

Summary | Real Analysis

Introduction— |

| \wedge | and | | \vee | or | | \rightarrow | then | | \implies | implies | | \Leftarrow | implied by | | \iff | if and only if | | \forall | for all | | \exists | there exists | | \sim | not |

Let's take $a \rightarrow b$.

1. Contrapositive or transposition: $\sim b \rightarrow \sim a$. This is equivalent to the original.
2. Inverse: $\sim a \rightarrow \sim b$. Does not depend on the original.
3. Converse: $b \rightarrow a$. Does not depend on the original.

$$a \rightarrow b \equiv \sim a \vee b \equiv \sim b \rightarrow \sim a$$

Required proofs

- $\sim \forall x P(x) \equiv \exists x \sim P(x)$
- $\sim \exists x P(x) \equiv \forall x \sim P(x)$
- $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$
- $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$
- $\exists x \forall y P(x, y) \implies \forall y \exists x P(x, y)$
- $(A \rightarrow C) \wedge (B \rightarrow C) \equiv (A \vee B) \rightarrow C$

Methods of proofs

1. Just proof what should be proven
2. Prove the contrapositive
3. Proof by contradiction
4. Proof by induction

Proof by contradiction

Suppose $a \implies b$ has to be proven. If $a \wedge \sim b$ is proven to be false, then, by proof by contradiction, $a \implies b$ can be trivially proven.

Logic behind proof by contradiction

$$a \wedge \sim b = F$$

$$\sim (a \wedge \sim b) = \sim F$$

$$\sim a \vee b = T$$

$$a \rightarrow b = T$$

$$a \implies b$$

Set theory

Zermelo-Fraenkel set theory with axiom of Choice(ZFC):9 axioms all together is being used here.

Definitions

- $x \in A^c \iff x \notin A$
- $x \in A \cup B \iff x \in A \vee x \in B$
- $x \in A \cap B \iff x \in A \wedge x \in B$
- $A \subset B = \forall x(x \in A \implies x \in B)$
- $A - B = A \cap B^c$
- $A = B \iff ((\forall z \in A \implies z \in B) \wedge (\forall z \in B \implies z \in A))$

Required proofs

- $(A \cap B)^c = A^c \cup B^c$
- $(A \cup B)^c = A^c \cap B^c$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \subset A \cup B$
- $A \cap B \subset A$

Set of Numbers

Sets of numbers

- Positive integers: $\mathbb{Z}^+ = \{1, 2, 3, 4, \dots\}$.
- Natural integers: $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$.
- Negative integers: $\mathbb{Z}^- = \{-1, -2, -3, -4, \dots\}$.
- Integers: $\mathbb{Z} = \mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+$.
- Rational numbers: $\mathbb{Q} = \left\{ \frac{p}{q} \mid q \neq 0 \wedge p, q \in \mathbb{Z} \right\}$.
- Irrational numbers: limits of sequences of rational numbers (which are not rational numbers)
- Real numbers: $\mathbb{R} = \mathbb{Q}^c \cup \mathbb{Q}$.

Complex numbers are not part of the study here.

Continued Fraction Expansion

The process

- Separate the integer part
- Find the inverse of the remaining part. Result will be greater than 1.
- Repeat the process for the remaining part.

Finite expansion

Take $\frac{420}{69}$ for example.

$$\frac{420}{69} = 6 + \frac{6}{69}$$

$$\frac{420}{69} = 6 + \frac{1}{\frac{69}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{3}{6}}$$

$$\frac{420}{69} = 6 + \frac{1}{11 + \frac{1}{2}}$$

As $\frac{420}{69}$ is finite, its continued fraction expansion is also finite. And it can be written as $\frac{420}{69} = [6; 11, 2]$.

Infinite expansion

For irrational numbers, the expansion will be infinite.

For example π :

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \dots}}}}$$

Conintued fraction expansion of π is $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \dots]$.

Field Axioms

Field Axioms of \mathbb{R}

$\mathbb{R} \neq \emptyset$ with two binary operations $+$ and \cdot satisfying the following properties

1. Closed under addition: $\forall a, b \in \mathbb{R}; a + b \in \mathbb{R}$
2. Commutative: $\forall a, b \in \mathbb{R}; a + b = b + a$

3. Associative: $\forall a, b, c \in \mathbb{R}; (a + b) + c = a + (b + c)$
4. Additive identity: $\exists 0 \in \mathbb{R} \forall a \in \mathbb{R}; a + 0 = 0 + a = a$
5. Additive inverse: $\forall a \in \mathbb{R} \exists (-a); a + (-a) = (-a) + a = 0$
6. Closed under multiplication: $\forall a, b \in \mathbb{R}; a \cdot b \in \mathbb{R}$
7. Commutative: $\forall a, b \in \mathbb{R}; a \cdot b = b \cdot a$
8. Associative: $\forall a, b, c \in \mathbb{R}; (a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. Multiplicative identity: $\exists 1 \in \mathbb{R} \forall a \in \mathbb{R}; a \cdot 1 = 1 \cdot a = a$
10. Multiplicative inverse: $\forall a \in \mathbb{R} - \{0\} \exists a^{-}; a \cdot a^{-} = a^{-} \cdot a = 1$
11. Multiplication is distributive over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$

① Field

Any set satisfying the above axioms with two binary operations (commonly $+$ and \cdot) is called a **field**. Written as $(\mathbb{R}, +, \cdot)$ is a **Field**. But $(\mathbb{R}, \cdot, +)$ is not a **field**.

