Summary | Matrices

Introduction

Revise Matrices unit from G.C.E. (A/L) Combined Mathematics and G.C.E. (O/L) Mathematics.

Types of matrices

Square matrix

Number of columns equal to number of rows.

> **Main diagonals of a square matrix**
>
> Formed by elements having equal subscripts.

Diagonal matrix

A square matrix whose only non-zero elements are main-diagonal elements. Denoted by D. Subset of triangular matrices.

Identity matrix or Unit matrix

A diagonal matrix whose diagonal elements are all equal to 1. Denoted by I. Subset of diagonal matrices.

Zero matrix / Null matrix

All elements are 0.

Column matrix (column vector)

Only 1 column.

Row matrix (row vector)

Only 1 row.

Triangular matrix

Upper triangular matrix or lower triangular matrix.

Upper triangular matrix

All elements below the main diagonal are 0. Subset of square matrices.

Lower triangular matrix

All elements above the main diagonal are 0. Subset of square matrices.

Matrix operations

Addition and subtraction

Order of the 2 matrices must be same. Matrix obtained by adding or subtracting corresponding elements.

Scalar multiplication

Matrix obtained by multiplying all elements by the scalar.

> **Note**
>
> Matrix multiplication is also defined.

Transpose

Matrix obtained from a given matrix by interchanging its rows and columns. Denoted by a superscript T, like A^T.

Properties

1. $(A^T)^T = A$
2. Distributive over addition: $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(A \times B)^T = B^T \times A^T$

More types of matrices

Symmetric matrix

If $A = A^T$. Subset of square matrices.

Skew-symmetric matrix

If $A = -A^T$. Subset of square matrices. All elements in main diagonal are 0.

> **Note**
>
> Any square matrix can be expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

Matrix multiplication

Defined only if the number of columns of the first matrix is equal to the number of rows of the second matrix.

If $A = (a_{ij})_{m\times p}$ and $B = (b_{ij})_{p\times n}$, then $A \times B = C = (c_{ij})_{m\times n}$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$.

> **Note**
>
> - Generally $A \times B \neq B \times A$.
> - $A \times B = 0 \not\Longrightarrow A = 0 \vee B = 0$
> - $A \neq 0 \wedge B \neq 0 \not\Longrightarrow A \times B \neq 0$

Properties of matrix multiplication

A, B, C, I matrices must be chosen so that below-mentioned product matrices are defined.

1. Associative: $A(BC) = (AB)C$
2. Right distributive over addition: $(A+B)C = AC + BC$
3. Left distributive over addition: $C(A+B) = CA + CB$
4. $AI = IA = A$; I is an identity matrix.

Determinant

Defined only for square matrices. Denoted by $|A|$.

For 2x2

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For higher order

Minor of an element

Suppose $A = (a_{ij})$.

Minor of an element a_{ij}, is the matrix obtained by deleting i^{th} row and j^{th} column of A. Denoted by M_{ij}.

Co-factor of an element

Suppose $A = (a_{ij})$.

Co-factor of an element a_{ij}, is defined as (commonly denoted as A_{ij}):

$$A_{ij} = (-1)^{i+j} \left| M_{ij} \right|$$

Definition

If $A = (a_{ij})_{n\times n}$ then the **determinant** of A is denoted by $|A|$ and is defined by:

$$|A| = \sum_{j=1}^{n} a_{ij} A_{ij}$$

where $1 \leq j \leq n$.

Properties of determinants

- $\left|A^T\right| = |A|$
- Every element of a row or column of a matrix is 0 then the value of its determinant is 0.
- If 2 columns or 2 rows of a matrix are identical then its determinant is 0.
- If A and B are two square matrices then $|AB| = |A||B|$.
- The value of the determinant of a matrix remains unchanged if a scalar multiple of a row or column is added to any other row or column.
- If a matrix B is obtained from a square matrix A by an interchange of two columns or rows:
  $|B| = -|A|$.
- If every entry in any row or column is multiplied by k, then the whole determinant is multiplied by k.

In relation with eigenvalues

For a $n \times n$ matrix A with n number of eigenvalues:

$$|A| = \prod_{i=1}^{n} \lambda_i$$

Adjoint

Suppose $A = (a_{ij})_{n\times n}$.

$$\mathrm{adj}A = (A_{ij})^T_{n\times n}$$

Where A_{ij} is the co-factor of a_{ij}.

Inverse

Suppose A and B are square matrices of the same order. If $AB = BA = I$ then B is called the inverse of A and is denoted by A^{-1}.

$$A^{-1} = \frac{\operatorname{adj} A}{|A|}$$

> ⓘ **Singular vs Non-singular**
>
> A square matrix is singular if $|A| = 0$. Otherwise non-singular or invertible matrices.

Properties of Inverse

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $A \operatorname{adj} A = \operatorname{adj} A \, A = |A| I$

> ⓘ **Orthogonal Matrix**
>
> A square matrix is orthogonal if $A^T = A^{-1}$.

> ⓘ **Orthogonal Matrix Pair**
>
> 2 column vectors v_1, v_2 are said to be orthogonal if $v_1 \cdot v_2 = 0$.

Elementary Transformations

- Interchange of any columns or rows
- Addition of multiple of any row or column to any other row or column
- Multiplication of each element of a column or a row by a non-zero constant

When a matrix B is obtained by applying elementary transformations to a matrix A, then A **is equivalent to** B. Denoted by $A \approx B$.

