

Summary | Matrices

Introduction

Revise Matrices unit from G.C.E. (A/L) Combined Mathematics and G.C.E. (O/L) Mathematics.

Types of matrices

Square matrix

Number of columns equal to number of rows.

Main diagonals of a square matrix

Formed by elements having equal subscripts.

Diagonal matrix

A square matrix whose only non-zero elements are main-diagonal elements. Denoted by D . Subset of triangular matrices.

Identity matrix or Unit matrix

A diagonal matrix whose diagonal elements are all equal to 1 . Denoted by I . Subset of diagonal matrices.

Zero matrix / Null matrix

All elements are 0 .

Column matrix (column vector)

Only 1 column.

Row matrix (row vector)

Only 1 row.

Triangular matrix

Upper triangular matrix or lower triangular matrix.

Upper triangular matrix

All elements below the main diagonal are **0**. Subset of square matrices.

Lower triangular matrix

All elements above the main diagonal are **0**. Subset of square matrices.

Matrix operations

Addition and subtraction

Order of the 2 matrices must be same. Matrix obtained by adding or subtracting corresponding elements.

Scalar multiplication

Matrix obtained by multiplying all elements by the scalar.

Note

[Matrix multiplication](#) is also defined.

Transpose

Matrix obtained from a given matrix by interchanging its rows and columns. Denoted by a superscript T, like A^T .

Properties

1. $(A^T)^T = A$
2. Distributive over addition: $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$
4. $(A \times B)^T = B^T \times A^T$

More types of matrices

Symmetric matrix

If $A = A^T$. Subset of square matrices.

Skew-symmetric matrix

If $A = -A^T$. Subset of square matrices. All elements in main diagonal are 0.

Note

Any square matrix can be expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

Matrix multiplication

Defined only if the number of columns of the first matrix is equal to the number of rows of the second matrix.

If $A = (a_{ij})_{m \times p}$ and $B = (b_{ij})_{p \times n}$, then $A \times B = C = (c_{ij})_{m \times n}$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ip}b_{pj}$.

Note

- Generally $A \times B \neq B \times A$.
- $A \times B = 0 \not\Rightarrow A = 0 \vee B = 0$
- $A \neq 0 \wedge B \neq 0 \not\Rightarrow A \times B \neq 0$

Properties of matrix multiplication

A, B, C, I matrices must be chosen so that below-mentioned product matrices are defined.

1. Associative: $A(BC) = (AB)C$
2. Right distributive over addition: $(A + B)C = AC + BC$
3. Left distributive over addition: $C(A + B) = CA + CB$
4. $AI = IA = A$; I is an identity matrix.

Determinant

Defined only for square matrices. Denoted by $|A|$.

For 2x2

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For higher order

Minor of an element

Suppose $A = (a_{ij})$.

Minor of an element a_{ij} , is the matrix obtained by deleting i^{th} row and j^{th} column of A .
Denoted by M_{ij} .

Co-factor of an element

Suppose $A = (a_{ij})$.

Co-factor of an element a_{ij} , is defined as (commonly denoted as A_{ij}):

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

Definition

If $A = (a_{ij})_{n \times n}$ then the **determinant** of A is denoted by $|A|$ and is defined by:

$$|A| = \sum_{j=1}^n a_{ij} A_{ij}$$

where $1 \leq j \leq n$.

Properties of determinants

- Every element of a row or column of a matrix is 0 then the value of its determinant is 0 .
- If 2 columns or 2 rows of a matrix are identical then its determinant is 0 .
- If A and B are two square matrices then $|AB| = |A||B|$.
- The value of the determinant of a matrix remains unchanged if a scalar multiple of a row or column is added to any other row or column.
- If a matrix B is obtained from a square matrix A by an interchange of two columns or rows:
 $|B| = -|A|$.
- If every entry in any row or column is multiplied by k , then the whole determinant is multiplied by k .

Adjoint

Suppose $A = (a_{ij})_{n \times n}$.

$$\text{adj}A = (A_{ij})_{n \times n}^T$$

Where A_{ij} is the [co-factor of](#) a_{ij} .

Inverse

Suppose A and B are square matrices of the same order. If $AB = BA = I$ then B is called the inverse of A and is denoted by A^{-1} .

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

① Singular vs Non-singular

A square matrix is singular if $|A| = 0$. Otherwise non-singular or invertible matrices.

Properties of Inverse

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $A \operatorname{adj} A = \operatorname{adj} A A = |A|I$

Elementary Transformations

- Interchange of any columns or rows
- Addition of multiple of any row or column to any other row or column
- Multiplication of each element of a column or a row by a non-zero constant

When a matrix B is obtained by applying elementary transformations to a matrix A , then A is **equivalent to** B . Denoted by $A \approx B$.

Theorem

The elementary row operations that reduce a given matrix A to the identity matrix, also transform the identity matrix to the inverse of A .