Required proofs

The below mentioned propositions can and should be proven using the above-mentioned axioms.

$a, b, c \in \mathbb{R}$.

- $a \cdot 0 = 0$
Hint: Start with $a(1 + 0)$
- $1 \neq 0$
- Additive identity (0) is unique
- Multiplicative identity (1) is unique
- Additive inverse ($-a$) is unique for a given a
- Multiplicative inverse (a^{-1}) is unique for a given a
- $a + b = 0 \implies b = -a$
- $a + c = b + c \implies a = b$
- $-(a + b) = (-a) + (-b)$
- $-(-a) = a$
- $ac = bc \implies a = b$
- $ab = 0 \implies a = 0 \vee b = 0$
- $-(ab) = (-a)b = a(-b)$
- $(-a)(-b) = ab$
- $a \neq 0 \implies (a^{-1})^{-1} = a$
- $a, b \neq 0 \implies ab^{-1} = a^{-1}b^{-1}$

Field or Not?

	Is field?	Reason (if not)
$(\mathbb{R}, +, \cdot)$	True	
$(\mathbb{R}, \cdot, +)$	False	Axiom 11 is invalid
$(\mathbb{Z}, +, \cdot)$	False	Multiplicative inverse doesn't exist
$(\mathbb{Q}, +, \cdot)$	True	
$(\mathbb{Q}^c, +, \cdot)$	False	$\sqrt{2} \cdot \sqrt{2} \notin \mathbb{Q}^c$
Boolean algebra	False	Additive inverse doesn't exist
$(\{0, 1\}, + \bmod 2, \cdot \bmod 2)$	True	

	Is field?	Reason (if not)
$(\{0, 1, 2\}, + \bmod 3, \cdot \bmod 3)$	True	
$(\{0, 1, 2, 3\}, + \bmod 4, \cdot \bmod 4)$	False	Multiplicative inverse doesn't exist

Completeness Axiom

Let A be a non empty subset of \mathbb{R} .

- u is the upper bound of A if: $\forall a \in A; a \leq u$
- A is bounded above if A has an upper bound
- Maximum element of A : $\max A = u$ if $u \in A$ and u is an upper bound of A
- Supremum of A $\sup A$, is the smallest upper bound of A
- Maximum is a supremum. Supremum is not necessarily a maximum.
- l is the lower bound of A if: $\forall a \in A; a \geq l$
- A is bounded below if A has a lower bound
- Minimum element of A : $\min A = l$ if $l \in A$ and l is a lower bound of A
- Infimum of A $\inf A$, is the largest lower bound of A
- Minimum is a infimum. Infimum is not necessarily a minimum.

Theorems

Let A be a non empty subset of \mathbb{R} .

- Say u is an upper bound of A . Then $u = \sup A$ iff: $\forall \epsilon > 0 \exists a \in A; a + \epsilon > u$
- Say l is a lower bound of A . Then $l = \inf A$ iff: $\forall \epsilon > 0 \exists a \in A; a - \epsilon < l$

Proof Hint

Prove the contrapositive. Use $\epsilon = \frac{1}{2}(L - \sup(A))$ for supremum proof.

Required proofs

- $\sup(a, b) = b$
- $\inf(a, b) = a$

Completeness property

A set A is said to have the completeness property **iff** every non-empty subset of A :

- Which is bounded below has a infimum in A
- Which is bounded above has a supremum in A

Both \mathbb{R}, \mathbb{Z} have the completeness property. \mathbb{Q} doesn't.

In addition to that:

- Every non empty subset of \mathbb{Z} which is bounded above has a maximum
- Every non empty subset of \mathbb{Z} which is bounded below has a minimum

Order Axioms

- **Trichotomy:** $\forall a, b \in \mathbb{R}$ exactly one of these holds: $a > b$, $a = b$, $a < b$
- **Transitivity:** $\forall a, b, c \in \mathbb{R}; a < b \wedge b < c \implies a < c$
- **Operation with addition:** $\forall a, b \in \mathbb{R}; a < b \implies a + c < b + c$
- **Operation with multiplication:** $\forall a, b, c \in \mathbb{R}; a < b \wedge 0 < c \implies ac < bc$