Theorem

The elementary row operations that reduce a given matrix A to the identity matrix, also transform the identity matrix to the inverse of A.

Augmented Matrix

Two matrices are written as a single matrix with a vertical line in-between. Denoted by $(A|B)$. Example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$$

Inverse using elementary row transformations

Let A be a square matrix with order $n \times n$.

- Start with $(A_{n \times n}|I_n)$
- Repeatedly add **row** transformations (not column) to both of the matrices until the LHS becomes an identity matrix.
  - Convert all elements outside the main diagonal to 0.
  - Convert elements on the main diagonal to 1 by multiplying by a constant.
- When LHS is an identity matrix, RHS is A^{-1}.

> ⚠ **TODO**
>
> What about singular matrices?

Echelon Form

A matrix is in row echelon form (or just “row echelon” form) **iff**:

- All rows having only zero entries are at the bottom.
- For all row that does not contain entirely zeros, the first non-zero entry is 1.
- For 2 successive non-zero rows, the leading 1 in the higher row is further left than the leading 1 in the lower row.

The process of reducing the augmented matrix to row Echelon form is known as **Gaussian elimination**.

Column echelon form

A matrix A is in column echelon form if A^{T} is in row echelon form.

System of Linear Equations

Any system of linear equations can be represented in matrix notation as shown below.

- $a_{11}x + a_{12}y + a_{13}z = b_1$
- $a_{21}x + a_{22}y + a_{23}z = b_2$
- $a_{31}x + a_{32}y + a_{33}z = b_3$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \implies AX = B$$

2 types based on B:

- $= 0$: Homogeneous system
- $\neq 0$: Non-homogeneous system

Consistent

When the system of equations has at least 1 solution. Otherwise inconsistent.

Rank

Number of non-zero rows of row echelon form of a matrix A. Denoted by $\mathrm{Rank}\ A$.

> **Note**
> $\mathrm{Rank}\ A \leq \mathrm{Rank}\ (A|B)$ is always true.

Relation with non-homogenous system of solutions

Consider the system: $A_{n \times n} X_{n \times 1} = Bn \times 1$.

- $|A| \neq 0 \iff \text{Rank } A = \text{Rank } (A|B) = n \iff \text{unique solution exists}$
- $|A| = 0 \implies \text{no solution} \vee \text{infinitely many solutions}$
- $\text{Rank } A < \text{Rank } (A|B) \implies \text{no solutions}$
- $\text{Rank } A = \text{Rank } (A|B) < n \implies \text{infinitely many solutions}$

Solutions of homogenous systems

Consider the system:

$$A_{m\times n}X_{n\times 1} = O_{m\times 1}$$

Any homogenous system is consistent, because

$$X = O$$

is always a solution.

- $\text{Rank } A = \text{Rank } (A|B) = n \iff \text{unique solution exists}$
- $\text{Rank } A = \text{Rank } (A|B) < n \implies \text{infinitely many solutions}$

Solution of non-homogenous systems

Method 1: Direct approach

Used when coefficient matrix A is invertible. It means the system has a unique set of solutions.

$$AX = B \implies X = A^{-1}B$$

Method 2: Cramer's Rule

Let $AX = B$, where A is the coefficient matrix and $X = (x_i)_{n\times 1}$.

$$x_i = \frac{|A_i|}{|A|}$$

Where A_i is the matrix obtained by replacing ith column in matrix A by B.

Method 3: Reducing to Echelon Form

Start with $(A|B)$. Convert the LHS to echelon form using elementary row transformations. The solution can be found now. If a contradiction is encountered while solving the equation, that means the system has no solutions.

Eigenvalues & Eigenvectors

Definitions

Characteristic Polynomial

Let A be a $n \times n$ matrix.

$$p(\lambda) = |A - \lambda I|$$

Eigenvalues

Roots of the equation $p(\lambda) = 0$ are the eigenvalues of A.

> **Note**
>
> Determinant of a matrix can be written in terms of all of its eigenvalues.

Eigenvectors

The column vectors satisfying the equation $(A - \lambda_i I)X_i$.

Normalized eigenvectors

An eigenvector with the magnitude (norm) of 1. Normalizing factor k of any eigenvector is:

$$\frac{1}{k} = \sqrt{\sum_{i=1}^{n} X_i^2}$$

Norm

Norm of a column or row matrix $W_{n \times n}$ is denoted by $||W||$ and defined as:

$$||W|| = \sqrt{\sum_{i=1}^{n} w_i^2}$$

Algebraic Multiplicity

If the characteristic polynomial consists of a factor of the form $(\lambda - \lambda_i)^r$ and $(\lambda - \lambda_i)^{r+1}$ is not a factor of the characteristic polynomial then r is the algebraic multiplicity of the eigenvalue λ.

Spectrum

Set of all eigenvalues.

Spectral Radius

$$R = \max\left\{|\lambda_i| \; where \; \lambda_i \in \text{Spectrum}\right\}$$

Linear Independence of Eigenvectors

Suppose $X_1, X_2, X_3, \ldots, X_n$ is a set of eigenvectors. $k_1, k_2, k_3, \ldots, k_n$ is a set of scalars.

All those eigenvectors are independent **iff**:

$$k_1X_1 + k_2X_2 + k_3X_3 + \cdots + k_nX_n = 0 \implies k_1 = k_2 = k_3 = \cdots = k_n = 0$$