Summary | Riemann Integration

Introduction

Interval

Let $I = [a, b]$. Length of the interval $|I| = b - a$.

Disjoint interval

When 2 intervals don't share any common numbers.

Almost disjoint interval

When 2 intervals are disjoint or intersect only at a common endpoint.

Riemann Integral

Let $f - [a, b] \to \mathbb{R}$ is a bounded (not necessarily continuous) function on a closed, bounded (compact) interval.

Riemann integral of f is: $\int_a^b f$

Definite integral

When a, b are constants.

Indefinite integral

When a is a constant but b is replaced with x.

Partition

Let I be a non-empty, compact interval (closed and bounded). A partition of I is a finite collection $\{I_1, I_2, \ldots, I_n\}$ of almost disjoint, non-empty, compact sub-intervals whose union is I.

A partition is determined by the endpoints of all sub-intervals:
$a = x_0 < x_1 < \cdots < x_n = b$.

A partition can be denoted by:

- its intervals - $P = \{I_1, I_2, \ldots, I_n\}$
- the endpoints of its intervals - $P = \{x_0, x_1, \ldots, x_n\}$

Riemann Sum

Let

- $f : [a, b] \to \mathbb{R}$ is a bounded function on the compact interval $I = [a, b]$ with $M = \sup_I f$ and $m = \inf_I f$.
- $P = \{I_1, I_2, \ldots, I_n\}$
- $M_k = \sup_{I_k} f = \sup \{f(x) : x \in [x_{k-1}, x_k]\}$
- $m_k = \inf_{I_k} f = \inf \{f(x) : x \in [x_{k-1}, x_k]\}$

Upper riemann sum

$$U(f;P)=\sum_{k=1}^{n} M_k|I_k|$$

Lower riemann sum

$$L(f;P)=\sum_{k=1}^{n} m_k|I_k|$$

$m_k < M_k \implies L(f;P) \le U(f;P)$

When P_1, P_2 are any 2 partitions of I: $L(f;P_1) \le U(f;P_2)$

Refinements

Q is called a refinement of $P \iff$ if P and Q are partitions of $[a,b]$ and $P \subseteq Q$.

When Q is a refinement of P:

$$L(f;P) \le L(f;Q) \le U(f;Q) \le U(f;P)$$

> ⓘ **Note**
>
> If P_1 and P_2 are partitions of $[a, b]$, then $Q = P_1 \cup P_2$ is a refinement of both P_1 and P_2. In that case:
>
> $$L(f; P_1) \leq L(f; Q) \leq U(f; Q) \leq U(f; P_2)$$

Upper & Lower integral

Let $\mathbb{P}$ be the collection of all possible partitions of the interval $[a, b]$.

Upper Integral

$$U(f) = \inf \{U(f; P); P \in \mathbb{P}\} = \overline{\int_a^b f}$$

Lower Integral

$$L(f) = \sup \{L(f; P); P \in \mathbb{P}\} = \underline{\int_a^b f}$$

For a bounded function f, always $L(f) \leq U(f)$

Riemann Integrable

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ **iff** $U(f) = L(f)$. In that case, the Riemann integral of f on $[a, b]$ is denoted by $\int_a^b f(x)\,\mathrm{d}x$.

Reimann Integrable or not

Function	Yes or No?	Proof hint
Unbounded	No	By definition
Constant	Yes	$\forall P\,(\textbf{any partition})\;\; L(f; P) = U(f; P)$
Monotonically increasing/decreasing	Yes	Take a partition such that $\Delta x < \delta = \frac{\epsilon}{f(b)-f(a)}$
Continuous	Yes	Take a partition such that $\Delta x < \delta = \frac{\epsilon}{2(b-a)}$

> **Note**
>
> If the set of points of discontinuity of a bounded function $f : [a, b] \to \mathbb{R}$ is finite, then f is Riemann integrable on $[a, b]$.

> **Note**
>
> If the set of points of discontinuity of a bounded function $f : [a, b] \to \mathbb{R}$ is finite number of limit points, then f is integrable on $[a, b]$.
>
> A function may have infinitely many discontinuous points, but if the set of all discontinuous points have finite number of limit points, then f is integrable on $[a, b]$.