Definitions

- $a < b \equiv b > a$
- $a \leq b \equiv a < b \vee a = b$
- $a \neq b \equiv a < b \vee a > b$
- $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$

Triangular inequalities

$$|a| - |b| \leq |a + b| \leq |a| + |b|$$

$$||a| - |b|| \leq |a + b|$$

Proof Hint

For first:

- Use $-|a| \leq a \leq |a|$

For second:

- Use the below substitutions in first conclusion
 - $a = a - b \wedge b = b$
 - $a = b - a \wedge b = a$

Required proofs

- $\forall a, b, c \in \mathbb{R}; a < b \wedge c < 0 \implies ac > bc$
- $1 > 0$
- $-|a| \leq a \leq |a|$
- Triangular inequalities

Theorems

- $\exists a \forall \epsilon > 0, a < \epsilon \implies a \leq 0$
- $\exists a \forall \epsilon > 0, 0 \leq a < \epsilon \implies a = 0$
- $\forall \epsilon > 0 \exists a, a < \epsilon \not\Rightarrow a \leq 0$

Caution

$\forall \epsilon > 0 \exists a, a < \epsilon \implies a \leq 0$ is **not** valid.

Relations

Definitions

- Cartesian Product of sets A, B $A \times B = \{(a, b) | a \in A, b \in B\}$
- Ordered pair $(a, b) = \left\{ \{a\}, \{a, b\} \right\}$

Relation

Let $A, B \neq \emptyset$. A relation $R : A \rightarrow B$ is a non-empty subset of $A \times B$.

- $a R b \equiv (a, b) \in R$
- Domain of R : $dom(R) = A$
- Codomain of R : $codom(R) = B$
- Range of R : $ran(R) = \{y | (x, y) \in R\}$
- $ran(R) \subseteq B$
- Pre-range of R : $preran(R) = \{x | (x, y) \in R\}$
- $preran(R) \subseteq A$
- $R(a) = \{b | (a, b) \in R\}$

Everywhere defined

R is everywhere defined $\iff A = dom(R) = preran(R)$

$\iff \forall a \in A, \exists b \in B; (a, b) \in R$

Onto

R is onto $\iff B = codom(R) = ran(R) \iff \forall b \in B \exists a \in A (a, b) \in R$

Aka. **surjection**.

Inverse

Inverse of a relation R :

$$R^{-1} = \{(b, a) | (a, b) \in R\}$$

Types of relation

one-many

$$\iff \exists a \in A, \exists b_1, b_2 \in B ((a, b_1), (a, b_2) \in R \wedge b_1 \neq b_2)$$

Not one-many

$$\iff \forall a \in A, \forall b_1, b_2 \in B ((a, b_1), (a, b_2) \in R \implies b_1 = b_2)$$

many-one

$$\iff \exists a_1, a_2 \in A, \exists b \in B ((a_1, b), (a_2, b) \in R \wedge a_1 \neq a_2)$$

Not many-one

$$\iff \forall a_1, a_2 \in A, \forall b \in B ((a_1, b), (a_2, b) \in R \implies a_1 = a_2)$$

many-many

iff R is **one-many** and **many-one**.

one-one

iff R is **not one-many** and **not many-one**. Aka. **injection**.

Bijection

When a relation is **onto** and **one-one**.

Functions

A function $f : A \rightarrow B$ is a relation $f : A \rightarrow B$ which is [everywhere defined](#) and [not one-many](#).

- $\text{dom}(f) = A = \text{preran}(f)$

Inverse

For a function $f : A \rightarrow B$ to have its inverse relation $f^{-1} : B \rightarrow A$ be also a function, we need:

- f is [onto](#)
- f is [not many-one](#) (in other words, f must be [one-one](#))

The above statement is true for all unrestricted function f that has an inverse f^{-1} :

$$f(f^{-1}(x)) = x = f^{-1}(f(x)) = x$$

Composition

Composition of relations

Let $R : A \rightarrow B$ and $S : B \rightarrow C$ are 2 relations. Composition can be defined when $\text{ran}(R) = \text{preran}(S)$.

Say $\text{ran}(R) = \text{preran}(S) = D$. Composition of the 2 relations is written as:

$$S \circ R = \{(a, c) \mid (a, b) \in R, (b, c) \in S, b \in D\}$$

Composition of functions

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be 2 functions where f is [onto](#).

$$g \circ f = \{(x, z) \mid (x, y) \in f, (y, z) \in g, y \in B\} = g(f(x))$$

Countability

A set A is countable **iff** $\exists f : A \rightarrow \mathbb{Z}^+$, where f is a one-one function.

Examples

- Countable: Any finite set, \mathbb{Z}, \mathbb{Q}
- Uncountable: \mathbb{R} , Any open/closed intervals in \mathbb{R} .

