# **Summary | Riemann Integration**

## Introduction

### Interval

Let I = [a, b]. Length of the interval |I| = b - a.

## **Disjoint interval**

When 2 intervals don't share any common numbers.

## Almost disjoint interval

When 2 intervals are disjoint or intersect only at a common endpoint.

## Riemann Integral

Let  $f-[a,b] o\mathbb{R}$  is a bounded (not necessarily continuous) function on a closed, bounded (compact) interval.

Riemann integral of f is:  $\int_a^b f$ 

## Definite integral

When a, b are constants.

## Indefinite integral

When a is a constant but b is replaced with x.

## **Partition**

Let I be a non-empty, compact interval (closed and bounded). A partition of I is a finite collection  $\{I_1,I_2,\ldots,I_n\}$  of almost disjoint, non-empty, compact sub-intervals whose union is I.

A partition is determined by the endpoints of all sub-intervals:  $a = x_0 < x_1 < \dots < x_n = b$ .

A partition can be denoted by:

- its intervals  $P = \{I_1, I_2, \ldots, I_n\}$
- ullet the endpoints of its intervals  $P=\{x_0,x_1,\ldots,x_n\}$

## Riemann Sum

Let

- $f:[a,b] o \mathbb{R}$  is a bounded function on the compact interval I=[a,b] with  $M=\sup_I f$  and  $m=\inf_I f$  .
- $P = \{I_1, I_2, \dots, I_n\}$
- $\bullet \hspace{0.2cm} M_k=\sup_{I_k}f=\sup\left\{f(x):x\in[x_{k-1},x_k]\right\}$
- $\bullet \hspace{0.3cm} m_k = \inf_{I_k} f = \inf \left\{ f(x) : x \in [x_{k-1}, x_k] \right\}$

## Upper riemann sum

$$U(f;P) = \sum_{k=1}^n M_k |I_k|$$

#### Lower riemann sum

$$L(f;P) = \sum_{k=1}^n m_k |I_k|$$

$$m_k < M_k \implies L(f;P) \le U(f;P)$$

When  $P_1, P_2$  are any 2 partitions of I:  $L(f; P_1) \leq U(f; P_2)$ 

## Refinements

Q is called a refinement of  $P\iff$  if P and Q are partitions of [a,b] and  $P\subseteq Q$ .

When Q is a refinement of P:

$$L(f; P) \le L(f; Q) \le U(f; Q) \le U(f; P)$$

(i) Note

If  $P_1$  and  $P_2$  are partitions of [a,b], then  $Q=P_1\cup P_2$  is a refinement of both  $P_1$  and  $P_2$ . In that case:

$$L(f; P_1) \leq L(f; Q) \leq U(f; Q) \leq U(f; P_2)$$

## **Upper & Lower integral**

Let  $\mathbb P$  be the collection of all possible partitions of the interval [a,b].

## **Upper Integral**

$$U(f)=\inf\left\{U(f;P);P\in\mathbb{P}
ight\}=\overline{\int_a^bf}$$

## Lower Integral

$$L(f)=\sup\left\{L(f;P);P\in\mathbb{P}
ight\}=\int_a^bf$$

For a bounded function f, always  $L(f) \leq U(f)$ 

# Riemann Integrable

A bounded function  $f:[a,b] \to \mathbb{R}$  is Riemann integrable on [a,b] iff U(f)=L(f). In that case, the Riemann integral of f on [a,b] is denoted by  $\int_a^b f(x) \,\mathrm{d}x$ .

## Reimann Integrable or not

Function	Yes or No?	Proof hint
Unbounded	No	By definition
Constant	Yes	$orall P  ext{ (any partition) } L(f;P) = U(f;P)$
Monotonically increasing/decreasing	Yes	Take a partition such that $\Delta x < \delta = rac{\epsilon}{f(b) - f(a)}$
Continuous	Yes	Take a partition such that $\Delta x < \delta = rac{\epsilon}{2(b-a)}$

### (i) Note

If the set of points of discontinuity of a bounded function  $f:[a,b] \to \mathbb{R}$  is finite, then f is Riemann integrable on [a,b].

### (i) Note

If the set of points of discontinuity of a bounded function  $f:[a,b] \to \mathbb{R}$  is finite number of limit points, then f is integrable on [a,b].

