# **Summary | Matrices**

## Introduction

Revise Matrices unit from G.C.E. (A/L) Combined Mathematics and G.C.E. (O/L) Mathematics.

## **Types of matrices**

### **Square matrix**

Number of columns equal to number of rows.

i Main diagonals of a square matrix

Formed by elements having equal subscripts.

### **Diagonal matrix**

A square matrix whose only non-zero elements are main-diagonal elements. Denoted by  $m{D}$ . Subset of triangular matrices.

### **Identity matrix or Unit matrix**

A diagonal matrix whose diagonal elements are all equal to  ${f 1}$ . Denoted by  ${f I}$ . Subset of diagonal matrices.

#### Zero matrix / Null matrix

All elements are 0.

### **Column matrix (column vector)**

Only 1 column.

### **Row matrix (row vector)**

Only 1 row.

### **Triangular matrix**

Upper triangular matrix or lower triangular matrix.

### **Upper triangular matrix**

All elements below the main diagonal are 0. Subset of square matrices.

#### Lower triangular matrix

All elements above the main diagonal are 0. Subset of square matrices.

## **Matrix operations**

#### **Addition and subtraction**

Order of the 2 matrices must be same. Matrix obtained by adding or subtracting corresponding elements.

### Scalar multiplication

Matrix obtained by multiplying all elements by the scalar.

### **Matrix multiplication**

Explained on a separate page: Matrix multiplication

(i) Note

Other operations are also defined in separate pages.

# **Matrix Multiplication**

Defined only when the number of columns of the first matrix is equal to the number of rows of the second matrix.

Suppose  $A=(a_{ij})_{m imes p}$  and  $B=(b_{ij})_{p imes n}$  .

$$A imes B = C = (c_{ij})_{m imes n} ~~ ext{where}~~ c_{ij} = \sum_{k=1}^p a_{ik} imes b_{kj}$$

## (i) Note

ullet Generally A imes B
eq B imes A

•  $A \times B = 0 \implies A = 0 \lor B = 0$ 

•  $A \neq 0 \land B \neq 0 \Longrightarrow A \times B \neq 0$ 

## **Properties of matrix multiplication**

A, B, C, I (Identity) matrices must be chosen so that below-mentioned product matrices are defined.

1. Associative: A(BC) = (AB)C

2. Right distributive over addition: (A+B)C=AC+BC

3. Left distributive over addition: C(A+B)=CA+CB

4. AI = IA = A

# **Transpose**

Matrix obtained from a given matrix by interchanging its rows and columns. Denoted by a superscript T, like  $A^T$ .

## **Properties**

1.  $(A^T)^T = A$ 

2. Distributive over addition:  $(A+B)^T=A^T+B^T$ 

з.  $(kA)^T = kA^T$ 

4.  $(A \times B)^T = B^T \times A^T$ 

## More types of matrices

## Symmetric matrix

If  $A = A^T$ . Subset of square matrices.

## **Skew-symmetric matrix**

If  $A=-A^T$ . Subset of square matrices. All elements in main diagonal are 0.

Any square matrix can be expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

### **Determinant**

Defined only for square matrices. Denoted by |A|.

#### For 2x2

$$|A| = egin{array}{c|c} a_{11} & a_{12} \ a_{21} & a_{22} \ \end{array} = a_{11}a_{22} - a_{12}a_{21}$$

### For higher order

#### Minor of an element

Suppose  $A = (a_{ij})$ .

Minor of an element  $a_{ij}$ , is the matrix obtained by deleting  $i^{\mathrm{th}}$  row and  $j^{\mathrm{th}}$  column of A. Denoted by  $M_{ij}$ .

#### Co-factor of an element

Suppose  $A=(a_{ij})$ .

Co-factor of an element  $a_{ij}$ , is defined as (commonly denoted as  $A_{ij}$ ):

$$A_{ij} = (-1)^{i+j} \, |M_{ij}|$$

#### Definition

If  $A=(a_{ij})_{n imes n}$  then the  $\operatorname{ extbf{determinant}}$  of A is denoted by |A| and is defined by:

$$|A| = \sum_{j=1}^n a_{ij} A_{ij}$$

where  $1 \leq j \leq n$ .

## **Properties of determinants**

- $|A^T| = |A|$
- ullet Every element of a row or column of a matrix is ullet then the value of its determinant is ullet .
- If 2 columns or 2 rows of a matrix are identical then its determinant is  $\mathbf{0}$ .
- If A and B are two square matrices then |AB|=|A||B| .
- The value of the determinant of a matrix remains unchanged if a scalar multiple of a row or column is added to any other row or column.
- If a matrix B is obtained from a square matrix A by an interchange of two columns or rows: |B| = -|A| .
- If every entry in any row or column is multiplied by  $m{k}$  , then the whole determinant is multiplied by  $m{k}$  .

