Summary | Real Analysis

Introduction— |

| $\wedge$ | and | | $\vee$ | or | | $\to$ | then | | $\implies$ | implies | | $\Leftarrow$ | implied by | | $\iff$ | if and only if | | $\forall$ | for all | | $\exists$ | there exists | | $\sim$ | not |

Let's take $a \to b$.

1. Contrapositive or transposition: $\sim b \to \sim a$. This is equivalent to the original.
2. Inverse: $\sim a \to \sim b$. Does not depend on the original.
3. Converse: $b \to a$. Does not depend on the original.

$$a \to b \equiv \sim a \vee b \equiv \sim b \to \sim a$$

Required proofs

- $\sim \forall x\, P(x) \equiv \exists x \sim P(x)$
- $\sim \exists x\, P(x) \equiv \forall x \sim P(x)$
- $\exists x\, \exists y P(x, y) \equiv \exists y\, \exists x P(x, y)$
- $\forall x\, \forall y P(x, y) \equiv \forall y\, \forall x P(x, y)$
- $\exists x\, \forall y P(x, y) \implies \forall y\, \exists x P(x, y)$
- $(A \to C) \wedge (B \to C) \equiv (A \vee B) \to C$

Methods of proofs

1. Just proof what should be proven
2. Prove the contrapositive
3. Proof by contradiction
4. Proof by induction

Proof by contradiction

Suppose $a \implies b$ has to be proven. If $a \wedge \sim b$ is proven to be false, then, by proof by contradiction, $a \implies b$ can be trivially proven.

Logic behind proof by contradiction

$$a \wedge \sim b = F$$

$$\sim (a \wedge \sim b) = \sim F$$

$$\sim a \vee b = T$$

$$a \rightarrow b = T$$

$$a \implies b$$

Set theory

Zermelo-Fraenkel set theory with axiom of Choice(ZFC):9 axioms all together is being used here.

Definitions

- $x \in A^{\mathrm{c}} \iff x \notin A$
- $x \in A \cup B \iff x \in A \vee x \in B$
- $x \in A \cap B \iff x \in A \wedge x \in B$
- $A \subset B = \forall x(x \in A \implies x \in B)$
- $A - B = A \cap B^{\mathrm{c}}$
- $A = B \iff ((\forall z \in A \implies z \in B) \wedge (\forall z \in B \implies z \in A))$

Required proofs

- $(A \cap B)^\mathrm{c} = A^\mathrm{c} \cup B^\mathrm{c}$
- $(A \cup B)^\mathrm{c} = A^\mathrm{c} \cap B^\mathrm{c}$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \subset A \cup B$
- $A \cap B \subset A$

Set of Numbers

Sets of numbers

- Positive integers: $\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$.
- Natural integers: $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$.
- Negative integers: $\mathbb{Z}^- = \{-1, -2, -3, -4, \dots\}$.
- Integers: $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$.
- Rational numbers: $\mathbb{Q} = \left\{ \frac{p}{q} \middle| q \neq 0 \land p, q \in \mathbb{Z} \right\}$.
- Irrational numbers: limits of sequences of rational numbers (which are not rational numbers)
- Real numbers: $\mathbb{R} = \mathbb{Q}^c \cup \mathbb{Q}$.

Complex numbers are not part of the study here.

Continued Fraction Expansion

The process

- Separate the integer part
- Find the inverse of the remaining part. Result will be greated than 1.
- Repeat the process for the remaining part.

Finite expansion

Take $\frac{420}{69}$ for example.

$$\frac{420}{69} = 6 + \frac{6}{69}$$

$$\frac{420}{69} = 6 + \frac{1}{\frac{69}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{3}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{1}{2}}$$

As $\frac{420}{69}$ is finite, its continued fraction expansion is also finite. And it can be written as $\frac{420}{69} = [6; 11, 2]$.

Infinite expansion

For irrational numbers, the expansion will be infinite.

For example π:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}$$

Conintued fraction expansion of π is $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots]$.