A function may have infinitely many discontinuous points, but if the set of all discontinuous points have finite number of limit points, then f is integrable on [a, b].

# **Cauchy Criterion**

#### **Theorem**

A bounded function  $f:[a,b]\to R$  is Riemann integrable **iff** for every  $\epsilon>0$  there exists a partition  $P_\epsilon$  of [a,b], which may depend on  $\epsilon$ , such that:

$$U(f,P\epsilon)-L(f,P\epsilon)\leq \epsilon$$

- To prove  $\implies$  : consider  $L(f) rac{\epsilon}{2} < L(f;P)$  and  $U(f;P) < U(f) + rac{\epsilon}{2}$
- ullet To prove  $\buildrel =$  : consider L(f;P) < L(f) and U(f) < U(f;P)

### (i) Note

 $f:[a,b] o\mathbb{R}$  is integrable on [a,b] when:

- ullet The set of points of discontinuity of a bounded function f is finite.
- ullet The set of points of discontinuity of a bounded function  $m{f}$  is finite number of limit points. (may have infinite number of discontinuities)

## Theorems on Integrability

### Theorem 1

Suppose  $f:[a,b] o \mathbb{R}$  is bounded, and integrable on [c,b] for all  $c\in (a,b)$ . Then f is integrable on [a,b]. Also valid for the other end.

#### (i) Proof Hint

- Isolate a partition on the required end.
- ullet Choose  $x_1$  or  $x_{n-1}$  such that  $\Delta x < rac{\epsilon}{4M}$  where M is an upper or lower bound.

### Theorem 2

Suppose  $f:[a,b] o \mathbb{R}$  is bounded, and continuous on [c,b] for all  $c\in(a,b)$ . Then f is integrable on [a,b]. Also valid for the other end.

⚠ TODO: Proof Hint

# **Properties of Integrals**

#### **Notation**

If a < b and f is integrable on [a, b], then:

$$\int_a^b f = -\int_b^a f$$

## **Properties**

Suppose f and g are integrable on [a, b].

#### **Addition**

f + g will be integrable on [a, b].

$$\int_a^b (f\pm g) = \int_a^b f\pm \int_a^b g$$

## i Proof Hint

- Prove f + g is integrable using:
  - $\circ \ \ sup(f+g) \leq \sup(f) + \sup(g)$
  - $\circ inf(f+g) \ge \inf(f) + \inf(g)$
- ullet Start with U(f+g) and show  $U(f+g) \leq U(f) + U(g)$
- ullet Start with L(f+g) and show  $L(f+g) \geq L(f) + L(g)$

## Constant multiplication

Suppose  $k \in \mathbb{R}$ . kf will be integrable [a,b].

$$\int_a^b kf = k \int_a^b f$$

- ullet Prove for  $k\geq 0$  . Use  $U-L<rac{\epsilon}{k}$
- Prove for k=-1
- ullet Using the above results, proof for  $\,k < 0\,$  is apparent

#### **Bounds**

If  $m \leq f(x) \leq M$  on [a, b]:

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

If  $f(x) \leq g(x)$  on [a,b]:

$$\int_a^b f \leq \int_a^b g$$

#### **Modulus**

|f| will be integrable on [a, b].

$$igg|\int_a^b figg| \leq \int_a^b |f|$$

### (i) Proof Hint

Start with  $-|f| \leq f \leq |f|$ . And integrate both sides.

## Multiple

fg will be integrable on [a, b].

- ullet Suppose f is bounded by k
- ullet Prove  $f^2$  is integrable (Use  $rac{\epsilon}{2k}$  )
- ullet fg is integrable because:

$$fg = \frac{1}{2} [(f+g)^2 - f^2 - g^2]$$

### Max, Min

 $\max(f,g)$  and  $\min(f,g)$  are integrable.

