Introduction to Real Analysis

Mathematical logic

Proposition

A statement in either true or false state.

Symbols

Symbol	Read as
$\wedge$	and
$\vee$	or
$\rightarrow$	then
$\implies$	implies
$\Leftarrow$	implied by
$\iff$	if and only if
$\forall$	for all
$\exists$	there exists
$\sim$	not

Let’s take $a \rightarrow b$.

1. Contrapositive or transposition:
   $$\sim b \rightarrow \sim a$$
   . This is equivalent to the original.
2. Inverse:
   $$\sim a \rightarrow \sim b$$
   . Does not depend on the original.
3. Converse:
   $$b \rightarrow a$$
   . Does not depend on the original.

$$a \rightarrow b \equiv \sim a \vee b \equiv \sim b \rightarrow \sim a$$

Examples

- $\sim \forall x P(x) \equiv \exists x \sim P(x)$
- $\sim \exists x P(x) \equiv \forall x \sim P(x)$
- $\exists x \exists y P(x,y) \equiv \exists y \exists x P(x,y)$
- $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$
- $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$

Methods of proofs

1. Just proof what should be proven
2. Prove the contrapositive.
3. Proof by contradiction

Proof by contradiction

Let's say we have to prove: $a \implies b$. We will prove $a \wedge \sim b$ to be false. Then by proof by contradiction, we can prove $a \implies b$.

Proof of proof by contradiction

$$a \wedge \sim b = F$$

$$\sim (a \wedge \sim b) = \sim F$$

$$\sim a \vee b = T$$

$$a \rightarrow b = T$$

$$a \implies b$$

Set theory

Zermelo-Fraenkel set theory with axiom of Choice(ZFC):9 axioms all together is being used here.

Definitions

- $x \in A^{c} \iff x \notin A$
- $x \in A \cup B \iff x \in A \vee x \in B$
- $x \in A \cap B \iff x \in A \wedge x \in B$
- $A \subset B = \forall x(x \in A \implies x \in B)$
- $A - B = A \cap B^{c}$
- $A = B \iff ((\forall z \in A \implies z \in B) \wedge (\forall z \in B \implies z \in A))$

Required proofs

- $(A \cap B)^{c} = A^{c} \cup B^{c}$
- $(A \cup B)^{c} = A^{c} \cap B^{c}$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \subset A \cup B$
- $A \cap B \subset A$

Set of Numbers

Sets of numbers

- Positive integers:
  $$\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$$
  .
- Natural integers:
  $$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$
  .
- Negative integers:
  $$\mathbb{Z}^- = \{-1, -2, -3, -4, \dots\}$$
  .
- Integers:
  $$\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$$
  .
- Rational numbers:
  $$\mathbb{Q} = \left\{ \frac{p}{q} \middle| q \neq 0 \land p, q \in \mathbb{Z} \right\}$$
  .
- Irrational numbers: limits of sequences of rational numbers (which are not rational numbers)
- Real numbers:
  $$\mathbb{R} = \mathbb{Q}^c \cup \mathbb{Q}$$
  .

Complex numbers are not part of the study here.

Continued Fraction Expansion

The process

- Separate the integer part
- Find the inverse of the remaining part. Result will be greated than 1.
- Repeat the process for the remaining part.

Finite expansion

Take $\frac{420}{69}$ for example.

$$\frac{420}{69} = 6 + \frac{6}{69}$$

$$\frac{420}{69} = 6 + \frac{1}{\frac{69}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{3}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{1}{2}}$$

As $\frac{420}{69}$ is finite, its continued fraction expansion is also finite. And it can be written as $\frac{420}{69} = [6; 11, 2]$.

Infinite expansion

For irrational numbers, the expansion will be infinite.

For example π:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}$$

Conintued fraction expansion of π is $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots]$.