Field Axioms

Field Axioms of $\mathbb{R}$

$\mathbb{R} \neq \emptyset$ with two binary operations $+$ and $\cdot$ satisfying the following properties

1. Closed under addition: $\forall a, b \in \mathbb{R}; a + b \in \mathbb{R}$
2. Commutative: $\forall a, b \in \mathbb{R}; a + b = b + a$

3. Associative: $\forall a, b, c \in \mathbb{R}; (a+b)+c = a+(b+c)$
4. Additive identity: $\exists 0 \in \mathbb{R}\, \forall a \in \mathbb{R}; a+0 = 0+a = a$
5. Additive inverse: $\forall a \in \mathbb{R}\, \exists (-a); a+(-a) = (-a)+a = 0$
6. Closed under multiplication: $\forall a, b \in \mathbb{R}; a \cdot b \in \mathbb{R}$
7. Commutative: $\forall a, b \in \mathbb{R}; a \cdot b = b \cdot a$
8. Associative: $\forall a, b, c \in \mathbb{R}; (a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. Multiplicative identity: $\exists 1 \in \mathbb{R}\, \forall a \in \mathbb{R}; a \cdot 1 = 1 \cdot a = a$
10. Multiplicative inverse: $\forall a \in \mathbb{R} - \{0\}\, \exists a^{-}; a \cdot a^{-} = a^{-} \cdot a = 1$
11. Multiplication is distributive over addition: $a \cdot (b+c) = a \cdot b + a \cdot c$

Field

Any set satisfying the above axioms with two binary operations (commonly $+$ and $\cdot$) is called a **field**. Written as $(\mathbb{R}, +, \cdot)$ is a Field. But $(\mathbb{R}, \cdot, +)$ is not a field.

Required proofs

The below mentioned propositions can and should be proven using the above-mentioned axioms. $a, b, c \in \mathbb{R}$.

- $a \cdot 0 = 0$
  Hint: Start with $a(1+0)$
- $1 \neq 0$
- Additive identity (0) is unique
- Multiplicative identity (1) is unique
- Additive inverse ($-a$) is unique for a given a
- Multiplicative inverse (a^{-1}) is unique for a given a
- $a + b = 0 \implies b = -a$
- $a + c = b + c \implies a = b$
- $-(a+b) = (-a) + (-b)$
- $-(-a) = a$
- $ac = bc \implies a = b$
- $ab = 0 \implies a = 0 \lor b = 0$
- $-(ab) = (-a)b = a(-b)$
- $(-a)(-b) = ab$
- $a \neq 0 \implies (a^{-1})^{-1} = a$
- $a, b \neq 0 \implies ab^{-1} = a^{-1}b^{-1}$

Field or Not?

	Is field?	**Reason (if not)**
$(\mathbb{R}, +, \cdot)$	True	
$(\mathbb{R}, \cdot, +)$	False	Axiom 11 is invalid
$(\mathbb{Z}, +, \cdot)$	False	Multiplicative inverse doesn’t exist
$(\mathbb{Q}, +, \cdot)$	True	
$(\mathbb{Q}^c, +, \cdot)$	False	$\sqrt{2} \cdot \sqrt{2} \notin \mathbb{Q}^c$
Boolean algebra	False	Additive inverse doesn’t exist
$(\{0,1\}, + \bmod 2, \cdot \bmod 2)$	True	
$(\{0,1,2\}, + \bmod 3, \cdot \bmod 3)$	True	
$(\{0,1,2,3\}, + \bmod 4, \cdot \bmod 4)$	False	Multiplicative inverse doesn’t exist

Completeness Axiom

Let A be a non empty subset of $\mathbb{R}$.

- u is the upper bound of A if: $\forall a \in A; a \leq u$
- A is bounded above if A has an upper bound
- Maximum element of A: $\max A = u$ if $u \in A$ and u is an upper bound of A
- Supremum of A $\sup A$, is the smallest upper bound of A
- Maximum is a supremum. Supremum is not necessarily a maximum.
- l is the lower bound of A if: $\forall a \in A; a \geq l$
- A is bounded below if A has a lower bound
- Minimum element of A: $\min A = l$ if $l \in A$ and l is a lower bound of A
- Infimum of A $\inf A$, is the largest lower bound of A
- Minimum is a infimum. Infimum is not necessarily a minimum.

Theorems

Let A be a non empty subset of $\mathbb{R}$.

- Say u is an upper bound of A. Then $u = \sup A$ **iff**:
  $\forall \epsilon > 0 \; \exists a \in A; \; a + \epsilon > u$
- Say l is a lower bound of A. Then $l = \inf A$ **iff**:
  $\forall \epsilon > 0 \; \exists a \in A; \; a - \epsilon < l$

Proof Hint

Prove the contrapositive. Use $\epsilon = \frac{1}{2}(L - sup(A))$ for supremum proof.

Required proofs

- $sup(a, b) = b$
- $inf(a, b) = a$

Completeness axioms of real numbers

- Every non empty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$
- Every non empty subset of $\mathbb{R}$ which is bounded below has a infimum in $\mathbb{R}$

Note

$\mathbb{Q}$ doesn't have the completeness property.