Where max and min functions are defined as:

$$\max(f,g) = \frac{1}{2}(|f-g|+f+g)$$

$$\min(f,g) = \tfrac{1}{2}(-|f-g|+f+g)$$

### **Additivity**

 $\iff f$  is Riemann integrable on [a,c] and [c,b] where  $c\in(a,b).$ 

## (i) Proof Hint

• ⇒ : Use Cauchy criterion after defining these:

$$\circ$$
  $P' = \{c\} \cap P$ 

$$\circ \ \ Q = P' \cap [a,c]$$

$$\circ R = P' \cap [c,b]$$

ullet : Use cauchy criterion on [a,c],[c,b] separately and then combine using a union partition

After the integrability is proven,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

- 1. Let  $\,Q\,$  be a partition on  $\,[a,c]\,$  and  $\,R\,$  be a partition on  $\,[c,b]\,$  . And  $\,P=Q\cap R\,$  .
- 2. Prove the below using Cauchy criteria:

$$\int_a^b f < L(f;P) + \epsilon \;\;\implies \;\; \int_a^b f \leq \int_a^c f + \int_c^b f$$

3. Prove the below using Cauchy criteria (by considering RHS):

$$\int_a^c f + \int_c^b f \le \int_a^b f$$

## Sequential Characterization of Integrability

A bounded function  $f:[a,b] o \mathbb{R}$  is Riemann integrable if and only if  $\exists\,\{P_n\}$  a sequence of partitions, such that:

$$\lim_{n o\infty} \left[ U(f;P_n) - L(f;P_n) 
ight] = 0$$

In that case:

$$\int_a^b f = \lim_{n o\infty} U(f;P_n) = \lim_{n o\infty} L(f;P_n)$$

Cauchy criteria and squeeze theorem is used for both side proof.

For  $\iff$ :

- Consider the limit definition.
- ullet Prove f is Riemann integrable on  $P_n$  by Cauchy criteria.
- ullet Use squeeze theorem for  $\,U(f;P_n)-U(f)\leq U(f;P_n)-L(f;P_n)\,$  to prove limit of upper sum
- · Prove limit of lower sum using the limit of upper sum

For  $\Longrightarrow$ : Consider the below, where  $n \in \mathbb{N}$ .

$$0 \leq U(f;P_n) - L(f;L_n) \leq \frac{1}{n}$$

### **Theorem**

Suppose f is Riemann integrable on [a,b] and  $\epsilon>0$ . Then  $\exists \epsilon>0 \forall P$ :

$$|P| < \delta \implies \left| \int_a^b f - \sum_{j=1}^n f(\zeta_j) I_j 
ight| < \epsilon$$

where  $\zeta_i \in [x_{i-1}, x_i], j=1, 2, \cdots, n$ .

## (i) Proof Hint

$$\int_a^b f - \epsilon \ < \ L(f;P) \ \le \ \sum_{j=1}^n f(\zeta_j) I_j \ \le \ U(f;P) \ < \ \overline{\int_a^b f} + \epsilon$$

## Intermediate Value Theorem for Integrals

Suppose f is a continuous function on [a,b]. Then  $\exists x \in (a,b)$ :

$$f(x) = rac{1}{b-a} \int_a^b f$$

### (i) Proof Hint

Suppose  $f_{
m max}=M=f(x_0)$  and  $f_{
m min}=m=f(y_0).$ 

When M=m: f is a constant function. Proof is trivial.

Otherwise:

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

Then there exists  $x \in (x_0, y_0)$ .

## **Generlized IVT**

Suppose f,g are continuous functions on [a,b] and  $g\geq 0$ . Then  $\exists x\in (a,b)$ :

$$f(x)\int_a^b g=\int_a^b fg$$

### (i) Proof Hint

Consider this and proof is similar to IVT.

$$mg \leq fg \leq Mg$$

## **Convergence Functions**

Convergence of functions is introduced in <u>Sequence of Functions | Real Analysis</u>.

## **Uniform Convergence Theorem**

Let  $f_n$  be a sequence of Riemann integrable functions on [a,b]. Suppose  $f_n$  converges to f uniformly. Then f is Riemann integrable on [a,b] and  $\forall x \in [a,b]$ :

$$\int_a^x f_n(x) dx$$
 converges to  $\int_a^x f(x) dx$  uniformly

and:

$$\lim_{n o\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

### (i) Proof Hint

- Consider  $\frac{\epsilon}{2(b-a)}$  in place of  $\epsilon$  .
- ullet Consider Cauchy criteria for  $f_N$  .
- ullet Prove  $f-f_N$  is Riemann integrable using Cauchy criteria.
- f is Riemann integrable as  $f=f_N+\left(f-f_N
  ight)$  .

