Summary | Fluid Mechanics

Introduction

Study of Fluids Mechanics include

- Fluid Statics - Equilibrium
- Fluid Kinematics - Physical aspects of motion
- Fluid Dynamics - Physical aspects of motion and causes of motion

Normal Forces

Forces acting perpendicular to the plane of a surface.

Shear Forces

Forces acting in the plane of a surface. When a fluid is at rest, no shear forces act on it.

Fluid

A fluid is defined as a substance which flows continuously under the action of shear forces no matter how small the forces may be.

Liquids and gasses are considered fluids. Unlike solids, fluids don’t show permanent resistance to deformation. Fluids are considered a continuum (continuously filled matter). We consider their bulk behaviours to solve fluid mechanics problems.

Properties at a point - defined for a fluid particle.

Fluid Particle

Very small volume of fluid containing the point concerned.

Average values of properties for fluid particles are considered. Properties are assumed to vary gradually between particles.

Properties of Fluids

Mass Density

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{m}{v}$$

At a point:

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v}$$

For liquids

Varies very slightly with temperature (negligible in calculations).

Example: Water

- $100 kgm^{-3}$ - at $4^{\circ} C$
- $995.7 kgm^{-3}$ - at $30^{\circ} C$

For gases

Highly dependent on pressure & temperature.

Specific Weight / Unit Weight

$$\omega = \gamma = \frac{\text{Weight}}{\text{Volume}} = \frac{w}{v}$$

Relative Density / Specific Density

$$s = \sigma = \frac{\text{Density of the substance}}{\text{Density of standard substance}}$$

For solids and liquids, water is the standard substance.

Bulk Modulus

$$k = \frac{\text{Change in pressure}}{\text{Change in volume, per volume}} = -\frac{\Delta p}{\frac{\Delta v}{v}} = -v\frac{\mathrm{d}p}{\mathrm{d}v}$$

In terms of the density:

$$k = \rho\frac{\mathrm{d}p}{\mathrm{d}\rho}$$

High bulk modulus means hard to compress.

Vapour Pressure

Vaporisation is when evaporation happens at the free surface of a liquid.

Vapour Pressure is the pressure due to liquid vapour just above the free surface of the liquid. Increases with temperature.

A liquid boils when: $\text{vapour pressure} = \text{external pressure on the liquid}$

Surface Tension

$$\sigma = \frac{\text{Tensile Force}}{\text{length}} = \frac{F}{L}$$

Negligible in many applications. Considered in small-scale applications. Causes capillary effect.

Viscosity

The force resisting the flow of a liquid.

In liquids, viscosity is mainly caused by inter-molecular attraction. Decreases slightly with temperature.

In gases, mainly due to momentum exchange between molecules. Increases with temperature.

Newton’s law of viscosity

In straight & parallel flow, the shear stress τ (as in $\frac{F}{A}$) between adjacent layers is proportional to the velocity gradient perpendicular to the layers.

$$\tau \propto \frac{\delta v}{\delta y} (= \text{velocity gradient})$$

As $\delta y \to 0$,

$$\tau = \mu \frac{\partial v}{\partial y}$$

Coefficient of dynamic viscosity

Above, μ is **coefficient of dynamic viscosity** or **coefficient of absolute viscosity** or **coefficient of viscosity**.

Fluids can be divided into 2 types:

- μ is a constant: Newtonian fluid
- μ is not a constant: Non-newtonian fluid (**not** focused on for s1)

Coefficient of kinematic viscosity

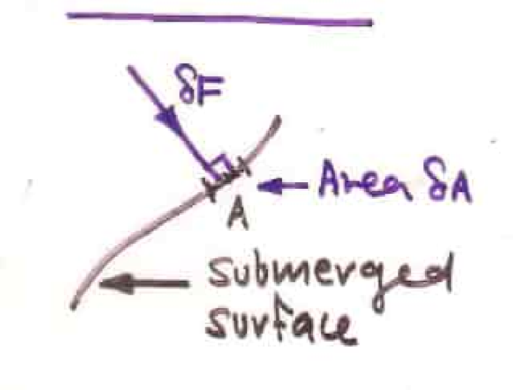
$$v = \frac{\mu}{\rho}$$

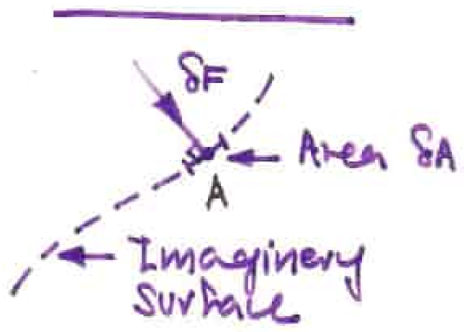
Pressure

A force is exerted on all surfaces in contact with a fluid. A scalar.

$$P = \frac{\text{Normal Force}}{\text{Area}} = \frac{F}{A}$$

Hydrostatic Pressure





At a point,

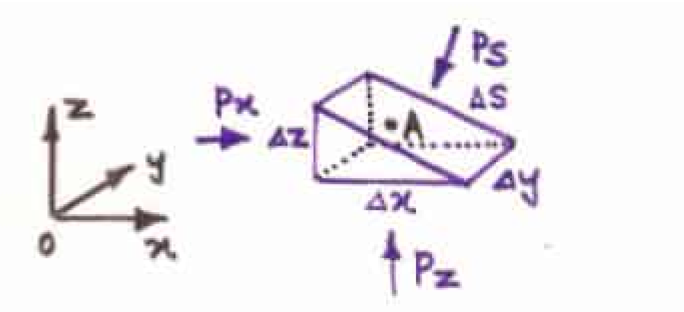
$$P = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}$$

Pascal's law

The hydrostatic pressure at a point is the same from all directions. Applies only for hydrostatic pressure.

