

# Summary | Real Analysis

## Introduction— |

|  $\wedge$  | and | |  $\vee$  | or | |  $\rightarrow$  | then | |  $\implies$  | implies | |  $\Leftarrow$  | implied by | |  $\iff$  | if and only if | |  $\forall$  | for all | |  $\exists$  | there exists | |  $\sim$  | not |

Let's take  $a \rightarrow b$ .

1. Contrapositive or transposition:  $\sim b \rightarrow \sim a$ . This is equivalent to the original.
2. Inverse:  $\sim a \rightarrow \sim b$ . Does not depend on the original.
3. Converse:  $b \rightarrow a$ . Does not depend on the original.

$$a \rightarrow b \equiv \sim a \vee b \equiv \sim b \rightarrow \sim a$$

## Required proofs

- $\sim \forall x P(x) \equiv \exists x \sim P(x)$
- $\sim \exists x P(x) \equiv \forall x \sim P(x)$
- $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$
- $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$
- $\exists x \forall y P(x, y) \implies \forall y \exists x P(x, y)$
- $(A \rightarrow C) \wedge (B \rightarrow C) \equiv (A \vee B) \rightarrow C$

## Methods of proofs

1. Just proof what should be proven
2. Prove the contrapositive
3. Proof by contradiction
4. Proof by induction

## Proof by contradiction

Suppose  $a \implies b$  has to be proven. If  $a \wedge \sim b$  is proven to be false, then, by proof by contradiction,  $a \implies b$  can be trivially proven.

## Logic behind proof by contradiction

$$a \wedge \sim b = F$$

$$\sim (a \wedge \sim b) = \sim F$$

$$\sim a \vee b = T$$

$$a \rightarrow b = T$$

$$a \implies b$$

## Set theory

Zermelo-Fraenkel set theory with axiom of Choice(ZFC):9 axioms all together is being used here.

## Definitions

- $x \in A^c \iff x \notin A$
- $x \in A \cup B \iff x \in A \vee x \in B$
- $x \in A \cap B \iff x \in A \wedge x \in B$
- $A \subset B = \forall x(x \in A \implies x \in B)$
- $A - B = A \cap B^c$
- $A = B \iff ((\forall z \in A \implies z \in B) \wedge (\forall z \in B \implies z \in A))$

## Required proofs

- $(A \cap B)^c = A^c \cup B^c$
- $(A \cup B)^c = A^c \cap B^c$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \subset A \cup B$
- $A \cap B \subset A$

## Set of Numbers

### Sets of numbers

- Positive integers:  $\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$ .
- Natural integers:  $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$ .
- Negative integers:  $\mathbb{Z}^- = \{-1, -2, -3, -4, \dots\}$ .
- Integers:  $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$ .
- Rational numbers:  $\mathbb{Q} = \left\{ \frac{p}{q} \mid q \neq 0 \wedge p, q \in \mathbb{Z} \right\}$ .
- Irrational numbers: limits of sequences of rational numbers (which are not rational numbers)
- Real numbers:  $\mathbb{R} = \mathbb{Q}^c \cup \mathbb{Q}$ .

Complex numbers are not part of the study here.

## Continued Fraction Expansion

### The process

- Separate the integer part
- Find the inverse of the remaining part. Result will be greater than 1.
- Repeat the process for the remaining part.

### Finite expansion

Take  $\frac{420}{69}$  for example.

$$\frac{420}{69} = 6 + \frac{6}{69}$$

$$\frac{420}{69} = 6 + \frac{1}{\frac{69}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{3}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{1}{2}}$$

As  $\frac{420}{69}$  is finite, its continued fraction expansion is also finite. And it can be written as  $\frac{420}{69} = [6; 11, 2]$ .

## Infinite expansion

For irrational numbers, the expansion will be infinite.

For example  $\pi$ :

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \dots}}}}$$

Conintued fraction expansion of  $\pi$  is  $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \dots]$ .

## Field Axioms

### Field Axioms of $\mathbb{R}$

$\mathbb{R} \neq \emptyset$  with two binary operations  $+$  and  $\cdot$  satisfying the following properties

1. Closed under addition:  $\forall a, b \in \mathbb{R}; a + b \in \mathbb{R}$
2. Commutative:  $\forall a, b \in \mathbb{R}; a + b = b + a$

3. Associative:  $\forall a, b, c \in \mathbb{R}; (a + b) + c = a + (b + c)$
4. Additive identity:  $\exists 0 \in \mathbb{R} \forall a \in \mathbb{R}; a + 0 = 0 + a = a$
5. Additive inverse:  $\forall a \in \mathbb{R} \exists (-a); a + (-a) = (-a) + a = 0$
6. Closed under multiplication:  $\forall a, b \in \mathbb{R}; a \cdot b \in \mathbb{R}$
7. Commutative:  $\forall a, b \in \mathbb{R}; a \cdot b = b \cdot a$
8. Associative:  $\forall a, b, c \in \mathbb{R}; (a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. Multiplicative identity:  $\exists 1 \in \mathbb{R} \forall a \in \mathbb{R}; a \cdot 1 = 1 \cdot a = a$
10. Multiplicative inverse:  $\forall a \in \mathbb{R} - \{0\} \exists a^{-}; a \cdot a^{-} = a^{-} \cdot a = 1$
11. Multiplication is distributive over addition:  $a \cdot (b + c) = a \cdot b + a \cdot c$

### ① Field

Any set satisfying the above axioms with two binary operations (commonly  $+$  and  $\cdot$ ) is called a **field**. Written as  $(\mathbb{R}, +, \cdot)$  is a **Field**. But  $(\mathbb{R}, \cdot, +)$  is not a **field**.