Completeness axioms of integers

- Every non empty subset of $\mathbb{Z}$ which is bounded above has a maximum
- Every non empty subset of $\mathbb{Z}$ which is bounded below has a minimum

Order Axioms

- **Trichotomy**: $\forall a, b \in \mathbb{R}$ exactly one of these holds: $a > b$, $a = b$, $a < b$
- **Transitivity**: $\forall a, b, c \in \mathbb{R}; a < b \land b < c \implies a < c$
- **Operation with addition**: $\forall a, b \in \mathbb{R}; a < b \implies a + c < b + c$
- **Operation with mutliplication**: $\forall a, b, c \in \mathbb{R}; a < b \land 0 < c \implies ac < bc$

Definitions

- $a < b \equiv b > a$
- $a \leq b \equiv a < b \lor a = b$
- $a \neq b \equiv a < b \lor a > b$
- $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$

Triangular inequalities

$$|a| - |b| \leq |a + b| \leq |a| + |b|$$

$$\big||a| - |b|\big| \leq |a + b|$$

Proof Hint

For first:

- Use $-|a| \leq a \leq |a|$

For second:

- Use the below substitutions in first conclusion
  - $a = a - b \quad \land \quad b = b$
  - $a = b - a \quad \land \quad b = a$

Required proofs

- $\forall a, b, c \in \mathbb{R}; a < b \land c < 0 \implies ac > bc$
- $1 > 0$
- $-|a| \leq a \leq |a|$
- Triangular inequalities

Theorems

- $\exists a \ \forall \epsilon > 0,\ a < \epsilon \implies a \leq 0$
- $\exists a \ \forall \epsilon > 0,\ 0 \leq a < \epsilon \implies a = 0$
- $\forall \epsilon > 0 \ \exists a, a < \epsilon \not\Longrightarrow a \leq 0$

Caution

$\forall \epsilon > 0 \ \exists a,\ a < \epsilon \implies a \leq 0$ is **not** valid.

Relations

Definitions

- Cartesian Product of sets A, B
  $A \times B = \{(a, b) | a \in A, b \in B\}$

- Ordered pair
  $(a, b) = \{\{a\}, \{a, b\}\}$

Relation

Let $A, B \neq \emptyset$. A relation $R : A \to B$ is a non-empty subset of $A \times B$.

- $a\, R\, b \equiv (a, b) \in R$
- Domain of R: $dom(R) = A$
- Codomain of R: $codom(R) = B$
- Range of R: $ran(R) = \{y | (x, y) \in R\}$
- $ran(R) \subseteq B$
- Pre-range of R: $preran(R) = \{x \mid (x, y) \in R\}$
- $preran(R) \subseteq A$
- $R(a) = \{b \mid (a, b) \in R\}$

Everywhere defined

R is everywhere defined
$\iff A = dom(R) = preran(R)$
$\iff \forall a \in A,\ \exists b \in B;\ (a, b) \in R$.

Onto

R is onto
$\iff B = codom(R) = ran(R)$
$\iff \forall b \in B\, \exists a \in A\, (a, b) \in R$

Aka. **surjection**.

Inverse

Inverse of R: $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

Types of relation

one-many

$$\iff \exists a \in A, \exists b_1, b_2 \in B\ ((a, b_1), (a, b_2) \in R \land b_1 \neq b_2)$$

Not one-many

$$\iff \forall a \in A, \forall b_1, b_2 \in B\ ((a, b_1), (a, b_2) \in R \implies b_1 = b_2)$$

many-one

$$\iff \exists a_1, a_2 \in A, \exists b \in B\ ((a_1, b), (a_2, b) \in R \land a_1 \neq a_2)$$

Not many-one

$$\iff \forall a_1, a_2 \in A, \forall b \in B\ ((a_1, b), (a_2, b) \in R \implies a_1 = a_2)$$

many-many

iff R is **one-many** and **many-one**.

one-one

iff R is **not one-many** and **not many-one**. Aka. **injection**.

Bijection

When a relation is **onto** and **one-one**.

Functions

A function $f : A \rightarrow B$ is a relation $f : A \rightarrow B$ which is [everywhere defined] and [not one-many].

- $dom(f) = A = preran(f)$

Inverse

For a function $f:\ A \to B$ to have its inverse relation $f^{-1}:\ B \to A$ be also a function, we need:

- f is onto
- f is not many-one (in other words, f must be one-one)

The above statement is true for all unrestricted function f that has an inverse f^{-1}:

$$f(f^{-1}(x)) = x = f^{-1}(f(x)) = x$$

Composition

Composition of relations

Let $R : A \to B$ and $S : B \to C$ are 2 relations. Composition can be defined when $\text{ran}(R) = \text{preran}(S)$.

