Summary | Riemann Integration

Introduction

Interval

Let $I = [a, b]$. Length of the interval $|I| = b - a$.

Disjoint interval

When 2 intervals don't share any common numbers.

Almost disjoint interval

When 2 intervals are disjoint or intersect only at a common endpoint.

Riemann Integral

Let $f - [a, b] \to \mathbb{R}$ is a bounded (not necessarily continuous) function on a closed, bounded (compact) interval.

Riemann integral of f is: $\int_a^b f$

Definite integral

When a, b are constants.

Indefinite integral

When a is a constant but b is replaced with x.

Partition

Let I be a non-empty, compact interval (closed and bounded). A partition of I is a finite collection $\{I_1, I_2, \ldots, I_n\}$ of almost disjoint, non-empty, compact sub-intervals whose union is I.

A partition is determined by the endpoints of all sub-intervals: $a = x_0 < x_1 < \cdots < x_n = b$.

A partition can be denoted by:

- its intervals - $P = \{I_1, I_2, \ldots, I_n\}$
- the endpoints of its intervals - $P = \{x_0, x_1, \ldots, x_n\}$

Riemann Sum

Let

- $f : [a, b] \to \mathbb{R}$ is a bounded function on the compact interval $I = [a, b]$ with $M = \sup_I f$ and $m = \inf_I f$.
- $P = \{I_1, I_2, \ldots, I_n\}$
- $M_k = \sup_{I_k} f = \sup \{f(x) : x \in [x_{k-1}, x_k]\}$
- $m_k = \inf_{I_k} f = \inf \{f(x) : x \in [x_{k-1}, x_k]\}$

Upper riemann sum

$$U(f; P) = \sum_{k=1}^{n} M_k |I_k|$$

Lower riemann sum

$$L(f; P) = \sum_{k=1}^{n} m_k |I_k|$$

$m_k < M_k \implies L(f; P) \le U(f; P)$

When P_1, P_2 are any 2 partitions of I: $L(f; P_1) \le U(f; P_2)$

Refinements

Q is called a refinement of $P \iff$ if P and Q are partitions of $[a, b]$ and $P \subseteq Q$.

When Q is a refinement of P:

$$L(f;P) \le L(f;Q) \le U(f;Q) \le U(f;P)$$

> ⓘ **Note**
>
> If P_1 and P_2 are partitions of $[a,b]$, then $Q = P_1 \cup P_2$ is a refinement of both P_1 and P_2. In that case:
>
> $$L(f;P_1) \le L(f;Q) \le U(f;Q) \le U(f;P_2)$$

Upper & Lower integral

Let $\mathbb{P}$ be the collection of all possible partitions of the interval $[a,b]$.

Upper Integral

$$U(f) = \inf \{U(f;P); P \in \mathbb{P}\} = \overline{\int_a^b} f$$

Lower Integral

$$L(f) = \sup \{L(f;P); P \in \mathbb{P}\} = \underline{\int_a^b} f$$

For a bounded function f, always $L(f) \le U(f)$

Riemann Integrable

A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ **iff** $U(f) = L(f)$. In that case, the Riemann integral of f on $[a,b]$ is denoted by $\int_a^b f(x)\, \mathrm{d}x$.

Reimann Integrable or not

Function	Yes or No?	Proof hint
Unbounded	No	By definition
Constant	Yes	$\forall P \, (\text{any partition}) \;\; L(f; P) = U(f; P)$
Monotonically increasing/decreasing	Yes	Take a partition such that $\Delta x < \delta = \frac{\epsilon}{f(b) - f(a)}$
Continuous	Yes	Take a partition such that $\Delta x < \delta = \frac{\epsilon}{2(b-a)}$

> ⓘ **Note**
>
> If the set of points of discontinuity of a bounded function $f : [a, b] \to \mathbb{R}$ is finite, then f is Riemann integrable on $[a, b]$.

> ⓘ **Note**
>
> If the set of points of discontinuity of a bounded function $f : [a, b] \to \mathbb{R}$ is finite number of limit points, then f is integrable on $[a, b]$.
>
> A function may have infinitely many discontinuous points, but if the set of all discontinuous points have finite number of limit points, then f is integrable on $[a, b]$.

Cauchy Criterion

Theorem

A bounded function $f : [a, b] \to R$ is Riemann integrable **iff** for every $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$, which may depend on ϵ, such that:

$$U(f, P\epsilon) - L(f, P\epsilon) \le \epsilon$$

> **ⓘ Proof Hint**
>
> - To prove $\implies$: consider $L(f) - \frac{\epsilon}{2} < L(f; P)$ and $< U(f) + \frac{\epsilon}{2}$
> - To prove $\impliedby$: consider $L(f; P) < L(f)$ and $U(f) < U(f; P)$

> **ⓘ Note**
>
> $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ when:
>
> - The set of points of discontinuity of a bounded function f is finite.
> - The set of points of discontinuity of a bounded function f is finite number of limit points. (may have infinite number of discontinuities)

Theorems on Integrability

Theorem 1

Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and integrable on $[c, b]$ for all $c \in (a, b)$. Then f is integrable on $[a, b]$. Also valid for the other end.

> **ⓘ Proof Hint**
>
> - Isolate a partition on the required end.
> - Choose x_1 or x_{n-1} such that $\Delta x < \frac{\epsilon}{4M}$ where M is an upper or lower bound.

Theorem 2

Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and continuous on $[c, b]$ for all $c \in (a, b)$. Then f is integrable on $[a, b]$. Also valid for the other end.