Proof

Consider the fluid element shown, containing the point A.



From the image: $\sin\theta = \frac{\Delta z}{\Delta s} \ \wedge \ \cos\theta = \frac{\Delta x}{\Delta s}$

For equilibrium:

$$P_x(\Delta y \Delta z) - P_s(\Delta y \Delta s)\sin\theta = 0 \implies P_x = P_s$$

$$P_z(\Delta x\Delta y) - P_s(\Delta y\Delta s)\cos\theta - \frac{1}{2}\Delta x\Delta y\Delta z\rho g = 0 \implies P_z = P_s + \frac{1}{2}\Delta z\rho g$$

As all $\Delta x, \Delta y, \Delta z$ approaches 0: $P_z = P_s$. Therefore $P_x = P_z = P_s$

Variation along directions

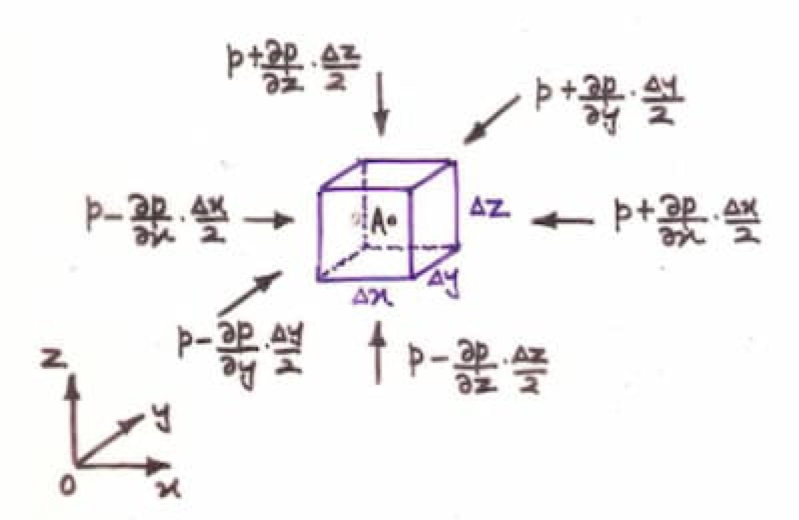
Proof

Let p be the pressure at the point $A \equiv (x, y, z)$.

$$p = f(x, y, z)$$

$$dp = \frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y + \frac{\partial p}{\partial z}\mathrm{d}z$$

By considering equilibrium of this fluid element containing A.



In the x direction,

$$\left(p - \frac{\partial p}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z - \left(p + \frac{\partial p}{\partial x}\frac{\Delta x}{2}\right)\Delta y\Delta z = 0$$

$$\frac{\partial p}{\partial x} = 0$$

Similarly $\frac{\partial p}{\partial y} = 0$ can be proven.

In the z direction,

$$\left(p - \frac{\partial p}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y - \left(p + \frac{\partial p}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y - \Delta x \Delta y \Delta z \rho g = 0$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$dp = -\rho g \, \mathrm{d}z$$

$$p = -\int \rho g \, \mathrm{d}z$$

> **For incompressible fluids**
>
> ρ is constant. $p = -\rho g \int \mathrm{d}z = -\rho g z + c = f(z)$.

Piezometric pressure

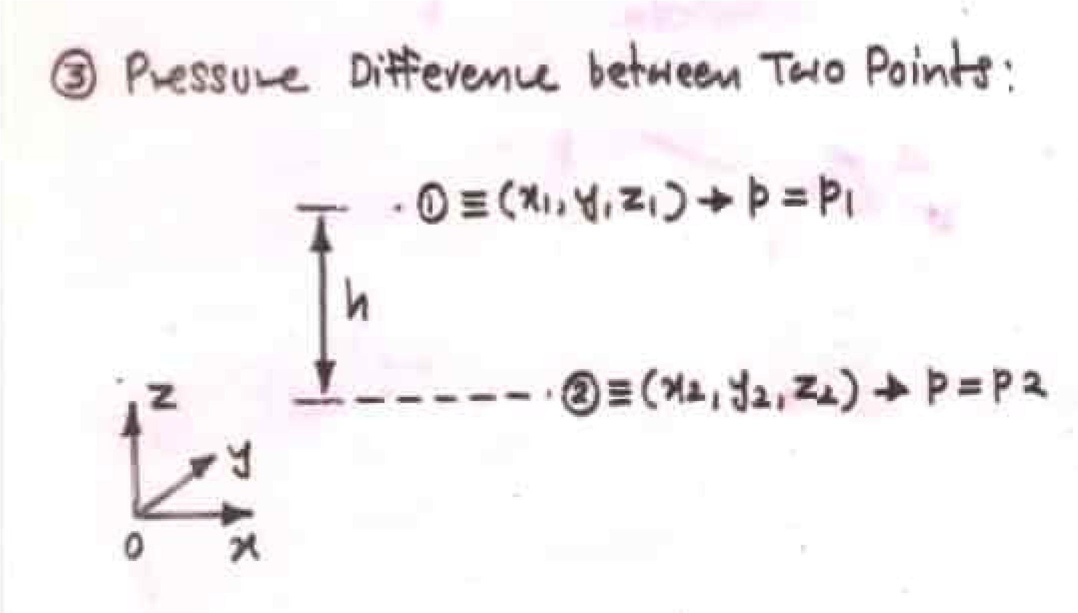
$$P = -\rho g z + c$$

$$P + \rho g z = c = P^*$$

Isobar

Surface of constant pressure.

