Introduction to Vectors

Revise Vectors unit from G.C.E (A/L) Combined Mathematics.

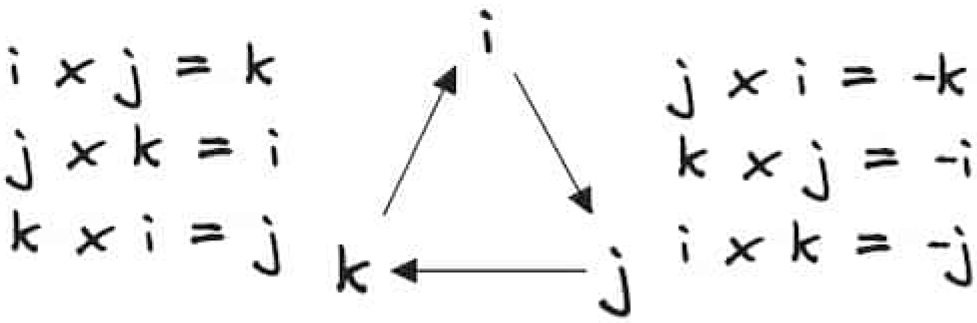
Cross Product

$$a \times b = |a||b|sin(\theta)n = \det \begin{pmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$$

n is the **unit normal vector** to a and b. Direction is based on the right hand rule.

$$a \times b = 0 \implies |a| = 0 \vee |b| = 0 \vee a \parallel b$$

Cross products between i, j, k are circular.



> **Note**
>
> Area of a parallelogram ABCD = $|\vec{AB} \times \vec{AD}|$.

Scalar Triple Product

$$[a, b, c] = a \cdot (b \times c) = \det \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}$$

$$[a, b, c] = a \cdot (b \times c) = (a \times b) \cdot c$$

$$[a, b, c] = [b, c, a] = [c, b, a]$$

$[a, b, c] = 0$ **iff** a, b, c are coplanar.

> **Note**
>
> Volume of a parallelepiped with a, b, c as adjacent edges = $[a, b, c]$
>
> Volume of a tetrahedron with a, b, c as adjacent edges = $\frac{1}{6}[a, b, c]$

Vector Triple Product

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

Vector Equation of Straight Lines

Line that passes through the point $\underline{r_0}$ and parallel to $\underline{v}$

Here $r_0 = (x_0, y_0, z_0)$ and $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$

Parametric equation

$$\underline{r} = \underline{r_0} + t\underline{v};\ t \in \mathbb{R}$$

Symmetric equation

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Line that passes through the point A and B

Here $A = (x_1, y_1, z_1)$, $B = (x_2, y_2, z_2)$. r_A and r_B are the position vectors of A and B.

Parametric equation

$$\underline{r} = (1 - t)\underline{r_A} + t\underline{r_B};\ t \in \mathbb{R}$$

Symmetric equation

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

> **Note**
>
> To show that two straight lines intersect in 3D space, it is **not** enough to show that the cross product of their parallel vectors is non-zero.
>
> **Existence of a point which satisfies both lines must be proven.**

Angle between two straight lines

Let $\alpha : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$, $\beta : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ be two lines.

$$cos\theta = \frac{(a_1\underline{i} + b_1\underline{j} + c_1\underline{k}) \cdot (a_2\underline{i} + b_2\underline{j} + c_2\underline{k})}{|a_1\underline{i} + b_1\underline{j} + c_1\underline{k}||a_2\underline{i} + b_2\underline{j} + c_2\underline{k}|}$$

Vector Equation of Planes

Plane that contains a point r_0 and is parallel to both $\underline{a}$ and $\underline{b}$

Here $\underline{r_0} = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$.

$$\underline{r} = \underline{r_0} + s\underline{a} + t\underline{b}\ ;\ s, t \in \mathbb{R}$$

Plane that contains a point r_0 and $\underline{n}$ is a normal

Here $\underline{r_0} = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$.

$$(\underline{r} - \underline{r_0}) \cdot \underline{n} = 0$$

Plane that contains 3 points r_0, r_1, r_2

Here $\underline{r_0}, \underline{r_1}, \underline{r_2}$ are the position vectors of r_0, r_1, r_2 respectively.

$$(\underline{r} - \underline{r_1})\ \cdot\ \Big[(\underline{r_1} - \underline{r_0}) \times (\underline{r_1} - \underline{r_2})\Big] = 0$$

Normal to a plane

Suppose $ax + by + cz = d$ is a plane.

$\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$ is a normal to the plane.

Angle between 2 planes

Consider the two planes:

- $A : a_1x + a_2y + a_3z = d$
- $B : b_1x + b_2y + b_3z = d'$

The angle between the planes ϕ is:

$$cos(\phi) = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}$$

Shortest distance to a point

Considering a plane $ax + by + cz = d$.

$$\text{distance} = \frac{|(\underline{r_1} - \underline{r_0}) \cdot \underline{n}|}{|\underline{n}|}$$

- $\underline{n}$
  is a normal to the plane
- $\underline{r_0}$
  is the position vector of a point on the plane
- $\underline{r_1}$
  is the position vector to the arbitrary point

Skew Lines

Two non-parallel lines in a 3-space that do not intersect.

Normal to 2 skew lines

Let l_1, l_2 be 2 skew lines.

$$l_1 : \frac{x - x_0}{a_0} = \frac{y - y_0}{b_0} = \frac{z - z_0}{c_0} \; ; \; l_2 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

The normal to both lines $\underline{n}$ is:

$$\underline{n} = \frac{\langle a_0, b_0, c_0 \rangle \times \langle a_1, b_1, c_1 \rangle}{|\langle a_0, b_0, c_0 \rangle \times \langle a_1, b_1, c_1 \rangle|}$$

Distance between 2 skew lines

$$\text{distance} = |\overrightarrow{AB} \cdot \underline{n}|$$

Here

- $\underline{n}$ is the normal to both l_1, l_2
- A and B are points lying on each line