Summary | Riemann Integration

Introduction

Interval

Let I = [a,b]. Length of the interval |I| = b - a.

Disjoint interval

When 2 intervals don't share any common numbers.

Almost disjoint interval

When 2 intervals are disjoint or intersect only at a common endpoint.

Riemann Integral

Let $f-[a,b] o \mathbb{R}$ is a bounded (not necessarily continuous) function on a closed, bounded (compact) interval.

Riemann integral of f is: $\int_a^b f$

Definite integral

When a, b are constants.

Indefinite integral

When a is a constant but b is replaced with x.

Partition

Let I be a non-empty, compact interval (closed and bounded). A partition of I is a finite collection $\{I_1,I_2,\ldots,I_n\}$ of almost disjoint, non-empty, compact sub-intervals whose union is I.

A partition is determined by the endpoints of all sub-intervals: $a = x_0 < x_1 < \cdots < x_n = b$.

A partition can be denoted by:

- ullet its intervals $P=\{I_1,I_2,\ldots,I_n\}$
- ullet the endpoints of its intervals $P=\{x_0,x_1,\ldots,x_n\}$

Riemann Sum

Let

- $f:[a,b] o \mathbb{R}$ is a bounded function on the compact interval I=[a,b] with $M=\sup_I f$ and $m=\inf_I f$.
- $P = \{I_1, I_2, \dots, I_n\}$
- $M_k = \sup_{I_k} f = \sup \left\{ f(x) : x \in [x_{k-1}, x_k]
 ight\}$
- $\bullet \hspace{0.3cm} m_k=\inf_{I_k}f=\inf\left\{f(x):x\in[x_{k-1},x_k]\right\}$

Upper riemann sum

$$U(f;P) = \sum_{k=1}^n M_k |I_k|$$

Lower riemann sum

$$L(f;P) = \sum_{k=1}^n m_k |I_k|$$

$$m_k < M_k \implies L(f;P) \le U(f;P)$$

When P_1, P_2 are any 2 partitions of I: $L(f; P_1) \leq U(f; P_2)$

Refinements

Q is called a refinement of $P\iff$ if P and Q are partitions of [a,b] and $P\subseteq Q$.

When $oldsymbol{Q}$ is a refinement of $oldsymbol{P}$:

$$L(f; P) \le L(f; Q) \le U(f; Q) \le U(f; P)$$

(i) Note

If P_1 and P_2 are partitions of [a,b], then $Q=P_1\cup P_2$ is a refinement of both P_1 and P_2 . In that case:

$$L(f; P_1) \leq L(f; Q) \leq U(f; Q) \leq U(f; P_2)$$

Upper & Lower integral

Let $\mathbb P$ be the collection of all possible partitions of the interval [a,b].

Upper Integral

$$U(f)=\inf\left\{U(f;P);P\in\mathbb{P}
ight\}=\overline{\int_a^bf}$$

Lower Integral

$$L(f)=\sup\left\{L(f;P);P\in\mathbb{P}
ight\}=\underline{\int_a^bf}$$

For a bounded function f, always $L(f) \leq U(f)$

Riemann Integrable

A bounded function $f:[a,b] o\mathbb{R}$ is Riemann integrable on [a,b] iff U(f)=L(f). In that case, the Riemann integral of f on [a,b] is denoted by $\int_a^b f(x)\,\mathrm{d}x$.

Reimann Integrable or not

Function	Yes or No?	Proof hint
Unbounded	No	By definition
Constant	Yes	$orall P ext{ (any partition) } L(f;P) = U(f;P)$
Monotonically increasing/decreasing	Yes	Take a partition such that $\Delta x < \delta = rac{\epsilon}{f(b) - f(a)}$
Continuous	Yes	Take a partition such that $\Delta x < \delta = rac{\epsilon}{2(b-a)}$

(i) Note

If the set of points of discontinuity of a bounded function $f:[a,b] \to \mathbb{R}$ is finite, then f is Riemann integrable on [a,b].

(i) Note

If the set of points of discontinuity of a bounded function $f:[a,b]\to\mathbb{R}$ is finite number of limit points, then f is integrable on [a,b].

