

# Summary | Vectors

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## Introduction

Revise Vectors unit from G.C.E (A/L) Combined Mathematics.

## Section formula

Suppose **O** is the reference point, and **P, Q** are 2 points.

If **R** divides the line segment **PQ** in the ratio **m : n** (both are positive and **m ≥ n**), the division can either be internal or external.

### Internally

$$\overrightarrow{OR} = \frac{m\overrightarrow{OQ} + n\overrightarrow{OP}}{m + n}$$

### Externally

$$\overrightarrow{OR} = \frac{m\overrightarrow{OQ} - n\overrightarrow{OP}}{m - n}$$

## Direction Cosines

Suppose  $\vec{p} = a\underline{i} + b\underline{j} + c\underline{k}$ . Direction cosines of **p** are  $\cos \alpha, \cos \beta, \cos \gamma$  where  $\alpha, \beta, \gamma$  are the angles **p** makes with **x, y, z** axes.

Unit vector in the direction of  $\vec{p} = \underline{i} \cos \alpha + \underline{j} \cos \beta + \underline{k} \cos \gamma$ . Because of this:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

## Direction Ratio

Ratio of the direction cosines is called as direction ratio.

$$\cos \alpha : \cos \beta : \cos \gamma$$

## Cross Product

$$a \times b = |a||b|\sin(\theta)n = \det \begin{pmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$$

$n$  is the **unit normal vector** to  $a$  and  $b$ . Direction is based on the right hand rule.

$$a \times b = 0 \implies |a| = 0 \vee |b| = 0 \vee a \parallel b$$

Cross products between  $i, j, k$  are circular.

$$\begin{array}{l} i \times j = k \\ j \times k = i \\ k \times i = j \end{array} \quad \begin{array}{c} i \\ \swarrow \quad \searrow \\ k \quad \quad j \\ \longleftarrow \end{array} \quad \begin{array}{l} j \times i = -k \\ k \times j = -i \\ i \times k = -j \end{array}$$

## Properties

- $a \times a = 0$
- $(a \times b) = -(b \times a)$
- $a \times (b + c) = (a \times b) + (a \times c)$

### Note

Area of a parallelogram  $ABCD = |\vec{AB} \times \vec{AD}|$

## Scalar Triple Product

$$[a, b, c] = a \cdot (b \times c) = \det \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}$$

$$[a, b, c] = a \cdot (b \times c) = (a \times b) \cdot c$$

$$[a, b, c] = [b, c, a] = [c, a, b] = -[a, c, b]$$

$[a, b, c] = 0$  iff  $a, b, c$  are coplanar. Swapping any 2 vectors will negate the product.

### Note

Volume of a parallelepiped with  $a, b, c$  as adjacent edges =  $[a, b, c]$

Volume of a tetrahedron with  $a, b, c$  as adjacent edges =  $\frac{1}{6} [a, b, c]$

## Vector Triple Product

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

Resulting vector lies in the plane that contains  $b$  and  $c$

## Vector Equation of Straight Lines

**Passes through a point & parallel to a vector**

Equation for a line that:

- passes through  $\underline{r_0} = \langle x_0, y_0, z_0 \rangle$
- is parallel to  $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$

### Parametric equation

$$\underline{r} = \underline{r_0} + t\underline{v}; \quad t \in \mathbb{R}$$

### Symmetric equation

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

### Passes through 2 points

Equation of a line passes through  $A = (x_1, y_1, z_1)$ ,  $B = (x_2, y_2, z_2)$ .  $\underline{r_A}$  and  $\underline{r_B}$  are the position vectors of  $A$  and  $B$ .

### Parametric equation

$$\underline{r} = (1 - t)\underline{r_A} + t\underline{r_B}; \quad t \in \mathbb{R}$$

### Symmetric equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

#### Note

To show that two straight lines intersect in 3D space, it is **not** enough to show that the cross product of their parallel vectors is non-zero.

**Also:** Existence of a point which satisfies both lines must be proven.

### Normal to 2 lines

Let  $\alpha, \beta$  be two lines.

$$\alpha : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}; \quad \beta : \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

Here  $\underline{v}_1 = \langle \underline{a}_1, \underline{b}_1, \underline{c}_1 \rangle$ ,  $\underline{v}_2 = \langle \underline{a}_2, \underline{b}_2, \underline{c}_2 \rangle$  are **2** vectors parallel to  $\alpha, \beta$  respectively.