### Required proofs

The below mentioned propositions can and should be proven using the above-mentioned axioms.

$a, b, c \in \mathbb{R}$ .

- $a \cdot 0 = 0$   
Hint: Start with  $a(1 + 0)$
- $1 \neq 0$
- Additive identity ( $0$ ) is unique
- Multiplicative identity ( $1$ ) is unique
- Additive inverse ( $-a$ ) is unique for a given  $a$
- Multiplicative inverse ( $a^{-1}$ ) is unique for a given  $a$
- $a + b = 0 \implies b = -a$
- $a + c = b + c \implies a = b$
- $-(a + b) = (-a) + (-b)$
- $-(-a) = a$
- $ac = bc \implies a = b$
- $ab = 0 \implies a = 0 \vee b = 0$
- $-(ab) = (-a)b = a(-b)$
- $(-a)(-b) = ab$
- $a \neq 0 \implies (a^{-1})^{-1} = a$
- $a, b \neq 0 \implies ab^{-1} = a^{-1}b^{-1}$

## Field or Not?

	Is field?	Reason (if not)
$(\mathbb{R}, +, \cdot)$	True	
$(\mathbb{R}, \cdot, +)$	False	Axiom 11 is invalid
$(\mathbb{Z}, +, \cdot)$	False	Multiplicative inverse doesn't exist
$(\mathbb{Q}, +, \cdot)$	True	
$(\mathbb{Q}^c, +, \cdot)$	False	$\sqrt{2} \cdot \sqrt{2} \notin \mathbb{Q}^c$
Boolean algebra	False	Additive inverse doesn't exist
$(\{0, 1\}, + \bmod 2, \cdot \bmod 2)$	True	

	Is field?	Reason (if not)
$(\{0, 1, 2\}, + \bmod 3, \cdot \bmod 3)$	True	
$(\{0, 1, 2, 3\}, + \bmod 4, \cdot \bmod 4)$	False	Multiplicative inverse doesn't exist

## Completeness Axiom

Let  $A$  be a non empty subset of  $\mathbb{R}$ .

- $u$  is the upper bound of  $A$  if:  $\forall a \in A; a \leq u$
- $A$  is bounded above if  $A$  has an upper bound
- Maximum element of  $A$ :  $\max A = u$  if  $u \in A$  and  $u$  is an upper bound of  $A$
- Supremum of  $A$   $\sup A$ , is the smallest upper bound of  $A$
- Maximum is a supremum. Supremum is not necessarily a maximum.
- $l$  is the lower bound of  $A$  if:  $\forall a \in A; a \geq l$
- $A$  is bounded below if  $A$  has a lower bound
- Minimum element of  $A$ :  $\min A = l$  if  $l \in A$  and  $l$  is a lower bound of  $A$
- Infimum of  $A$   $\inf A$ , is the largest lower bound of  $A$
- Minimum is a infimum. Infimum is not necessarily a minimum.

## Theorems

Let  $A$  be a non empty subset of  $\mathbb{R}$ .

- Say  $u$  is an upper bound of  $A$ . Then  $u = \sup A$  iff:  $\forall \epsilon > 0 \exists a \in A; a + \epsilon > u$
- Say  $l$  is a lower bound of  $A$ . Then  $l = \inf A$  iff:  $\forall \epsilon > 0 \exists a \in A; a - \epsilon < l$

### Proof Hint

Prove the contrapositive. Use  $\epsilon = \frac{1}{2}(L - \sup(A))$  for supremum proof.

## Required proofs

- $\sup(a, b) = b$
- $\inf(a, b) = a$

## Completeness property

A set  $A$  is said to have the completeness property **iff** every non-empty subset of  $A$ :

- Which is bounded below has a infimum in  $A$
- Which is bounded above has a supremum in  $A$

Both  $\mathbb{R}, \mathbb{Z}$  have the completeness property.  $\mathbb{Q}$  doesn't.

In addition to that:

- Every non empty subset of  $\mathbb{Z}$  which is bounded above has a maximum
- Every non empty subset of  $\mathbb{Z}$  which is bounded below has a minimum

## Order Axioms

- **Trichotomy:**  $\forall a, b \in \mathbb{R}$  exactly one of these holds:  $a > b$ ,  $a = b$ ,  $a < b$
- **Transitivity:**  $\forall a, b, c \in \mathbb{R}; a < b \wedge b < c \implies a < c$
- **Operation with addition:**  $\forall a, b \in \mathbb{R}; a < b \implies a + c < b + c$
- **Operation with multiplication:**  $\forall a, b, c \in \mathbb{R}; a < b \wedge 0 < c \implies ac < bc$

## Definitions

- $a < b \equiv b > a$
- $a \leq b \equiv a < b \vee a = b$
- $a \neq b \equiv a < b \vee a > b$
- $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$

## Triangular inequalities

$$|a| - |b| \leq |a + b| \leq |a| + |b|$$



$$||a| - |b|| \leq |a + b|$$

### Proof Hint

For first:

- Use  $-|a| \leq a \leq |a|$

For second:

- Use the below substitutions in first conclusion
  - $a = a - b \wedge b = b$
  - $a = b - a \wedge b = a$

## Required proofs

- $\forall a, b, c \in \mathbb{R}; a < b \wedge c < 0 \implies ac > bc$
- $1 > 0$
- $-|a| \leq a \leq |a|$
- Triangular inequalities

## Theorems

- $\exists a \forall \epsilon > 0, a < \epsilon \implies a \leq 0$
- $\exists a \forall \epsilon > 0, 0 \leq a < \epsilon \implies a = 0$
- $\forall \epsilon > 0 \exists a, a < \epsilon \not\Rightarrow a \leq 0$

### Caution

$\forall \epsilon > 0 \exists a, a < \epsilon \implies a \leq 0$  is **not** valid.

