

# Summary | Real Analysis

## Introduction— |

|  $\wedge$  | and | |  $\vee$  | or | |  $\rightarrow$  | then | |  $\implies$  | implies | |  $\Leftarrow$  | implied by | |  $\iff$  | if and only if | |  
 $\forall$  | for all | |  $\exists$  | there exists | |  $\sim$  | not |

Let's take  $a \rightarrow b$ .

1. Contrapositive or transposition:  $\sim b \rightarrow \sim a$ . This is equivalent to the original.
2. Inverse:  $\sim a \rightarrow \sim b$ . Does not depend on the original.
3. Converse:  $b \rightarrow a$ . Does not depend on the original.

$$a \rightarrow b \equiv \sim a \vee b \equiv \sim b \rightarrow \sim a$$

## Examples

- $\sim \forall x P(x) \equiv \exists x \sim P(x)$
- $\sim \exists x P(x) \equiv \forall x \sim P(x)$
- $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$
- $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$
- $\exists x \forall y P(x, y) \implies \forall y \exists x P(x, y)$

## Methods of proofs

1. Just proof what should be proven
2. Prove the contrapositive.
3. Proof by contradiction

## Proof by contradiction

Let's say we have to prove:  $a \implies b$ . We will prove  $a \wedge \sim b$  to be false. Then by proof by contradiction, we can prove  $a \implies b$ .

##### Proof of proof by contradiction

$$a \wedge \sim b = F$$

$$\sim (a \wedge \sim b) = \sim F$$

$$\sim a \vee b = T$$

$$a \rightarrow b = T$$

$$a \implies b$$

## Set theory

Zermelo-Fraenkel set theory with axiom of Choice(ZFC):9 axioms all together is being used here.

### Definitions

- $x \in A^c \iff x \notin A$
- $x \in A \cup B \iff x \in A \vee x \in B$
- $x \in A \cap B \iff x \in A \wedge x \in B$
- $A \subset B = \forall x(x \in A \implies x \in B)$
- $A - B = A \cap B^c$
- $A = B \iff ((\forall z \in A \implies z \in B) \wedge (\forall z \in B \implies z \in A))$

### Required proofs

- $(A \cap B)^c = A^c \cup B^c$
- $(A \cup B)^c = A^c \cap B^c$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \subset A \cup B$
- $A \cap B \subset A$

## Set of Numbers

### Sets of numbers

- Positive integers:  $\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$ .
- Natural integers:  $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$ .
- Negative integers:  $\mathbb{Z}^- = \{-1, -2, -3, -4, \dots\}$ .
- Integers:  $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$ .
- Rational numbers:  $\mathbb{Q} = \left\{ \frac{p}{q} \mid q \neq 0 \wedge p, q \in \mathbb{Z} \right\}$ .
- Irrational numbers: limits of sequences of rational numbers (which are not rational numbers)
- Real numbers:  $\mathbb{R} = \mathbb{Q}^c \cup \mathbb{Q}$ .

Complex numbers are not part of the study here.

## Continued Fraction Expansion

### The process

- Separate the integer part

- Find the inverse of the remaining part. Result will be greater than 1.
- Repeat the process for the remaining part.

## Finite expansion

Take  $\frac{420}{69}$  for example.

$$\frac{420}{69} = 6 + \frac{6}{69}$$

$$\frac{420}{69} = 6 + \frac{1}{\frac{69}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{3}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{1}{2}}$$

As  $\frac{420}{69}$  is finite, its continued fraction expansion is also finite. And it can be written as  $\frac{420}{69} = [6; 11, 2]$ .

## Infinite expansion

For irrational numbers, the expansion will be infinite.

For example  $\pi$ :

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \dots}}}}$$

Continued fraction expansion of  $\pi$  is  $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \dots]$ .

## Field Axioms

### Field Axioms of $\mathbb{R}$

$\mathbb{R} \neq \emptyset$  with two binary operations  $+$  and  $\cdot$  satisfying the following properties

1. Closed under addition:  $\forall a, b \in \mathbb{R}; a + b \in \mathbb{R}$
2. Commutative:  $\forall a, b \in \mathbb{R}; a + b = b + a$