Cauchy Criterion

Theorem

A bounded function $f : [a, b] \to R$ is Riemann integrable **iff** for every $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$, which may depend on ϵ, such that:

$$U(f, P\epsilon) - L(f, P\epsilon) \leq \epsilon$$

Proof Hint

- To prove $\implies$: consider $L(f) - \frac{\epsilon}{2} < L(f; P)$ and $U(f; P) < U(f) + \frac{\epsilon}{2}$
- To prove $\impliedby$: consider $L(f; P) < L(f)$ and $U(f) < U(f; P)$

Note

$f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ when:

- The set of points of discontinuity of a bounded function f is finite.
- The set of points of discontinuity of a bounded function f is finite number of limit points. (may have infinite number of discontinuities)

Theorems on Integrability

Theorem 1

Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and integrable on $[c, b]$ for all $c \in (a, b)$. Then f is integrable on $[a, b]$. Also valid for the other end.

> **Proof Hint**
>
> - Isolate a partition on the required end.
> - Choose x_1 or x_{n-1} such that $\Delta x < \frac{\epsilon}{4M}$ where M is an upper or lower bound.

Theorem 2

Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and continuous on $[c, b]$ for all $c \in (a, b)$. Then f is integrable on $[a, b]$. Also valid for the other end.

> **TODO: Proof Hint**

Properties of Integrals

Notation

If $a < b$ and f is integrable on $[a, b]$, then:

$$\int_a^b f = -\int_b^a f$$

Properties

Suppose f and g are integrable on $[a, b]$.

Addition

$f + g$ will be integrable on $[a, b]$.

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$$

Proof Hint

- Prove $f + g$ is integrable using:
  - $sup(f + g) \leq \sup(f) + \sup(g)$
  - $inf(f + g) \geq \inf(f) + \inf(g)$
- Start with $U(f + g)$ and show $U(f + g) \leq U(f) + U(g)$
- Start with $L(f + g)$ and show $L(f + g) \geq L(f) + L(g)$

Constant multiplication

Suppose $k \in \mathbb{R}$. kf will be integrable $[a, b]$.

$$\int_a^b kf = k \int_a^b f$$

Proof Hint

- Prove for $k \geq 0$. Use $U - L < \frac{\epsilon}{k}$
- Prove for $k = -1$
- Using the above results, proof for $k < 0$ is apparent

Bounds

If $m \leq f(x) \leq M$ on $[a, b]$:

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

If $f(x) \leq g(x)$ on $[a,b]$:

$$\int_a^b f \leq \int_a^b g$$

Modulus

$|f|$ will be integrable on $[a,b]$.

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

> **Proof Hint**
>
> Start with $-|f| \leq f \leq |f|$. And integrate both sides.

Multiple

fg will be integrable on $[a,b]$.

Proof Hint

- Suppose f is bounded by k
- Prove f^2 is integrable (Use $\frac{\epsilon}{2k}$)
- fg is integrable because:

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$$

Max, Min

$\max(f, g)$ and $\min(f, g)$ are integrable.

Where $\max$ and $\min$ functions are defined as:

$$\max(f, g) = \frac{1}{2}(|f-g| + f + g)$$

$$\min(f, g) = \frac{1}{2}(-|f-g| + f + g)$$

Additivity

$\iff$ f is Riemann integrable on $[a, c]$ and $[c, b]$ where $c \in (a, b)$.

Proof Hint

- $\implies$: Use Cauchy criterion after defining these:
  - $P' = \{c\} \cap P$
  - $Q = P' \cap [a, c]$
  - $R = P' \cap [c, b]$
- $\impliedby$: Use cauchy criterion on $[a, c], [c, b]$ separately and then combine using a union partition

After the integrability is proven,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof Hint

1. Let Q be a partition on $[a, c]$ and R be a partition on $[c, b]$. And $P = Q \cap R$.

2. Prove the below using Cauchy criteria:

$$\int_a^b f < L(f; P) + \epsilon \implies \int_a^b f \leq \int_a^c f + \int_c^b f$$

3. Prove the below using Cauchy criteria (by considering RHS):

$$\int_a^c f + \int_c^b f \leq \int_a^b f$$

Sequential Characterization of Integrability

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable **iff** $\exists \{P_n\}$ a sequence of partitions, such that:

$$\lim_{n \to \infty} \left[U(f; P_n) - L(f; P_n) \right] = 0$$

In that case:

$$\int_a^b f = \lim_{n \to \infty} U(f; P_n) = \lim_{n \to \infty} L(f; P_n)$$

Proof Hint

Cauchy criteria and squeeze theorem is used for both side proof.