Transitive property

Say $B \subset A$.

$$A \text{ is countable} \implies B \text{ is countable}$$

B is not countable $\implies A$ is not countable

Limits

$\lim_{x \rightarrow a} f(x) = L$ iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$

Defining δ in terms of a given ϵ is enough to prove a limit.

One sided limits

$\lim_{x \rightarrow a^+} f(x) = L$ iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < x - a < \delta \implies |f(x) - L| < \epsilon)$$

$\lim_{x \rightarrow a^-} f(x) = L$ iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (-\delta < x - a < 0 \implies |f(x) - L| < \epsilon)$$

$\lim_{x \rightarrow a} f(x) = L^+$ iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies 0 \leq f(x) - L < \epsilon)$$

$\lim_{x \rightarrow a} f(x) = L^-$ iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies -\epsilon < f(x) - L \leq 0)$$

Limits including infinite

$\lim_{x \rightarrow \infty} f(x) = L$ iff:

$$\forall \epsilon > 0 \exists N > 0 \forall x (x > N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \rightarrow -\infty} f(x) = L$ **iff:**

$$\forall \epsilon > 0 \exists N > 0 \forall x (x < -N \implies |f(x) - L| < \epsilon)$$

$\lim_{x \rightarrow a} f(x) = \infty$ **iff:**

$$\forall M > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies f(x) > M)$$

$\lim_{x \rightarrow a} f(x) = -\infty$ **iff:**

$$\forall M > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \implies f(x) < -M)$$

Indeterminate forms

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $\infty \cdot 0$
- $\infty - \infty$
- ∞^0
- 0^0
- 1^∞

Continuity

A function f is continuous at a **iff:**

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (|x - a| < \delta \implies |f(x) - f(a)| < \epsilon)$$

One-side continuous

A function f is continuous from right at a **iff:**

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

A function f is continuous from left at a **iff**:

$$\lim_{x \rightarrow a^-} f(x) = f(a)$$

Continuous on an open interval

A function f is continuous in (a, b) **iff** f is continuous on every $c \in (a, b)$.

Continuous on a closed interval

A function f is continuous in $[a, b]$ **iff** f is:

- continuous on every $c \in (a, b)$
- right-continuous at a
- left-continuous at b

Uniformly continuous

Suppose a function f is continuous on (a, b) . f is uniformly continuous on (a, b) **iff**:

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

If a function f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$.

 **Todo**

Is this section correct? I am not 100% sure.

Continuity Theorems

Extreme Value Theorem

If f is continuous on $[a, b]$, f has a maximum and a minimum in $[a, b]$.

Proof Hint

Proof is quite hard.

Intermediate Value Theorem

Let f is continuous on $[a, b]$. If $\exists u$ such that $f(a) > u > f(b)$ or $f(a) < u < f(b)$: $\exists c \in (a, b)$ such that $f(c) = u$.

Proof Hint

Proof the case when $u = 0$. Otherwise define a new function $g(x)$ such that middle part of the above inequality has a 0 in the place of u .

Sandwich (or Squeeze) Theorem

Let:

- For some $\delta > 0$: $\forall x (0 < |x - a| < \delta \implies f(x) \leq g(x) \leq h(x))$
- $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \in \mathbb{R}$

Then $\lim_{x \rightarrow a} g(x) = L$.

Note

Works for any kind of x limits.

"No sudden changes"

Positive

Let f be continuous on a and $f(a) > 0$

$$\implies \exists \delta > 0; \forall x (|x - a| < \delta \implies f(x) > 0)$$

① Proof Hint

$$\text{Take } \epsilon = \frac{f(a)}{2}$$

Negative

Let f be continuous on a and $f(a) < 0$

$$\implies \exists \delta > 0; \forall x (|x - a| < \delta \implies f(x) < 0)$$

① Proof Hint

$$\text{Take } \epsilon = -\frac{f(a)}{2}$$

Differentiability

A function f is differentiable at a **iff**:

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'(a)$$

$f'(a)$ is called the derivative of f at a .

One-side differentiable

Left differentiable

A function f is left-differentiable at a **iff**:

$$\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_-(a)$$

Right differentiable

A function f is right-differentiable at a **iff**:

$$\lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = L \in \mathbb{R} = f'_+(a)$$

Differentiability implies continuity

$$f \text{ is differentiable at } a \implies f \text{ is continuous at } a$$

① Proof Hint

Use $\delta = \min(\delta_1, \frac{\epsilon}{1+|f'(a)|})$.

① Note

Suppose f is differentiable at a . Define g :

$$g(x) = \begin{cases} \frac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

g is continuous at a .

Properties of differentiation

Addition

$$\frac{d}{dx}(f \pm g) = f' \pm g'$$

Multiplication

$$\frac{d}{dx}(fg) = fg' + fg'$$

Division

$$\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{gf' - fg'}{g^2}$$

Composition

$$\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$$

Power

$$\frac{d}{dx} f^n = n f^{n-1}(x) f'(x)$$

Extreme Values

Suppose $f : [a, b] \rightarrow \mathbb{R}$, and $F = f([a, b]) = \left\{ f(x) \mid x \in [a, b] \right\}$. Minimum and maximum values of f are called the extreme values.

