Summary | Riemann Integration

Introduction

Interval

Let $I = [a, b]$. Length of the interval $|I| = b - a$.

Disjoint interval

When 2 intervals don't share any common numbers.

Almost disjoint interval

When 2 intervals are disjoint or intersect only at a common endpoint.

Riemann Integral

Let $f - [a, b] \to \mathbb{R}$ is a bounded (not necessarily continuous) function on a closed, bounded (compact) interval.

Riemann integral of f is: $\int_a^b f$

Definite integral

When a, b are constants.

Indefinite integral

When a is a constant but b is replaced with x.

Partition

Let I be a non-empty, compact interval (closed and bounded). A partition of I is a finite collection $\{I_1, I_2, \ldots, I_n\}$ of almost disjoint, non-empty, compact sub-intervals whose union is I.

A partition is determined by the endpoints of all sub-intervals: $a = x_0 < x_1 < \cdots < x_n = b$.

A partition can be denoted by:

- its intervals - $P = \{I_1, I_2, \ldots, I_n\}$
- the endpoints of its intervals - $P = \{x_0, x_1, \ldots, x_n\}$

Riemann Sum

Let

- $f : [a, b] \to \mathbb{R}$ is a bounded function on the compact interval $I = [a, b]$ with $M = \sup_I f$ and $m = \inf_I f$.
- $P = \{I_1, I_2, \ldots, I_n\}$
- $M_k = \sup_{I_k} f = \sup \{f(x) : x \in [x_{k-1}, x_k]\}$
- $m_k = \inf_{I_k} f = \inf \{f(x) : x \in [x_{k-1}, x_k]\}$

Upper riemann sum

$$U(f; P) = \sum_{k=1}^{n} M_k |I_k|$$

Lower riemann sum

$$L(f; P) = \sum_{k=1}^{n} m_k |I_k|$$

$m_k < M_k \implies L(f; P) \le U(f; P)$

When P_1, P_2 are any 2 partitions of I: $L(f; P_1) \le U(f; P_2)$

Refinements

Q is called a refinement of $P \iff$ if P and Q are partitions of $[a, b]$ and $P \subseteq Q$.

When Q is a refinement of P:

$$L(f;P) \le L(f;Q) \le U(f;Q) \le U(f;P)$$

> **Note**
>
> If P_1 and P_2 are partitions of $[a,b]$, then $Q = P_1 \cup P_2$ is a refinement of both P_1 and P_2. In that case:
>
> $$L(f;P_1) \le L(f;Q) \le U(f;Q) \le U(f;P_2)$$

Upper & Lower integral

Let $\mathbb{P}$ be the collection of all possible partitions of the interval $[a,b]$.

Upper Integral

$$U(f) = \inf\{U(f;P); P \in \mathbb{P}\} = \overline{\int_a^b} f$$

Lower Integral

$$L(f) = \sup\{L(f;P); P \in \mathbb{P}\} = \underline{\int_a^b} f$$

For a bounded function f, always $L(f) \le U(f)$

Riemann Integrable

A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ **iff** $U(f) = L(f)$. In that case, the Riemann integral of f on $[a,b]$ is denoted by $\int_a^b f(x)\,\mathrm{d}x$.

Reimann Integrable or not

Function	Yes or No?	Proof hint
Unbounded	No	By definition
Constant	Yes	$\forall P\,(\textbf{any partition})\;\; L(f;P) = U(f;P)$
Monotonically increasing/decreasing	Yes	Take a partition such that $\Delta x < \delta = \frac{\epsilon}{f(b)-f(a)}$
Continuous	Yes	Take a partition such that $\Delta x < \delta = \frac{\epsilon}{2(b-a)}$

Note

If the set of points of discontinuity of a bounded function $f : [a, b] \to \mathbb{R}$ is finite, then f is Riemann integrable on $[a, b]$.

Note

If the set of points of discontinuity of a bounded function $f : [a, b] \to \mathbb{R}$ is finite number of limit points, then f is integrable on $[a, b]$.

A function may have infinitely many discontinuous points, but if the set of all discontinuous points have finite number of limit points, then f is integrable on $[a, b]$.

Cauchy Criterion

Theorem

A bounded function $f : [a, b] \to R$ is Riemann integrable **iff** for every $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$, which may depend on ϵ, such that:

$$U(f, P\epsilon) - L(f, P\epsilon) \leq \epsilon$$

Proof Hint

- To prove $\implies$: consider $L(f) - \frac{\epsilon}{2} < L(f;P)$ and $U(f;P) < U(f) + \frac{\epsilon}{2}$
- To prove $\impliedby$: consider $L(f;P) < L(f)$ and $U(f) < U(f;P)$

Note

$f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ when:

- The set of points of discontinuity of a bounded function f is finite.
- The set of points of discontinuity of a bounded function f is finite number of limit points. (may have infinite number of discontinuities)

Theorems on Integrability

Theorem 1

Suppose $f : [a,b] \to \mathbb{R}$ is bounded, and integrable on $[c,b]$ for all $c \in (a,b)$. Then f is integrable on $[a,b]$. Also valid for the other end.