A function may have infinitely many discontinuous points, but if the set of all discontinuous points have finite number of limit points, then f is integrable on [a, b].

Cauchy Criterion

Theorem

A bounded function $f:[a,b] \to R$ is Riemann integrable iff for every $\epsilon>0$ there exists a partition P_ϵ of [a,b], which may depend on ϵ , such that:

$$U(f, P\epsilon) - L(f, P\epsilon) \le \epsilon$$

- To prove \implies : consider $L(f) rac{\epsilon}{2} < L(f;P)$ and $< U(f) + rac{\epsilon}{2}$
- ullet To prove $\buildrel =$: consider L(f;P) < L(f) and U(f) < U(f;P)

(i) Note

 $f:[a,b] o\mathbb{R}$ is integrable on [a,b] when:

- ullet The set of points of discontinuity of a bounded function $oldsymbol{f}$ is finite.
- ullet The set of points of discontinuity of a bounded function $m{f}$ is finite number of limit points. (may have infinite number of discontinuities)

Theorems on Integrability

Theorem 1

Suppose $f:[a,b]\to\mathbb{R}$ is bounded, and integrable on [c,b] for all $c\in(a,b)$. Then f is integrable on [a,b]. Also valid for the other end.

(i) Proof Hint

- Isolate a partition on the required end.
- ullet Choose x_1 or x_{n-1} such that $\Delta x < rac{\epsilon}{4M}$ where M is an upper or lower bound.

Theorem 2

Suppose $f:[a,b]\to\mathbb{R}$ is bounded, and continuous on [c,b] for all $c\in(a,b)$. Then f is integrable on [a,b]. Also valid for the other end.

Properties of Integrals

Notation

If a < b and f is integrable on [a, b], then:

$$\int_a^b f = - \int_b^a f$$

Properties

Suppose f and g are integrable on [a, b].

Addition

f + g will be integrable on [a, b].

$$\int_a^b (f\pm g) = \int_a^b f\pm \int_a^b g$$

(i) Proof Hint

- Prove f+g is integrable using:
 - $\circ sup(f+g) \leq \sup(f) + \sup(g)$
 - $\circ \ \inf(f+g) \geq \inf(f) + \inf(g)$
- ullet Start with U(f+g) and show $U(f+g) \leq U(f) + U(g)$
- ullet Start with L(f+g) and show $L(f+g)\geq L(f)+L(g)$

Constant multiplication

Suppose $k \in \mathbb{R}$. kf will be integrable [a,b].

$$\int_a^b kf = k \int_a^b f$$

- ullet Prove for $k\geq 0$. Use $U-L<rac{\epsilon}{k}$
- ullet Prove for k=-1
- ullet Using the above results, proof for $\,k < 0\,$ is apparent

Bounds

If $m \leq f(x) \leq M$ on [a,b]:

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

If $f(x) \leq g(x)$ on [a,b]:

$$\int_a^b f \le \int_a^b g$$

Modulus

|f| will be integrable on [a,b].

$$igg|\int_a^b figg| \leq \int_a^b |f|$$

(i) Proof Hint

Start with $-|f| \leq f \leq |f|$. And integrate both sides.

Multiple

fg will be integrable on [a, b].

- ullet Suppose $oldsymbol{f}$ is bounded by $oldsymbol{k}$
- ullet Prove f^2 is integrable (Use $rac{\epsilon}{2k}$)
- $oldsymbol{\cdot}$ fg is integrable because:

$$fg=rac{1}{2}igl[(f+g)^2-f^2-g^2igr]$$

Max, Min

 $\max(f,g)$ and $\min(f,g)$ are integrable.

Where **max** and **min** functions are defined as:

$$\max(f,g) = \frac{1}{2}(|f-g| + f + g)$$

$$\min(f,g) = \frac{1}{2}(-|f-g|+f+g)$$

Additivity

 $\iff f$ is Riemann integrable on [a,c] $\mathrm{and}\ [c,b]$ where $c\in(a,b)$.

