Introduction to Riemann Integration

Interval

Let $I = [a, b]$. Length of the interval $|I| = b - a$.

Disjoint interval

When 2 intervals don’t share any common numbers.

Almost disjoint interval

When 2 intervals are disjoint or intersect only at a common endpoint.

Riemann Integral

Let $f - [a, b] \to \mathbb{R}$ is a bounded (not necessarily continuous) function on a closed, bounded (compact) interval.

Riemann integral of f is: $\int_a^b f$

Definite integral

When a, b are constants.

Indefinite integral

When a is a constant but b is replaced with x.

Partition

Let I be a non-empty, compact interval (closed and bounded). A partition of I is a finite collection $\{I_1, I_2, \ldots, I_n\}$ of almost disjoint, non-empty, compact sub-intervals whose union is I.

A partition is determined by the endpoints of all sub-intervals: $a = x_0 < x_1 < \cdots < x_n = b$.

A partition can be denoted by:

- its intervals - $P = \{I_1, I_2, \ldots, I_n\}$
- the endpoints of its intervals - $P = \{x_0, x_1, \ldots, x_n\}$

Riemann Sum

Let

- $f : [a, b] \to \mathbb{R}$ is a bounded function on the compact interval $I = [a, b]$ with $M = \sup_I f$ and $m = \inf_I f$.
- $P = \{I_1, I_2, \ldots, I_n\}$
- $M_k = \sup_{I_k} f = \sup \{f(x) : x \in [x_{k-1}, x_k]\}$
- $m_k = \inf_{I_k} f = \inf \{f(x) : x \in [x_{k-1}, x_k]\}$

Upper riemann sum

$$U(f; P) = \sum_{k=1}^{n} M_k |I_k|$$

Lower riemann sum

$$L(f; P) = \sum_{k=1}^{n} m_k |I_k|$$

$m_k < M_k \implies L(f; P) \le U(f; P)$

When P_1, P_2 are any 2 partitions of I: $L(f; P_1) \le U(f; P_2)$

Refinements

Q is called a refinement of $P \iff$ if P and Q are partitions of $[a, b]$ and $P \subseteq Q$.

When Q is a refinement of P:

$$L(f; P) \leq L(f; Q) \leq U(f; Q) \leq U(f; P)$$

> **Note**
>
> If P_1 and P_2 are partitions of $[a, b]$, then $Q = P_1 \cup P_2$ is a refinement of both P_1 and P_2. In that case:
>
> $$L(f; P_1) \leq L(f; Q) \leq U(f; Q) \leq U(f; P_2)$$

Upper & Lower integral

Let $\mathbb{P}$ be the collection of all possible partitions of the interval $[a, b]$.

Upper Integral

$$U(f) = \inf\{U(f; P); P \in \mathbb{P}\} = \overline{\int_a^b} f$$

Lower Integral

$$L(f) = \sup\{L(f; P); P \in \mathbb{P}\} = \underline{\int_a^b} f$$

For a bounded function f, always $L(f) \leq U(f)$

Riemann Integrable

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ **iff** $U(f) = L(f)$. In that case, the Riemann integral of f on $[a, b]$ is denoted by $\int_a^b f(x)\,\mathrm{d}x$.

An unbounded function is not Riemann integrable.

Cauchy Criterion

Theorem

A bounded function $f : [a, b] \to R$ is Riemann integrable **iff** for every $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$, which may depend on ϵ, such that:

$$U(f, P\epsilon) - L(f, P\epsilon) \leq \epsilon$$