Normal to both lines:  $\underline{v}_1 \times \underline{v}_2$ . Unit normal to both lines can be found by:

$$\frac{\underline{v}_1 \times \underline{v}_2}{|\underline{v}_1 \times \underline{v}_2|}$$

## Angle between 2 straight lines

Using the  $\alpha, \beta$  lines mentioned above:

$$\cos \theta = \frac{\underline{v}_1 \cdot \underline{v}_2}{|\underline{v}_1| \cdot |\underline{v}_2|} = \frac{(\underline{a}_1 \underline{i} + \underline{b}_1 \underline{j} + \underline{c}_1 \underline{k}) \cdot (\underline{a}_2 \underline{i} + \underline{b}_2 \underline{j} + \underline{c}_2 \underline{k})}{|\underline{a}_1 \underline{i} + \underline{b}_1 \underline{j} + \underline{c}_1 \underline{k}| \cdot |\underline{a}_2 \underline{i} + \underline{b}_2 \underline{j} + \underline{c}_2 \underline{k}|}$$

Here  $\underline{v}_1, \underline{v}_2$  are **2** vectors parallel to  $\alpha, \beta$  respectively.

## Shortest distance to a point

Suppose  $\underline{x}_1$  and  $\underline{x}_2$  lie on a line. Shortest distance to the point  $P$  is:

$$d^2 = \frac{\left| (\underline{x}_2 - \overrightarrow{OP}) \times (\underline{x}_1 - \overrightarrow{OP}) \right|^2}{\left| \underline{x}_2 - \underline{x}_1 \right|^2}$$

## Vector Equation of Planes

### Contains a point and parallel to 2 vectors

Suppose a plane:

- is parallel to both  $\underline{a}$  and  $\underline{b}$
- contains  $\underline{r}_0 = \underline{x}_0 \underline{i} + \underline{y}_0 \underline{j} + \underline{z}_0 \underline{k}$

Equation for the plane is:

$$\underline{r} = \underline{r}_0 + s \underline{a} + t \underline{b} ; s, t \in \mathbb{R}$$

## Contains a point and normal is given

Suppose a plane:

- contains  $\underline{r}_0 = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$
- has a normal  $\underline{n}$

Equation for the plane is:

$$(\underline{r} - \underline{r}_0) \cdot \underline{n} = 0$$

## Contains 3 points

Suppose a plane contains  $\underline{r}_0, \underline{r}_1, \underline{r}_2$  ( $\underline{r}_0, \underline{r}_1, \underline{r}_2$  are the position vectors of respectively).

$$(\underline{r} - \underline{r}_1) \cdot [(\underline{r}_1 - \underline{r}_0) \times (\underline{r}_1 - \underline{r}_2)] = 0$$

## Normal to a plane

Suppose  $ax + by + cz = d$  is a plane.  $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$  is a normal to the plane.

## Angle between 2 planes

Consider the two planes:

- $A : a_1x + a_2y + a_3z = d$
- $B : b_1x + b_2y + b_3z = d'$

The angle between the planes  $\phi$  is given by:

$$\cos(\phi) = \frac{\underline{n}_A \cdot \underline{n}_B}{|\underline{n}_A| \cdot |\underline{n}_B|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}$$

Here  $\underline{n}_A, \underline{n}_B$  are normal to the planes  $A, B$ .

## Shortest distance to a point

Considering a plane  $ax + by + cz = d$ .

$$\text{distance} = \frac{|(\underline{r_1} - \underline{r_0}) \cdot \underline{n}|}{|\underline{n}|}$$

- $\underline{n}$  is a normal to the plane
- $\underline{r_0}$  is the position vector of a point on the plane
- $\underline{r_1}$  is the position vector to the arbitrary point

## Skew Lines

Two non-parallel lines in a 3-space that do not intersect.

### Normal to 2 skew lines

Let  $l_1, l_2$  be 2 skew lines.

$$l_1 : \frac{x - x_0}{a_0} = \frac{y - y_0}{b_0} = \frac{z - z_0}{c_0} ; \quad l_2 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

The unit normal to both lines  $\underline{n}$  is:

$$\underline{n} = \frac{\langle a_0, b_0, c_0 \rangle \times \langle a_1, b_1, c_1 \rangle}{|\langle a_0, b_0, c_0 \rangle \times \langle a_1, b_1, c_1 \rangle|}$$

### Distance between 2 skew lines

$$\text{distance} = |\overrightarrow{AB} \cdot \underline{n}|$$

Here

- $\underline{n}$  is the normal to both  $l_1, l_2$
- $A$  and  $B$  are points lying on each line