Say $\text{ran}(R) = \text{preran}(S) = D$. Composition of the 2 relations is written as:

$$S \circ R = \{(a, c) \mid (a, b) \in R,\ (b, c) \in S,\ b \in D\}$$

Composition of functions

Let $f : A \to B$ and $g : B \to C$ be 2 functions where f is onto.

$$g \circ f = \{(x, z) \mid (x, y) \in f,\ (y, z) \in g,\ y \in B\} = g(f(x))$$

Countability

A set A is countable **iff** $\exists f : A \to Z^+$, where f is a one-one function.

Examples

- Countable: Any finite set, $\mathbb{Z}, \mathbb{Q}$
- Uncountable: $\mathbb{R}$, Any open/closed intervals in $\mathbb{R}$.

Transitive property

Say $B \subset A$.

$$A \text{ is countable} \implies B \text{ is countable}$$

$$B \text{ is not countable} \implies A \text{ is not countable}$$

Limits

$\lim\limits_{x \to a} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

Defining δ in terms of a given ϵ is enough to prove a limit.

One sided limits

$\lim\limits_{x \to a^+} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < x - a < \delta \implies |f(x) - L| < \epsilon)$$

$\lim\limits_{x \to a^-} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (-\delta < x - a < 0 \implies |f(x) - L| < \epsilon)$$

$\lim\limits_{x \to a} f(x) = L^+$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies 0 \leq f(x) - L < \epsilon)$$

$\lim\limits_{x \to a} f(x) = L^-$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies -\epsilon < f(x) - L \leq 0)$$

Limits including infinite

$\lim_{x \to \infty} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists N > 0 \; \forall x \; (x > N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to -\infty} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists N > 0 \; \forall x \; (x < -N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to a} f(x) = \infty$ **iff**:

$$\forall M > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies f(x) > M)$$

$\lim_{x \to a} f(x) = -\infty$ **iff**:

$$\forall M > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies f(x) < -M)$$

Indeterminate forms

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $\infty \cdot 0$
- $\infty - \infty$
- ∞^0
- 0^0
- 1^∞

Continuity

A function f is continuous at a **iff**:

$$\lim_{x \to a} f(x) = f(a)$$

$$\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \ (|x-a| < \delta \implies |f(x) - f(a)| < \epsilon)$$

One-side continuous

A function f is continuous from right at a **iff**:

$$\lim_{x \to a^+} f(x) = f(a)$$

A function f is continuous from left at a **iff**:

$$\lim_{x \to a^-} f(x) = f(a)$$

Continuous on an open interval

A function f is continuous in (a, b) **iff** f is continuous on every $c \in (a, b)$.

Continuous on a closed interval

A function f is continuous in $[a, b]$ **iff** f is:

- continuous on every $c \in (a, b)$
- right-continuous at a
- left-continuous at b

Uniformly continuous

Suppose a function f is continuous on (a, b). f is uniformly continuous on (a, b) **iff**:

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ s.t. } |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

If a function f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$.

> **Todo**
>
> Is this section correct? I am not 100% sure.

Continuity Theorems

Extreme Value Theorem

If f is continuous on $[a, b]$, f has a maximum and a minimum in $[a, b]$.

> **Proof Hint**
>
> Proof is quite hard.

Intermediate Value Theorem

Let f is continuous on $[a, b]$. If $\exists u$ such that $f(a) > u > f(b)$ or $f(a) < u < f(b)$: $\exists c \in (a, b)$ such that $f(c) = u$.

> **Proof Hint**
>
> Proof the case when $u = 0$. Otherwise define a new function $g(x)$ such that middle part of the above inequality has a 0 in the place of u.

Sandwich (or Squeeze) Theorem

Let:

- For some $\delta > 0$: $\forall x(0 < |x - a| < \delta \implies f(x) \leq g(x) \leq h(x))$
- $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \in \mathbb{R}$

Then $\lim_{x \to a} g(x) = L$.

> **Note**
>
> Works for any kind of x limits.

”No sudden changes”

Positive

Let f be continuous on a and $f(a) > 0$

$$\implies \exists \delta > 0; \forall x \left(|x-a| < \delta \implies f(x) > 0 \right)$$

> **Proof Hint**
>
> Take $\epsilon = \frac{f(a)}{2}$

Negative

Let f be continuous on a and $f(a) < 0$

$$\implies \exists \delta > 0; \forall x \left(|x-a| < \delta \implies f(x) < 0 \right)$$

> **Proof Hint**
>
> Take $\epsilon = -\frac{f(a)}{2}$

Differentiability

A function f is differentiable at a **iff**:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'(a)$$

$f'(a)$ is called the derivative of f at a.