Pressure difference between 2 points



$$P_1 = -\rho g z_1 + c$$

$$P_2 = -\rho g z_2 + c$$

$$P_2 - P_1 = -\rho g(z_2 - z_1) = -\rho g(-h) = h\rho g$$

$$P_2 = P_1 + h\rho g$$

In a homogenous, incompressible fluid in equilibrium:

1. Piezometric pressure is constant throughout the fluid
2. Pressure varies linearly with depth only
3. Isobars are horizontal

Pressure

Atmospheric Pressure

Pressure exerted by atmospheric air.

Gauge Pressure

Measured in respect to atmospheric pressure.

Absolute Pressure

Measured in respect to perfect vaccum.

$$\text{Absolute Pressure} = \text{Armospheric Pressure} + \text{Guage Pressure}$$

Pressure diagram

A diagram showing the variation of pressure along a submerged surface.

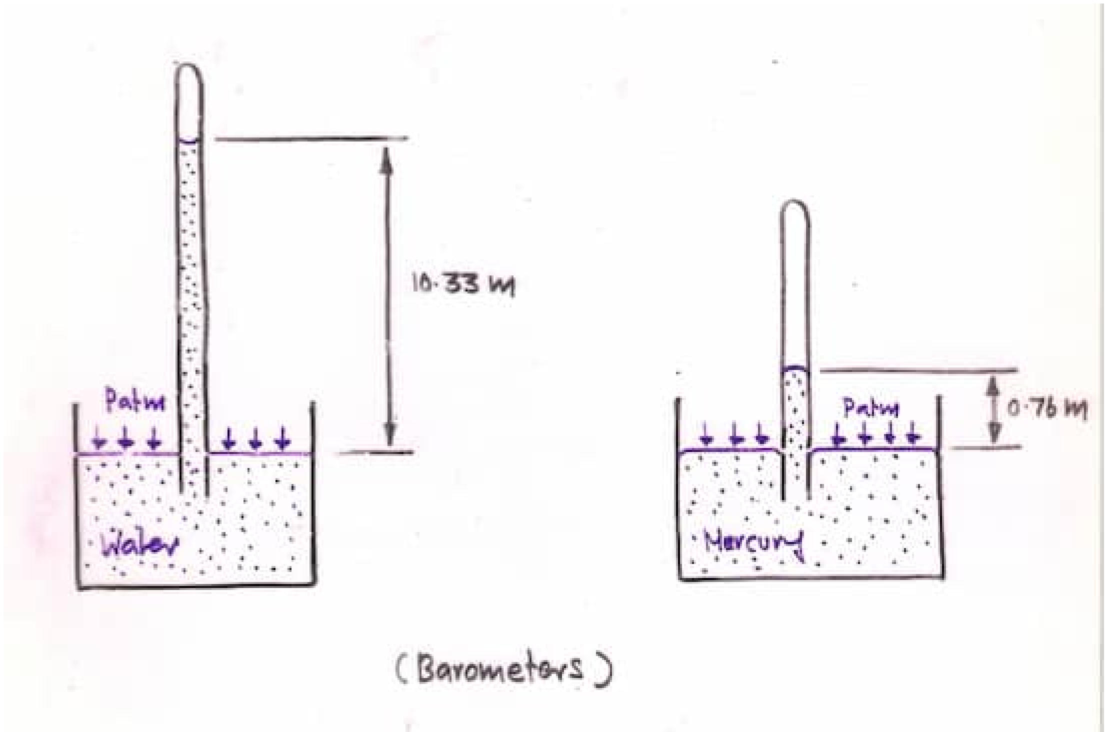
Pressure Head

Height of a particular fluid column that will produce the pressure at a point.

$$\text{Pressure head} = h = \frac{p}{\gamma}$$

Pressure Measurements

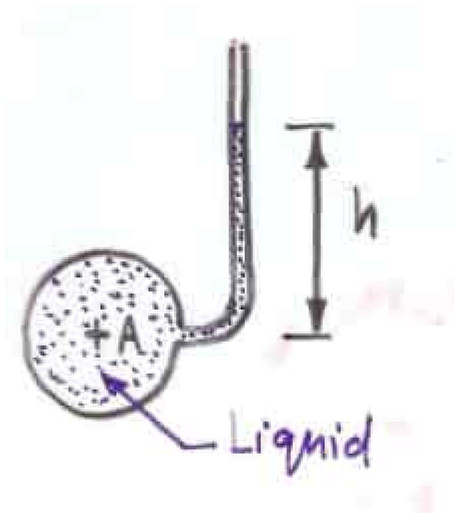
Barometer



Piezometer

Open-ended tube connected to a vessel / pipe containing a liquid.

Measures the pressure head of a liquid.



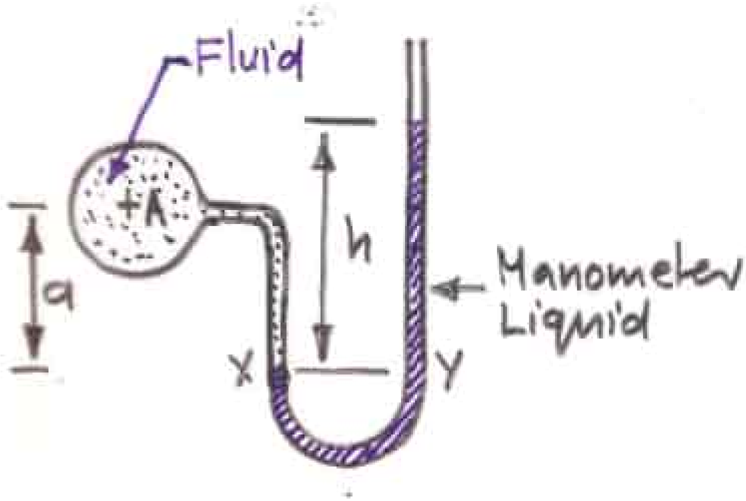
Advantages

- Simple
- Small pressure differences can be measured

Disadvantages

- Only for liquids
- Long tube required to measure even moderate pressures

Manometer



Hydrostatic principle is used here. Measures absolute pressue. Manometer liquid should not mix with the liquid in which the pressure is to be measured.

