

# Introduction to Vectors

Revise Vectors unit from G.C.E (A/L) Combined Mathematics.

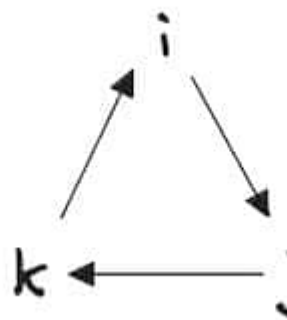
## Cross Product

$$a \times b = |a||b|\sin(\theta)n = \det \begin{pmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix}$$

$n$  is the **unit normal vector** to  $a$  and  $b$ . Direction is based on the right hand rule.

$$a \times b = 0 \implies |a| = 0 \vee |b| = 0 \vee a \parallel b$$

Cross products between  $i, j, k$  are circular.


$$\begin{array}{l} i \times j = k \\ j \times k = i \\ k \times i = j \end{array} \quad \begin{array}{l} j \times i = -k \\ k \times j = -i \\ i \times k = -j \end{array}$$

### Note

Area of a parallelogram ABCD =  $|\vec{AB} \times \vec{AD}|$ .

## Scalar Triple Product

$$[a, b, c] = a \cdot (b \times c) = \det \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}$$

$$[a, b, c] = a \cdot (b \times c) = (a \times b) \cdot c$$

$$[a, b, c] = [b, c, a] = [c, a, b]$$

$[a, b, c] = 0$  **iff**  $a, b, c$  are coplanar.

**Note**

Volume of a parallelepiped with  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  as adjacent edges  $= [\mathbf{a}, \mathbf{b}, \mathbf{c}]$

Volume of a tetrahedron with  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  as adjacent edges  $= \frac{1}{6} [\mathbf{a}, \mathbf{b}, \mathbf{c}]$

## Vector Triple Product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

# Vector Equation of Straight Lines

**Line that passes through the point  $\underline{r_0}$  and parallel to  $\underline{v}$**

Here  $\underline{r_0} = (x_0, y_0, z_0)$  and  $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$

**Parametric equation**

$$\underline{r} = \underline{r_0} + t\underline{v}; t \in \mathbb{R}$$

**Symmetric equation**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

**Line that passes through the point  $A$  and  $B$**

Here  $A = (x_1, y_1, z_1)$ ,  $B = (x_2, y_2, z_2)$ .  $\underline{r_A}$  and  $\underline{r_B}$  are the position vectors of  $A$  and  $B$ .

**Parametric equation**

$$\underline{r} = (1 - t)\underline{r_A} + t\underline{r_B}; t \in \mathbb{R}$$

**Symmetric equation**

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

## Note

To show that two straight lines intersect in 3D space, it is **not** enough to show that the cross product of their parallel vectors is non-zero.

**Existence of a point which satisfies both lines must be proven.**

**Angle between two straight lines**

Let  $\alpha : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ ,  $\beta : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$  be two lines.

$$\cos\theta = \frac{(a_1\underline{i} + b_1\underline{j} + c_1\underline{k}) \cdot (a_2\underline{i} + b_2\underline{j} + c_2\underline{k})}{|a_1\underline{i} + b_1\underline{j} + c_1\underline{k}||a_2\underline{i} + b_2\underline{j} + c_2\underline{k}|}$$

# Vector Equation of Planes

## Plane that contains a point $\underline{r}_0$ and is parallel to both $\underline{a}$ and $\underline{b}$

Here  $\underline{r}_0 = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$ .

$$\underline{r} = \underline{r}_0 + s\underline{a} + t\underline{b} ; s, t \in \mathbb{R}$$

## Plane that contains a point $\underline{r}_0$ and $\underline{n}$ is a normal

Here  $\underline{r}_0 = x_0\underline{i} + y_0\underline{j} + z_0\underline{k}$ .

$$(\underline{r} - \underline{r}_0) \cdot \underline{n} = 0$$

## Plane that contains 3 points $\underline{r}_0, \underline{r}_1, \underline{r}_2$

Here  $\underline{r}_0, \underline{r}_1, \underline{r}_2$  are the position vectors of  $\underline{r}_0, \underline{r}_1, \underline{r}_2$  respectively.

$$(\underline{r} - \underline{r}_1) \cdot \left[ (\underline{r}_1 - \underline{r}_0) \times (\underline{r}_1 - \underline{r}_2) \right] = 0$$

## Normal to a plane

Suppose  $ax + by + cz = d$  is a plane.

$\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$  is a normal to the plane.

## Angle between 2 planes

Consider the two planes:

- $A : a_1x + a_2y + a_3z = d$
- $B : b_1x + b_2y + b_3z = d'$

The angle between the planes  $\phi$  is:

$$\cos(\phi) = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}}$$

## Shortest distance to a point

Considering a plane  $ax + by + cz = d$ .

$$\text{distance} = \frac{|(\underline{r_1} - \underline{r_0}) \cdot \underline{n}|}{|\underline{n}|}$$

- $\underline{n}$  is a normal to the plane
- $\underline{r_0}$  is the position vector of a point on the plane
- $\underline{r_1}$  is the position vector to the arbitrary point

# Skew Lines

Two non-parallel lines in a 3-space that do not intersect.

## Normal to 2 skew lines

Let  $l_1, l_2$  be 2 skew lines.

$$l_1 : \frac{x - x_0}{a_0} = \frac{y - y_0}{b_0} = \frac{z - z_0}{c_0} ; \quad l_2 : \frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

The normal to both lines  $\underline{n}$  is:

$$\underline{n} = \frac{\langle a_0, b_0, c_0 \rangle \times \langle a_1, b_1, c_1 \rangle}{|\langle a_0, b_0, c_0 \rangle \times \langle a_1, b_1, c_1 \rangle|}$$

## Distance between 2 skew lines

$$\text{distance} = \left| \overrightarrow{AB} \cdot \underline{n} \right|$$

Here

- $\underline{n}$  is the normal to both  $l_1, l_2$
- $A$  and  $B$  are points lying on each line