# Relations

## Definitions

- Cartesian Product of sets  $A, B$   $A \times B = \{(a, b) | a \in A, b \in B\}$
- Ordered pair  $(a, b) = \left\{ \{a\}, \{a, b\} \right\}$

## Relation

Let  $A, B \neq \emptyset$ . A relation  $R : A \rightarrow B$  is a non-empty subset of  $A \times B$ .

- $a R b \equiv (a, b) \in R$
- Domain of  $R$ :  $dom(R) = A$
- Codomain of  $R$ :  $codom(R) = B$
- Range of  $R$ :  $ran(R) = \{y | (x, y) \in R\}$
- $ran(R) \subseteq B$
- Pre-range of  $R$ :  $preran(R) = \{x | (x, y) \in R\}$
- $preran(R) \subseteq A$
- $R(a) = \{b | (a, b) \in R\}$

## Everywhere defined

$R$  is everywhere defined  $\iff A = dom(R) = preran(R)$

$\iff \forall a \in A, \exists b \in B; (a, b) \in R$

## Onto

$R$  is onto  $\iff B = codom(R) = ran(R) \iff \forall b \in B \exists a \in A (a, b) \in R$

Aka. **surjection**.

## Inverse

Inverse of a relation  $R$ :

$$R^{-1} = \{(b, a) | (a, b) \in R\}$$

## Types of relation

### one-many

$$\iff \exists a \in A, \exists b_1, b_2 \in B ((a, b_1), (a, b_2) \in R \wedge b_1 \neq b_2)$$

### Not one-many

$$\iff \forall a \in A, \forall b_1, b_2 \in B ((a, b_1), (a, b_2) \in R \implies b_1 = b_2)$$

### many-one

$$\iff \exists a_1, a_2 \in A, \exists b \in B ((a_1, b), (a_2, b) \in R \wedge a_1 \neq a_2)$$

### Not many-one

$$\iff \forall a_1, a_2 \in A, \forall b \in B ((a_1, b), (a_2, b) \in R \implies a_1 = a_2)$$

### many-many

iff  $R$  is **one-many** and **many-one**.

### one-one

iff  $R$  is **not one-many** and **not many-one**. Aka. **injection**.

### Bijection

When a relation is **onto** and **one-one**.

## Functions

A function  $f : A \rightarrow B$  is a relation  $f : A \rightarrow B$  which is [everywhere defined](#) and [not one-many](#).

- $\text{dom}(f) = A = \text{preran}(f)$

### Inverse

For a function  $f : A \rightarrow B$  to have its inverse relation  $f^{-1} : B \rightarrow A$  be also a function, we need:

- $f$  is [onto](#)
- $f$  is [not many-one](#) (in other words,  $f$  must be [one-one](#))

The above statement is true for all unrestricted function  $f$  that has an inverse  $f^{-1}$ :

$$f(f^{-1}(x)) = x = f^{-1}(f(x)) = x$$

## Composition

### Composition of relations

Let  $R : A \rightarrow B$  and  $S : B \rightarrow C$  are 2 relations. Composition can be defined when  $\text{ran}(R) = \text{preran}(S)$ .

Say  $\text{ran}(R) = \text{preran}(S) = D$ . Composition of the 2 relations is written as:

$$S \circ R = \{(a, c) \mid (a, b) \in R, (b, c) \in S, b \in D\}$$

### Composition of functions

Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be 2 functions where  $f$  is [onto](#).

$$g \circ f = \{(x, z) \mid (x, y) \in f, (y, z) \in g, y \in B\} = g(f(x))$$

## Countability

A set  $A$  is countable **iff**  $\exists f : A \rightarrow \mathbb{Z}^+$ , where  $f$  is a one-one function.

### Examples

- Countable: Any finite set,  $\mathbb{Z}, \mathbb{Q}$
- Uncountable:  $\mathbb{R}$ , Any open/closed intervals in  $\mathbb{R}$ .

### Transitive property

Say  $B \subset A$ .

$$A \text{ is countable} \implies B \text{ is countable}$$

$B$  is not countable  $\implies A$  is not countable

## Limits

$\lim_{x \rightarrow a} f(x) = L$  iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

Defining  $\delta$  in terms of a given  $\epsilon$  is enough to prove a limit.