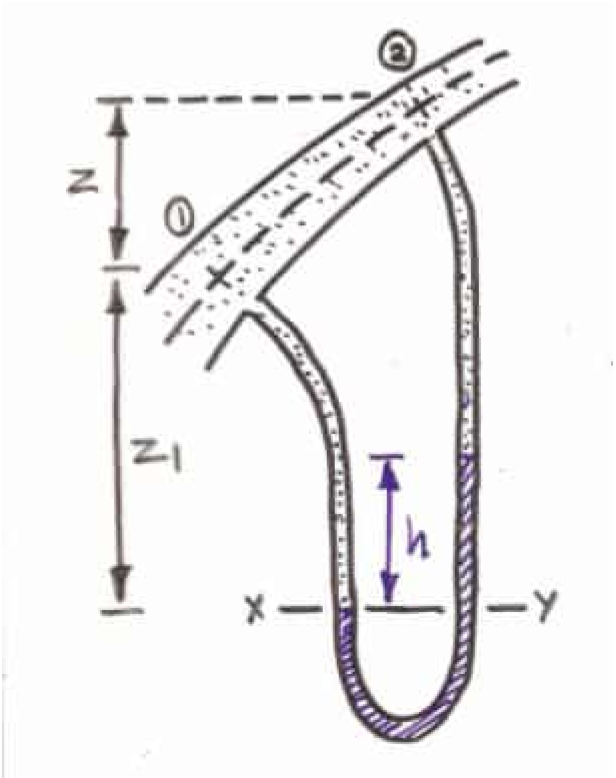
$$P_x = P_y$$

$$P_A + a\rho g = P_{\text{atm}} + h\rho_m g$$

$$P_A = P_{\text{atm}} + h\rho_m - a\rho g$$

If $P_{\text{atm}} = 0$, $P_A = h\rho_m - a\rho g$. That's gauge pressure.

Differential Manometer



Used to measure pressure difference between two points. Can be used for both liquids and gases. Difficult to measure small pressure differences (because small displacement of manometer liquid).

Pressure Gauges

Bourdon Pressure Gauge

Advantages

- Easy to use
- Wide range of pressures can be measured

Disadvantages

- Not very accurate
- Needs to be calibrated regularly

Hydrostatic Thrust

On a Plane Surface

Acts **normal to the surface** on the point on the surface known as **the Centre of Pressure** with a magnitude of:

$$\text{Thrust} = \text{submerged area} \times P_c$$

C is the centroid of the submerged area. P_c is the pressure at the centroid.

Centre of Pressure

$$y_p = y_c + \frac{I_{\text{cc}}}{A \cdot y_c}$$

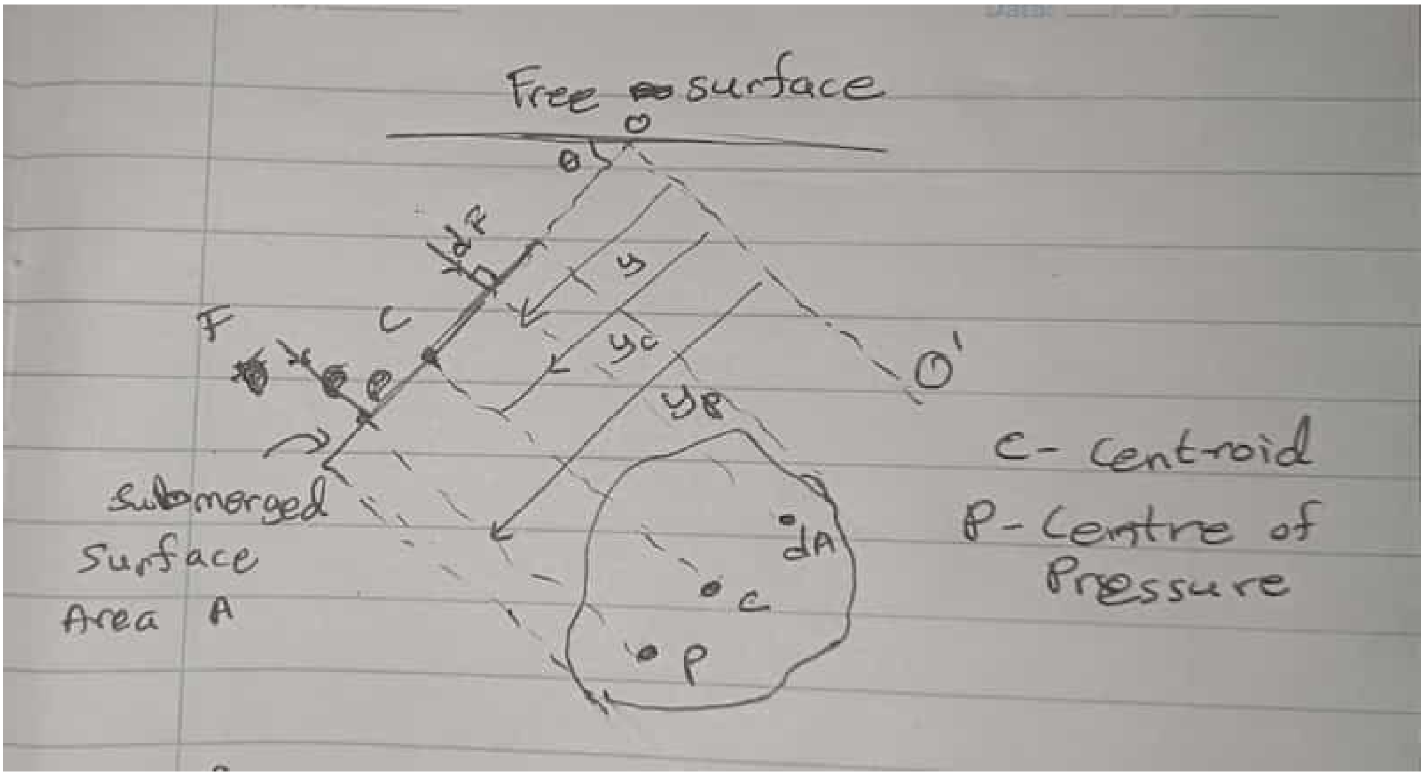
Here:

- A - Total submerged area
- y_p - Distance to centre of pressure measured along the submerged surface from the free surface
- y_c - Distance to C measured along the submerged surface from the free surface
- I_{cc} - Second moment of submerged area about the centroidal axis parallel to the free surface

For a plane surface

$$\frac{\text{Hydrostatic thrust}}{\text{Unit length}} = \text{Area of the pressure diagram}$$

Proof



Note

OO' is the free surface (waterline plane). **It is not a surface inside the fluid**. It's drawn like that for ease of reference.

Direction

All forces acting on the surface is normal to the surface. Therefore F is normal to the surface.

Magnitude

$$F = \int_A \mathrm{d}F = \int_A p\mathrm{d}A = \int_A y sin\theta\rho g\,\mathrm{d}A$$

$$F = sin\theta\rho g \int_A y\,\mathrm{d}A = sin\theta\rho g \cdot Ay_c = A \cdot y_c sin\theta\rho g$$

$$F = AP_c$$

Line of action

$$F \cdot y_p = \int_A y \, \mathrm{d}F$$

$$y_p = \frac{\int_A y \, \mathrm{d}F}{\int_A \mathrm{d}F} = \frac{\int_A y(y sin\theta \rho g) \, \mathrm{d}A}{\int_A y sin\theta \rho g \, \mathrm{d}A} = \frac{\int_A y^2 \, \mathrm{d}A}{\int_A y \, \mathrm{d}A}$$

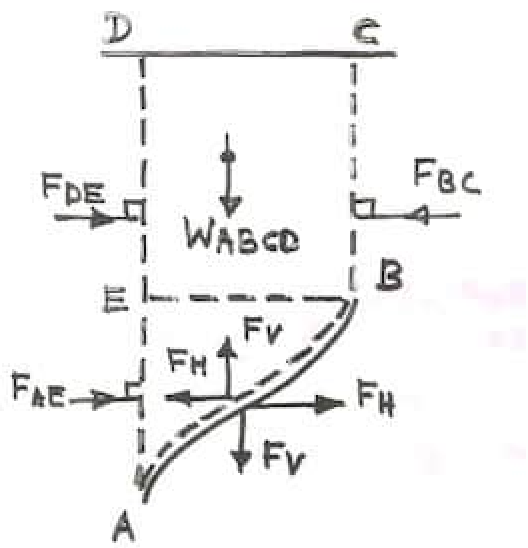
$$y_p = \frac{I_{oo}}{A y_c} = y_c + \frac{I_{cc}}{A y_c}$$

On a Curved Surface

$$F_x = \text{Thrust exerted on the vertical projection of the submerged surface}$$

$$F_y = \text{Weight of the fluid above submerged surface}$$

Proof



For the equilibrium of the fluid volume $ABCDA$.

$$F_y = W_{ABCDA}$$

For the equilibrium of the fluid volume $ABEA$.

$$F_x = F_{AE}$$

> **Tensile stress in pipe**
>
> For a pipe with inner diameter d and thickness t containing a liquid under pressure p, experiences a tensile stress $f = \frac{pd}{2t}$.

Centre of Pressure Derivations

Shape	**Description**	y_p
Rectangle or Parallelogram	Base b. Height h. Base is at the free surface.	$\frac{2h}{3}$
Triangle	Base b. Height h. Base is at the free surface.	$\frac{5h}{6}$

Buoyancy

Thrust exerted on a submerged object in a liquid. Direction is vertically upwards. Line of action passes through center of buoyancy.

$$u = \text{weight of the fluid volume displaced} = v\rho g$$

Here:

- u - the upthrust
- v - the submerged volume
- ρ - density of the fluid

Center of buoyancy

Center of gravity of the displaced fluid volume. **NOT** the center of gravity of the submerged object.

Proof

Forces exerted on the submerged object is equivalent to forces exerted on the displaced volume before it was displaced.

Consider the equilibrium of displaced volume before it was displaced:

$$\text{Weight} = W = \text{Resultant force exerted by surrounding liquid} = F$$

F must be equal to W, opposite to W and acts through G of the considered volume of fluid.

Stability of fully submerged bodies

Equilibrium type	Description
Stable	B is above G
Unstable	B is below G
Neutral	$B \equiv G$

Stability of floating bodies

Suppose a body of weight W acting through the centre of gravity G is floating in a fluid is at equilibrium. The buoyancy U acts through the centre of buoyancy B.

Metacentre

Intersection point between the line of action of U through B AND the axis BG. Denoted by M.

For small displacements M is fixed in position relative to the body.

Stability conditions

Equilibrium type	Description	Condition
Stable	M is above G	$GM > 0$
Unstable	M is below G	$GM < 0$

Equilibrium type	Description	Condition
Neutral	$M \equiv G$	$GM = 0$

Metacentric height

The distance GM. Measured upwards from G.