Augmented Matrix

Two matrices are written as a single matrix with a vertical line in-between. Denoted by $(A|B)$. Example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right]$$

Inverse using elementary row transformations

Let A be a square matrix with order $n \times n$.

- Start with $(A_{n \times n} | I_n)$
- Repeatedly add **row** transformations (not column) to both of the matrices until the **LHS** becomes an identity matrix.
 - Convert all elements outside the main diagonal to **0**.
 - Convert elements on the main diagonal to **1** by multiplying by a constant.
- When **LHS** is an identity matrix, **RHS** is A^{-1} .

⚠️ TODO

What about singular matrices?

Echelon Form

A matrix is in row echelon form (or just “row echelon” form) **iff**:

- All rows having only zero entries are at the bottom.
- For all row that does not contain entirely zeros, the first non-zero entry is 1.
- For 2 successive non-zero rows, the leading 1 in the higher row is further left than the leading 1 in the lower row.

The process of reducing the augmented matrix to row Echelon form is known as **Gaussian elimination**.

Column echelon form

A matrix A is in column echelon form if A^T is in row echelon form.

System of Linear Equations

Any system of linear equations can be represented in matrix notation as shown below.

- $a_{11}x + a_{12}y + a_{13}z = b_1$
- $a_{21}x + a_{22}y + a_{23}z = b_2$
- $a_{31}x + a_{32}y + a_{33}z = b_3$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \implies AX = B$$

2 types based on B :

- $= 0$: [Homogeneous system](#)
- $\neq 0$: [Non-homogeneous system](#)

Consistent

When the system of equations has at least 1 solution. Otherwise inconsistent.

Rank

Number of non-zero rows of row echelon form of a matrix A . Denoted by **Rank** A .

Note

Rank $A \leq \text{Rank } (A|B)$ is always true.

Relation with non-homogenous system of solutions

Consider the system: $A_{n \times n} X_{n \times 1} = B_{n \times 1}$.

- $|A| \neq 0 \iff \text{Rank } A = \text{Rank } (A|B) = n \iff$ unique solution exists
- $|A| = 0 \implies$ no solution \vee infinitely many solutions
- $\text{Rank } A < \text{Rank } (A|B) \implies$ no solutions
- $\text{Rank } A = \text{Rank } (A|B) < n \implies$ infinitely many solutions

Solutions of homogenous systems

Consider the system:

$$A_{m \times n} X_{n \times 1} = O_{m \times 1}$$

Any homogenous system is consistent, because

$$X = O$$

is always a solution.

- $\text{Rank } A = \text{Rank } (A|B) = n \iff$ unique solution exists
- $\text{Rank } A = \text{Rank } (A|B) < n \implies$ infinitely many solutions

Solution of non-homogenous systems

Method 1: Direct approach

Used when coefficient matrix A is invertible. It means the system has a unique set of solutions.

$$AX = B \implies X = A^{-1}B$$

Method 2: Cramer's Rule

Let $AX = B$, where A is the coefficient matrix and $X = (x_i)_{n \times 1}$.

$$x_i = \frac{|A_i|}{|A|}$$

Where A_i is the matrix obtained by replacing i th column in matrix A by B .

Method 3: Reducing to Echelon Form

Start with $(A|B)$. Convert the **LHS** to echelon form using elementary row transformations. The solution can be found now. If a contradiction is encountered while solving the equation, that means the system has no solutions.

Eigenvalues & Eigenvectors

Definitions

Characteristic Polynomial

Let A be a $n \times n$ matrix.

$$p(\lambda) = |A - \lambda I|$$

Eigenvalues

Roots of the equation $p(\lambda) = 0$ are the eigenvalues of A .

Eigenvectors

The column vectors satisfying the equation $(A - \lambda_i I)X_i$.

Normalized eigenvectors

An eigenvector with the magnitude (norm) of **1**. Normalizing factor of any eigenvector is:

$$\frac{1}{\sqrt{\sum_{i=1}^n X_i^2}}$$

Norm

Norm of a column or row matrix $W_{n \times n}$ is denoted by $\|W\|$ and defined as:

$$\|W\| = \sqrt{\sum_{i=1}^n w_i^2}$$

Algebraic Multiplicity

If the characteristic polynomial consists of a factor of the form $(\lambda - \lambda_i)^r$ and $(\lambda - \lambda_i)^{r+1}$ is not a factor of the characteristic polynomial then ***r*** is the algebraic multiplicity of the eigenvalue λ .

Spectrum

Set of all eigenvalues.

Spectral Radius

$$R = \max \left\{ |\lambda_i| \text{ where } \lambda_i \in \text{Spectrum} \right\}$$

Linear Independence of Eigenvectors

Suppose $X_1, X_2, X_3, \dots, X_n$ is a set of eigenvectors. $k_1, k_2, k_3, \dots, k_n$ is a set of scalars.

All those eigenvectors are independent **iff**:

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$$k_1X_1 + k_2X_2 + k_3X_3 + \cdots + k_nX_n = 0 \implies k_1 = k_2 = k_3 = \cdots = k_n = 0$$

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