## One sided limits

$\lim_{x \rightarrow a^+} f(x) = L$  iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < x - a < \delta \implies |f(x) - L| < \epsilon)$$

$\lim_{x \rightarrow a^-} f(x) = L$  iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (-\delta < x - a < 0 \implies |f(x) - L| < \epsilon)$$

$\lim_{x \rightarrow a} f(x) = L^+$  iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies 0 \leq f(x) - L < \epsilon)$$

$\lim_{x \rightarrow a} f(x) = L^-$  iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies -\epsilon < f(x) - L \leq 0)$$

## Limits including infinite

$\lim_{x \rightarrow \infty} f(x) = L$  iff:

$$\forall \epsilon > 0 \exists N > 0 \forall x (x > N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \rightarrow -\infty} f(x) = L$  **iff**:

$$\forall \epsilon > 0 \exists N > 0 \forall x (x < -N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \rightarrow a} f(x) = \infty$  **iff**:

$$\forall M > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies f(x) > M)$$

$\lim_{x \rightarrow a} f(x) = -\infty$  **iff**:

$$\forall M > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies f(x) < -M)$$

## Indeterminate forms

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $\infty \cdot 0$
- $\infty - \infty$
- $\infty^0$
- $0^0$
- $1^\infty$

## Continuity

A function  $f$  is continuous at  $a$  **iff**:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (|x - a| < \delta \implies |f(x) - f(a)| < \epsilon)$$

## One-side continuous

A function  $f$  is continuous from right at  $a$  **iff**:

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

A function  $f$  is continuous from left at  $a$  **iff**:

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

## Continuous on an open interval

A function  $f$  is continuous in  $(a, b)$  **iff**  $f$  is continuous on every  $c \in (a, b)$ .

## Continuous on a closed interval

A function  $f$  is continuous in  $[a, b]$  **iff**  $f$  is:

- continuous on every  $c \in (a, b)$
- right-continuous at  $a$
- left-continuous at  $b$

## Uniformly continuous

Suppose a function  $f$  is continuous on  $(a, b)$ .  $f$  is uniformly continuous on  $(a, b)$  **iff**:

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

If a function  $f$  is continuous on  $[a, b]$ ,  $f$  is uniformly continuous on  $[a, b]$ .

 **Todo**

Is this section correct? I am not 100% sure.

## Continuity Theorems

### Extreme Value Theorem

If  $f$  is continuous on  $[a, b]$ ,  $f$  has a maximum and a minimum in  $[a, b]$ .

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**Proof Hint**

Proof is quite hard.

## Intermediate Value Theorem

Let  $f$  be continuous on  $[a, b]$ . If  $\exists u$  such that  $f(a) > u > f(b)$  or  $f(a) < u < f(b)$ :  $\exists c \in (a, b)$  such that  $f(c) = u$ .

**Proof Hint**

Proof the case when  $u = 0$ . Otherwise define a new function  $g(x)$  such that middle part of the above inequality has a 0 in the place of  $u$ .

## Sandwich (or Squeeze) Theorem

Let:

- For some  $\delta > 0$ :  $\forall x (0 < |x - a| < \delta \implies f(x) \leq g(x) \leq h(x))$
- $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \in \mathbb{R}$

Then  $\lim_{x \rightarrow a} g(x) = L$ .

**Note**

Works for any kind of  $x$  limits.

## "No sudden changes"

### Positive

Let  $f$  be continuous on  $a$  and  $f(a) > 0$

$$\implies \exists \delta > 0; \forall x (|x - a| < \delta \implies f(x) > 0)$$



📘 Proof Hint

$$\text{Take } \epsilon = \frac{f(a)}{2}$$

## Negative

Let  $f$  be continuous on  $a$  and  $f(a) < 0$

$$\implies \exists \delta > 0; \forall x (|x - a| < \delta \implies f(x) < 0)$$

📘 Proof Hint

$$\text{Take } \epsilon = -\frac{f(a)}{2}$$

## Differentiability

A function  $f$  is differentiable at  $a$  **iff**:

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'(a)$$

$f'(a)$  is called the derivative of  $f$  at  $a$ .

## One-side differentiable

### Left differentiable

A function  $f$  is left-differentiable at  $a$  **iff**:

$$\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_-(a)$$

### Right differentiable

A function  $f$  is right-differentiable at  $a$  **iff**:

$$\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_+(a)$$

## Differentiability implies continuity

$f$  is differentiable at  $a \implies f$  is continuous at  $a$

### Proof Hint

Use  $\delta = \min(\delta_1, \frac{\epsilon}{1+|f'(a)|})$ .

### Note

Suppose  $f$  is differentiable at  $a$ . Define  $g$ :

$$g(x) = \begin{cases} \frac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

$g$  is continuous at  $a$ .

## Properties of differentiation

### Addition

$$\frac{d}{dx}(f \pm g) = f' \pm g'$$

### Multiplication

$$\frac{d}{dx}(fg) = fg' + fg'$$

## Division

$$\frac{d}{dx} \left( \frac{f}{g} \right) = \frac{gf' - fg'}{g^2}$$

## Composition

$$\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$$

## Power

$$\frac{d}{dx} f^n = n f^{n-1}(x) f'(x)$$

# Extreme Values

Suppose  $f : [a, b] \rightarrow \mathbb{R}$ , and  $F = f([a, b]) = \left\{ f(x) \mid x \in [a, b] \right\}$ . Minimum and maximum values of  $f$  are called the extreme values.

## Maximum

Maximum of the function  $f$  is  $f(c)$  where  $c \in [a, b]$  **iff**:

$$\forall x \in [a, b], f(c) \geq f(x)$$

aka. **Global Maximum**. Maximum doesn't exist always.

## Local Maximum

A Local maximum of the function  $f$  is  $f(c)$  where  $c \in [a, b]$  **iff**:

$$\exists \delta \forall x (0 < |x - c| < \delta \implies f(c) \geq f(x))$$

Global maximum is obviously a local maximum.