### In relation with eigenvalues

For a  $n \times n$  matrix A with n number of <u>eigenvalues</u>:

$$|A| = \prod_{i=1}^n \lambda_i$$

# **Adjoint**

Suppose  $A=(a_{ij})_{n\times n}$ .

$$\mathrm{adj}A = (A_{ij})_{n \times n}^T$$

Where  $A_{ij}$  is the  ${ ext{co-factor of}}\ a_{ij}$ .

## Inverse

Suppose A and B are square matrices of the same order. If AB=BA=I then B is called the inverse of A and is denoted by  $A^{-1}$ .

$$A^{-1} = \frac{\operatorname{adj} A}{|A|}$$

## Singular or Non-singular

A square matrix is singular iff ert A ert = 0. Otherwise its non-singular or invertible.

## **Properties of Inverse**

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $A(\operatorname{adj} A) = (\operatorname{adj} A) A = |A|I$

# **Elementary Transformations**

- · Interchange of any columns or rows
- Addition of multiple of any row or column to any other row or column
- Multiplication of each element of a column or a row by a non-zero constant

When a matrix B is obtained by applying elementary transformations to a matrix A, then A is equivalent to B. Denoted by  $A \approx B$ .

### **Theorem**

The elementary row operations that reduce a given matrix A to the identity matrix, also transform the identity matrix to the inverse of A.

## **Augmented Matrix**

Two matrices are written as a single matrix with a vertical line in-between. Denoted by (A | B). Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

## Inverse using elementary row transformations

Let A be a square matrix with order  $n \times n$ .

- Start with  $(A_{n \times n} | I_n)$
- ullet Repeatedly add  $oldsymbol{row}$  transformations (not column) to both of the matrices until the LHS becomes an identity matrix.
  - $\circ$  Convert all elements outside the main diagonal to 0.
  - $\circ$  Convert elements on the main diagonal to 1 by multiplying by a constant.
- ullet When LHS is an identity matrix, RHS is  $A^{-1}$  .

**⚠** TODO

What about singular matrices?

## **Echelon Form**

A matrix is in row echelon form (or just "row echelon" form) iff:

- All rows having only zero entries are at the bottom.
- For all row that does not contain entirely zeros, the first non-zero entry is 1.
- For 2 successive non-zero rows, the leading 1 in the higher row is further left than the leading 1 in the lower row.

The process of reducing the augmented matrix to row Echelon form is known as **Gaussian** elimination.

### Column echelon form

A matrix A is in column echelon form if  $A^{\mathbf{T}}$  is in row echelon form.

# **System of Linear Equations**

Any system of linear equations can be represented in matrix notation as shown below.

• 
$$a_{11}x + a_{12}y + a_{13}z = b_1$$

• 
$$a_{21}x + a_{22}y + a_{23}z = b_2$$

• 
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$egin{pmatrix} a_{11} & a_{12} & a_{13} \ a_{21} & a_{22} & a_{23} \ a_{31} & a_{32} & a_{33} \ \end{pmatrix} egin{pmatrix} x \ y \ z \ \end{pmatrix} = egin{pmatrix} b_1 \ b_2 \ b_3 \ \end{pmatrix} \implies AX = B$$

2 types based on  $\boldsymbol{B}$ :

• = 0: Homogeneous system

•  $\neq 0$ : Non-homogeneous system

### **Consistent**

When the system of equations has at least 1 solution. Otherwise inconsistent.

## Rank

Number of non-zero rows of row echelon form of a matrix A. Denoted by  $\operatorname{\mathbf{Rank}} A$ .

(i) Note

Rank  $A \leq \text{Rank } (A|B)$  is always true.

## Relation with non-homogenous system of solutions

Consider the system:  $A_{n\times n}X_{n\times 1}=Bn\times 1$ .

- $|A| \neq 0 \iff \operatorname{Rank} A = \operatorname{Rank} (A|B) = n \iff \text{unique solution exists}$
- $|A|=0 \implies$  no solution  $\lor$  infinitely many solutions
- Rank  $A < \text{Rank } (A|B) \implies \text{no solutions}$
- Rank  $A = \text{Rank } (A|B) < n \implies \text{infinitely many solutions}$

# **Solutions of Homogenous Systems**

Consider the system:

$$A_{m \times n} X_{n \times 1} = O_{m \times 1}$$

Any homogenous system is consistent, because X = O is always a solution.