Metacentric radius

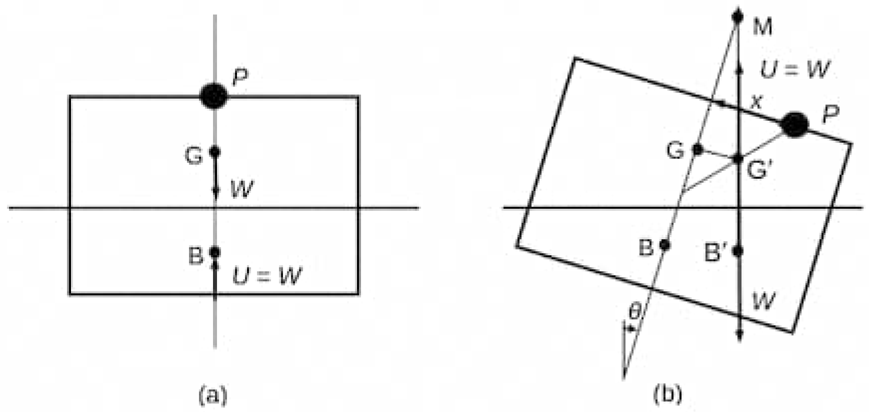
The distance BM. Measured upwards from B.

> **Note**
>
> Metacentric height and metacentric radius are related by: $BM = BG + GM$.

Determination of metacentric height

Experimental value

The metacentric height of a floating body can be determined experimentally by shifting a known weight by a known distance and measuring the angle of tilt.



(a) Initial equilibrium position

(b) Tilted equilibrium position

In the above picture

- P - a small mass
- G - initial centre of mass
- B - initial centre of buoyancy
- W - total weight of floating body
- U - upthrust exerted on floating body
- G' - new centre of mass
- B' - new centre of buoyancy
- x - small displacement applied to P

Considering the shift in centre of gravity:

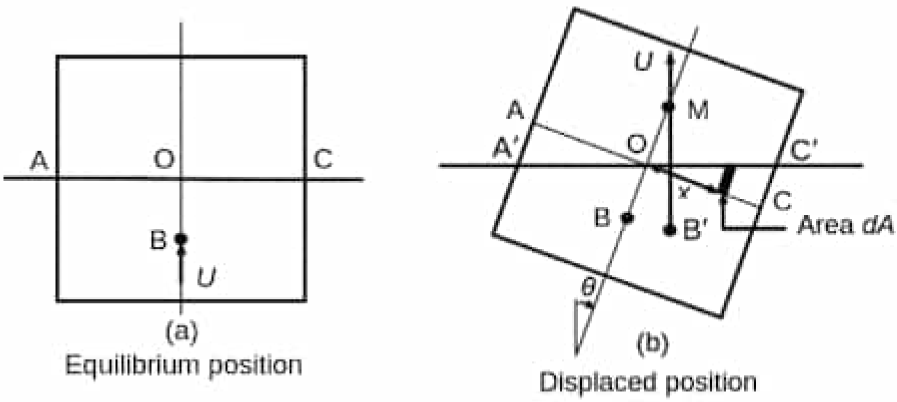
$$W(GG') = Px + 0(W - P) \implies GG' = \frac{Px}{W}$$

When θ is very small: $GG' = \frac{Px}{W} \approx (GM)\theta \implies GM \approx \frac{Px}{W\theta}$

$$GM = \lim_{\theta \to 0} \frac{Px}{W\theta}$$

Theoretical value

If the shape of the submerged volume is known, the metacentric height can theoretically be determined.



Rotation is about centroidal axis of waterline plane

As the submerged volume remains unchanged during angular displacement, it can be derived that the rotation occurs about the centroidal axis of the waterline plane.

$$\int_O^C x\tan\theta\,\mathrm{d}A = \int_O^A x\tan\theta\,\mathrm{d}A \implies \int_O^C x\,\mathrm{d}A = 0 = A\overline{x}$$

Here,

- A - area of waterline plane
- $\overline{x}$ - distance to the centroid from axis OO

Equation for metacentric radius

Considering the shift in centre of buoyancy:

$$U(BB') = \int_O^C x\theta\rho g\cdot x\,\mathrm{d}A - \int_O^A x\theta\rho g\cdot x\,\mathrm{d}A$$

$$V\rho g(BB') = \theta\rho g\left(\int_O^C x^2\,\mathrm{d}A - \int_O^A x^2\,\mathrm{d}A\right)$$

$$V(BB') = \theta\left(\int_A^C x^2\,\mathrm{d}A\right) = I\theta$$

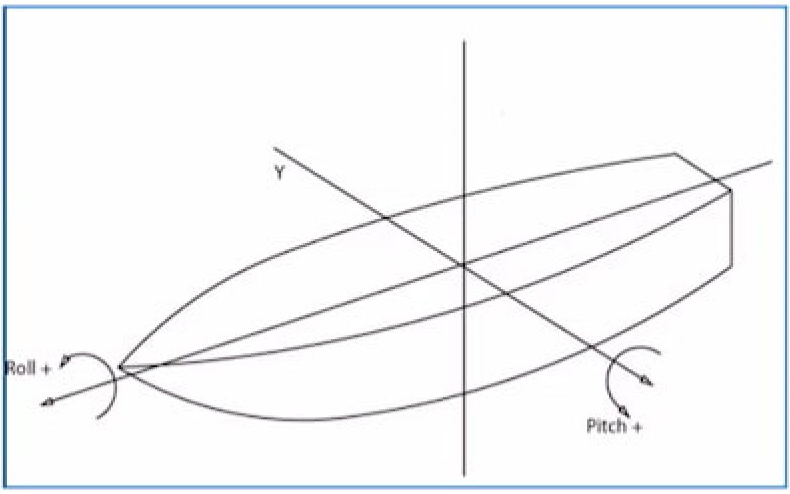
Here

- V - submerged volume
- I - second moment of area of the waterline plane about the centroidal axis OO

$$BB' = \frac{I\theta}{V} \approx (BM)\theta \implies BM = \frac{I}{V}$$

> **Note**
>
> This result is restricted to small angular displacements — usually up to about 8° — and the restriction is particularly important when the sides of the floating body are not vertical.