The above statement can be simplified when  $c = a$  or  $c = b$ .

When  $c = a$ :

$$\exists \delta \forall x (0 < x - c < \delta \implies f(c) \geq f(x))$$

When  $c = b$ :

$$\exists \delta \forall x (-\delta < x - c < 0 \implies f(c) \geq f(x))$$

## Minimum

Minimum of the function  $f$  is  $f(c)$  where  $c \in [a, b]$  **iff**:

$$\forall x \in [a, b], f(c) \leq f(x)$$

aka. **Global Minimum**. Minimum doesn't exist always.

## Local Minimum

$$\exists \delta \forall x (0 < |x - c| < \delta \implies f(c) \leq f(x))$$

Global minimum is obviously a local maximum.

The above statement can be simplified when  $c = a$  or  $c = b$ .

When  $c = a$ :

$$\exists \delta \forall x (0 < x - c < \delta \implies f(c) \leq f(x))$$

When  $c = b$ :

$$\exists \delta \forall x (-\delta < x - c < 0 \implies f(c) \leq f(x))$$

## Special cases

### $f$ is continuous

Then by [Extreme Value Theorem](#), we know  $f$  has a minimum and maximum in  $[a, b]$ .

### $f$ is differentiable

- If  $f(a)$  is a local maximum:  $f'_+(a) \leq 0$
- If  $f(b)$  is a local maximum:  $f'_-(b) \geq 0$
- $c \in (a, b)$  and If  $f(c)$  is a local maximum:  $f'(c) = 0$
- If  $f(a)$  is a local minimum:  $f'_+(a) \geq 0$
- If  $f(b)$  is a local minimum:  $f'_-(b) \leq 0$
- $c \in (a, b)$  and If  $f(c)$  is a local minimum:  $f'(c) = 0$

## Critical point

$c \in [a, b]$  is called a critical point **iff**:

$$f'(c) = 0 \quad \vee \quad f'(c) \text{ is undefined}$$

## Other Theorems

### Rolle's Theorem

Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . And  $f(a) = f(b)$ . Then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$

#### Proof Hint

By [Extreme Value Theorem](#), maximum and minimum exists for  $f$ .

Consider 2 cases:

1. Both minimum and maximum exist at  $a$  and  $b$ .
2. One of minimum or maximum occurs in  $(a, b)$ .

## Mean Value Theorem

Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

### Proof Hint

- Define  $g(x) = f(x) - \left( \frac{f(a)-f(b)}{a-b} \right)x$
- $g(a)$  will be equal to  $g(b)$
- Use Rolle's Theorem for  $g$

## Cauchy's Mean Value Theorem

Let  $f$  and  $g$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and  $\forall x \in (a, b) \ g'(x) \neq 0$  Then:

$$\exists c \in (a, b) \text{ s.t. } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

### Proof Hint

- Define  $h(x) = f(x) - \left( \frac{f(a)-f(b)}{g(a)-g(b)} \right)g(x)$
- $h(a)$  will be equal to  $h(b)$
- Use Rolle's Theorem for  $h$

Mean value theorem can be obtained from this when  $g(x) = x$ .

## Generalized MVT for Riemann Integrals

Let  $f, g$  be continuous on  $[a, b]$  ( $\implies f, g$  are integrable), and  $g$  does not change sign on  $(a, b)$ .

Then  $\exists \zeta \in (a, b)$  such that:

$$\int_a^b f(x)g(x)dx = f(\zeta) \int_a^b g(x)dx$$

### 📌 Proof Hint

- Use [Extreme value theorem](#) for  $f$
- Multiply by  $g(x)$ . Then integrate. Then divide by  $\int_a^b g(x)$ .
- Use intermediate value theorem to find  $f(\zeta)$

## L'Hopital's Rule

### 📌 Note

Be careful with the pronunciation.

- It's not "Hospital's Rule", there are no "s"
- It's not "Hopital's Rule" either, there is a "L"

L'Hopital's Rule can be used when all of these conditions are met. (here  $\delta$  is some positive number).

Select the appropriate  $x$  ranges.

1. Either of these conditions must be satisfied

- $f(a) = g(a) = 0$
- $\lim f(x) = \lim g(x) = 0$
- $\lim f(x) = \lim g(x) = \infty$

2.  $f, g$  are continuous on  $x \in [a, a + \delta]$

3.  $f, g$  are differentiable on  $x \in (a, a + \delta)$

4.  $g'(x) \neq 0$  on  $x \in (a, a + \delta)$

5.  $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$

Then:  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$

**Note**

L'Hopital's rule can be proven using Cauchy's Mean Value Theorem.

It is valid for all types of "x limits".

## Higher Order Derivatives

Suppose  $f$  is a function defined on  $(a, b)$ .  $f$  is  $n$  times differentiable or  $n$ -th differentiable **iff**:

$$\lim_{x \rightarrow a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a} = L \in \mathbb{R} = f^{(n)}(a)$$

Here  $f^{(n)}$  denotes  $n$ -th derivative of  $f$ . And  $f^{(0)}$  means the function itself.

$f^{(n)}(a)$  is the  $n$ -th derivative of  $f$  at  $a$ .