- Rank  $A = \text{Rank}(A|B) = n \iff \text{unique solution exists}$
- Rank  $A = \operatorname{Rank}(A|B) < n \implies \text{infinitely many solutions}$

# **Solution of Non-homogenous Systems**

## **Method 1: Direct approach**

Used when coefficient matrix  $m{A}$  is invertible. It means the system has a unique set of solutions.

$$AX = B \implies X = A^{-1}B$$

### Method 2: Cramer's Rule

Let AX=B, where A is the coefficient matrix and  $X=(x_i)_{n imes 1}$ .

$$x_i = rac{|A_i|}{|A|}$$

Where  $A_i$  is the matrix obtained by replacing ith column in matrix A by B.

# **Method 3: Reducing to Echelon Form**

Start with (A|B). Convert the **LHS** to echelon form using elementary row transformations. The solution can be found now. If a contradiction is encountered while solving the equation, that means the system has no solutions.

# **Eigenvalues & Eigenvectors**

### **Definitions**

### **Characteristic Polynomial**

Let A be a  $n \times n$  matrix.

$$p(\lambda) = |A - \lambda I|$$

### **Eigenvalues**

Roots of the equation  $p(\lambda)=0$  are the eigenvalues of A.

### (i) Note

- <u>Determinant of a matrix</u> can be written in terms of all of its eigenvalues.
- ullet If  $\lambda$  is an eigenvalue of A , then  $\lambda^2$  is an eigenvalue of  $A^2$

### **Eigenvectors**

The column vectors satisfying the equation  $(A - \lambda_i I) X_i$ .

#### Normalized eigenvectors

An eigenvector with the magnitude (norm) of  ${f 1}$ . Normalizing factor  ${m k}$  of any eigenvector is:

$$\frac{1}{k} = \sqrt{\sum_{i=1}^n X_i^2}$$

#### Norm

Norm of a column or row matrix  $W_{n \times n}$  is denoted by ||W|| and defined as:

$$||W|| = \sqrt{\sum_{i=1}^n w_i^2}$$

### **Algebraic Multiplicity**

If the characteristic polynomial consists of a factor of the form  $(\lambda - \lambda_i)^r$  and  $(\lambda - \lambda_i)^{r+1}$  is not a factor of the characteristic polynomial then r is the algebraic multiplicity of the eigenvalue  $\lambda$ .

### **Spectrum**

Set of all eigenvalues.

### **Spectral Radius**

$$R = \max \left\{ |\lambda_i| \ where \ \lambda_i \in ext{Spectrum} 
ight\}$$

## **Linear Independence of Eigenvectors**

Suppose  $X_1, X_2, X_3, \ldots, X_n$  is a set of eigenvectors.  $k_1, k_2, k_3, \ldots, k_n$  is a set of scalars.

All those eigenvectors are independent iff:

$$k_1X_1 + k_2X_2 + k_3X_3 + \cdots + k_nX_n = 0 \implies k_1 = k_2 = k_3 = \cdots = k_n = 0$$

## For special matrices

### Real symmetric matrix

Suppose  $\boldsymbol{A}$  is a symmetric matrix with all real entries. Then:

- The eigenvalues of A are all real:  $orall \lambda \in S_A, (\lambda_i \in \mathbb{R})$
- ullet The eigenvectors of A (corresponding to distinct values of  $\lambda$ ) are mutually orthogonal

## Upper triangular matrix

The eigenvalues are the diagonal entries

# **Orthogonal & Orthonormal Vectors**

Consider 2 column matrices  $v_1$  and  $v_2$ :

$$v_1 = egin{pmatrix} a_1 \ dots \ a_n \end{pmatrix} \ \wedge \ v_2 = egin{pmatrix} b_1 \ dots \ b_n \end{pmatrix}$$

### **Product**

The product of  $v_1$  and  $v_2$  is defined as:

$$v_1 \cdot v_2 = \sum_{k=1}^n a_k b_k = v_2 \cdot v_1 = v_1^T v_2$$

## Orthogonal

### **Orthogonal matrix**

For a square matrix  $oldsymbol{A}$  with real entries, it is orthogonal  $\inf A^{-1} = A^T$ 

### **Orthogonal vectors**

 $v_1$  and  $v_2$  are orthogonal iff  $v_1 \cdot v_2 = 0$ .

For a set of n column vectors, they are orthogonal iff they are pairwise orthogonal. That is:

$$orall i,j \in \{1,\ldots,n\} \wedge i 
eq j, (v_i \cdot v_j = 0)$$

(i) Note

 $v_1,v_2$  are orthogonal  $\implies v_1,v_2$  are linearly independent.

Converse is **not** true.