Field Axioms

Field Axioms of $\mathbb{R}$

$\mathbb{R} \neq \emptyset$ with two binary operations $+$ and $\cdot$ satisfying the following properties

1. Closed under addition:
   $\forall a, b \in \mathbb{R}; a + b \in \mathbb{R}$
2. Commutative:
   $\forall a, b \in \mathbb{R}; a + b = b + a$
3. Associative:
   $\forall a, b, c \in \mathbb{R}; (a + b) + c = a + (b + c)$
4. Additive identity:
   $\exists 0 \in \mathbb{R}\, \forall a \in \mathbb{R}; a + 0 = 0 + a = a$
5. Additive inverse:
   $\forall a \in \mathbb{R}\, \exists (-a); a + (-a) = (-a) + a = 0$
6. Closed under multiplication:
   $\forall a, b \in \mathbb{R}; a \cdot b \in \mathbb{R}$
7. Commutative:
   $\forall a, b \in \mathbb{R}; a \cdot b = b \cdot a$
8. Associative:
   $\forall a, b, c \in \mathbb{R}; (a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. Multiplicative identity:
   $\exists 1 \in \mathbb{R}\, \forall a \in \mathbb{R}; a \cdot 1 = 1 \cdot a = a$
10. Multiplicative inverse:
    $\forall a \in \mathbb{R} - \{0\}\, \exists a^{-}; a \cdot a^{-} = a^{-} \cdot a = 1$
11. Multiplication is distributive over addition:
    $a \cdot (b + c) = a \cdot b + a \cdot c$

> **Field**
>
> Any set satisfying the above axioms with two binary operations (commonly $+$ and $\cdot$) is called a **field**. Written as $(\mathbb{R}, +, \cdot)$ is a Field. But $(\mathbb{R}, \cdot, +)$ is not a field.

Required proofs

The below mentioned propositions can and should be proven using the above-mentioned axioms. $a, b, c \in \mathbb{R}$.

- $a \cdot 0 = 0$

  Hint: Start with
  $a(1 + 0)$
- $1 \neq 0$
- Additive identity (
  0

) is unique

- Multiplicative identity (1) is unique
- Additive inverse ($-a$) is unique for a given a
- Multiplicative inverse (a^{-1}) is unique for a given a
- $a+b=0 \implies b=-a$
- $a+c=b+c \implies a=b$
- $-(a+b)=(-a)+(-b)$
- $-(-a)=a$
- $ac=bc \implies a=b$
- $ab=0 \implies a=0 \vee b=0$
- $-(ab)=(-a)b=a(-b)$
- $(-a)(-b)=ab$
- $a \neq 0 \implies (a^{-1})^{-1}=a$
- $a,b \neq 0 \implies ab^{-1}=a^{-1}b^{-1}$

Field or Not?

	Is field?	Reason (if not)
$(\mathbb{R},+,\cdot)$	True	
$(\mathbb{R},\cdot,+)$	False	Axiom 11 is invalid
$(\mathbb{Z},+,\cdot)$	False	Multiplicative inverse doesn’t exist
$(\mathbb{Q},+,\cdot)$	True	
$(\mathbb{Q}^c,+,\cdot)$	False	$\sqrt{2}\cdot\sqrt{2} \notin \mathbb{Q}^c$
Boolean algebra	False	Additive inverse doesn’t exist
$(\{0,1\},+ \bmod 2,\cdot \bmod 2)$	True	
$(\{0,1,2\},+ \bmod 3,\cdot \bmod 3)$	True	
$(\{0,1,2,3\},+ \bmod 4,\cdot \bmod 4)$	False	Multiplicative inverse doesn’t exist

Completeness Axiom

Let A be a non empty subset of $\mathbb{R}$.

- u is the upper bound of A if: $\forall a \in A; a \leq u$
- A is bounded above if A has an upper bound
- Maximum element of A : $\max A = u$ if $u \in A$ and u is an upper bound of A
- Supremum of A $\sup A$, is the smallest upper bound of A
- Maximum is a supremum. Supremum is not necessarily a maximum.
- l is the lower bound of A if: $\forall a \in A; a \geq l$
- A is bounded below if A has a lower bound
- Minimum element of A : $\min A = l$ if $l \in A$ and l is a lower bound of A
- Infimum of A $\inf A$

, is the largest lower bound of A

- Minimum is a infimum. Infimum is not necessarily a minimum.

Theorems

Let A be a non empty subset of $\mathbb{R}$.

- Say u is an upper bound of A. Then $u = \sup A$ **iff**: $\forall \epsilon > 0 \; \exists a \in A; \; a + \epsilon > u$
- Say l is a lower bound of A. Then $l = \inf A$ **iff**: $\forall \epsilon > 0 \; \exists a \in A; \; a - \epsilon < l$

Required proofs

- $sup(a, b) = b$
- $inf(a, b) = a$

Completeness axioms of real numbers

- Every non empty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$
- Every non empty subset of $\mathbb{R}$ which is bounded below has a infimum in $\mathbb{R}$

> **Note**
>
> $\mathbb{Q}$ doesn't have the completeness property.

