Computational Fluid Dynamics
http://www.nd.edu/~gtryggva/CFD-Course/

# Vortex Methods

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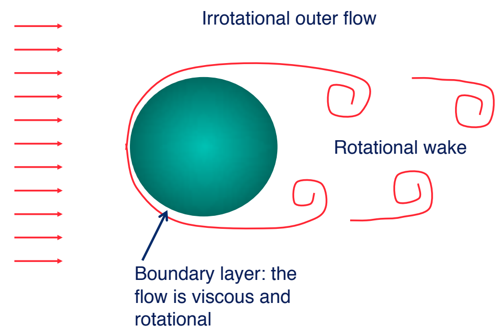
Computational Fluid Dynamics
Flow over a body

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Computational Fluid Dynamics

# Integral Solutions to the Poisson Equation

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Computational Fluid Dynamics

To find a solution to the Poisson equation

$$\nabla^2\phi = \sigma$$

We start by considering a point source at the origin.

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi}{\partial r} = \sigma\delta(r)$$

Except at the origin the RHS is zero so we can integrate

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\phi}{dr} = 0 \Rightarrow d\left(r^2\frac{d\phi}{dr}\right) = 0 \Rightarrow \frac{d\phi}{dr} = \frac{C}{r^2} \Rightarrow \phi = -\frac{C}{r^2}$$

To evaluate the constant we integrate the equation over a small sphere

$$\int_V \nabla^2\phi dv = \oint_S \nabla\phi\cdot\mathbf{n}\,ds = \oint_S \frac{\partial\phi}{\partial n}ds = 4\pi r^2\frac{\partial\phi}{\partial r} = 4\pi r^2\left(\frac{-C}{r^2}\right) = -4\pi C = \sigma$$

The solution is therefore: $\phi = \dfrac{-\sigma}{4\pi r}$

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Computational Fluid Dynamics
Vortex Methods

$$\nabla^2\phi = \sigma$$

To solve the equation with a distributed source we integrate

$$\phi(\boldsymbol{x}) = \frac{-1}{4\pi}\int_A \frac{\sigma(\boldsymbol{x}')}{r}dv'$$

The solution in a three-dimensional unbounded domain

The solution in a two-dimensional unbounded domain is

$$r = |\boldsymbol{x}-\boldsymbol{x}'|$$

$$\phi(\boldsymbol{x}) = \frac{1}{2\pi}\int_A \sigma(\boldsymbol{x}')\ln r da'$$

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Vortex Methods

$\nabla^2\psi = -\omega$ Solution $\psi(\boldsymbol{x}) = \dfrac{-1}{2\pi}\displaystyle\int_A \omega(\boldsymbol{x}')\ln r da'$

Velocity for 2D flow

$$\boldsymbol{u}(\boldsymbol{x}) = \left(-\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial x}\right) = \frac{-1}{2\pi}\int_A \omega(\boldsymbol{x}')\left(-\frac{\partial\ln r}{\partial y},\frac{\partial\ln r}{\partial x}\right)da'$$

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{-1}{2\pi}\int_A \frac{\omega(\boldsymbol{x}')\big(-(y-y'),(x-x')\big)}{r^2}da' = \frac{1}{2\pi}\int_A \frac{\mathbf{k}\times\mathbf{r}}{r^2}\omega(\boldsymbol{x}')da'$$

since

$$\frac{\partial\ln r}{\partial y} = \frac{\partial}{\partial y}\ln\left(\sqrt{(x-x')^2+(y-y')^2}\right) = \frac{(y-y')}{(x-x')^2+(y-y')^2}$$

# Why Vortex Methods?



Helmholtz decomposition:
Any vector field can be written as a sum of

$$\boldsymbol{u} = \nabla\phi + \nabla\times\Psi$$

Take divergence

$$\nabla\cdot\boldsymbol{u} = \nabla\cdot\nabla\phi = \nabla^2\phi = 0$$

Take the curl

$$\nabla\times\boldsymbol{u} = \nabla\times\left(\nabla\times\Psi\right) = \omega$$

By a Gauge transform this can be written as

$$\nabla^2\Psi = -\omega$$



For incompressible flow with constant density and viscosity, taking the curl of the momentum equation yields:

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u}\nabla\cdot\omega = (\omega\cdot\nabla)\boldsymbol{u} + \nu\nabla^2\omega$$

or:

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)\boldsymbol{u} + \nu\nabla^2\omega$$

Helmholtz's theorem:
Inviscid Irrotational flow remains irrotational



In two-dimensions: $\Psi = (0,0,\psi) \quad \omega = (0,0,\omega)$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

or:

$$\frac{D\omega}{Dt} = \nu\nabla^2\omega \qquad \nabla^2\psi = -\omega$$

Zero viscosity: $\frac{D\omega}{Dt} = 0$ The vorticity of a fluid particle does not change!