> **⚠ TODO: Proof Hint**
>

Properties of Integrals

Notation

If $a < b$ and f is integrable on $[a, b]$, then:

$$\int_a^b f = -\int_b^a f$$

Properties

Suppose f and g are integrable on $[a, b]$.

Addition

$f + g$ will be integrable on $[a, b]$.

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$$

> ### ⓘ Proof Hint
>
> - Prove $f + g$ is integrable using:
>   - $sup(f + g) \leq \sup(f) + \sup(g)$
>   - $inf(f + g) \geq \inf(f) + \inf(g)$
> - Start with $U(f + g)$ and show $U(f + g) \leq U(f) + U(g)$
> - Start with $L(f + g)$ and show $L(f + g) \geq L(f) + L(g)$

Constant multiplication

Suppose $k \in \mathbb{R}$. kf will be integrable $[a, b]$.

$$\int_a^b kf = k \int_a^b f$$

Bounds

If $m \leq f(x) \leq M$ on $[a, b]$:

$$m(b - a) \leq \int_a^b f \leq M(b - a)$$

If $f(x) \leq g(x)$ on $[a, b]$:

$$\int_a^b f \leq \int_a^b g$$

Modulus

$|f|$ will be integrable on $[a, b]$.

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Multiple

fg will be integrable on $[a, b]$.

Max, Min

$\max(f,g)$ and $\min(f,g)$ are integrable.

Where $\max$ and $\min$ functions are defined as:

$$\max(f,g) = \tfrac{1}{2}\left(|f-g| + f + g\right)$$

$$\min(f,g) = \tfrac{1}{2}\left(-|f-g| + f + g\right)$$

Additivity

$\Longleftarrow$ f is Riemann integrable on $[a,c]$ **and** $[c,b]$ where $c \in (a,b)$.

After the integrability is proven,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Sequential Characterization of Integrability

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if $\exists \{P_n\}$ a sequence of partitions, such that:

$$\lim_{n \to \infty} \left[U(f; P_n) - L(f; P_n) \right] = 0$$

In that case:

$$\int_a^b f = \lim_{n \to \infty} U(f; P_n) = \lim_{n \to \infty} L(f; P_n)$$

> ### ⓘ Proof Hint
>
> Cauchy criteria and squeeze theorem is used for both side proof.
>
> For $\Longleftarrow$:
>
> - Consider the limit definition.
> - Prove f is Riemann integrable on P_n by Cauchy criteria.
> - Use squeeze theorem for $U(f; P_n) - U(f) \leq U(f; P_n) - L(f; P_n)$ to prove limit of upper sum
> - Prove limit of lower sum using the limit of upper sum
>
> For $\Longrightarrow$: Consider the below, where $n \in \mathbb{N}$.
>
> $$0 \leq U(f; P_n) - L(f; L_n) \leq \frac{1}{n}$$

Theorem

Suppose f is Riemann integrable on $[a, b]$ and $\epsilon > 0$. Then $\exists \epsilon > 0 \forall P$:

$$|P| < \delta \implies \left| \int_a^b f - \sum_{j=1}^n f(\zeta_j) I_j \right| < \epsilon$$

where $\zeta_j \in [x_{j-1}, x_j], j = 1, 2, \cdots, n$.

> ### ⓘ Proof Hint
>
> $$\underline{\int_a^b} f - \epsilon \; < \; L(f; P) \; \leq \; \sum_{j=1}^n f(\zeta_j) I_j \; \leq \; U(f; P) \; < \; \overline{\int_a^b} f + \epsilon$$

Intermediate Value Theorem for Integrals

Suppose f is a continuous function on $[a, b]$. Then $\exists x \in (a, b)$:

$$f(x) = \frac{1}{b - a} \int_a^b f$$

> **ⓘ Proof Hint**
>
> Suppose $f_{\max} = M = f(x_0)$ and $f_{\min} = m = f(y_0)$.
>
> When $M = m$: f is a constant function. Proof is trivial.
>
> Otherwise:
>
> $$m(b - a) \leq \int_a^b f \leq M(b - a)$$
>
> Then there exists $x \in (x_0, y_0)$.

Generlized IVT

Suppose f, g are continuous functions on $[a, b]$ and $g \geq 0$. Then $\exists x \in (a, b)$:

$$f(x) \int_a^b g = \int_a^b fg$$

> **ⓘ Proof Hint**
>
> Consider this and proof is similar to IVT.
>
> $$mg \leq fg \leq Mg$$

Sequence of Functions

Types of Convergence

Uniformly convergence

$$\forall \epsilon > 0 \; \exists N \; \forall x \in [a, b] \;\; \text{s.t.} \; \left| f_n(x) - f(x) \right| < \epsilon$$

Here N depends on ϵ only.

Pointwise convergence

$$\forall \epsilon > 0 \; \exists N \; \forall x \in [a, b] \;\; \text{s.t.} \; \left| f_n(x) - f(x) \right| < \epsilon$$

Here N depends on ϵ, x.

Theorem

Let f_n be a sequence of Riemann integrable functions on $[a, b]$. Suppose f_n convergex to f uniformly. Then f is Riemann integrable on $[a, b]$ and:

$$\lim_{n \to \infty} \int_a^b f_n(x) \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x$$

> ⓘ **Proof Hint**
>
> - Consider $\frac{\epsilon}{2(b-a)}$ in place of ϵ.
> - Consider Cauchy criteria for f_N.
> - Prove $f - f_N$ is Riemann integrable using Cauchy criteria.
> - f is Riemann integrable as $f = f_N + (f - f_N)$.