3. Associative:  $\forall a, b, c \in \mathbb{R}; (a + b) + c = a + (b + c)$
4. Additive identity:  $\exists 0 \in \mathbb{R} \forall a \in \mathbb{R}; a + 0 = 0 + a = a$
5. Additive inverse:  $\forall a \in \mathbb{R} \exists (-a); a + (-a) = (-a) + a = 0$
6. Closed under multiplication:  $\forall a, b \in \mathbb{R}; a \cdot b \in \mathbb{R}$
7. Commutative:  $\forall a, b \in \mathbb{R}; a \cdot b = b \cdot a$
8. Associative:  $\forall a, b, c \in \mathbb{R}; (a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. Multiplicative identity:  $\exists 1 \in \mathbb{R} \forall a \in \mathbb{R}; a \cdot 1 = 1 \cdot a = a$
10. Multiplicative inverse:  $\forall a \in \mathbb{R} - \{0\} \exists a^{-1}; a \cdot a^{-1} = a^{-1} \cdot a = 1$
11. Multiplication is distributive over addition:  $a \cdot (b + c) = a \cdot b + a \cdot c$

## Field

Any set satisfying the above axioms with two binary operations (commonly  $+$  and  $\cdot$ ) is called a **field**. Written as  $(\mathbb{R}, +, \cdot)$  is a **Field**. But  $(\mathbb{R}, \cdot, +)$  is **not a field**.

## Required proofs

The below mentioned propositions can and should be proven using the above-mentioned axioms.  $a, b, c \in \mathbb{R}$ .

- $a \cdot 0 = 0$   
Hint: Start with  $a(1 + 0)$
- $1 \neq 0$
- Additive identity ( $0$ ) is unique
- Multiplicative identity ( $1$ ) is unique
- Additive inverse ( $-a$ ) is unique for a given  $a$
- Multiplicative inverse ( $a^{-1}$ ) is unique for a given  $a$
- $a + b = 0 \implies b = -a$
- $a + c = b + c \implies a = b$
- $-(a + b) = (-a) + (-b)$
- $-(-a) = a$
- $ac = bc \implies a = b$
- $ab = 0 \implies a = 0 \vee b = 0$
- $-(ab) = (-a)b = a(-b)$
- $(-a)(-b) = ab$
- $a \neq 0 \implies (a^{-1})^{-1} = a$
- $a, b \neq 0 \implies ab^{-1} = a^{-1}b^{-1}$

## Field or Not?

	Is field?	Reason (if not)
$(\mathbb{R}, +, \cdot)$	True	

	Is field?	Reason (if not)
$(\mathbb{R}, \cdot, +)$	False	Axiom 11 is invalid
$(\mathbb{Z}, +, \cdot)$	False	Multiplicative inverse doesn't exist
$(\mathbb{Q}, +, \cdot)$	True	
$(\mathbb{Q}^c, +, \cdot)$	False	$\sqrt{2} \cdot \sqrt{2} \notin \mathbb{Q}^c$
Boolean algebra	False	Additive inverse doesn't exist
$(\{0, 1\}, + \bmod 2, \cdot \bmod 2)$	True	
$(\{0, 1, 2\}, + \bmod 3, \cdot \bmod 3)$	True	
$(\{0, 1, 2, 3\}, + \bmod 4, \cdot \bmod 4)$	False	Multiplicative inverse doesn't exist

## Completeness Axiom

Let  $A$  be a non empty subset of  $\mathbb{R}$ .

- $u$  is the upper bound of  $A$  if:  $\forall a \in A; a \leq u$
- $A$  is bounded above if  $A$  has an upper bound
- Maximum element of  $A$ :  $\max A = u$  if  $u \in A$  and  $u$  is an upper bound of  $A$
- Supremum of  $A$   $\sup A$ , is the smallest upper bound of  $A$
- Maximum is a supremum. Supremum is not necessarily a maximum.
- $l$  is the lower bound of  $A$  if:  $\forall a \in A; a \geq l$
- $A$  is bounded below if  $A$  has a lower bound
- Minimum element of  $A$ :  $\min A = l$  if  $l \in A$  and  $l$  is a lower bound of  $A$
- Infimum of  $A$   $\inf A$ , is the largest lower bound of  $A$
- Minimum is a infimum. Infimum is not necessarily a minimum.

## Theorems

Let  $A$  be a non empty subset of  $\mathbb{R}$ .

- Say  $u$  is an upper bound of  $A$ . Then  $u = \sup A$  iff:  
 $\forall \epsilon > 0 \exists a \in A; a + \epsilon > u$
- Say  $l$  is a lower bound of  $A$ . Then  $l = \inf A$  iff:  
 $\forall \epsilon > 0 \exists a \in A; a - \epsilon < l$

## Required proofs

- $\sup(a, b) = b$
- $\inf(a, b) = a$

## Completeness axioms of real numbers

- Every non empty subset of  $\mathbb{R}$  which is bounded above has a supremum in  $\mathbb{R}$
- Every non empty subset of  $\mathbb{R}$  which is bounded below has a infimum in  $\mathbb{R}$

### Note

$\mathbb{Q}$  doesn't have the completeness property.