### **Orthonormal**

For a set of n column vectors, they are orthonormal iff:

- They are pairwise orthogonal AND
- ullet For all n column vectors their norm is 1  $orall i \in \{1,\ldots,n\}, ||v_i|| = 1$

## **Properties of orthogonal matrices**

- Product of 2 orthogonal matrices of the same order is also an orthogonal matrix
- The columns or rows of an orthogonal matrix form an orthogonal set of vectors

## **Trace**

Suppose  $A=(a_{ij})_{n imes n}$  is an square matrix. Trace of A is the sum of the diagonal entries.

$$\operatorname{trace}(A) = \operatorname{Tr}(A) = \sum_{i=1}^n a_{ii}$$

Trace is also equal to the sum of eigenvalues.

$$\operatorname{trace}(A) = \sum \lambda_i ext{ where } \lambda_i \in \operatorname{spectrum of } A$$

# Diagonalization

#### Similar matrices

2 square matrices  $m{A}$  and  $m{B}$  of the same order, are similar **iff** there exists an invertible matrix  $m{P}$  such that:

$$B = P^{-1}AP$$

Similarity of 2 matrices is commutative.

Similar matrices have the set of eigenvalues.

If A and B are similar, then  $A^2$  and  $B^2$  are similar.

### **Definition**

A matrix A is **diagonalizable** if it is similar to a <u>diagonal matrix</u>.

$$\exists D, P \text{ s.t. } D = P^{-1}AP$$

Here:

- $oldsymbol{D}$  is a diagonal matrix
- $oldsymbol{P}$  is an invertible matrix

## **Steps**

- ullet Find eigenvalues of  $A_{n imes n}$  :  $\lambda_1,\lambda_2,\ldots,\lambda_n$
- Find corresponding eigenvectors:  $X_1, X_2, \ldots, X_n$
- ullet Construct  $oldsymbol{P}$  by joining the eigenvectors as columns

$$P = (X_1 X_2 \dots X_n)_{n imes n} \ \wedge \ D = egin{pmatrix} \lambda_1 & & & \ & \ddots & \ & & \lambda_n \end{pmatrix}$$

(i) Note

Order of those eigenvectors is  ${f not}$  a problem. Here the matrix  ${m P}$  differs based on the order, and hence is not unique.

### (i) Real symmetric matrix

Suppose  $A_{n \times n}$  is a **real symmetric matrix** . If it has **distinct** eigenvalues then it has n mutually orthogonal linearly-independent eigenvectors.

Hence the diagonalizing matrix  $m{P}$  (formed by using the normalized eigenvectors) is an **orthogonal matrix**.

### **Uses**

### **Finding integer powers**

Suppose  $A_{n \times n}$  is diagonalizable, and  $k \in \mathbb{R}$ .

$$A = P^{-1}DP \implies A^k = P^{-1}D^kP$$

# **Cayley-Hamilton Theorem**

If  $p(\lambda)$  is the characteristic polynomial of the matrix  $A_{n imes n}$ , then p(A) = O

### Uses

- Easily compute the inverse of a matrix
- Easily express higher powers of a matrix in terms of its lower powers

## **Matrix Norms**

Let  $A_{n \times n}$ . A norm of A is denoted by ||A||.

### **Definitions**

Suppose  $A=(a_{ij})_{m imes n}$  for all the definitions below.

#### 1-norm

Maximum of the absolute column sums.

$$\|A\|_1 = \max igg\{ \sum_{i=1}^m |a_{ij}| \; ; \; j \in [1,n] igg\}$$

#### 2-norm

Square root of the sum of all elements squared. Also called as Euclidean norm.

$$\left( \left\| A 
ight\|_2 
ight)^2 = \left( \left\| A 
ight\|_E 
ight)^2 = \sum_{i=1}^m \sum_{j=1}^n \left( a_{ij} 
ight)^2$$

## **Infinity norm**

Maximum of the row sums.

$$\left\Vert A
ight\Vert _{\infty}=\max\left\{ \sum_{j=1}^{n}\leftert a_{ij}
ightert ;\;i\in\left[1,m
ight] 
ight\}$$

### (i) Note

For any matrix  $X \in \mathbb{R}^n$ :

$$\|X\|_{\infty} \leq \|X\|_2 \leq \|X\|_1$$

#### **Vector norm**

Norm defined for column vectors.

## **Properties of Norms**

Works for all types of norms.

Suppose A, B are  $m \times n$  ordered.

1. 
$$||A|| \geq 0$$

2. 
$$||A||=0 \iff A=0$$

- 3.  $\|kA\|=|k| imes\|A\|$
- 4.  $\|A+B\| \leq \|A\| + \|B\|$  (triangle inequality)
- 5.  $||AB|| \le ||A|| \times ||B||$

## **Unit Ball**

A unit ball in  $\mathbb{R}^n$  with respect to a norm  $\|\|$ .

$$\left\{X\mid X\in\mathbb{R}^n,\; \|X\|\leq 1\right\}$$

## **Unit disc**

When n=2, unit ball is also called the unit disc.

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