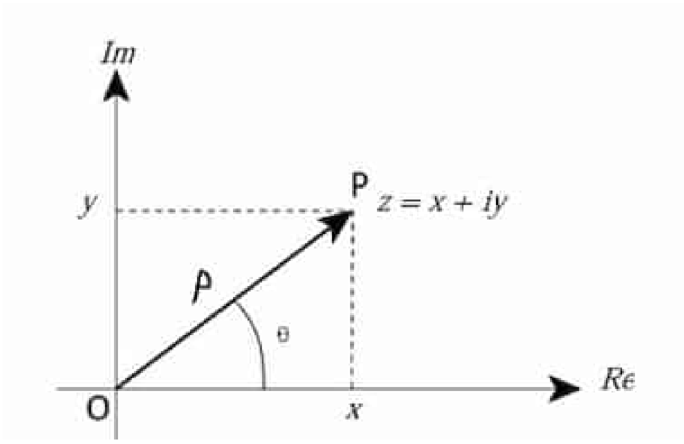
Summary | Complex Numbers

Introduction

Representation methods



The methods are:

- Cartesian representation: $z = x + iy$
- Polar representation: $z = pe^{i\theta}$

Here:

- $x = p\cos\theta$ - real part
- $y = p\sin\theta$ - imaginary part
- $p = \sqrt{x^2 + y^2}$ - modulus
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ - arg angle

Euler's Formula

For $x \in \mathbb{R}$:

$$e^{ix} = \cos x + i\sin x$$

Proof Hint

Use power series for e^x, $\cos x$, $\sin x$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Euler’s Identity

One of the most beautiful equations in mathematics.

$$e^{i\pi} + 1 = 0$$

Roots of Unity

n-th roots of unity (1) are the complex numbers that satisfy the equation, $z^n = 1$. There are n distinct solutions.

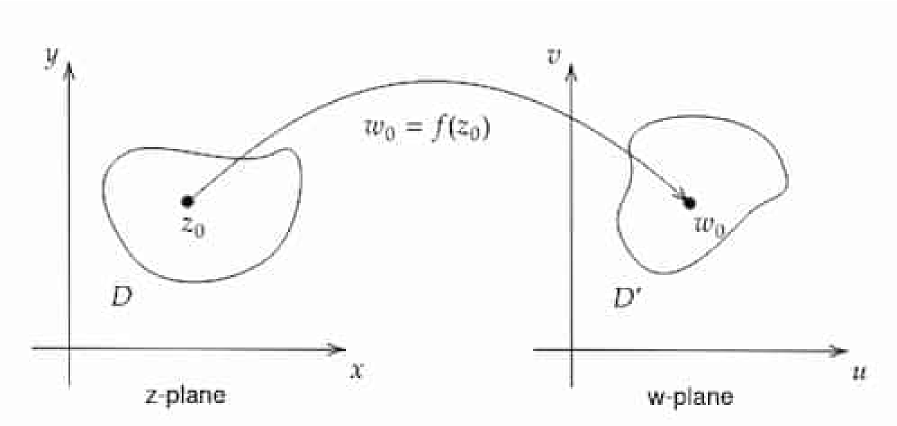
$$z = \exp\left(i\left(\frac{2m\pi}{n}\right)\right) \quad \text{where} \quad m \in \mathbb{Z} \cup [0, n)$$

The solution can be written as $1, w, w^2, w^3, \ldots, w^{n-1}$.

1 is called the trivial solution. Other solutions are primitive n-th roots.

Complex Functions

Suppose $w = f(z)$ where $z, w \in \mathbb{C}$. Input and output points are denoted in 2 separate complex planes.



Here:

- D - domain of f
- D' - codomain of f

Image

Image of f is the set:

$$\{f(z) \mid z \in D\}$$

Cartesian form

$$f(z) = u(x, y) + iv(x, y)$$

Here

$$u, v$$

are real functions.

Limit of Complex Functions

$\lim_{z \to z_0} f(z) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall z \; (0 < |z - z_0| < \delta \implies |f(z) - L| < \epsilon)$$

Complex limit properties are similar to real limits.

Difference from real functions

For real functions, when considering the limit at a point, we could approach the point either from the left or from the right.

For complex functions, the point can be approached along any path in the complex plane. The distance $|z - z_0|$ decreases to 0.

Real and imaginary limits

Suppose $f(z) = u(x, y) + iv(x, y)$, $\lim_{(x,y) \to (x_0, y_0)} u(x, y) = L_1$ and $\lim_{(x,y) \to (x_0, y_0)} v(x, y) = L_2$, where $z_0 = x_0 + iy_0$ and $z = x + iy$. Then $\lim_{z \to z_0} f(z) = L_1 + iL_2$.

Continuity

$f(z)$ is continuous at z_0 **iff**:

$$\lim_{z \to z_0} f(z) = f(z_0)$$

$$\iff \forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon)$$

Differentiability

A complex function f is differentiable at z_0 **iff**:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L = f'(z_0)$$

$f'(z_0)$ is called the derivative of f at z_0. The rules for differentiation in real functions can also be applied to complex functions. So, go through [Differentiability — Real Analysis].

Singular point

If $f(z)$ is not differentiable at z_0 then z_0 is called a singular point of $f(z)$.

Neighbourhood

Suppose $z_0 \in \mathbb{C}$. A neighborhood of z_0 is the region contained in the circle $|z - z_0| = r > 0$.

Analytic

A function f is said to be analytic at z_0 **iff** it is differentiable throughout a neighbourhood of z_0.

Analytic implies differentiable

$$f \text{ is analytic at } z_0 \implies f \text{ is differentiable at } z_0$$

Cauchy Riemann Equations

The set of equations mentioned below are the Cauchy Riemann Equations, where u, v are functions of x, y.

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial v}{\partial y} = v_y \quad \wedge \quad \frac{\partial u}{\partial y} = u_y = -\frac{\partial v}{\partial x} = -v_x$$

Theorem 1

Suppose $f(z) = u(x, y) + iv(x, y)$, and f is differentiable at z_0. Then

- All partial derivatives u_x, u_y, v_x, v_y exist
- They satisfy the Cauchy Riemann equations

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

Note

Contrapositive is useful when proving f is **not** differentiable at z_0.

Theorem 2

Suppose $f(z) = u(x, y) + iv(x, y)$. All partial derivatives exist, and they are all continuous at z_0. Then f is differentiable at z_0. And:

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

Theorem 3

If f is analytic at z_0, then its first-order partial derivatives are continuous in a neighbourhood of z_0.

Entire Functions

A complex function that is differentiable everywhere. Entire functions are analytic everywhere.

Examples:

- polynomial functions
- e^z

Counter examples:

- Rational functions are not entire functions