# Summary | Vectors

## Introduction

Revise Vectors unit from G.C.E (A/L) Combined Mathematics.

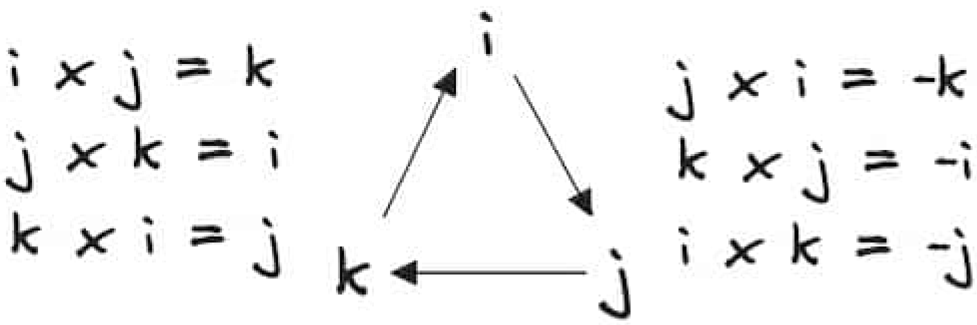
### Cross Product

$$a \times b = |a||b|sin(\theta)n = \det \begin{pmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$$

$n$ is the **unit normal vector** to $a$ and $b$. Direction is based on the right hand rule.

$$a \times b = 0 \implies |a| = 0 \lor |b| = 0 \lor a \parallel b$$

Cross products between $i$, $j$, $k$ are circular.




> **Note**
>
> Area of a parallelogram ABCD = $|\vec{AB} \times \vec{AD}|$.

### Scalar Triple Product

$$[a, b, c] = a \cdot (b \times c) = \det \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}$$

$$[a, b, c] = a \cdot (b \times c) = (a \times b) \cdot c$$

$$[a, b, c] = [b, c, a] = [c, b, a]$$

$[a, b, c] = 0$ **iff** $a$, $b$, $c$ are coplanar.

**Note**

Volume of a parallelepiped with $a$, $b$, $c$ as adjacent edges = $[a, b, c]$

Volume of a tetrahedron with $a$, $b$, $c$ as adjacent edges = $\frac{1}{6}[a, b, c]$

### Vector Triple Product

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

## Vector Equation of Straight Lines

### Line that passes through the point $\underline{r_0}$ and parallel to $\underline{v}$

Here $r_0 = (x_0, y_0, z_0)$ and $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$

#### Parametric equation

$$\underline{r} = \underline{r_0} + t\underline{v}; \; t \in \mathbb{R}$$

### Symmetric equation

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

## Line that passes through the point $A$ and $B$

Here $A=(x_1,y_1,z_1)$, $B=(x_2,y_2,z_2)$. $r_A$ and $r_B$ are the position vectors of $A$ and $B$.

### Parametric equation

$$\underline{r}=(1-t)\underline{r_A}+t\underline{r_B};\ t\in\mathbb{R}$$

### Symmetric equation

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

> **Note**
>
> To show that two straight lines intersect in 3D space, it is **not** enough to show that the cross product of their parallel vectors is non-zero.
>
> **Existence of a point which satisfies both lines must be proven.**

## Angle between two straight lines

Let $\alpha:\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$, $\beta:\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ be two lines.

$$cos\theta=\frac{(a_1\underline{i}+b_1\underline{j}+c_1\underline{k})\cdot(a_2\underline{i}+b_2\underline{j}+c_2\underline{k})}{|a_1\underline{i}+b_1\underline{j}+c_1\underline{k}||a_2\underline{i}+b_2\underline{j}+c_2\underline{k}|}$$

# Vector Equation of Planes

## Plane that contains a point $r_0$ and is parallel to both $\underline{a}$ and $\underline{b}$

Here $\underline{r_0} = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$.

$$\underline{r} = \underline{r_0} + s\underline{a} + t\underline{b}\ ;\ s, t \in \mathbb{R}$$

## Plane that contains a point $r_0$ and $\underline{n}$ is a normal

Here $\underline{r_0} = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$.

$$(\underline{r} - \underline{r_0}) \cdot \underline{n} = 0$$

## Plane that contains 3 points $r_0, r_1, r_2$

Here $\underline{r_0}, \underline{r_1}, \underline{r_2}$ are the position vectors of $r_0, r_1, r_2$ respectively.

$$(\underline{r} - \underline{r_1})\ \cdot\ \left[(\underline{r_1} - \underline{r_0}) \times (\underline{r_1} - \underline{r_2})\right] = 0$$

## Normal to a plane

Suppose $ax + by + cz = d$ is a plane.

$\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$ is a normal to the plane.

## Angle between 2 planes

Consider the two planes:

- $A : a_1x + a_2y + a_3z = d$
- $B : b_1x + b_2y + b_3z = d'$

The angle between the planes $\phi$ is:

$$cos(\phi) = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}$$

### Shortest distance to a point

Considering a plane $ax + by + cz = d$.

$$\text{distance} = \frac{|(\underline{r_1} - \underline{r_0}) \cdot \underline{n}|}{|\underline{n}|}$$

- $\underline{n}$ is a normal to the plane
- $\underline{r_0}$ is the position vector of a point on the plane
- $\underline{r_1}$ is the position vector to the arbitrary point

## Skew Lines

Two non-parallel lines in a 3-space that do not intersect.

### Normal to 2 skew lines

Let $l_1, l_2$ be 2 skew lines.

$$l_1 : \frac{x - x_0}{a_0} = \frac{y - y_0}{b_0} = \frac{z - z_0}{c_0} \;\; ; \;\; l_2 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

The normal to both lines $\underline{n}$ is:

$$\underline{n} = \frac{\langle a_0, b_0, c_0 \rangle \times \langle a_1, b_1, c_1 \rangle}{|\langle a_0, b_0, c_0 \rangle \times \langle a_1, b_1, c_1 \rangle|}$$

### Distance between 2 skew lines

$$\text{distance} = |\overrightarrow{AB} \cdot \underline{n}|$$

Here

- $\underline{n}$ is the normal to both $l_1, l_2$
- $A$ and $B$ are points lying on each line