For $\Longleftarrow$:

- Consider the limit definition.
- Prove f is Riemann integrable on P_n by Cauchy criteria.
- Use squeeze theorem for $U(f;P_n) - U(f) \leq U(f;P_n) - L(f;P_n)$ to prove limit of upper sum
- Prove limit of lower sum using the limit of upper sum

For $\Longrightarrow$: Consider the below, where $n \in \mathbb{N}$.

$$0 \leq U(f;P_n) - L(f;L_n) \leq \frac{1}{n}$$

Theorem

Suppose f is Riemann integrable on $[a,b]$ and $\epsilon > 0$. Then $\exists \epsilon > 0 \forall P$:

$$|P| < \delta \implies \left| \int_a^b f - \sum_{j=1}^{n} f(\zeta_j) I_j \right| < \epsilon$$

where $\zeta_j \in [x_{j-1}, x_j], j = 1, 2, \cdots, n$.

Proof Hint

$$\underline{\int_a^b} f - \epsilon \ < \ L(f;P) \ \leq \ \sum_{j=1}^{n} f(\zeta_j)I_j \ \leq \ U(f;P) \ < \ \overline{\int_a^b} f + \epsilon$$

Intermediate Value Theorem for Integrals

Suppose f is a continuous function on $[a, b]$. Then $\exists x \in (a, b)$:

$$f(x) = \frac{1}{b-a}\int_a^b f$$

Proof Hint

Suppose $f_{\max} = M = f(x_0)$ and $f_{\min} = m = f(y_0)$.

When $M = m$: f is a constant function. Proof is trivial.

Otherwise:

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

Then there exists $x \in (x_0, y_0)$.

Generlized IVT

Suppose f, g are continuous functions on $[a, b]$ and $g \geq 0$. Then $\exists x \in (a, b)$:

$$f(x) \int_a^b g = \int_a^b fg$$

Proof Hint

Consider this and proof is similar to IVT.

$$mg \leq fg \leq Mg$$

Convergence Functions

Convergence of functions is introduced in [Sequence of Functions | Real Analysis].

Uniform Convergence Theorem

Let f_n be a sequence of Riemann integrable functions on $[a, b]$. Suppose f_n converges to f uniformly. Then f is Riemann integrable on $[a, b]$ and $\forall x \in [a, b]$:

$$\int_a^x f_n(x)\, \mathrm{d}x \text{ converges to } \int_a^x f(x)\, \mathrm{d}x \text{ uniformly}$$

and:

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Proof Hint

- Consider $\frac{\epsilon}{2(b-a)}$ in place of ϵ.
- Consider Cauchy criteria for f_N.
- Prove $f - f_N$ is Riemann integrable using Cauchy criteria.
- f is Riemann integrable as $f = f_N + (f - f_N)$.

When f_n converges to f pointwise, we cannot be sure if f is Riemann integrable or not. An example where f is not Riemann integrable:

$$\lim_{n\to\infty} u_n = \begin{cases} 1 & x = q_k \text{ where } k \le n \\ 0 & \text{otherwise} \end{cases}$$

Here q_k is the enumeration of rational numbers in $[0, 1]$.

Dominated Convergence Theorem

Let f_n be a sequence of Riemann integrable functions on $[a, b]$. Suppose f_n converges to f pointwise where f is Riemann integrable on $[a, b]$. If
$\exists M > 0\ \forall n\ \forall x \in [a, b] \text{ s.t. } |f_n(x)| \le M$:

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Monotone Convergence Theorem

Let f_n be a sequence of Riemann integrable functions on $[a, b]$, and they are monotone (all increasing or decreasing, like $f_1 \leq f_2 \cdots \leq f_n$). Suppose f_n converges to f pointwise where f is Riemann integrable on $[a, b]$. If $\exists M > 0 \ \forall n \ \forall x \in [a, b] \text{ s.t. } |f_n(x)| \leq M$:

$$\lim_{n \to \infty} \int_a^b f_n(x) \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x$$

Can be proven from the dominated convergence theorem.