Maximum

Maximum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\forall x \in [a, b], f(c) \geq f(x)$$

aka. **Global Maximum**. Maximum doesn't exist always.

Local Maximum

A Local maximum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\exists \delta \forall x (0 < |x - c| < \delta \implies f(c) \geq f(x))$$

Global maximum is obviously a local maximum.

The above statement can be simplified when $c = a$ or $c = b$.

When $c = a$:

$$\exists \delta \forall x (0 < x - c < \delta \implies f(c) \geq f(x))$$

When $c = b$:

$$\exists \delta \forall x (-\delta < x - c < 0 \implies f(c) \geq f(x))$$

Minimum

Minimum of the function f is $f(c)$ where $c \in [a, b]$ **iff**:

$$\forall x \in [a, b], f(c) \leq f(x)$$

aka. **Global Minimum**. Minimum doesn't exist always.

Local Minimum

$$\exists \delta \forall x (0 < |x - c| < \delta \implies f(c) \leq f(x))$$

Global minimum is obviously a local maximum.

The above statement can be simplified when $c = a$ or $c = b$.

When $c = a$:

$$\exists \delta \forall x (0 < x - c < \delta \implies f(c) \leq f(x))$$

When $c = b$:

$$\exists \delta \forall x (-\delta < x - c < 0 \implies f(c) \leq f(x))$$

Special cases

f is continuous

Then by [Extreme Value Theorem](#), we know f has a minimum and maximum in $[a, b]$.

f is differentiable

- If $f(a)$ is a local maximum: $f'_+(a) \leq 0$
- If $f(b)$ is a local maximum: $f'_-(b) \geq 0$
- $c \in (a, b)$ and If $f(c)$ is a local maximum: $f'(c) = 0$
- If $f(a)$ is a local minimum: $f'_+(a) \geq 0$
- If $f(b)$ is a local minimum: $f'_-(b) \leq 0$
- $c \in (a, b)$ and If $f(c)$ is a local minimum: $f'(c) = 0$

Critical point

$c \in [a, b]$ is called a critical point **iff**:

$$f'(c) = 0 \quad \vee \quad f'(c) \text{ is undefined}$$

Other Theorems

Rolle's Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b) . And $f(a) = f(b)$. Then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$

Proof Hint

By [Extreme Value Theorem](#), maximum and minimum exists for f .

Consider 2 cases:

1. Both minimum and maximum exist at a and b .
2. One of minimum or maximum occurs in (a, b) .

Mean Value Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b) . Then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof Hint

- Define $g(x) = f(x) - \left(\frac{f(a)-f(b)}{a-b} \right)x$
- $g(a)$ will be equal to $g(b)$
- Use Rolle's Theorem for g

Cauchy's Mean Value Theorem

Let f and g be continuous on $[a, b]$ and differentiable on (a, b) , and $\forall x \in (a, b) \ g'(x) \neq 0$ Then:

$$\exists c \in (a, b) \text{ s.t. } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof Hint

- Define $h(x) = f(x) - \left(\frac{f(a)-f(b)}{g(a)-g(b)} \right)g(x)$
- $h(a)$ will be equal to $h(b)$
- Use Rolle's Theorem for h

Mean value theorem can be obtained from this when $g(x) = x$.

Generalized MVT for Riemann Integrals

Let f, g be continuous on $[a, b]$ ($\implies f, g$ are integrable), and g does not change sign on (a, b) .

Then $\exists \zeta \in (a, b)$ such that:

$$\int_a^b f(x)g(x)dx = f(\zeta) \int_a^b g(x)dx$$

📌 Proof Hint

- Use [Extreme value theorem](#) for f
- Multiply by $g(x)$. Then integrate. Then divide by $\int_a^b g(x)$.
- Use intermediate value theorem to find $f(\zeta)$

L'Hopital's Rule

📌 Note

Be careful with the pronunciation.

- It's not "Hospital's Rule", there are no "s"
- It's not "Hopital's Rule" either, there is a "L"

L'Hopital's Rule can be used when all of these conditions are met. (here δ is some positive number).

Select the appropriate x ranges.

1. Either of these conditions must be satisfied

- $f(a) = g(a) = 0$
- $\lim f(x) = \lim g(x) = 0$
- $\lim f(x) = \lim g(x) = \infty$

2. f, g are continuous on $x \in [a, a + \delta]$

3. f, g are differentiable on $x \in (a, a + \delta)$

4. $g'(x) \neq 0$ on $x \in (a, a + \delta)$

5. $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$

Then: $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$

Note

L'Hopital's rule can be proven using Cauchy's Mean Value Theorem.