One-side differentiable

Left differentiable

A function f is left-differentiable at a **iff**:

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_-(a)$$

Right differentiable

A function f is right-differentiable at a **iff**:

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_+(a)$$

Differentiability implies continuity

$$f \text{ is differentiable at } a \implies f \text{ is continuous at } a$$

Proof Hint

Use $\delta = min(\delta_1, \frac{\epsilon}{1+|f'(a)|})$.

Note

Suppose f is differentiable at a. Define g:

$$g(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

g is continuous at a.

Extreme Values

Suppose $f : [a, b] \to \mathbb{R}$, and $F = f([a, b]) = \left\{ f(x) \mid x \in [a, b] \right\}$. Minimum and maximum values of f are called the extreme values.

Maximum

Maximum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\forall x \in [a, b],\ f(c) \geq f(x)$$

aka. **Global Maximum**. Maximum doesn't exist always.

Local Maximum

A Local maximum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\exists \delta \;\; \forall x \, (0 < |x - c| < \delta \implies f(c) \geq f(x))$$

Global maximum is obviously a local maximum.

The above statement can be simplified when $c = a$ or $c = b$.

When $c = a$:

$$\exists \delta \;\; \forall x \, (0 < x - c < \delta \implies f(c) \geq f(x))$$

When $c = b$:

$$\exists \delta \;\; \forall x \, (-\delta < x - c < 0 \implies f(c) \geq f(x))$$

Minimum

Minimum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\forall x \in [a, b], \;\; f(c) \leq f(x)$$

aka. **Global Minimum**. Minimum doesn't exist always.

Local Minimum

$$\exists \delta \;\; \forall x \, (0 < |x - c| < \delta \implies f(c) \leq f(x))$$

Global minimum is obviously a local maximum.

The above statement can be simplified when $c = a$ or $c = b$.

When $c = a$:

$$\exists \delta \;\; \forall x \, (0 < x - c < \delta \implies f(c) \leq f(x))$$

When $c = b$:

$$\exists \delta \ \forall x \left(-\delta < x - c < 0 \implies f(c) \leq f(x)\right)$$

Special cases

f is continuous

Then by Extreme Value Theorem, we know f has a minimum and maximum in $[a, b]$.

f is differentiable

- If $f(a)$ is a local maximum: $f'_+(a) \leq 0$
- If $f(b)$ is a local maximum: $f'_-(b) \geq 0$
- $c \in (a, b)$ and If $f(c)$ is a local maximum: $f'(c) = 0$
- If $f(a)$ is a local minimum: $f'_+(a) \geq 0$
- If $f(b)$ is a local minimum: $f'_-(b) \leq 0$
- $c \in (a, b)$ and If $f(c)$ is a local minimum: $f'(c) = 0$

Critical point

$c \in [a, b]$ is called a critical point **iff**:

$$f'(c) = 0 \quad \lor \quad f'(c) \text{ is undefined}$$

Other Theorems

Rolle's Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). And $f(a) = f(b)$. Then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$

Proof Hint

By Extreme Value Theorem, maximum and minimum exists for f.

Consider 2 cases:

1. Both minimum and maximum exist at a and b.
2. One of minimum or maximum occurs in (a, b).

Mean Value Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof Hint

- Define $g(x) = f(x) - \left(\frac{f(a)-f(b)}{a-b}\right)x$
- $g(a)$ will be equal to $g(b)$
- Use Rolle's Theorem for g

Cauchy's Mean Value Theorem

Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and $\forall x \in (a, b)\ g'(x) \neq 0$ Then:

$$\exists c \in (a, b) \text{ s.t. } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof Hint

- Define $h(x) = f(x) - \left(\frac{f(a)-f(b)}{g(a)-g(b)}\right) g(x)$
- $h(a)$ will be equal to $h(b)$
- Use Rolle's Theorem for h

Mean value theorem can be obtained from this when $g(x) = x$.

Generalized MVT for Riemann Integrals

Let f, g be continuous on $[a, b]$ ($\implies$ f, g are integrable), and g does not change sign on (a, b). Then $\exists \zeta \in (a, b)$ such that:

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\zeta)\int_a^b g(x)\mathrm{d}x$$

Proof Hint

- Use Extreme value theorem for f
- Multiply by $g(x)$. Then integrate. Then divide by $\int_a^b g(x)$.
- Use intermediate value theorem to find $f(\zeta)$

L'Hopital's Rule

Note

Be careful with the pronunciation.