Types of tilting



- Pitching - tilting about transverse axis
- Rolling - tilting about longitudinal axis

Time period of oscillation

Below equation can be derived by using $T = I\ddot{\theta}$ (for small θ):

$$T = -W(\mathrm{GM})\sin\theta = I_G\ddot{\theta}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{W(\mathrm{GM})}{I_G}\theta = -\frac{Mg(\mathrm{GM})}{Mk^2}\theta$$

Here

- k - Radius of gyration
- M - Total mass
- I_G - Moment of inertia of the floating body about G

Period of time of oscillation is given by:

$$T = \frac{2\pi k}{\sqrt{g(\mathrm{GM})}}$$

Liquid cargo in a vessel

- Liquid cargo in a vessel reduces its geocentric height.
- When the cargo is contained in 1 compartment: $\Delta \mathrm{GM}_1 = \frac{\rho_1 I_1}{\rho v}$
- When the liquid cargo is contained in n compartments: $\Delta \mathrm{GM} = \frac{1}{n^2}(\Delta \mathrm{GM}_1)$

Relative Equilibrium

When a fluid-contained vessel moves with a constant acceleration it will be transmitted to the fluid. The fluid particles will move to a new position and remain in such position in equilibrium, relative to the vessel. Such equilibrium is known as the Relative Equilibrium of a fluid.

Under linear acceleration

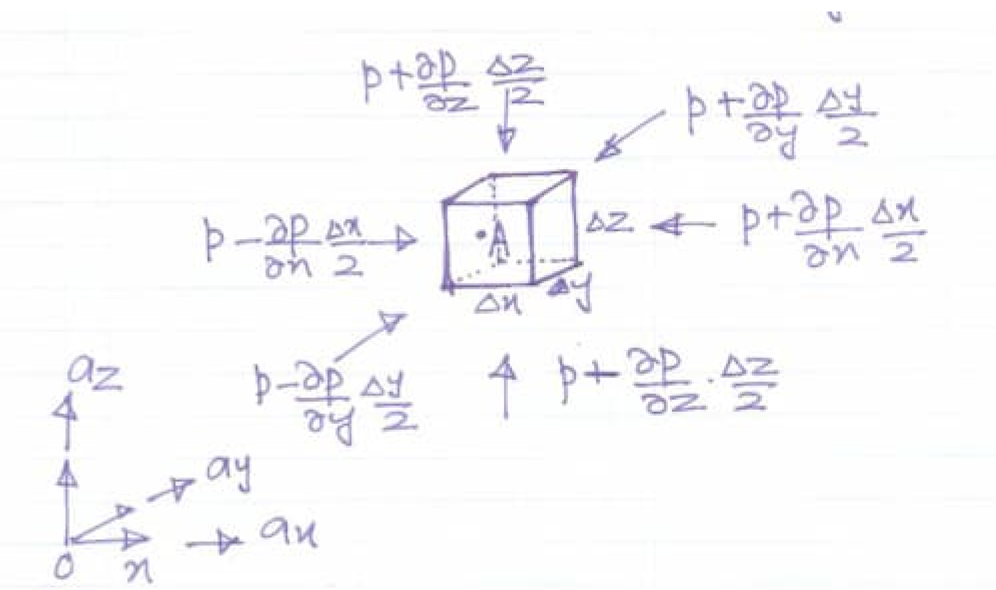
No flow of the fluid (relative to the fluid particles). No shear forces, and all forces are normal to the surface they act on. Hence, fluid statics equations can be used in relative equilibrium.

Variation of pressure

Let $P = f(x, y, z)$.

$$\mathrm{d}p = \frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y + \frac{\partial p}{\partial z}\mathrm{d}z$$

Consider the fluid element containing point A which is under an acceleration of a_x, a_y, a_z in the x, y, z directions.



By applying Newton's second law of motion in all 3 directions:

$$\frac{\partial p}{\partial x} = -\rho a_x \ \wedge \ \frac{\partial p}{\partial y} = -\rho a_y$$

$$\frac{\partial p}{\partial z} = -\rho(a_z + g)$$

Substituting all the terms:

$$\mathrm{d}p = -\rho a_x \mathrm{d}x - \rho a_y \mathrm{d}y - \rho(a_z + g)\mathrm{d}z$$

Integrating both sides:

$$P = -\rho a_x x - \rho a_y y - \rho(a_z + g)z + c_1$$

Shape of free surface

On the free surface $P = 0$ as gauge pressure is considered.

$$\rho a_x x + \rho a_y y + \rho(a_z + g)z = c_1$$

Free surface is a plane surface in 3D.

Inclination with horizontal plane

> **Caution**
>
> I am unsure whether this section is 100% correct.

Suppose a vessel is in acceleration in a_x, a_z in x, z directions and $a_y = 0$.