## Completeness axioms of integers

- Every non empty subset of  $\mathbb{Z}$  which is bounded above has a maximum
- Every non empty subset of  $\mathbb{Z}$  which is bounded below has a minimum

## Two important theorems

- $\exists a \forall \epsilon > 0, a < \epsilon \implies a \leq 0$
- $\forall \epsilon > 0 \exists a, a < \epsilon \not\implies a \leq 0$

## Order Axioms

- **Trichotomy:**  $\forall a, b \in \mathbb{R}$  exactly one of these holds:  $a > b$ ,  $a = b$ ,  $a < b$
- **Transitivity:**  $\forall a, b, c \in \mathbb{R}; a < b \wedge b < c \implies a < c$
- **Operation with addition:**  $\forall a, b \in \mathbb{R}; a < b \implies a + c < b + c$
- **Operation with multiplication:**  $\forall a, b, c \in \mathbb{R}; a < b \wedge 0 < c \implies ac < bc$

## Definitions

- $a < b \equiv b > a$
- $a \leq b \equiv a < b \vee a = b$
- $a \neq b \equiv a < b \vee a > b$
- $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$

## Triangular inequalities

$$|a| - |b| \leq |a + b| \leq |a| + |b|$$

$$||a| - |b|| \leq |a + b|$$

## Required proofs

- $\forall a, b, c \in \mathbb{R}; a < b \wedge c < 0 \implies ac > bc$
- $1 > 0$

- $-|a| \leq a \leq |a|$
- Triangular inequalities

## Theorems

- $\exists a \forall \epsilon > 0, a < \epsilon \implies a \leq 0$
- $\exists a \forall \epsilon > 0, 0 \leq a < \epsilon \implies a = 0$

### ⚠ Caution

$\forall \epsilon > 0 \exists a, a < \epsilon \implies a \leq 0$  is **not** valid.

Let  $A$  be a non-empty subset of  $\mathbb{R}$  which is bounded above and has an upper bound  $u$ .

$$u = \sup A \iff \forall \epsilon > 0 \exists a \in A, a > u - \epsilon$$

Let  $A$  be a non-empty subset of  $\mathbb{R}$  which is bounded below and has a lower bound  $m$ .

$$m = \inf A \iff \forall \epsilon > 0 \exists a \in A, a < m + \epsilon$$

## Relations

### Definitions

- Cartesian Product of sets  $A, B$   
 $A \times B = \{(a, b) | a \in A, b \in B\}$
- Ordered pair  
 $(a, b) = \{\{a\}, \{a, b\}\}$

### Relation

Let  $A, B \neq \emptyset$ . A relation  $R: A \rightarrow B$  is a non-empty subset of  $A \times B$ .

- $a R b \equiv (a, b) \in R$
- Domain of  $R$ :  $dom(R) = A$
- Codomain of  $R$ :  $codom(R) = B$
- Range of  $R$ :  $ran(R) = \{y | (x, y) \in R\}$
- $ran(R) \subseteq B$
- Pre-range of  $R$ :  $preran(R) = \{x | (x, y) \in R\}$
- $preran(R) \subseteq A$
- $R(a) = \{b | (a, b) \in R\}$

## Everywhere defined

$R$  is everywhere defined

$$\begin{aligned} \iff A &= \text{dom}(R) = \text{preran}(R) \\ \iff \forall a \in A, \exists b \in B; (a, b) \in R. \end{aligned}$$

## Onto

$R$  is onto

$$\begin{aligned} \iff B &= \text{codom}(R) = \text{ran}(R) \\ \iff \forall b \in B \exists a \in A (a, b) \in R \end{aligned}$$

Aka. **surjection**.

## Inverse

Inverse of  $R$ :  $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

## Types of relation

### one-many

$$\iff \exists a \in A, \exists b_1, b_2 \in B ((a, b_1), (a, b_2) \in R \wedge b_1 \neq b_2)$$

##### Not one-many

$$\iff \forall a \in A, \forall b_1, b_2 \in B ((a, b_1), (a, b_2) \in R \implies b_1 = b_2)$$

### many-one

$$\iff \exists a_1, a_2 \in A, \exists b \in B ((a_1, b), (a_2, b) \in R \wedge a_1 \neq a_2)$$

##### Not many-one

$$\iff \forall a_1, a_2 \in A, \forall b \in B ((a_1, b), (a_2, b) \in R \implies a_1 = a_2)$$

### many-many

**iff**  $R$  is **one-many** and **many-one**.

### one-one

**iff**  $R$  is **not one-many** and **not many-one**. Aka. **injection**.

## Bijection

When a relation is **onto** and **one-one**.