**Note**

$$f \text{ is } n\text{-th differentiable at } a \implies f^{(n-1)} \text{ is continuous at } a$$

## Taylor's Theorem

Let  $f$  is  $n + 1$  differentiable on  $(a, b)$ . Let  $c, x \in (a, b)$ . Then  $\exists \zeta \in (c, x)$  s.t. :

$$f(x) = f(c) + \sum_{k=1}^n \frac{f^{(k)}(c)}{k!} (x - c)^k + \frac{f^{(n+1)}(\zeta)}{(n+1)!} (x - c)^{n+1}$$

[Mean value theorem](#) can be derived from Taylor's theorem when  $n = 0$ .



**Proof Hint**

$$F(t) = f(t) + \sum_{k=1}^n \frac{f^{(k)}(t)}{k!} (x - t)^k$$

$$G(t) = (x - t)^{n+1}$$

- Define  $F, G$  as mentioned above
- Consider the interval  $[c, x]$
- Use [Cauchy's mean value theorem](#) for  $F, G$  after making sure the conditions are met.

The above equation can be written like:

$$f(x) = T_n(x, c) + R_n(x, c)$$

## Taylor Polynomial

This part of the above equation is called the Taylor polynomial. Denoted by  $T_n(x, c)$ .

$$T_n(x, c) = f(c) + \sum_{k=1}^n \frac{f^{(k)}(c)}{k!} (x - c)^k$$

## Remainder

Denoted by  $R_n(x, c)$ .

$$R_n(x, c) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} (x - c)^{n+1}$$

## Integral form of the remainder

$$R_n(x, c) = \frac{1}{n!} \int_c^x f^{(n+1)}(t) (x - t)^n dt$$

### Proof Hint

- Method 1: Use integration by parts and mathematical induction.
- Method 2: Use [Generalized MVT for Riemann Integrals](#) where:
  - $F = f^{(n+1)}$
  - $G = (x - t)^n$

## Second derivative test

When  $n = 1$ :

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(\zeta)}{2!}(x - c)^2$$

$$f(x) - \text{Tangent line} = \frac{f''(\zeta)}{2!}(x - c)^2$$

From this:  $f''(c) > 0 \implies$  a local minimum is at  $c$ . Converse is **not** true.

## Sequence

A sequence on a set  $A$  is a function  $u : \mathbb{Z}^+ \rightarrow A$ .

Image of the  $n$  is written as  $u_n$ . A sequence is indicated by one of these ways:

$$\left\{ u_n \right\}_{n=1}^{\infty} \quad \text{or} \quad \left\{ u_n \right\} \quad \text{or} \quad \left( u_n \right)_{n=1}^{\infty}$$

### Increasing or Decreasing

A sequence  $(u_n)$  is

- Increasing **iff**  $u_n \geq u_m$  for  $n > m$
- Decreasing **iff**  $u_n \leq u_m$  for  $n > m$
- Monotone **iff** either increasing or decreasing
- Strictly increasing **iff**  $u_n > u_m$  for  $n > m$
- Strictly decreasing **iff**  $u_n < u_m$  for  $n > m$

## Convergence

### Converging

A sequence  $(u_n)_{n=1}^{\infty}$  is converging (to  $L \in \mathbb{R}$ ) **iff**:  $\lim_{n \rightarrow \infty} u_n = L$

$$\forall \epsilon > 0 \exists N \in \mathbb{Z}^+ \forall n (n > N \implies |u_n - L| < \epsilon)$$

#### Note

$$\forall x \in \mathbb{R} \quad \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

### Diverging

A sequence is diverging **iff** it is not converging.

$$\lim_{n \rightarrow \infty} u_n = \begin{cases} \infty \\ -\infty \\ \text{undefined,} \end{cases} \quad \text{when } u_n \text{ is oscillating}$$

## Convergence test

All converging sequences are bounded.

### Increasing and bounded above

Let  $(u_n)$  be increasing and bounded above. Then  $(u_n)$  is converging (to  $\sup \{u_n\}$ ).

---

### Proof Hint

- $\{u_n\}$  has a  $\sup u_n (= s)$
- Prove:  $\lim_{n \rightarrow \infty} u_n = s^-$

## Decreasing and bounded below

Let  $(u_n)$  be decreasing and bounded below. Then  $(u_n)$  is converging (to  $\inf \{u_n\}$ ).

### Proof Hint

- $\{u_n\}$  has a  $\inf u_n (= l)$
- Prove:  $\lim_{n \rightarrow \infty} u_n = l^+$

## Newton's method of finding roots

Suppose  $f$  is a function. To find its roots:

- Select a point  $x_0$
- Draw a tangent at  $x_0$
- Choose  $x_1$  which is where the tangent meets  $y = 0$
- Continue this process repeatedly

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

## Cauchy Sequence

A sequence  $u : \mathbb{Z}^+ \rightarrow A$  is Cauchy **iff**:

$$\forall \epsilon > 0 \exists N \in \mathbb{Z}^+ \forall m, n; m, n > N \implies |u_n - u_m| < \epsilon$$

## Complete

A set  $A$  is complete **iff**:

$\forall u : \mathbb{Z}^+ \rightarrow A; u \text{ converges to } L \in A$

IMPORTANT:  $\mathbb{Q}$  is **not** complete because:

$$\sum_{k=1}^{\infty} \frac{1}{k!} = e - 1 \notin \mathbb{Q}$$

IMPORTANT:  $\mathbb{R}$  is complete.

**Proof Hint**

Proof is quite hard.