It is valid for all types of "x limits".

Higher Order Derivatives

Suppose f is a function defined on (a, b) . f is n times differentiable or n -th differentiable **iff**:

$$\lim_{x \rightarrow a} \frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a} = L \in \mathbb{R} = f^{(n)}(a)$$

Here $f^{(n)}$ denotes n -th derivative of f . And $f^{(0)}$ means the function itself.

$f^{(n)}(a)$ is the n -th derivative of f at a .

Note

$$f \text{ is } n\text{-th differentiable at } a \implies f^{(n-1)} \text{ is continuous at } a$$

Taylor's Theorem

Let f is $n + 1$ differentiable on (a, b) . Let $c, x \in (a, b)$. Then $\exists \zeta \in (c, x)$ s.t. :

$$f(x) = f(c) + \sum_{k=1}^n \frac{f^{(k)}(c)}{k!} (x - c)^k + \frac{f^{(n+1)}(\zeta)}{(n+1)!} (x - c)^{n+1}$$

[Mean value theorem](#) can be derived from Taylor's theorem when $n = 0$.

Proof Hint

$$F(t) = f(t) + \sum_{k=1}^n \frac{f^{(k)}(t)}{k!} (x - t)^k$$

$$G(t) = (x - t)^{n+1}$$

- Define F, G as mentioned above
- Consider the interval $[c, x]$
- Use [Cauchy's mean value theorem](#) for F, G after making sure the conditions are met.

The above equation can be written like:

$$f(x) = T_n(x, c) + R_n(x, c)$$

Taylor Polynomial

This part of the above equation is called the Taylor polynomial. Denoted by $T_n(x, c)$.

$$T_n(x, c) = f(c) + \sum_{k=1}^n \frac{f^{(k)}(c)}{k!} (x - c)^k$$

Remainder

Denoted by $R_n(x, c)$.

$$R_n(x, c) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} (x - c)^{n+1}$$

Integral form of the remainder

$$R_n(x, c) = \frac{1}{n!} \int_c^x f^{(n+1)}(t) (x - t)^n dt$$

Proof Hint

- Method 1: Use integration by parts and mathematical induction.
- Method 2: Use [Generalized MVT for Riemann Integrals](#) where:
 - $F = f^{(n+1)}$
 - $G = (x - t)^n$

Second derivative test

When $n = 1$:

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(\zeta)}{2!}(x - c)^2$$

$$f(x) - \text{Tangent line} = \frac{f''(\zeta)}{2!}(x - c)^2$$

From this: $f''(c) > 0 \implies$ a local minimum is at c . Converse is **not** true.

Sequence

A sequence on a set A is a function $u : \mathbb{Z}^+ \rightarrow A$.

Image of the n is written as u_n . A sequence is indicated by one of these ways:

$$\left\{ u_n \right\}_{n=1}^{\infty} \quad \text{or} \quad \left\{ u_n \right\} \quad \text{or} \quad \left(u_n \right)_{n=1}^{\infty}$$

Increasing or Decreasing

A sequence (u_n) is

- Increasing **iff** $u_n \geq u_m$ for $n > m$
- Decreasing **iff** $u_n \leq u_m$ for $n > m$
- Monotone **iff** either increasing or decreasing
- Strictly increasing **iff** $u_n > u_m$ for $n > m$
- Strictly decreasing **iff** $u_n < u_m$ for $n > m$

Convergence

Converging

A sequence $(u_n)_{n=1}^{\infty}$ is converging (to $L \in \mathbb{R}$) **iff**: $\lim_{n \rightarrow \infty} u_n = L$

$$\forall \epsilon > 0 \exists N \in \mathbb{Z}^+ \forall n (n > N \implies |u_n - L| < \epsilon)$$

Note

$$\forall x \in \mathbb{R} \quad \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

Diverging

A sequence is diverging **iff** it is not converging.

$$\lim_{n \rightarrow \infty} u_n = \begin{cases} \infty \\ -\infty \\ \text{undefined,} \end{cases} \quad \text{when } u_n \text{ is oscillating}$$

Convergence test

All converging sequences are bounded.

Increasing and bounded above

Let (u_n) be increasing and bounded above. Then (u_n) is converging (to $\sup \{u_n\}$).

Proof Hint

- $\{u_n\}$ has a $\sup u_n (= s)$
- Prove: $\lim_{n \rightarrow \infty} u_n = s^-$

Decreasing and bounded below

Let (u_n) be decreasing and bounded below. Then (u_n) is converging (to $\inf \{u_n\}$).