Completeness axioms of integers

- Every non empty subset of $\mathbb{Z}$ which is bounded above has a maximum

- Every non empty subset of $\mathbb{Z}$ which is bounded below has a minimum

Two important theorems

- $\exists a \, \forall \epsilon > 0, a < \epsilon \implies a \leq 0$
- $\forall \epsilon > 0 \, \exists a, a < \epsilon \not\Longrightarrow a \leq 0$

Order Axioms

- **Trichotomy**:
  $\forall a, b \in \mathbb{R}$
  exactly one of these holds:
  $a > b$
  ,
  $a = b$
  ,
  $a < b$
- **Transitivity**:
  $\forall a, b, c \in \mathbb{R}; a < b \land b < c \implies a < c$
- **Operation with addition**:
  $\forall a, b \in \mathbb{R}; a < b \implies a + c < b + c$
- **Operation with mutliplication**:
  $\forall a, b, c \in \mathbb{R}; a < b \land 0 < c \implies ac < bc$

Definitions

- $a < b \equiv b > a$
- $a \leq b \equiv a < b \lor a = b$
- $a \neq b \equiv a < b \lor a > b$
- $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$

Triangular inequalities

$$|a| - |b| \leq |a + b| \leq |a| + |b|$$

$$||a| - |b|| \leq |a + b|$$

Required proofs

- $\forall a, b, c \in \mathbb{R}; a < b \land c < 0 \implies ac > bc$
- $1 > 0$
- $-|a| \leq a \leq |a|$
- Triangular inequalities

Theorems

- $\exists a \; \forall \epsilon > 0, \; a < \epsilon \implies a \leq 0$
- $\exists a \; \forall \epsilon > 0, \; 0 \leq a < \epsilon \implies a = 0$

Caution

> $\forall \epsilon > 0 \, \exists a, \, a < \epsilon \implies a \leq 0$ is **not** valid.

Let A be a non-empty subset of $\mathbb{R}$ which is bounded above and has an upper bound u.

$$u = \sup A \iff \forall \epsilon > 0 \, \exists a \in A, \, a > u - \epsilon$$

Let A be a non-empty subset of $\mathbb{R}$ which is bounded below and has an lower bound m.

$$m = \inf A \iff \forall \epsilon > 0 \, \exists a \in A, \, a < m + \epsilon$$

Relations

Definitions

- Cartesian Product of sets A, B
  $A \times B = \{(a,b)|a \in A, b \in B\}$
- Ordered pair
  $(a,b) = \{\{a\},\{a,b\}\}$

Relation

Let $A, B \neq \emptyset$. A relation $R : A \to B$ is a non-empty subset of $A \times B$.

- $a\,R\,b \equiv (a,b) \in R$
- Domain of R: $dom(R) = A$
- Codomain of R: $codom(R) = B$
- Range of R: $ran(R) = \{y|(x,y) \in R\}$
- $ran(R) \subseteq B$
- Pre-range of R: $preran(R) = \{x \,|\, (x,y) \in R\}$
- $preran(R) \subseteq A$
- $R(a) = \{b \,|\, (a,b) \in R\}$

Everywhere defined

R is everywhere defined
$\iff A = dom(R) = preran(R)$
$\iff \forall a \in A,\ \exists b \in B;\ (a,b) \in R$.

Onto

R is onto
$\iff B = codom(R) = ran(R)$
$\iff \forall b \in B\, \exists a \in A\, (a,b) \in R$

Aka. **surjection**.

Inverse

Inverse of R: $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

Types of relation

one-many

$$\iff \exists a \in A, \exists b_1, b_2 \in B \ ((a, b_1), (a, b_2) \in R \land b_1 \neq b_2)$$

Not one-many

$$\iff \forall a \in A, \forall b_1, b_2 \in B \ ((a, b_1), (a, b_2) \in R \implies b_1 = b_2)$$

many-one

$$\iff \exists a_1, a_2 \in A, \exists b \in B \ ((a_1, b), (a_2, b) \in R \land a_1 \neq a_2)$$

Not many-one

$$\iff \forall a_1, a_2 \in A, \forall b \in B \ ((a_1, b), (a_2, b) \in R \implies a_1 = a_2)$$

many-many

iff R is **one-many** and **many-one**.

one-one

iff R is **not one-many** and **not many-one**. Aka. **injection**.