- It's not "Hospital's Rule", there are no "s"
- It's not "Hopital's Rule" either, there is a "L'"

L'Hopital's Rule can be used when all of these conditions are met. (here δ is some positive number). Select the appropriate x ranges.

1. Either of these conditions must be satisfied

- $f(a) = g(a) = 0$
- $\lim f(x) = \lim g(x) = 0$
- $\lim f(x) = \lim g(x) = \infty$

2. f, g are continuous on $x \in [a, a+\delta]$
3. f, g are differentiable on $x \in (a, a+\delta)$
4. $g'(x) \neq 0$ on $x \in (a, a+\delta)$
5. $\lim\limits_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$

Then: $\lim\limits_{x \to a^+} \frac{f(x)}{g(x)} = L$

> **Note**
>
> L'Hopital's rule can be proven using Cauchy's Mean Value Theorem.
>
> It is valid for all types of "x limits".

Higher Order Derivatives

Suppose f is a function defined on (a, b). f is n times differentiable or n-th differentiable **iff**:

$$\lim_{x \to a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a} = L \in \mathbb{R} = f^{(n)}(a)$$

Here $f^{(n)}$ denotes n-th derivative of f. And $f^{(0)}$ means the function itself.

$f^{(n)}(a)$ is the n-th derivative of f at a.

> **Note**
>
> $$f \text{ is } n\text{-th differentiable at } a \implies f^{(n-1)} \text{ is continuous at } a$$

Taylor's Theorem

Let f is $n+1$ differentiable on (a,b). Let $c,x \in (a,b)$. Then $\exists \zeta \in (c,x)$ s.t. :

$$f(x) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}$$

Mean value theorem can be derived from taylor's theorem when $n=0$.

> **Proof Hint**
>
> $$F(t) = f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$
>
> $$G(t) = (x-t)^{n+1}$$
>
> - Define F, G as mentioned above
> - Consider the interval $[c,x]$
> - Use Cauchy's mean value theorem for F, G after making sure the conditions are met.

The above equation can be written like:

$$f(x) = T_n(x,c) + R_n(x,c)$$

Taylor Polynomial

This part of the above equation is called the Taylor polynomial. Denoted by $T_n(x,c)$.

$$T_n(x,c) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Remainder

Denoted by $R_n(x,c)$.

$$R_n(x,c) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-c)^{n+1}$$

Integral form of the remainder

$$R_n(x,c) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n \mathrm{d}t$$

> **Proof Hint**
>
> - Method 1: Use integration by parts and mathematical induction.
> - Method 2: Use Generalized MVT for Riemann Integrals where:
>   - $F = f^{(n+1)}$
>   - $G = (x-t)^n$

Second derivative test

When $n = 1$:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(\zeta)}{2!}(x-c)^2$$

$$f(x) - \text{Tangent line} = \frac{f''(\zeta)}{2!}(x-c)^2$$

From this: $f''(c) > 0 \implies$ a local minimum is at c. Converse is **not** true.

Sequences

A sequence on a set A is a function $u : \mathbb{Z}^+ \to A$.

Image of the n is written as u_n. A sequence is indicated by one of these ways:

$$\left\{u_n\right\}_{n=1}^{\infty} \text{ or } \left\{u_n\right\} \text{ or } \left(u_n\right)_{n=1}^{\infty}$$

Increasing or Decreasing

A sequence (u_n) is

- Increasing **iff** $u_n \geq u_m$ for $n > m$
- Decreasing **iff** $u_n \leq u_m$ for $n > m$
- Monotone **iff** either increasing or decreasing
- Strictly increasing **iff** $u_n > u_m$ for $n > m$
- Strictly decreasing **iff** $u_n < u_m$ for $n > m$

Subsequence

Suppose $u : \mathbb{Z}^+ \to \mathbb{R}$ be a sequence and $v : \mathbb{Z}^+ \to \mathbb{Z}^+$ be an increasing sequence. Then $u \circ v : \mathbb{Z}^+ \to \mathbb{R}$ is a subsequence of u.