## Bounded

All Cauchy sequences are bounded. (has an upper bound).

**Proof Hint**

- Consider the Cauchy definition
- Take  $n > m = N + 1 > N$

## Subsequence

Suppose  $u : \mathbb{Z}^+ \rightarrow \mathbb{R}$  be a sequence and  $v : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  be an increasing sequence. Then  $u \circ v : \mathbb{Z}^+ \rightarrow \mathbb{R}$  is a subsequence of  $u$ .

## Existence of subsequence

Every sequence has a monotonic subsequence.

---

### Proof Hint

- Let  $n \in \mathbb{Z}^+$  be called “good” **iff**  $\forall m > n, u_n > u_m$ .
- Suppose  $u_n$  has infinitely many “good” points. That implies  $u_n$  has a decreasing subsequence.
- Suppose  $u_n$  has finitely many “good” points. Let  $N$  is the maximum of those.  
 $\forall n_1 > N, n_1$  is not ”good” That implies  $u_n$  has a increasing subsequence.

## Bolzano-Weierstrass

Every bounded sequence on  $\mathbb{R}$  has a converging subsequence.

### Proof Hint

From the above theorem, there is a monotonic subsequence  $u_{n_k}$  which is also bounded. Bounded monotone sequences converge.

### Note

For a set  $A$ , all 3 statements are equivalent:

- $A$  has the [completeness property](#)
- $A$  is [complete](#)
- [Bolzano-Weierstrass theorem](#) on  $A$

## Theorem 1

Suppose  $u_n$  is a sequence converging to  $L$ , and  $u_{n_k}$  is a subsequence of  $u_n$ . Then  $u_{n_k}$  is converging to  $L$ .

$$\lim_{n \rightarrow \infty} u_n = L \implies \lim_{n_k \rightarrow \infty} u_{n_k} = L$$

### Proof Hint

Note that  $n_k \geq k$ .

## Theorem 2

Suppose  $u_n$  is a sequence diverging to  $\infty$ , and  $u_{n_k}$  is a subsequence of  $u_n$ . Then  $u_{n_k}$  is diverging to  $\infty$ .

$$\lim_{n \rightarrow \infty} u_k = \infty \implies \lim_{n_k \rightarrow \infty} u_{n_k} = \infty$$

### Proof Hint

Note that  $n_k \geq k$ .

## Subsequence of a cauchy sequence

If  $u_n$  is Cauchy and  $u_{n_k}$  is a subsequence converging to  $L$ , then  $u_n$  converges to  $L$ .

## Series

Let  $(u_n)$  be a sequence, and a series (a new sequence) can be defined from it such that:

$$s_n = \sum_{k=1}^n u_k$$

## Convergence

If  $(s_n)$  is converging:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n u_k = \sum_{k=1}^{\infty} u_k = S \in \mathbb{R}$$

## Absolutely Converging

$\sum_{k=1}^n u_k$  is absolutely converging **iff**  $\sum_{k=1}^n |u_k|$  is converging.

$$\sum_{k=1}^n |u_k| \text{ is converging} \implies \sum_{k=1}^n u_k \text{ is converging}$$

### Proof Hint

Use this inequality:

$$0 \leq |u_k| - u_k \leq 2|u_k|$$

## Conditionally Converging

$\sum_{k=1}^n u_k$  is conditionally converging **iff**:

$$\sum_{k=1}^n |u_k| \text{ is diverging} \quad \text{and} \quad \sum_{k=1}^n u_k \text{ is converging}$$

## Theorem 1

$$\sum_{k=1}^n u_k \text{ is converging} \implies \lim_{k \rightarrow \infty} u_k = 0$$

The converse is more useful:

$$\lim_{k \rightarrow \infty} u_k \neq 0 \implies \sum_{k=1}^n u_k \text{ is diverging}$$

## Convergence Tests

### Direct Comparison Test

Let  $0 < u_k < v_k$ .



$$\sum_{k=1}^{\infty} v_k \text{ is converges} \implies \sum_{k=1}^{\infty} u_k \text{ is converges}$$

### ① Proof Hint

- Note that  $\sum_{k=1}^n u_k$  and  $\sum_{k=1}^n v_k$  are increasing
- Show that  $\sum_{k=1}^{\infty} v_k$  converges to its supremum  $v$  which is an upper bound of  $\sum_{k=1}^n u_k$

### ① Example

Proving the convergence of  $\sum_{k=1}^{\infty} \frac{1}{k!}$ , by using  $k! \geq 2^{k-1}$  for all  $k \geq 0$ .

## Limit Comparison Test

Let  $0 < u_k, v_k$  and  $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = R$ .

$$R > 0 \implies \left( \sum_{n=1}^{\infty} u_n \text{ is converging} \iff \sum_{n=1}^{\infty} v_n \text{ is converging} \right)$$

$$R = 0 \implies \left( \sum_{n=1}^{\infty} v_n \text{ is converging} \implies \sum_{n=1}^{\infty} u_n \text{ is converging} \right)$$

$$R = \infty \implies \left( \sum_{n=1}^{\infty} v_n \text{ is diverging} \implies \sum_{n=1}^{\infty} u_n \text{ is diverging} \right)$$

### Proof Hint

Only possibilities are  $R = 0$ ,  $R > 0$ ,  $R = \infty$ .