## Functions

A function  $f: A \rightarrow B$  is a relation  $f: A \rightarrow B$  which is [everywhere defined](#) and [not one-many](#).

- $\text{dom}(f) = A = \text{preran}(f)$



## Inverse

For a function  $f : A \rightarrow B$  to have its inverse relation  $f^{-1} : B \rightarrow A$  be also a function, we need:

- $f$  is [onto](#)
- $f$  is [not many-one](#) (in other words,  $f$  must be [one-one](#))

The above statement is true for all unrestricted function  $f$  that has an inverse  $f^{-1}$ :

$$f(f^{-1}(x)) = x = f^{-1}(f(x)) = x$$

## Composition

### Composition of relations

Let  $R : A \rightarrow B$  and  $S : B \rightarrow C$  are 2 relations. Composition can be defined when  $\text{ran}(R) = \text{preran}(S)$ .

Say  $\text{ran}(R) = \text{preran}(S) = D$ . Composition of the 2 relations is written as:

$$S \circ R = \{(a, c) \mid (a, b) \in R, (b, c) \in S, b \in D\}$$

### Composition of functions

Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be 2 functions where  $f$  is [onto](#).

$$g \circ f = \{(x, z) \mid (x, y) \in f, (y, z) \in g, y \in B\} = g(f(x))$$

## Countability

A set  $A$  is countable **iff**  $\exists f : A \rightarrow \mathbb{Z}^+$ , where  $f$  is a one-one function.

### Examples

- Countable: Any finite set,  $\mathbb{Z}, \mathbb{Q}$
- Uncountable:  $\mathbb{R}$ , Any open/closed intervals in  $\mathbb{R}$ .

### Transitive property

Say  $B \subset A$ .

$$A \text{ is countable} \implies B \text{ is countable}$$

$$B \text{ is not countable} \implies A \text{ is not countable}$$

# Limits

$$\lim_{x \rightarrow a} f(x) = L \text{ iff:}$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

Defining  $\delta$  in terms of a given  $\epsilon$  is enough to prove a limit.

## One sided limits

$$\lim_{x \rightarrow a^+} f(x) = L \text{ iff:}$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < x - a < \delta \implies |f(x) - L| < \epsilon)$$

$$\lim_{x \rightarrow a^-} f(x) = L \text{ iff:}$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (-\delta < x - a < 0 \implies |f(x) - L| < \epsilon)$$

$$\lim_{x \rightarrow a} f(x) = L^+ \text{ iff:}$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies 0 \leq f(x) - L < \epsilon)$$

$$\lim_{x \rightarrow a} f(x) = L^- \text{ iff:}$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies -\epsilon < f(x) - L \leq 0)$$

## Limits including infinite

$$\lim_{x \rightarrow \infty} f(x) = L \text{ iff:}$$

$$\forall \epsilon > 0 \exists N > 0 \forall x (x > N \implies |f(x) - L| < \epsilon)$$

$$\lim_{x \rightarrow -\infty} f(x) = L \text{ iff:}$$

$$\forall \epsilon > 0 \exists N > 0 \forall x (x < -N \implies |f(x) - L| < \epsilon)$$

$$\lim_{x \rightarrow a} f(x) = \infty \text{ iff:}$$

$$\forall M > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies f(x) > M)$$

$\lim_{x \rightarrow a} f(x) = -\infty$  **iff**:

$$\forall M > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies f(x) < -M)$$

## Indeterminate forms

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $\infty \cdot 0$
- $\infty - \infty$
- $\infty^0$
- $0^0$
- $1^\infty$

## Continuity

A function  $f$  is continuous at  $a$  **iff**:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (|x - a| < \delta \implies |f(x) - f(a)| < \epsilon)$$

## One-side continuous

A function  $f$  is continuous from right at  $a$  **iff**:

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

A function  $f$  is continuous from left at  $a$  **iff**:

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

## Continuous on an open interval

A function  $f$  is continuous in  $(a, b)$  **iff**  $f$  is continuous on every  $c \in (a, b)$ .

## Continuous on a closed interval

A function  $f$  is continuous in  $[a, b]$  **iff**  $f$  is:

- continuous on every  $c \in (a, b)$
- right-continuous at  $a$

- left-continuous at  $b$

## Continuity Theorems

### Extreme Value Theorem

If  $f$  is continuous on  $[a, b]$ ,  $f$  has a maximum and a minimum in  $[a, b]$ .

### Intermediate Value Theorem

Let  $f$  is continuous on  $[a, b]$ . If  $\exists u$  such that  $f(a) > u > f(b)$  or  $f(a) < u < f(b)$ :  
 $\exists c \in (a, b)$  such that  $f(c) = u$ .

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