Summary | Matrices

Introduction

Revise Matrices unit from G.C.E. (A/L) Combined Mathematics and G.C.E. (O/L) Mathematics.

Types of matrices

Square matrix

Number of columns equal to number of rows.

> **Main diagonals of a square matrix**
>
> Formed by elements having equal subscripts.

Diagonal matrix

A square matrix whose only non-zero elements are main-diagonal elements. Denoted by D. Subset of triangular matrices.

Identity matrix or Unit matrix

A diagonal matrix whose diagonal elements are all equal to 1. Denoted by I. Subset of diagonal matrices.

Zero matrix / Null matrix

All elements are 0.

Column matrix (column vector)

Only 1 column.

Row matrix (row vector)

Only 1 row.

Triangular matrix

Upper triangular matrix or lower triangular matrix.

####### Upper triangular matrix

All elements below the main diagonal are 0. Subset of square matrices.

####### Lower triangular matrix

All elements above the main diagonal are 0. Subset of square matrices.

Matrix operations

Addition and subtraction

Order of the 2 matrices must be same. Matrix obtained by adding or subtracting corresponding elements.

Scalar multiplication

Matrix obtained by multiplying all elements by the scalar.

> **Note**
>
> Matrix multiplication is also defined.

Transpose

Matrix obtained from a given matrix by interchanging its rows and columns. Denoted by a superscript T, like A^T.

Properties

1. $(A^T)^T = A$
2. Distributive over addition: $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(A \times B)^T = B^T \times A^T$

More types of matrices

Symmetric matrix

If $A = A^T$. Subset of square matrices.

Skew-symmetric matrix

If $A = -A^T$. Subset of square matrices. All elements in main diagonal are 0.

> **Note**
>
> Any square matrix can be expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

Matrix multiplication

Defined only if the number of columns of the first matrix is equal to the number of rows of the second matrix.

If $A = (a_{ij})_{m \times p}$ and $B = (b_{ij})_{p \times n}$, then $A \times B = C = (c_{ij})_{m \times n}$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$.

> **Note**
>
> - Generally $A \times B \neq B \times A$.
> - $A \times B = 0 \not\Rightarrow A = 0 \vee B = 0$
> - $A \neq 0 \wedge B \neq 0 \not\Rightarrow A \times B \neq 0$

Properties of matrix multiplication

A, B, C, I matrices must be chosen so that below-mentioned product matrices are defined.

1. Associative: $A(BC) = (AB)C$
2. Right distributive over addition: $(A + B)C = AC + BC$
3. Left distributive over addition: $C(A + B) = CA + CB$
4. $AI = IA = A$; I is an identity matrix.

Determinant

Defined only for square matrices. Denoted by $|A|$.

For 2x2

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For higher order

####### Minor of an element

Suppose $A = (a_{ij})$.

Minor of an element a_{ij}, is the matrix obtained by deleting i^{th} row and j^{th} column of A. Denoted by M_{ij}.

####### Co-factor of an element

Suppose $A = (a_{ij})$.

Co-factor of an element a_{ij}, is defined as (commonly denoted as A_{ij}):

$$A_{ij} = (-1)^{i+j}\,|M_{ij}|$$

####### Definition

If $A = (a_{ij})_{n\times n}$ then the **determinant** of A is denoted by $|A|$ and is defined by:

$$|A| = \sum_{j=1}^{n} a_{ij}A_{ij}$$

where $1 \leq j \leq n$.

Properties of determinants

- Every element of a row or column of a matrix is 0 then the value of its determinant is 0.
- If 2 columns or 2 rows of a matrix are identical then its determinant is 0.
- If A and B are two square matrices then $|AB| = |A||B|$.
- The value of the determinant of a matrix remains unchanged if a scalar multiple of a row or column is added to any other row or column.
- If a matrix B is obtained from a square matrix A by an interchange of two columns or rows:
  $|B| = -|A|$.
- If every entry in any row or column is multiplied by k, then the whole determinant is multiplied by k.

Adjoint

Suppose $A = (a_{ij})_{n\times n}$.

$$\text{adj}A = (A_{ij})^T_{n\times n}$$

Where A_{ij} is the co-factor of a_{ij}.

Inverse

Suppose A and B are square matrices of the same order. If $AB = BA = I$ then B is called the inverse of A and is denoted by A^{-1}.

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

Singular vs Non-singular

A square matrix is singular if $|A| = 0$. Otherwise non-singular or invertible matrices.

Properties of Inverse

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $A\,\mathrm{adj}A = \mathrm{adj}A\,A = |A|I$

Elementary Transformations

- Interchange of any columns or rows
- Addition of multiple of any row or column to any other row or column
- Multiplication of each element of a column or a row by a non-zero constant

When a matrix B is obtained by applying elementary transformations to a matrix A, then A **is equivalent to** B. Denoted by $A \approx B$.

Theorem

The elementary row operations that reduce a given matrix A to the identity matrix, also transform the identity matrix to the inverse of A.

Augmented Matrix

Two matrices are written as a single matrix with a vertical line in-between. Denoted by $(A|B)$. Example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$$

Inverse using elementary row transformations

Let A be a square matrix with order $n \times n$.

- Start with $(A_{n\times n}|I_n)$
- Repeatedly add **row** transformations (not column) to both of the matrices until the LHS becomes an identity matrix.
  - Convert all elements outside the main diagonal to 0.
  - Convert elements on the main diagonal to 1 by multiplying by a constant.
- When LHS is an identity matrix, RHS is A^{-1}.

> ⚠ **TODO**
>
> What about singular matrices?

System of Linear Equations

Any system of linear equations can be represented in matrix notation as shown below.

- $a_{11}x + a_{12}y + a_{13}z = b_1$
- $a_{21}x + a_{22}y + a_{23}z = b_2$
- $a_{31}x + a_{32}y + a_{33}z = b_3$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \implies AX = B$$

2 types based on B:

- $= 0$: Homogeneous
- $\neq 0$: Non-homogeneous

Solution of non-homogenous systems

Method 1: Direct approach

Used when coefficient matrix A is invertible. It means the system has a unique set of solutions.

$$AX = B \implies X = A^{-1}B$$

Method 2: Cramer's Rule

Let $AX = B$, where A is the coefficient matrix and $X = (x_i)_{n \times 1}$.

$$x_i = \frac{|A_i|}{|A|}$$

Where A_i is the matrix obtained by replacing ith column in matrix A by B.

Method 1: Reducing to Echelon Form

Start with $(A|B)$. Convert the LHS to echelon form using elementary row transformations. The solution can be found now. If a contradiction is encountered while solving the equation, that means the system has no solutions.

####### Echelon Form

A matrix is in echelon form **iff**:

- All rows having only zero entries are at the bottom.
- For all row that does not contain entirely zeros, the first non-zero entry is 1.
- For 2 successive non-zero rows, the leading 1 in the higher row is further left than the leading 1 in the lower row.

The process of reducing the augmented matrix to row Echelon form is known as **Gaussian elimination**.

Solution of homogenous systems

TODO