Convergence

Converging

A sequence $(u_n)_{n=1}^{\infty}$ is converging (to $L \in \mathbb{R}$) **iff**: $\lim_{n\to\infty} u_n = L$

$$\forall \epsilon > 0 \; \exists N \in \mathbb{Z}^+ \; \forall n \; (n > N \implies |u_n - L| < \epsilon)$$

> **Note**
>
> $$\forall x \in \mathbb{R} \quad \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Diverging

A sequence is diverging **iff** it is not converging.

$$\lim_{n\to\infty} u_n = \begin{cases} \infty \\ -\infty \\ \text{undefined}, & \text{when } u_n \text{ is osciallating} \end{cases}$$

Convergence test

All converging sequences are bounded.

Increasing and bounded above

Let $\left(u_n\right)$ be increasing and bounded above. Then $\left(u_n\right)$ is converging (to $\sup \left\{u_n\right\}$).

Proof Hint

- $\left\{u_n\right\}$ has a $\sup u_n (= s)$
- Prove: $\lim\limits_{n\to\infty} u_n = s^-$

Decreasing and bounded below

Let $\left(u_n\right)$ be decreasing and bounded below. Then $\left(u_n\right)$ is converging (to $\inf \left\{u_n\right\}$).

Proof Hint

- $\left\{u_n\right\}$ has a $\inf u_n (= l)$
- Prove: $\lim\limits_{n\to\infty} u_n = l^+$

Newton's method of finding roots

Suppose f is a function. To find its roots:

- Select a point x_0
- Draw a tangent at x_0
- Choose x_1 which is where the tangent meets $y = 0$
- Continue this process repeatedly

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Theorems

Existence of subsequence

Every sequence has a monotone subsequence.

Bolzano-Weistrass

Every bounded sequence has a converging sequence.

Proof Hint

Using the above theorem and the fact that bounded monotone sequences converge.

Cauchy Sequence

A sequence $u : \mathbb{Z}^+ \to A$ is Cauchy **iff**:

$$\forall \epsilon > 0 \, \exists N \in \mathbb{Z}^+ \, \forall m, n; m, n > N \implies |u_n - u_m| < \epsilon$$

Complete

A set A is complete **iff**:

$$\forall u : \mathbb{Z}^+ \to A; \ u \text{ converges to } L \in A$$

Q is not complete

Q is **not** complete because:

$$\sum_{k=1}^{\infty} \frac{1}{k!} = e - 1 \notin \mathbb{Q}$$

R is complete

$\mathbb{R}$ is complete.

Proof Hint

Proof is quite hard.

Bounded

All Cauchy sequences are bounded. (has an upper bound).

Proof Hint

- Consider the Cauchy definition
- Take $n > m = N + 1 > N$

Series

Let (u_n) be a sequence, and a series (a new sequence) can be defined from it such that:

$$s_n = \sum_{k=1}^{n} u_k$$

Convergence

If (s_n) is converging:

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \sum_{k=1}^{n} u_k = \sum_{k=1}^{\infty} u_k = S \in \mathbb{R}$$

Absolutely Converging

$\sum_{k=1}^{n} u_k$ is absolutely converging **iff** $\sum_{k=1}^{n} |u_k| \text{ is converging}$.

$$\sum_{k=1}^{n} |u_k| \text{ is converging} \implies \sum_{k=1}^{n} u_k \text{ is converging}$$

Proof Hint

Use this inequality:

$$0 \leq \ |u_k| - u_k \ \leq 2|u_k|$$

Conditionally Converging

$\sum_{k=1}^{n} u_k$ is condtionally converging **iff**:

$$\sum_{k=1}^{n} |u_k| \text{ is diverging} \quad \text{and} \sum_{k=1}^{n} u_k \text{ is converging}$$

Theorem 1

$$\sum_{k=1}^{n} u_k \text{ is converging} \implies \lim_{k \to \infty} u_k = 0$$

The converse is more useful:

$$\lim_{k \to \infty} u_k \neq 0 \implies \sum_{k=1}^{n} u_k \text{ is diverging}$$

Convergence Tests

Direct Comparison Test

Let $0 < u_k < v_k$.

$$\sum_{k=1}^{\infty} v_k \text{ is converges} \implies \sum_{k=1}^{\infty} u_k \text{ is converges}$$

Proof Hint

- Note that $\sum_{k=1}^{n} u_k$ and $\sum_{k=1}^{n} v_k$ are increasing
- Show that $\sum_{k=1}^{\infty} v_k$ converges to its supremum v which is an upper bound of $\sum_{k=1}^{n} u_k$

Example

Proving the convergence of $\sum_{k=1}^{\infty} \frac{1}{k!}$, by using $k! \geq 2^{k-1}$ for all $k \geq 0$.