Fundamental Theorem of Calculus

Theorem I

If g is continuous on $[a, b]$ that is differentiable on (a, b) and if g' is integrable on $[a, b]$ then

$$\int_a^b g' = g(b) - g(a)$$

> **Proof Hint**
>
> Consider a general partition and use [Mean Value Theorem] on each parition.

Integration by parts

Suppose u, v are continuous functions on $[a, b]$ that are differentiable on (a, b). If u' and v' are Riemann integrable on $[a, b]$:

$$\int_a^b u(x)v'(x)\,\mathrm{d}x + \int_a^b u'(x)v(x)\,\mathrm{d}x = u(b)v(b) - u(a)v(a)$$

Proof Hint

Consider $g = uv$ and use FTC I.

Theorem II

Suppose f is an Riemann integrable function on $[a, b]$. For $x \in (a, b)$.

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

- $F(x)$ is uniformly continuous on $[a, b]$
- f is continuous at $x_0 \in (a, b) \implies F$ is differentiable and $F'(x_0) = f(x_0)$

Proof Hint

For the first point:

- Consider 2 points in the interval $x, y\,(> x)$ such that $|x - y| < \delta = \frac{\epsilon}{M}$
- Show $|F(y) - F(x)| \leq \epsilon$

For the second point: Consider the continuity definition of f and prove is quite trivial.

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| < \epsilon$$

Theorem

Suppose f is Riemann integrable on an open interval I containing the values of differentiable functions a, b. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{a(x)}^{b(x)} f(t)\, \mathrm{d}t = f(b(x))b'(x) - f(a(x))a'(x)$$

> ⓘ **Proof Hint**
>
> Can be done using FTC I and II. Proof is quite trivial.

Theorem - Change of Variable

Suppose u is a differentiable function on an open interval J such that u' is continuous. Let I be an open interval such that $\forall x \in J,\ u(x) \in I$.

If f is continuous on I, then $f \circ u$ is continuous on J and:

$$\int_a^b (f \circ u)(x)\, u'(x)\, \mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\, \mathrm{d}u$$

Improper Riemann Integrals

Iniitally Riemann integrals are defined only for **bounded** functions defined on a set of **compact** intervals.

Type 1

A function that is **not** integrable at one endpoint of the interval.

Suppose $f : (a, b] \to \mathbb{R}$ is integrable on $[c, b]$ $\forall c \in (a, b)$.

$$\int_a^b f = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f$$

Can be similarly defined on the other endpoint. The above integral converges **iff** the limit exists and finite. Otherwise diverges.

Examples

$$\int_0^1 \frac{1}{x^p}\,\mathrm{d}x = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{x^p}\,\mathrm{d}x$$

The above integral converges (to $\frac{1}{p-1}$) **iff** $0 < p < 1$. When $p \geq 1$, it diverges to ∞.

Type 2

A function defined on unbounded interval (including ∞).

Suppose $f : [a, \infty) \to \mathbb{R}$ is integrable on $[a, r] \forall r > a$.

$$\int_a^\infty f = \lim_{r \to \infty} \int_a^r f$$

Can be similarly defined on the other endpoint. The above integral converges **iff** the limit exists and finite. Otherwise diverges.

Examples

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \lim_{r\to\infty}\int_1^r \frac{1}{x^p}\,\mathrm{d}x$$

The above integral converges (to $\frac{1}{p-1}$) **iff** $p > 1$. When $0 < p \leq 1$, it diverges to ∞.

Type 3

A function that is undefined at finite number of points. The integral can be split into multiple integrals of type 1. Similarly integrals from $-\infty$ to ∞ can be defined.

> **Note**
>
> The integral can be split into multiple integrals only when all those integrals exist.

Convergence of improper integrals is similar to the convergence of series.

Absolute convergence test

$$\int_a^b |f| \text{ converges} \implies \int_a^b f \text{ converges}$$

Gamma function

Defined as below for $n > 0$:

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,\mathrm{d}x$$

Aka. Eulerian integral of the second kind.

Convergence

$\Gamma(n)$ is convergent **iff** $n > 0$.

