

# Summary | Riemann Integration

---

## Introduction

### Interval

Let  $I = [a, b]$ . Length of the interval  $|I| = b - a$ .

### Disjoint interval

When 2 intervals don't share any common numbers.

### Almost disjoint interval

When 2 intervals are disjoint or intersect only at a common endpoint.

## Riemann Integral

Let  $f : [a, b] \rightarrow \mathbb{R}$  is a bounded (not necessarily continuous) function on a closed, bounded (compact) interval.

Riemann integral of  $f$  is:  $\int_a^b f$

### Definite integral

When  $a, b$  are constants.

### Indefinite integral

When  $a$  is a constant but  $b$  is replaced with  $x$ .

## Partition

Let  $I$  be a non-empty, compact interval (closed and bounded). A partition of  $I$  is a finite collection  $\{I_1, I_2, \dots, I_n\}$  of almost disjoint, non-empty, compact sub-intervals whose union is  $I$ .

A partition is determined by the endpoints of all sub-intervals:

$$a = x_0 < x_1 < \cdots < x_n = b.$$

A partition can be denoted by:

- its intervals -  $P = \{I_1, I_2, \dots, I_n\}$
- the endpoints of its intervals -  $P = \{x_0, x_1, \dots, x_n\}$

## Riemann Sum

Let

- $f : [a, b] \rightarrow \mathbb{R}$  is a bounded function on the compact interval  $I = [a, b]$  with  $M = \sup_I f$  and  $m = \inf_I f$ .
- $P = \{I_1, I_2, \dots, I_n\}$
- $M_k = \sup_{I_k} f = \sup \{f(x) : x \in [x_{k-1}, x_k]\}$
- $m_k = \inf_{I_k} f = \inf \{f(x) : x \in [x_{k-1}, x_k]\}$

### Upper riemann sum

$$U(f; P) = \sum_{k=1}^n M_k |I_k|$$

### Lower riemann sum

$$L(f; P) = \sum_{k=1}^n m_k |I_k|$$

$$m_k < M_k \implies L(f; P) \leq U(f; P)$$

When  $P_1, P_2$  are any 2 partitions of  $I$ :  $L(f; P_1) \leq U(f; P_2)$

## Refinements

$Q$  is called a refinement of  $P \iff$  if  $P$  and  $Q$  are partitions of  $[a, b]$  and  $P \subseteq Q$ .

When  $Q$  is a refinement of  $P$ :

$$L(f; P) \leq L(f; Q) \leq U(f; Q) \leq U(f; P)$$

### **Note**

If  $P_1$  and  $P_2$  are partitions of  $[a, b]$ , then  $Q = P_1 \cup P_2$  is a refinement of both  $P_1$  and  $P_2$ . In that case:

$$L(f; P_1) \leq L(f; Q) \leq U(f; Q) \leq U(f; P_2)$$

## Upper & Lower integral

Let  $\mathbb{P}$  be the collection of all possible partitions of the interval  $[a, b]$ .

### Upper Integral

$$U(f) = \inf \{U(f; P); P \in \mathbb{P}\} = \overline{\int_a^b f}$$

### Lower Integral

$$L(f) = \sup \{L(f; P); P \in \mathbb{P}\} = \underline{\int_a^b f}$$

For a bounded function  $f$ , always  $L(f) \leq U(f)$

## Riemann Integrable

A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable on  $[a, b]$  **iff**  $U(f) = L(f)$ . In that case, the Riemann integral of  $f$  on  $[a, b]$  is denoted by  $\int_a^b f(x) \, dx$ .

### Reimann Integrable or not

| Function  | Yes or No? | Proof hint    |
|-----------|------------|---------------|
| Unbounded | No         | By definition |

| Function                            | Yes or No? | Proof hint  |
|-------------------------------------|------------|---|
| Constant                            | Yes        | $\forall P$ (any partition) $L(f; P) = U(f; P)$                             |
| Monotonically increasing/decreasing | Yes        | Take a partition such that $\Delta x < \delta = \frac{\epsilon}{f(b)-f(a)}$ |
| Continuous                          | Yes        | Take a partition such that $\Delta x < \delta = \frac{\epsilon}{b-a}$       |

### Note

If the set of points of discontinuity of a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is finite, then  $f$  is Riemann integrable on  $[a, b]$ .

### Note

If the set of points of discontinuity of a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is finite number of limit points, then  $f$  is integrable on  $[a, b]$ .

A function may have infinitely many discontinuous points, but if the set of all discontinuous points have finite number of limit points, then  $f$  is integrable on  $[a, b]$ .

## Cauchy Criterion

### Theorem

A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable **iff** for every  $\epsilon > 0$  there exists a partition  $P_\epsilon$  of  $[a, b]$ , which may depend on  $\epsilon$ , such that:

$$U(f, P_\epsilon) - L(f, P_\epsilon) \leq \epsilon$$

### Proof Hint

- To prove  $\implies$  : consider  $L(f) - \frac{\epsilon}{2}$  and  $U(f) + \frac{\epsilon}{2}$
- To prove  $\impliedby$  : consider  $L(f; P) < L(f) \wedge U(f) < U(f; P)$