For  $R > 0$ :

- Consider limit definition with  $\epsilon = \frac{L}{2}$
- Direct comparison test can be used for the 2 set of inequalities

For  $R = 0$ :

- Consider limit definition with  $\epsilon = 1$
- Direct comparison test can be used now

For  $R = \infty$ :

- Consider limit definition with  $M = 1$
- Direct comparison test can be used now

## Integral Test

Let  $u(x) > 0$ , decreasing and integrable on  $[1, M]$  for all  $M > 1$ . Then:

$$\sum_{n=1}^{\infty} u_n \text{ is converging} \iff \int_1^{\infty} u(x) \, dx \text{ is converging}$$

### Proof Hint

As  $u(x)$  is decreasing, it is apparent that it is integrable.

Make use of this inequality:

$$s_n - u_1 \leq \int_1^n u(x) \, dx \leq s_n - u_n$$

For  $\Leftarrow$  :

- Note that  $s_n$  is increasing
- Show that  $s_n$  is bounded above by  $\int_1^\infty u(x) \, dx + u_1$

For  $\Rightarrow$  :

- Define  $F(n) = \int_1^n u(x) \, dx$
- Note that  $F(n)$  is increasing
- Note that  $\lim_{n \rightarrow \infty} u_n = 0$
- Show that  $F(n)$  is bounded above by  $\lim_{n \rightarrow \infty} s_n$

### Note

$$\sum_{n=1}^{\infty} u_n \text{ is converging} \implies \lim_{k \rightarrow \infty} u_k = 0$$

$$\int_1^{\infty} u(x) \, dx \text{ is converging} \implies \lim_{k \rightarrow \infty} u(k) = 0$$

## Ratio Test

Let  $u(x) > 0$  and  $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = L$ .

$$L < 1 \implies \sum_{n=1}^{\infty} u_n \text{ is converging}$$

$$L > 1 \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}$$

#### ① Proof Hint

- Consider the limit definition with  $\epsilon = \frac{1}{2}(1 - L)$
- Show that:  $\frac{1}{2}(3L - 2) < \frac{u_{k+1}}{u_k} < \frac{1}{2}(1 + L)$
- Use  $\sum_{k=1}^{\infty} r^k$  is converging **iff**  $r < 1$

## Root Test

Let  $u(x) > 0$  and  $\lim_{n \rightarrow \infty} u_n^{1/n} = L$ .

$$L < 1 \implies \sum_{n=1}^{\infty} u_n \text{ is converging}$$

$$(L > 1 \vee L = \infty) \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}$$

#### ① Proof Hint

For  $L < 1 \vee L > 1$ : Consider the limit definition with  $\epsilon = \frac{1}{2}(1 - L)$

For  $L = \infty$ : Consider the limit definition with  $M > 1$

## Riemann Zeta Function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

Convergence of this function can be derived using [integral test](#).

This function converges **iff**  $s > 1$ . And it converges to:

$$\frac{1}{s-1}$$

Otherwise it diverges.

## Alternating Series

Suppose  $u_k > 0$ . An alternating series is:

$$\sum_{k=1}^n (-1)^{k-1} u_k = u_1 - u_2 + u_3 - u_4 + \dots$$

## Convergence

If  $\forall k \ u_k > 0$ , decreasing and  $\lim_{n \rightarrow \infty} u_n = 0$ . Then

$$\sum_{k=1}^n (-1)^{k-1} u_k \text{ is converging}$$

---

## Proof

For odd-indexed elements:

$$s_{2m+3} \leq s_{2m+1} \leq s_1 = u_1$$

For even-indexed elements:

$$s_{2m+2} \geq s_{2m} \geq s_2 = u_1 - u_2$$

Combining these 2:

$$0 \leq u_1 - u_2 \leq s_2 \leq s_{2m} \leq s_{2m+1} \leq s_1 = u_1$$

$s_{2m}$  is bounded above by  $u_1$  and increasing.  $s_{2m+1}$  is bounded below by 0 and decreasing. So both converges.

$$\lim_{m \rightarrow \infty} (s_{2m+1} - s_{2m}) = \lim_{m \rightarrow \infty} u_{2m+1} = 0$$

$$\implies \lim_{m \rightarrow \infty} s_{2m+1} = \lim_{m \rightarrow \infty} s_{2m} = s$$

Both converges to the same number. ∴

## Power Series

A series of the form:

$$\sum_{n=0}^{\infty} a_n (x - c)^n$$

Here:

- $x$  - a variable
- $c$  - a constant

## Radius of convergence

Maximum radius of  $x$  in where the series converges.

$$R = \sup \{ r \mid \text{series converges for } |x - c| < r \}$$

## Range of convergence

$(c - R, c + R)$  is the range of convergence.

## Taylor Series

Let  $f$  be infinitely many times differentiable on  $(a, b)$  and  $c, x \in (a, b)$ .

If  $\lim_{n \rightarrow \infty} R_n(x) = 0$  for  $x \in (c - R, c + R) \subset (a, b)$ , then Taylor series of  $f$  at  $c$  is given by:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$$

## Examples

$e^x$

Range of convergence is  $\mathbb{R}$ .

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$\ln(1+x)$

Range of convergence is  $(-1, 1]$ .

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots$$

### **sin x**

Range of convergence is  $\mathbb{R}$ .

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

### **cos x**

Range of convergence is  $\mathbb{R}$ .

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

#### **📌 Note**

Next, sequence of functions and their convergence were discussed in class. This section is included under [Riemann Integration](#), because it was discussed there first.