When  $f_n$  converges to f pointwise, we cannot be sure if f is Riemann integrable or not. An example where f is not Riemann integrable:

$$\lim_{n o \infty} u_n = \left\{egin{array}{ll} 1 & x = q_k ext{ where } k \leq n \ 0 & ext{otherwise} \end{array}
ight.$$

Here  $q_k$  is the enumeration of rational numbers in [0,1].

## **Dominated Convergence Theorem**

Let  $f_n$  be a sequence of Riemann integrable functions on [a,b]. Suppose  $f_n$  converges to f pointwise where f is Riemann integrable on [a,b]. If  $\exists M>0 \ \forall n \ \forall x\in [a,b] \ \mathrm{s.t.} \ |f_n(x)|\leq M$ :

$$\lim_{n o\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

### **Monotone Convergence Theorem**

Let  $f_n$  be a sequence of Riemann integrable functions on [a,b], and they are monotone (all increasing or decreasing, like  $f_1 \leq f_2 \cdots \leq f_n$ ). Suppose  $f_n$  converges to f pointwise where f is Riemann integrable on [a,b]. If  $\exists M>0 \ \forall n \ \forall x\in [a,b] \ \mathrm{s.t.} \ |f_n(x)|\leq M$ :

$$\lim_{n o\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

Can be proven from the dominated convergence theorem.

## **Fundamental Theorem of Calculus**

#### Theorem I

If g is continuous on [a,b] that is differentiable on (a,b) and if  $g^\prime$  is integrable on [a,b] then

$$\int_a^b g' = g(b) - g(a)$$

i Proof Hint

Consider a general partition and use <u>Mean Value Theorem</u> on each parition.

## Integration by parts

Suppose u,v are continuous functions on [a,b] that are differentiable on (a,b). If u' and v' are Riemann integrable on [a,b]:

$$\int_a^b u(x)v'(x)\,\mathrm{d}x + \int_a^b u'(x)v(x)\,\mathrm{d}x = u(b)v(b) - u(a)v(a)$$

### (i) Proof Hint

Consider g = uv and use <u>FTC I</u>.

### Theorem II

Suppose f is an Riemann integrable function on [a,b]. For  $x\in(a,b)$ .

$$F(x) = \int_a^x f(t) \, \mathrm{d}t$$

- ullet F(x) is uniformly continuous on [a,b]
- ullet f is continuous at  $x_0 \in (a,b) \implies F$  is differentiable and  $F'(x_0) = f(x_0)$

### (i) Proof Hint

For the first point:

- ullet Consider 2 points in the interval  $x,y\,(>x)$  such that  $|x-y|<\delta=rac{\epsilon}{M}$
- Show  $|F(y) F(x)| \le \epsilon$

For the second point: Consider the continuity definition of  $m{f}$  and prove is quite trivial.

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|<\epsilon$$

### **Theorem**

Suppose f is Riemann integrable on an open interval I containing the values of differentiable functions a,b. Then:

$$rac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)}f(t)\,\mathrm{d}t=f(b(x))b'(x)-f(a(x))a'(x)$$

(i) Proof Hint

Can be done using FTC I and II. Proof is quite trivial.

## Theorem - Change of Variable

Suppose u is a differentiable function on an open interval J such that u' is continuous. Let I be an open interval such that  $\forall x \in J, \ u(x) \in I$ .

If f is continuous on I, then  $f \circ u$  is continuous on J and:

$$\int_a^b (f\circ u)(x)\,u'(x)\,\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u$$

## Improper Riemann Integrals

Riemann integral is defined only for **bounded** functions defined on a set of **compact** intervals.

### Type 1

A function that is **not** integrable at one endpoint of the interval.

Suppose  $f:(a,b] o \mathbb{R}$  is integrable on  $[c,b]\ orall c\in (a,b).$ 

$$\int_a^b f = \lim_{\epsilon o 0} \, \int_{a+\epsilon}^b f \, .$$

Can be similarly defined on the other endpoint. The above integral converges **iff** the limit exists and finite. Otherwise diverges.

#### **Examples**

$$\int_0^1 \frac{1}{x^p} \, \mathrm{d}x = \lim_{\epsilon \to 0^+} \int_{\epsilon}^1 \frac{1}{x^p} \, \mathrm{d}x$$

The above integral converges (to  $\frac{1}{p-1}$ ) iff  $0 . When <math>p \geq 1$ , it diverges to  $\infty$ .