(i) Proof Hint

• \implies : Use Cauchy criterion after defining these:

$$\circ \ P' = \{c\} \cap P$$

$$\circ \ Q = P' \cap [a,c]$$

$$\circ R = P' \cap [c,b]$$

ullet : Use cauchy criterion on [a,c],[c,b] separately and then combine using a union partition

After the integrability is proven,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

- 1. Let Q be a partition on [a,c] and R be a partition on [c,b] . And $P=Q\cap R$.
- 2. Prove the below using Cauchy criteria:

$$\int_a^b f < L(f;P) + \epsilon \quad \Longrightarrow \quad \int_a^b f \leq \int_a^c f + \int_c^b f$$

3. Prove the below using Cauchy criteria (by considering RHS):

$$\int_a^c f + \int_c^b f \leq \int_a^b f$$

Sequential Characterization of Integrability

A bounded function $f:[a,b] \to \mathbb{R}$ is Riemann integrable if and only if $\exists \, \{P_n\}$ a sequence of partitions, such that:

$$\lim_{n o\infty} \left[U(f;P_n) - L(f;P_n)
ight] = 0$$

In that case:

$$\int_a^b f = \lim_{n o \infty} U(f;P_n) = \lim_{n o \infty} L(f;P_n)$$

Cauchy criteria and squeeze theorem is used for both side proof.

For \iff :

- Consider the limit definition.
- ullet Prove f is Riemann integrable on P_n by Cauchy criteria.
- Use squeeze theorem for $U(f;P_n)-U(f)\leq U(f;P_n)-L(f;P_n)$ to prove limit of upper sum
- Prove limit of lower sum using the limit of upper sum

For \Longrightarrow : Consider the below, where $n \in \mathbb{N}$.

$$0 \leq U(f;P_n) - L(f;L_n) \leq \frac{1}{n}$$

Theorem

Suppose f is Riemann integrable on [a,b] and $\epsilon>0$. Then $\exists \epsilon>0 orall P$:

$$|P| < \delta \implies \left| \int_a^b f - \sum_{j=1}^n f(\zeta_j) I_j
ight| < \epsilon$$

where $\zeta_j \in [x_{j-1}, x_j], j=1,2,\cdots,n$.

(i) Proof Hint

$$\underbrace{\int_a^b f - \epsilon}_{} < L(f;P) \ \leq \ \sum_{j=1}^n f(\zeta_j) I_j \ \leq \ U(f;P) \ < \ \overline{\int_a^b f} + \epsilon$$

Intermediate Value Theorem for Integrals

Suppose f is a continuous function on [a,b]. Then $\exists x \in (a,b)$:

$$f(x)=rac{1}{b-a}\int_a^b f$$

(i) Proof Hint

Suppose $f_{
m max}=M=f(x_0)$ and $f_{
m min}=m=f(y_0)$.

When M=m: f is a constant function. Proof is trivial.

Otherwise:

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

Then there exists $x \in (x_0, y_0)$.

Generlized IVT

Suppose f,g are continuous functions on [a,b] and $g\geq 0$. Then $\exists x\in (a,b)$:

$$f(x)\int_a^b g = \int_a^b fg$$

(i) Proof Hint

Consider this and proof is similar to IVT.

$$mg \leq fg \leq Mg$$

Sequence of Functions

Types of Convergence

Uniformly convergence

$$orall \epsilon > 0 \; \exists N \; orall x \in [a,b] \; ext{ s.t. } \left| f_n(x) - f(x)
ight| < \epsilon$$

Here N depends on ϵ only.

Pointwise convergence

$$orall \epsilon > 0 \; \exists N \; orall x \in [a,b] \; ext{ s.t. } \left| f_n(x) - f(x)
ight| < \epsilon$$

Here N depends on ϵ, x .

Theorem

Let f_n be a sequence of Riemann integrable functions on [a,b]. Suppose f_n convergex to f uniformly. Then f is Riemann integrable on [a,b] and:

$$\lim_{n o\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

(i) Proof Hint

- ullet Consider $rac{\epsilon}{2(b-a)}$ in place of ϵ .
- ullet Consider Cauchy criteria for f_N .
- ullet Prove $f-f_N$ is Riemann integrable using Cauchy criteria.
- f is Riemann integrable as $f=f_N+(f-f_N)$.

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