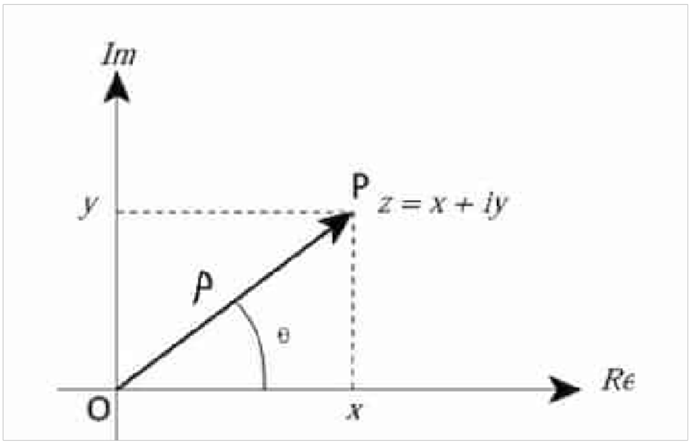
Summary | Complex Numbers

Introduction

Representation methods



The methods are:

- Cartesian representation: $z = x + iy$
- Polar representation: $z = pe^{i\theta}$

Here:

- $x = p\cos\theta$ - real part
- $y = p\sin\theta$ - imaginary part
- $p = \sqrt{x^2 + y^2}$ - modulus
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ - arg angle

Euler's Formula

For $x \in \mathbb{R}$:

$$e^{ix} = \cos x + i\sin x$$

ⓘ **Proof Hint**

Use Taylor series for $e^x, \cos x, \sin x$.

Euler's Identity

One of the most beautiful equations in mathematics.

$$e^{i\pi} + 1 = 0$$

Roots of Unity

n-th roots of unity (1) are the complex numbers that satisfy the equation, $z^n = 1$. There are n distinct solutions.

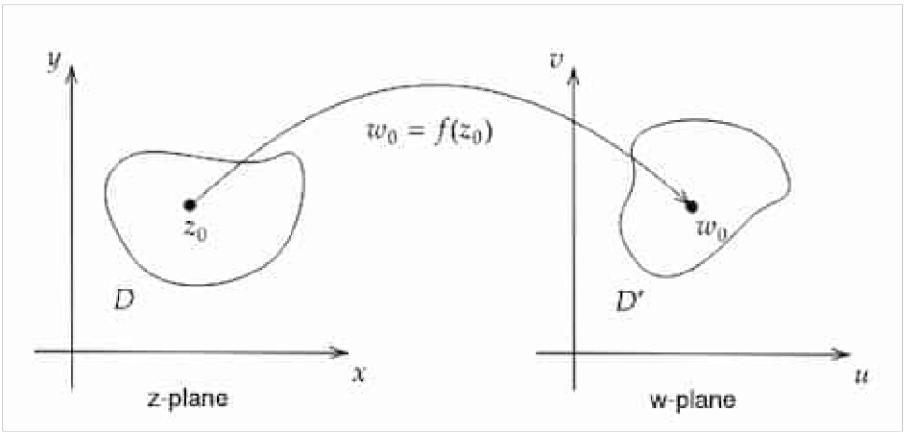
$$z = \exp\left(i\left(\frac{2m\pi}{n}\right)\right) \quad \text{where} \quad m \in \mathbb{Z} \cup [0, n)$$

The solution can be written as $1, w, w^2, w^3, \ldots, w^{n-1}$.

1 is called the trivial solution. Other solutions are called as primitive n-th roots.

Complex Functions

Suppose $w = f(z)$ where $z, w \in \mathbb{C}$. Input and output points are marked in 2 separate complex planes.



Here:

- D - domain of f
- D' - codomain of f

Image

Image of f is the set:

$$\{f(z) \mid z \in D\}$$

Cartesian form

$$f(z) = u(x, y) + iv(x, y)$$

Here u, v are real functions.

Limits

$\lim_{z \to z_0} f(z) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall z \; \left(0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon\right)$$

Complex limit properties are similar to real limits.

Difference from real functions

For real functions, when considering the limit at a point, the limit could be be approaching the point either from left or right.

For complex functions, the point can be approached along any path in the complex plane. The distance $|z - z_0|$ decreases to 0.

Disproving limits

One way of disproving a complex limit is to choose 2 different paths and showing the limits on each path are different. This is similar to showing the right and left limits are different in real analysis.

Real and imaginary limits

Suppose $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, $z = x + iy$, and:

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = L_1 \qquad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = L_2$$

(The real part and imaginary part limits to L_1, L_2), Then:

$$\lim_{z\to z_0} f(z) = L_1 + iL_2$$

Important limits

$$\lim_{z\to 0} \frac{z}{\overline{z}} \text{ doesn't exist}$$

The above limit is important as it shows up in many questions.

Continuity

$f(z)$ is continuous at z_0 **iff**:

$$\lim_{z\to z_0} f(z) = f(z_0)$$

$$\iff \forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \, \big(|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon \big)$$

Differentiability

A complex function f is differentiable at z_0 **iff**:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L = f'(z_0)$$

$f'(z_0)$ is called the derivative of f at z_0. The rules for differentiation in real functions can also be applied to complex functions. So, go through Differentiability — Real Analysis.

Singular point

If $f(z)$ is not differentiable at z_0 then z_0 is called a singular point of $f(z)$.

Neighbourhood

Suppose $z_0 \in \mathbb{C}$. A neighborhood of z_0 is the region contained in the circle $|z - z_0| = r > 0$.

Analytic

A function f is said to be analytic at z_0 **iff** it is differentiable throughout a neighbourhood of z_0.

Analytic implies differentiable

$$f \text{ is analytic at } z_0 \implies f \text{ is differentiable at } z_0$$

Cauchy Riemann Equations

The set of equations mentioned below are the Cauchy Riemann Equations, where u, v are functions of x, y.

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial v}{\partial y} = v_y \quad \wedge \quad \frac{\partial u}{\partial y} = u_y = -\frac{\partial v}{\partial x} = -v_x$$

Theorem 1

Suppose $f(z) = u(x, y) + iv(x, y)$, and f is differentiable at z_0. Then

- All partial derivatives u_x, u_y, v_x, v_y exist
- They satisfy the Cauchy Riemann equations

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

> **Note**
>
> Contrapositive is useful when proving f is **not** differentiable at z_0.

Theorem 2

Suppose $f(z) = u(x, y) + iv(x, y)$. All partial derivatives exist, and they are all continuous at z_0. Then f is differentiable at z_0. And:

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

Theorem 3

If f is analytic at z_0, then its first-order partial derivatives are continuous in a neighbourhood of z_0.

Entire Functions

A complex function that is differentiable everywhere. Entire functions are analytic everywhere.

Examples:

- polynomial functions
- e^z

Counter examples:

- Rational functions are not entire functions