Proof Hint

- $\{u_n\}$ has a $\inf u_n (= l)$
- Prove: $\lim_{n \rightarrow \infty} u_n = l^+$

Newton's method of finding roots

Suppose f is a function. To find its roots:

- Select a point x_0
- Draw a tangent at x_0
- Choose x_1 which is where the tangent meets $y = 0$
- Continue this process repeatedly

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Cauchy Sequence

A sequence $u : \mathbb{Z}^+ \rightarrow A$ is Cauchy **iff**:

$$\forall \epsilon > 0 \exists N \in \mathbb{Z}^+ \forall m, n; m, n > N \implies |u_n - u_m| < \epsilon$$

Complete

A set A is complete **iff**:

$\forall u : \mathbb{Z}^+ \rightarrow A; u \text{ converges to } L \in A$

IMPORTANT: \mathbb{Q} is **not** complete because:

$$\sum_{k=1}^{\infty} \frac{1}{k!} = e - 1 \notin \mathbb{Q}$$

IMPORTANT: \mathbb{R} is complete.

Proof Hint

Proof is quite hard.

Bounded

All Cauchy sequences are bounded. (has an upper bound).

Proof Hint

- Consider the Cauchy definition
- Take $n > m = N + 1 > N$

Subsequence

Suppose $u : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be a sequence and $v : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ be an increasing sequence. Then $u \circ v : \mathbb{Z}^+ \rightarrow \mathbb{R}$ is a subsequence of u .

Existence of subsequence

Every sequence has a monotonic subsequence.

Proof Hint

- Let $n \in \mathbb{Z}^+$ be called “good” **iff** $\forall m > n, u_n > u_m$.
- Suppose u_n has infinitely many “good” points. That implies u_n has a decreasing subsequence.
- Suppose u_n has finitely many “good” points. Let N is the maximum of those.
 $\forall n_1 > N, n_1$ is not ”good” That implies u_n has a increasing subsequence.

Bolzano-Weierstrass

Every bounded sequence on \mathbb{R} has a converging subsequence.

Proof Hint

From the above theorem, there is a monotonic subsequence u_{n_k} which is also bounded. Bounded monotone sequences converge.

Theorem

If u_n is Cauchy and u_{n_k} is a subsequence converging to L , then u_n converges to L .

Note

For a set A , all 3 statements are equivalent:

- A has the [completeness property](#).
- A is [complete](#)
- [Bolzano-Weierstrass theorem](#) on A

Series

Let (u_n) be a sequence, and a series (a new sequence) can be defined from it such that:

$$s_n = \sum_{k=1}^n u_k$$

Convergence

If (s_n) is converging:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n u_k = \sum_{k=1}^{\infty} u_k = S \in \mathbb{R}$$

Absolutely Converging

$\sum_{k=1}^n u_k$ is absolutely converging **iff** $\sum_{k=1}^n |u_k|$ is converging.

$$\sum_{k=1}^n |u_k| \text{ is converging} \implies \sum_{k=1}^n u_k \text{ is converging}$$

Proof Hint

Use this inequality:

$$0 \leq |u_k| - u_k \leq 2|u_k|$$

Conditionally Converging

$\sum_{k=1}^n u_k$ is conditionally converging **iff**:

$$\sum_{k=1}^n |u_k| \text{ is diverging} \quad \text{and} \quad \sum_{k=1}^n u_k \text{ is converging}$$

Theorem 1

$$\sum_{k=1}^n u_k \text{ is converging} \implies \lim_{k \rightarrow \infty} u_k = 0$$

The converse is more useful:

$$\lim_{k \rightarrow \infty} u_k \neq 0 \implies \sum_{k=1}^n u_k \text{ is diverging}$$

Convergence Tests

Direct Comparison Test

Let $0 < u_k < v_k$.

$$\sum_{k=1}^{\infty} v_k \text{ is converges} \implies \sum_{k=1}^{\infty} u_k \text{ is converges}$$

① Proof Hint

- Note that $\sum_{k=1}^n u_k$ and $\sum_{k=1}^n v_k$ are increasing
- Show that $\sum_{k=1}^{\infty} v_k$ converges to its supremum v which is an upper bound of $\sum_{k=1}^n u_k$

① Example

Proving the convergence of $\sum_{k=1}^{\infty} \frac{1}{k!}$, by using $k! \geq 2^{k-1}$ for all $k \geq 0$.

Limit Comparison Test

Let $0 < u_k, v_k$ and $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = R$.

$$R > 0 \implies \left(\sum_{n=1}^{\infty} u_n \text{ is converging} \iff \sum_{n=1}^{\infty} v_n \text{ is converging} \right)$$

$$R = 0 \implies \left(\sum_{n=1}^{\infty} v_n \text{ is converging} \implies \sum_{n=1}^{\infty} u_n \text{ is converging} \right)$$

$$R = \infty \implies \left(\sum_{n=1}^{\infty} v_n \text{ is diverging} \implies \sum_{n=1}^{\infty} u_n \text{ is diverging} \right)$$

Proof Hint

Only possibilities are $R = 0$, $R > 0$, $R = \infty$.

