for some $x_0 \in [a, b]$
3. $s'_n(x) = \sum_{k=1}^{n} u'_k(x)$ converges to $f(x)$ uniformly on $[a, b]$

Then:

1. $s_n(x)$ converges to $s(x)$ uniformly on $[a, b]$
2. $s(x)$ is differentiable on $[a, b]$
3. $s'(x) = f(x)$ **OR** in other words $s_n'(x)$ converges to $s'(x)$ uniformly

In that case, differentiation and infinite sum can be interchanged:

$$\sum_{k=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} u_k(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{\infty} u_k(x)$$

For power series

For any power series, inside the range of convergence, conditions for the above theorem is valid and thus the conclusions are valid.

Suppose $s_n = \sum_{k=1}^{n} a_k(x-c)^k$, and R is the radius of convergence. For $|x - c| \leq p < R$:

$$s'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{\infty} a_k(x-c)^k = \sum_{k=0}^{\infty} k a_k (x-c)^{k-1}$$

> ⓘ **Note**
>
> When a power series is differentiated: At the boundaries of the range of convergence which is a closed interval, the convergence might be lost.
>
> When a power series is integral: At the boundaries of the range of convergence which is an open interval, the convergence might occur.

Riemann Zeta Function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

Convergence of this function can be derived using integral test. The above-mentioned series is also referred to as p-series.

This function converges **iff** $s > 1$. And it converges to:

$$\frac{1}{s-1}$$

Extension

The ζ function can be extended to the set $\mathbb{C} - \{1\}$.

Ramanujan Sum

$$\zeta(-1) = -\frac{1}{12}$$

Which is why, it's used as below:

$$1 + 2 + 3 + 4 + 5 + \cdots = -\frac{1}{12}$$

This is known as the Ramanujan sum of the diverging series.

Riemann Hypothesis

The ζ function has its zeros only at negative even integers and complex numbers with real part $\frac{1}{2}$.

One of the most important unsolved problems in mathematics.