### Type 2

A function defined on unbounded interval (including  $\infty$ ).

Suppose  $f:[a,\infty) o\mathbb{R}$  is integrable on  $[a,r] \forall r>a$ .

$$\int_a^\infty f = \lim_{r \to \infty} \int_a^r f$$

Can be similarly defined on the other endpoint. The above integral converges **iff** the limit exists and finite. Otherwise diverges.

#### **Examples**

$$\int_1^\infty rac{1}{x^p}\,\mathrm{d}x = \lim_{r o\infty} \int_1^r rac{1}{x^p}\,\mathrm{d}x$$

The above integral converges (to  $rac{1}{p-1}$  ) iff p>1. When  $0< p\leq 1$ , it diverges to  $\infty$ .

### Type 3

A function that is undefined at finite number of points. The integral can be split into multiple integrals of type 1. Similarly integrals from  $-\infty$  to  $\infty$  can be defined.

The integral can be split into multiple integrals only when all those integrals exist.

Convergence of improper integrals is similar to the convergence of series.

## Absolute convergence test

$$\int_a^b |f| \text{ converges } \implies \int_a^b f \text{ converges}$$

## **Beta function**

Beta function is defined as below, for m, n > 0:

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, \mathrm{d}x$$

Aka. Eulerian integral of the first kind.

(i) Note

For

$$m, n \leq 0$$

, the gamma function is divergent.

## **Properties**

## **Symmetrical**

From the definition:

$$B(m,n) = B(n,m)$$

(i)	<b>Proof Hint</b>	
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Proof is trivial.

# Gamma function

Defined as below for n>0:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, \mathrm{d}x$$

Aka. Eulerian integral of the second kind.

## Convergence

 $\Gamma(n)$  is convergent **iff** n>0.

Direct comparison test is used. Proved in 3 cases:

- ullet Case 1: for positive integer n
  - o Consider the lemma 2's limit definition
  - $\circ$  Take  $\epsilon=1$
  - $\circ~$  Use the convergence of  $\,\int_0^\infty e^{-x/2}\,\mathrm{d}x$
- Case 2: for n>1 non-integers
  - $\circ$  Use  $\lfloor n 
    floor < n < \lfloor n 
    floor + 1$
  - $\circ$  Use  $x^{y-1}e^{-x} < x^{\lfloor n 
    floor}e^{-x}$
  - $\circ \ \Gamma(|n|+1)$  is converging from case 1
- ullet Case 3: for 0 < n < 1.
  - o Proof is similar to case 1
  - $\circ~$  But  $\int_0^N e^{-x} x^{n-1} \, \mathrm{d}x$  is an improper
  - o Prove that it is also converging

## **Properties**

Proofs are required for each property mentioned below.

- $\Gamma(1) = 1$
- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma(n+1)=n!$

## **Extension of gamma function**

 $\Gamma(n)$  function can be extended for negative non-integers using:

$$\Gamma(n) = rac{\Gamma(n+1)}{n}$$

This cannot be used to define  $\Gamma(0)$  because of the denominator. And through induction,  $\Gamma$  function cannot be defined for negative integers.

### Lemmas

Lemma 1

$$orall s>0 \ \int_0^\infty e^{-sx}\,\mathrm{d}x \ \mathrm{converges}$$

Lemma 2

$$orall n \in \mathbb{Z}^+ \lim_{x o \infty} rac{x^{n-1}}{e^{x/2}} = 0$$

## **Transformations**

Alternate forms of

$$\Gamma(n)$$

.

Form 1

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \,\mathrm{d}x$$

(i)	Proof	Hint
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Use  $x^n = t$ .

$\bigcirc$	Note
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One of the most frequently used integrals in mathematics:

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = rac{\sqrt{\pi}}{2}$$

#### Form 2

$$\int_0^\infty e^{-kx} x^{n-1} \,\mathrm{d}x = rac{\Gamma(n)}{k^n}$$

### (i) Proof Hint

Use x = kt.

### Form 3

$$\Gamma(n) = \int_0^1 \ln(rac{1}{x})^{n-1} \,\mathrm{d}x$$

### (i) Proof Hint

Use  $e^{-x} = t$ .

#### Form 4

$$\Gamma(n)=2\int_0^\infty e^{-x^2}x^{2n-1}\,\mathrm{d}x$$

Use  $x=t^2$ .

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