> **Proof Hint**
>
> Direct comparison test is used. Proved in 3 cases:
>
> - Case 1: for positive integer n
>   - Consider the lemma 2's limit definition
>   - Take $\epsilon = 1$
>   - Use the convergence of $\int_0^\infty e^{-x/2}\,\mathrm{d}x$
> - Case 2: for $n > 1$ non-integers
>   - Use $\lfloor n \rfloor < n < \lfloor n \rfloor + 1$
>   - Use $x^{y-1}e^{-x} \leq x^{\lfloor n \rfloor}e^{-x}$
>   - $\Gamma(\lfloor n \rfloor + 1)$ is converging from case 1
> - Case 3: for $0 < n < 1$.
>   - Proof is similar to case 1
>   - But $\int_0^N e^{-x}x^{n-1}\,\mathrm{d}x$ is an improper
>   - Prove that it is also converging

Properties

Proofs are required for each property mentioned below.

- $\Gamma(1) = 1$
- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma(n+1) = n!$

Extension of gamma function

$\Gamma(n)$ function can be extended for negative non-integers using:

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

This cannot be used to define $\Gamma(0)$ because of the denominator. And through induction, Γ function cannot be defined for negative integers.

Lemmas

Lemma 1

$$\forall s > 0 \int_0^\infty e^{-sx}\,\mathrm{d}x \text{ converges}$$

Lemma 2

$$\forall n \in \mathbb{Z}^+ \lim_{x\to\infty} \frac{x^{n-1}}{e^{x/2}} = 0$$

Transformations

Alternate forms of $\Gamma(n)$.

Form 1

$\forall n > 0$:

$$\Gamma(n) = \frac{1}{n} \int_0^\infty e^{-x^{1/n}} \, \mathrm{d}x$$

Proof Hint

Use $x^n = t$.

Note

One of the most frequently used integrals in mathematics:

$$\int_0^\infty e^{-x^2} \, \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

Form 2

$$\int_0^\infty e^{-kx} x^{n-1} \, \mathrm{d}x = \frac{\Gamma(n)}{k^n}$$

Proof Hint

Use $x = kt$.

Form 3

$$\Gamma(n) = \int_0^1 \ln\left(\frac{1}{x}\right)^{n-1} \mathrm{d}x$$

> **Proof Hint**
>
> Use $e^{-x} = t$.

Form 4

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} \, \mathrm{d}x$$

> **Proof Hint**
>
> Use $x = t^2$.

Beta function

Beta function is defined as below, for $m, n > 0$:

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} \, \mathrm{d}x$$

Aka. Eulerian integral of the first kind.

Note

For $m, n \leq 0$, the beta function is divergent.

Properties

Symmetrical

From the definition:

$$B(m,n) = B(n,m)$$

Proof Hint

Use $t = 1 - x$.

Transformations

Form 1

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(x+1)^{m+n}} \, \mathrm{d}x = \int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}} \, \mathrm{d}x$$

Proof Hint

Use $x = \frac{1}{1+t}$ in the definition.

Form 2

$$B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,\mathrm{d}x$$

ⓘ Proof Hint

Use $x = \frac{1}{t}$ in Form 1.

Form 3

$$\int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\,\mathrm{d}x = \frac{B(m,n)}{a^m b^n}$$

ⓘ Proof Hint

Use $x = \frac{a}{b}t$ in Form 1.

Form 4

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}(\theta)\cos^{2n-1}(\theta)}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}\,\mathrm{d}x = \frac{B(m,n)}{2a^m b^n}$$

ⓘ Proof Hint

Use $x = \tan\theta$ in Form 3.

Form 5

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,\mathrm{d}x = \frac{B(m,n)}{a^n(1+a)^m}$$

Proof Hint

Use the substituition in the definition.

$$x = \frac{t(1+a)}{t+a}$$

Form 6

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,\mathrm{d}x = (b-a)^{m+n-1}B(m,n)$$

Proof Hint

Use $x = at + b(1-t)$ in the definition.

Form 7

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+(b-a)x)^{m+n}}\,\mathrm{d}x = \frac{B(m,n)}{a^n b^m}$$

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}}\,\mathrm{d}x = \frac{B(m,n)}{(b+c)^n b^m}$$

Proof Hint

I don't know.

Relation with gamma function

$\forall m, n > 0.$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$