Limit Comparison Test

Let $0 < u_k, v_k$ and $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = R$.

$$R > 0 \implies \left(\sum_{n=1}^{\infty} u_n \text{ is converging} \iff \sum_{n=1}^{\infty} v_n \text{ is converging}\right)$$

$$R = 0 \implies \left(\sum_{n=1}^{\infty} v_n \text{ is converging} \implies \sum_{n=1}^{\infty} u_n \text{ is converging}\right)$$

$$R = \infty \implies \left(\sum_{n=1}^{\infty} v_n \text{ is diverging} \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}\right)$$

Proof Hint

Only possibilities are $R = 0, R > 0, R = \infty$.

For $R > 0$:

- Consider limit definition with $\epsilon = \frac{L}{2}$
- Direct comparison test can be used for the 2 set of inequalities

For $R = 0$:

- Consider limit definition with $\epsilon = 1$
- Direct comparison test can be used now

For $R = \infty$:

- Consider limit definition with $M = 1$
- Direct comparison test can be used now

Integral Test

Let $u(x) > 0$, decreasing and integrable on $[1, M]$ for all $M > 1$. Then:

$$\sum_{n=1}^{\infty} u_n \text{ is converging} \iff \int_1^{\infty} u(x)\,\mathrm{d}x \text{ is converging}$$

Proof Hint

As $u(x)$ is decreasing, it is apparent that it is integrable.

Make use of this inequality:

$$s_n - u_1 \leq \int_1^n u(x)\,\mathrm{d}x \leq s_n - u_n$$

For $\Longleftarrow$:

- Note that s_n is increasing
- Show that s_n is bounded above by $\int_1^\infty u(x)\,\mathrm{d}x + u_1$

For $\Longrightarrow$:

- Define $F(n) = \int_1^n u(x)\,\mathrm{d}x$
- Note that $F(n)$ is increasing
- Note that $\lim_{n\to\infty} u_n = 0$
- Show that $F(n)$ is bounded above by $\lim_{n\to\infty} s_n$

Note

$$\sum_{n=1}^{\infty} u_n \text{ is converging} \implies \lim_{k\to\infty} u_k = 0$$

$$\int_1^\infty u(x)\,\mathrm{d}x \text{ is converging} \implies \lim_{k\to\infty} u(k) = 0$$

Ratio Test

Let $u(x) > 0$ and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$.

$$L < 1 \implies \sum_{n=1}^{\infty} u_n \text{ is converging}$$

$$L > 1 \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}$$

> **Proof Hint**
>
> - Consider the limit definition with $\epsilon = \frac{1}{2}(1 - L)$
> - Show that: $\frac{1}{2}(3L - 2) < \frac{u_{k+1}}{u_k} < \frac{1}{2}(1 + L)$
> - Use $\sum_{k=1}^{\infty} r^k$ is converging **iff** $r < 1$

Root Test

Let $u(x) > 0$ and $\lim\limits_{n\to\infty} u_n^{1/n} = L$.

$$L < 1 \implies \sum_{n=1}^{\infty} u_n \text{ is converging}$$

$$(L > 1 \lor L = \infty) \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}$$

> **Proof Hint**
>
> For $L < 1 \lor L > 1$: Consider the limit definition with $\epsilon = \frac{1}{2}(1 - L)$
>
> For $L = \infty$: Consider the limit definition with $M > 1$

Riemann Zeta Function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

Convergence of this function can be derived using integral test.

This function converges **iff** $s > 1$. And it converges to:

$$\frac{1}{s-1}$$

Otherwise it diverges.

Alternating Series

Suppose $u_k > 0$. An alternating series is:

$$\sum_{k=1}^{n} (-1)^{k-1} u_k = u_1 - u_2 + u_3 - u_4 + \cdots$$

Convergence

If $\forall k \; u_k > 0$, decreasing and $\lim_{n \to \infty} u_n = 0$. Then

$$\sum_{k=1}^{n} (-1)^{k-1} u_k \text{ is converging}$$

Proof

For odd-indexed elements:

$$s_{2m+3} \leq s_{2m+1} \leq s_1 = u_1$$

For even-indexed elements:

$$s_{2m+2} \geq s_{2m} \geq s_2 = u_1 - u_2$$

Combining these 2:

$$0 \leq u_1 - u_2 \leq s_2 \leq s_{2m} \leq s_{2m+1} \leq s_1 = u_1$$

s_{2m} is bounded above by u_1 and increasing. s_{2m+1} is bounded below by 0 and decreasing. So both converges.

$$\lim_{m\to\infty} (s_{2m+1} - s_{2m}) = \lim_{m\to\infty} u_{2m+1} = 0$$

$$\implies \lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} = s$$

Both converges to the same number. :::