Bijection

When a relation is **onto** and **one-one**.

Functions

A function $f : A \to B$ is a relation $f : A \to B$ which is everywhere defined and not one-many.

- $dom(f) = A = preran(f)$

Inverse

For a function $f : A \to B$ to have its inverse relation $f^{-1} : B \to A$ be also a function, we need:

- f
  is onto
- f
  is not many-one (in other words,
  f
  must be one-one)

The above statement is true for all unrestricted function f that has an inverse f^{-1}:

$$f(f^{-1}(x)) = x = f^{-1}(f(x)) = x$$

Composition

Composition of relations

Let $R : A \to B$ and $S : B \to C$ are 2 relations. Composition can be defined when $\text{ran}(R) = \text{preran}(S)$.

Say $\text{ran}(R) = \text{preran}(S) = D$. Composition of the 2 relations is written as:

$$S \circ R = \{(a, c) \mid (a, b) \in R,\ (b, c) \in S,\ b \in D\}$$

Composition of functions

Let $f : A \to B$ and $g : B \to C$ be 2 functions where f is onto.

$$g \circ f = \{(x, z) \mid (x, y) \in f,\ (y, z) \in g,\ y \in B\} = g(f(x))$$

Countability

A set A is countable **iff** $\exists f : A \to Z^+$, where f is a one-one function.

Examples

- Countable: Any finite set,
  $\mathbb{Z}, \mathbb{Q}$
- Uncountable:
  $\mathbb{R}$
  , Any open/closed intervals in
  $\mathbb{R}$
  .

Transitive property

Say $B \subset A$.

$$A \text{ is countable } \implies B \text{ is countable}$$

$$B \text{ is not countable } \implies A \text{ is not countable}$$

Limits

$\lim_{x \to a} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

Defining δ in terms of a given ϵ is enough to prove a limit.

One sided limits

$\lim_{x \to a^+} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < x - a < \delta \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to a^-} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (-\delta < x - a < 0 \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to a} f(x) = L^+$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies 0 \leq f(x) - L < \epsilon)$$

$\lim_{x \to a} f(x) = L^-$ **iff**:

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (0 < |x - a| < \delta \implies -\epsilon < f(x) - L \leq 0)$$

Limits including infinite

$\lim_{x \to \infty} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists N > 0 \; \forall x \; (x > N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to -\infty} f(x) = L$ **iff**:

$$\forall \epsilon > 0 \; \exists N > 0 \; \forall x \; (x < -N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \to a} f(x) = \infty$ **iff**:

$$\forall M > 0\ \exists \delta > 0\ \forall x\ (0 < |x - a| < \delta \implies f(x) > M)$$

$\lim_{x \to a} f(x) = -\infty$ **iff**:

$$\forall M > 0\ \exists \delta > 0\ \forall x\ (0 < |x - a| < \delta \implies f(x) < -M)$$

Indeterminate forms

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $\infty \cdot 0$
- $\infty - \infty$
- ∞^0
- 0^0
- 1^∞

Continuity

A function f is continuous at a **iff**:

$$\lim_{x \to a} f(x) = f(a)$$

$$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; (|x - a| < \delta \implies |f(x) - L| < \epsilon)$$

One-side continuous

A function f is continuous from right at a **iff**:

$$\lim_{x \to a^+} f(x) = f(a)$$

A function f is continuous from left at a **iff**:

$$\lim_{x \to a^-} f(x) = f(a)$$

Continuous on an open interval

A function f is continuous in (a, b) **iff** f is continuous on every $c \in (a, b)$.

Continuous on a closed interval

A function f is continuous in $[a, b]$ **iff** f is:

- continuous on every $c \in (a, b)$
- right-continuous at a
- left-continuous at b

Continuity Theorems

Extreme Value Theorem

If f is continuous on $[a, b]$, f has a maximum and a minimum in $[a, b]$.

Intermediate Value Theorem

Let f is continuous on $[a, b]$. If $\exists u$ such that $f(a) > u > f(b)$ or $f(a) < u < f(b)$:
$\exists c \in (a, b)$ such that $f(c) = u$.