For inviscid unbounded flows, the motion everywhere, is completely determined by the vorticity. Thus, the evolution of the flow can be predicted by following the motion of the vorticity containing fluid ONLY.

$$\frac{D\omega}{Dt} = 0 \qquad \boldsymbol{u}(\boldsymbol{x}) = \frac{1}{2\pi}\int_A \frac{\mathbf{k}\times\mathbf{r}}{r^2}\omega(\boldsymbol{x}')\,da' \qquad \text{2D}$$

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)\boldsymbol{u} \qquad \boldsymbol{u}(\boldsymbol{x}) = \frac{1}{4\pi}\int_V \frac{\omega\times\mathbf{r}}{r^3}\,dv' \qquad \text{3D}$$



# Point Vortex Methods for 2D Flows

## Computational Fluid Dynamics
## Vortex Methods

Euler Equation for two-dimensional inviscid incompressible flow

$$\frac{d\omega}{dt}=0$$

The vorticity of each material particle is constant

$$\left.\begin{aligned}\nabla^2\psi &= -\omega \\ \boldsymbol{u} &= \nabla\times(\psi\boldsymbol{k})\end{aligned}\right\}\quad \boldsymbol{u}(\boldsymbol{x})=\frac{1}{2\pi}\int_A \boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}')\omega(\boldsymbol{x}')dA'$$

where $\boldsymbol{K}$ is the appropriate velocity kernal

## Computational Fluid Dynamics
## Vortex Methods

The velocity depends only on the vorticity distribution

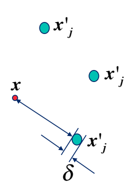
$$\boldsymbol{u}(\boldsymbol{x})=\frac{1}{2\pi}\int_A \boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}')\omega(\boldsymbol{x}')dA'$$

For unbounded domain

$$\boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}')=\frac{\boldsymbol{k}\times(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^2}=\frac{(-(y-y'),(x-x'))}{r^2}$$

## Computational Fluid Dynamics
## Vortex Methods

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x}) &= \frac{1}{2\pi}\int_A \boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}')\omega(\boldsymbol{x}')dA' \\ &= \frac{1}{2\pi}\sum_{j=1}^{N}\int_{A_j}\boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}')\omega(\boldsymbol{x}')dA' \\ &= \frac{1}{2\pi}\sum_{j=1}^{N}\boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}')\int_{A_j}\omega(\boldsymbol{x}')dA' \\ &= \frac{1}{2\pi}\sum_{j=1}^{N}\boldsymbol{K}(\boldsymbol{x},\boldsymbol{x}')\Gamma_j\end{aligned}$$




## Computational Fluid Dynamics
## Vortex Methods

$$\frac{d\boldsymbol{x}_i}{dt}=\frac{1}{2\pi}\sum_{j=1}^{N}\boldsymbol{K}(\boldsymbol{x}_i,\boldsymbol{x}_j)\Gamma_j$$

Point vortices in an unbounded domain. The motion of each point is determined by

$$\frac{d}{dt}(x_i,y_i)=\frac{1}{2\pi}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{\Gamma_j(-(y_i-y_j),(x_i-x_j))}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}$$

## Computational Fluid Dynamics
## Vortex Methods

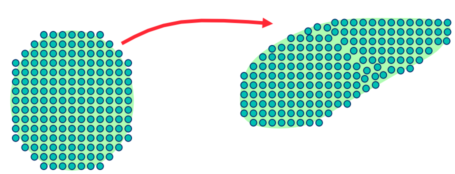
Generally, the point vorticies are too singular to make a practical numerical method and the point vortices must be regularized. The simplest way is to find the velocity by

$$\frac{d}{dt}(x_i,y_i)=\frac{1}{2\pi}\sum_{j=1}^{N}\frac{\Gamma_j(-(y_i-y_j),(x_i-x_j))}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2+\delta^2}}$$