If θ_x, θ_y are the angles in x, y directions.

$$\tan(\theta_x) = \frac{\mathrm{d}z}{\mathrm{d}x} \ \wedge \ \tan(\theta_y) = \frac{\mathrm{d}z}{\mathrm{d}y}$$

Differentiating the equation of the free surface with respect to x.

$$\rho a_x + \rho(a_z + g)\frac{\mathrm{d}z}{\mathrm{d}x} = 0 \implies \tan(\theta_x) = \frac{-a_x}{a_z + g}$$

And similarly for y:

$$\rho a_y + \rho(a_z + g)\frac{\mathrm{d}z}{\mathrm{d}y} = 0 \implies \tan(\theta_y) = \frac{-a_y}{a_z + g}$$

Under Horizontal Acceleration

$$a_x \neq 0 \ \wedge \ a_y = a_z = 0$$

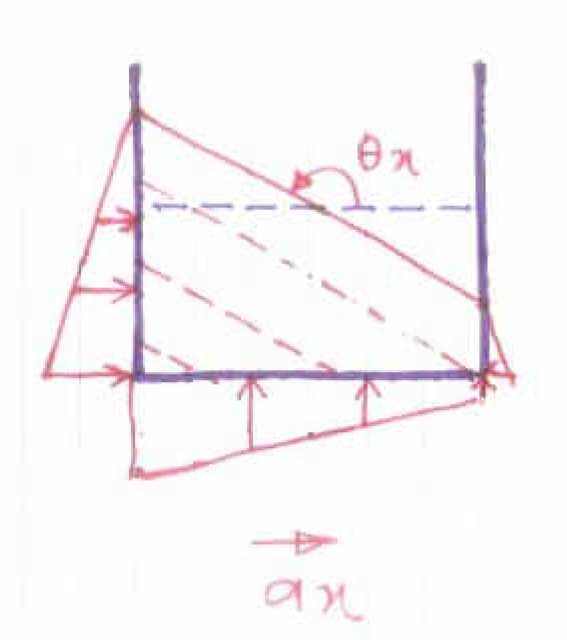
Equation of the free surface

$$\rho a_x x + \rho g z = c_1$$

Is a straight line in x, z axes. The straight line is at an inclination of θ_x:

$$\tan(\theta_x) = \frac{-a_x}{g}$$

Vertical Pressure Distribution



Under Vertical Acceleration

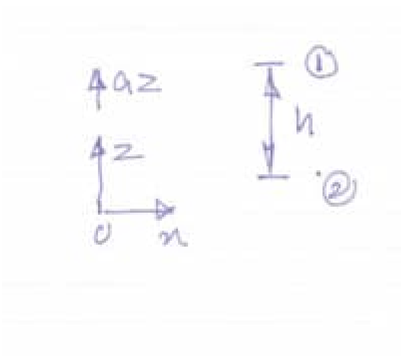
$$a_z \neq 0 \ \wedge \ a_x = a_y = 0$$

Equation of the free surface

$$\rho(a_z + g)z = c_1$$

Horizontal straight line.

Vertical Pressure Distribution



$$P_1 = -\rho(a_z + g)z_1 + c_1$$

$$P_2 = -\rho(a_z + g)z_2 + c_1$$

$$P_2 - P_1 = -\rho(a_z + g)(z_2 - z_1)$$

$$P_2 = h\rho(a_z + g)$$

Here:

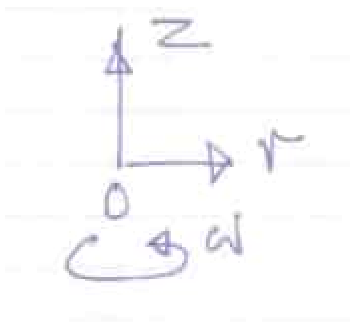
- $h\rho g$ - hydrostatic pressure
- $h\rho a_z$ - due to a_z

Varies only in z direction. Increases with height. Isobars are horizontal.

Forced Vortex Motion

Another type of relative equilibrium. If a fluid-contained vessel is rotating at a constant angular velocity, the fluid will reach a relative equilibrium position and rotate with the vessel. Under this condition, the fluid is said to be in Forced Vortex Motion.

> **Note**
>
> For S1, forced vortex motion, only about vertical axis, is required.



$$P = \frac{1}{2}\rho\omega^2 r^2 - \rho g z + c$$

Here:

- ω - angular velocity

> ⚠ **TODO**
>
> Explain how to derive the above equation.

Equation of the free surface

On the free surface $P = 0$.

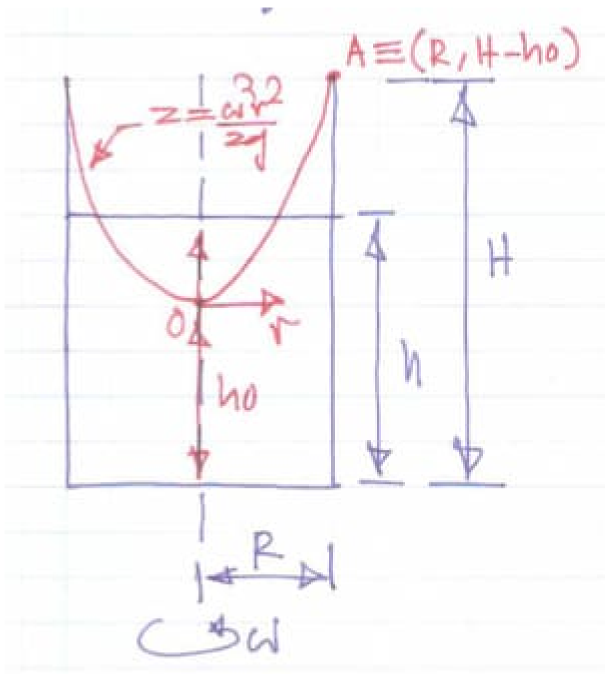
$$z = \frac{\omega^2}{2g} r^2 + c_0$$

The free surface is **parabolic**. The constant part c_0 can be found by a known point of the free surface. For ease of calculations, the axes can be chosen so that the free surface passes through $(0, 0)$. In that case, $c_0 = 0$.

Vertical Pressure Distribution

Pressure increases linearly with height. Increases exponentially with radial distance. Isobars are parabolic.

Volume of the fluid



Total volume of the fluid is:

$$V = \pi R^2 h_0 + \frac{1}{2}\pi R^2 (H - h_0) = \pi R^2 h$$

$$\implies V = \frac{1}{2}\pi R^2 (H + h_0) = \pi R^2 h$$

$$\implies H + h_0 = 2h$$