For $R > 0$:

- Consider limit definition with $\epsilon = \frac{L}{2}$
- Direct comparison test can be used for the 2 set of inequalities

For $R = 0$:

- Consider limit definition with $\epsilon = 1$
- Direct comparison test can be used now

For $R = \infty$:

- Consider limit definition with $M = 1$
- Direct comparison test can be used now

Integral Test

Let $u(x) > 0$, decreasing and integrable on $[1, M]$ for all $M > 1$. Then:

$$\sum_{n=1}^{\infty} u_n \text{ is converging} \iff \int_1^{\infty} u(x) \, dx \text{ is converging}$$

Proof Hint

As $u(x)$ is decreasing, it is apparent that it is integrable.

Make use of this inequality:

$$s_n - u_1 \leq \int_1^n u(x) \, dx \leq s_n - u_n$$

For \Leftarrow :

- Note that s_n is increasing
- Show that s_n is bounded above by $\int_1^\infty u(x) \, dx + u_1$

For \Rightarrow :

- Define $F(n) = \int_1^n u(x) \, dx$
- Note that $F(n)$ is increasing
- Note that $\lim_{n \rightarrow \infty} u_n = 0$
- Show that $F(n)$ is bounded above by $\lim_{n \rightarrow \infty} s_n$

Note

$$\sum_{n=1}^{\infty} u_n \text{ is converging} \implies \lim_{k \rightarrow \infty} u_k = 0$$

$$\int_1^{\infty} u(x) \, dx \text{ is converging} \implies \lim_{k \rightarrow \infty} u(k) = 0$$

Ratio Test

Let $u(x) > 0$ and $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = L$.

$$L < 1 \implies \sum_{n=1}^{\infty} u_n \text{ is converging}$$

$$L > 1 \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}$$

① Proof Hint

- Consider the limit definition with $\epsilon = \frac{1}{2}(1 - L)$
- Show that: $\frac{1}{2}(3L - 2) < \frac{u_{k+1}}{u_k} < \frac{1}{2}(1 + L)$
- Use $\sum_{k=1}^{\infty} r^k$ is converging **iff** $r < 1$

Root Test

Let $u(x) > 0$ and $\lim_{n \rightarrow \infty} u_n^{1/n} = L$.

$$L < 1 \implies \sum_{n=1}^{\infty} u_n \text{ is converging}$$

$$(L > 1 \vee L = \infty) \implies \sum_{n=1}^{\infty} u_n \text{ is diverging}$$

① Proof Hint

For $L < 1 \vee L > 1$: Consider the limit definition with $\epsilon = \frac{1}{2}(1 - L)$

For $L = \infty$: Consider the limit definition with $M > 1$

Riemann Zeta Function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

Convergence of this function can be derived using [integral test](#).

This function converges **iff** $s > 1$. And it converges to:

$$\frac{1}{s-1}$$

Otherwise it diverges.

Alternating Series

Suppose $u_k > 0$. An alternating series is:

$$\sum_{k=1}^n (-1)^{k-1} u_k = u_1 - u_2 + u_3 - u_4 + \dots$$

Convergence

If $\forall k \ u_k > 0$, decreasing and $\lim_{n \rightarrow \infty} u_n = 0$. Then

$$\sum_{k=1}^n (-1)^{k-1} u_k \text{ is converging}$$

Proof

For odd-indexed elements:

$$s_{2m+3} \leq s_{2m+1} \leq s_1 = u_1$$

For even-indexed elements:

$$s_{2m+2} \geq s_{2m} \geq s_2 = u_1 - u_2$$

Combining these 2:

$$0 \leq u_1 - u_2 \leq s_2 \leq s_{2m} \leq s_{2m+1} \leq s_1 = u_1$$

s_{2m} is bounded above by u_1 and increasing. s_{2m+1} is bounded below by 0 and decreasing. So both converges.

$$\lim_{m \rightarrow \infty} (s_{2m+1} - s_{2m}) = \lim_{m \rightarrow \infty} u_{2m+1} = 0$$

$$\implies \lim_{m \rightarrow \infty} s_{2m+1} = \lim_{m \rightarrow \infty} s_{2m} = s$$

Both converges to the same number. ∴

Power Series

A series of the form:

$$\sum_{n=0}^{\infty} a_n (x - c)^n$$

Here:

- x - a variable
- c - a constant

Radius of convergence

Maximum radius of x in where the series converges.

$$R = \sup \{r \mid \text{series converges for } |x - c| < r\}$$

Range of convergence

$(c - R, c + R)$ is the range of convergence.

Taylor Series

Let f be infinitely many times differentiable on (a, b) and $c, x \in (a, b)$.

If $\lim_{n \rightarrow \infty} R_n(x) = 0$ for $x \in (c - R, c + R) \subset (a, b)$, then Taylor series of f at c is given by:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$$

Taylor series can be used to define e^x , $\cos x$, $\sin x$, $\ln(1 + x)$ and more.