Proof Hint

- Isolate a partition on the required end.
- Choose x_1 or x_{n-1} such that $\Delta x < \frac{\epsilon}{4M}$ where M is an upper or lower bound.

Theorem 2

Suppose $f : [a,b] \to \mathbb{R}$ is bounded, and continuous on $[c,b]$ for all $c \in (a,b)$. Then f is integrable on $[a,b]$. Also valid for the other end.

TODO: Proof Hint

Properties of Integrals

Notation

If $a < b$ and f is integrable on $[a, b]$, then:

$$\int_a^b f = -\int_b^a f$$

Properties

Suppose f and g are integrable on $[a, b]$.

Addition

$f + g$ will be integrable on $[a, b]$.

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$$

Proof Hint

- Prove $f + g$ is integrable using:
  - $sup(f + g) \leq \sup(f) + \sup(g)$
  - $inf(f + g) \geq \inf(f) + \inf(g)$
- Start with $U(f + g)$ and show $U(f + g) \leq U(f) + U(g)$
- Start with $L(f + g)$ and show $L(f + g) \geq L(f) + L(g)$

Constant multiplication

Suppose $k \in \mathbb{R}$. kf will be integrable $[a, b]$.

$$\int_a^b kf = k \int_a^b f$$

Proof Hint

- Prove for $k \geq 0$. Use $U - L < \frac{\epsilon}{k}$
- Prove for $k = -1$
- Using the above results, proof for $k < 0$ is apparent

Bounds

If $m \leq f(x) \leq M$ on $[a, b]$:

$$m \leq \int_a^b f \leq M$$

If $f(x) \leq g(x)$ on $[a, b]$:

$$\int_a^b f \leq \int_a^b g$$

Modulus

$|f|$ will be integrable on $[a, b]$.

$$\left| \int_a^b f \right| = \int_a^b |f|$$

Proof Hint

Start with $-|f| \leq f \leq |f|$. And integrate both sides.

Multiple

fg will be integrable on $[a, b]$.

Proof Hint

- Suppose f is bounded by k
- Prove f^2 is integrable (Use $\frac{\epsilon}{2k}$)
- fg is integrable because:

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$$

Max, Min

$\max(f, g)$ and $\min(f, g)$ are integrable.

Where $\max$ and $\min$ functions are defined as:

$$\max(f, g) = \frac{1}{2}(|f-g| + f + g)$$

$$\min(f, g) = \frac{1}{2}(-|f-g| + f + g)$$

Additivity

$\iff$ f is Riemann integrable on $[a, c]$ and $[c, b]$ where $c \in (a, b)$.

Proof Hint

- $\implies$: Use Cauchy criterion after defining these:
  - $P' = \{c\} \cap P$
  - $Q = P' \cap [a, c]$
  - $R = P' \cap [c, b]$
- $\impliedby$: Use cauchy criterion on $[a, c], [c, b]$ separately and then combine using a union partition

After the integrability is proven,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof Hint

1. Let Q be a partition on $[a, c]$ and R be a partition on $[c, b]$. And $P = Q \cap R$.
2. Prove the below using Cauchy criteria:

$$\int_a^b f < L(f; P) + \epsilon \implies \int_a^b f \leq \int_a^c f + \int_c^b f$$

3. Prove the below using Cauchy criteria (by considering RHS):

$$\int_a^c f + \int_c^b f \leq \int_a^b f$$

Sequential Characterization of Integrability

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if $\exists \{P_n\}$ a sequence of partitions, such that:

$$\lim_{n\to\infty} \Big[U(f; P_n) - L(f; P_n)\Big] = 0$$

In that case:

$$\int_a^b f = \lim_{n\to\infty} U(f; P_n) = \lim_{n\to\infty} L(f; P_n)$$

Proof Hint

Cauchy criteria and squeeze theorem is used for both side proof.

For $\Longleftarrow$:

- Consider the limit definition.
- Prove f is Riemann integrable on P_n by Cauchy criteria.
- Use squeeze theorem for $U(f;P_n) - U(f) \le U(f;P_n) - L(f;P_n)$ to prove limit of upper sum
- Prove limit of lower sum using the limit of upper sum

For $\Longrightarrow$: Consider the below, where $n \in \mathbb{N}$.

$$0 \le U(f;P_n) - L(f;L_n) \le \frac{1}{n}$$

Theorem

Suppose f is Riemann integrable on $[a,b]$ and $\epsilon > 0$. Then $\exists \epsilon > 0 \forall P$:

$$|P| < \delta \implies \left| \int_a^b f - \sum_{j=1}^{n} f(\zeta_j) I_j \right| < \epsilon$$

where $\zeta_j \in [x_{j-1}, x_j], j = 1, 2, \cdots, n$.

Proof Hint

$$\underline{\int_a^b} f - \epsilon \; < \; L(f;P) \; \le \; \sum_{j=1}^{n} f(\zeta_j) I_j \; \le \; U(f;P) \; < \; \overline{\int_a^b} f + \epsilon$$