Numerical parameter

## Computational Fluid Dynamics
## Vortex Methods

Use point vortices to approximate a smooth distribution of vorticity



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Vortex Methods

Small viscosity can be added to vortex methods either by "random walk" or localized averaging

In three-dimension it is necessary to account also for stretching and tilting of vortex lines, but the basic methodology still works

Computational Fluid Dynamics

# The $N^2$ Problem

Computational Fluid Dynamics
Vortex Methods

The $N^2$ problem:
To find the velocity of each vortex, it is necessary to sum over all the other vortices. This leads to large computational times when the number of vortices, N, is large

Remedies:
Grid based Vortex-In-Cell methods
Fast summation methods

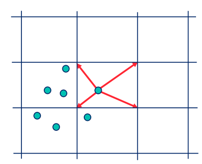
Computational Fluid Dynamics
Vortex Methods

Vortex-In-Cell

Advect point vortices, but solve

$$\nabla^2\psi = -\omega$$

to find the velocities



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Vortex Methods

Fast summation method

A vortex far away from a group of vortices "sees" the group as a single large vortex

By grouping the vortices together in an intelligent way, it is possible to reduce the operation count significantly

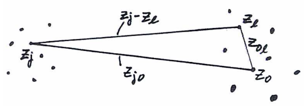
Computational Fluid Dynamics

Multipole expansion

For 2D flow we can rewrite the governing equations using complex numbers:

$$\frac{dx_j}{dt} - i\frac{dy_j}{dt} = q^* = \frac{1}{2\pi i}\sum_{\substack{l=1 \\ j\neq l}}^{N} \frac{\Gamma_l}{z_j - z_l} \qquad z = x + iy$$

Multipole expansion




$$q_j^* = \frac{1}{2\pi i}\sum_{l=1}^{N}\frac{\Gamma_l}{z_j - z_l}$$

$$z_j - z_l = z_j - z_0 + z_0 - z_l = z_{j0} + z_{0l}$$

$$q_j^* = \frac{1}{2\pi i}\sum_{l=1}^{N}\frac{\Gamma_l}{z_{jo} + z_{ol}}$$

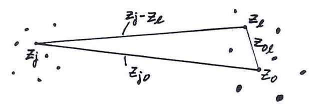
$$\frac{1}{z_{jo} + z_{ol}} = \frac{1}{z_{jo}} - \frac{z_{ol}}{z_{jo}^2} + \frac{z_{ol}^2}{z_{jo}^3} - \frac{z_{ol}^3}{z_{jo}^4} + .......$$

$$q_j^* = \frac{1}{2\pi i}\sum_{l=1}^{N}\Gamma_l\left\{\frac{1}{z_{jo}} - \frac{z_{ol}}{z_{jo}^2} + \frac{z_{ol}^2}{z_{jo}^3} - \frac{z_{ol}^3}{z_{jo}^4} + .......\right\}$$

$$= \frac{1}{2\pi i}\left\{\frac{1}{z_{jo}}\sum_{l=1}^{N}\Gamma_l - \frac{1}{z_{jo}^2}\sum_{l=1}^{N}\Gamma_l z_{ol} + \frac{1}{z_{jo}^3}\sum_{l=1}^{N}\Gamma_l z_{ol}^2 - \frac{1}{z_{jo}^4}\sum_{l=1}^{N}\Gamma_l z_{ol}^3 - .......\right\}$$



Multipole expansion




$$q_j^* = \frac{1}{2\pi i}\left\{\frac{1}{z_{jo}}\sum_{l=1}^{N}\Gamma_l - \frac{1}{z_{jo}^2}\sum_{l=1}^{N}\Gamma_l z_{ol} + \frac{1}{z_{jo}^3}\sum_{l=1}^{N}\Gamma_l z_{ol}^2 - \frac{1}{z_{jo}^4}\sum_{l=1}^{N}\Gamma_l z_{ol}^3 - .......\right\}$$

The first term is the point vortex approximation but higher accuracy can be obtained using more terms.



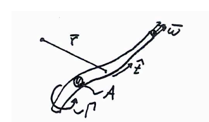
# Vortex Methods for 3D Flows



3D vortex methods

Filament method

For a thin filament:




$$\mathbf{u}(\mathbf{x}) = \frac{-\Gamma}{4\pi}\int_C \frac{\mathbf{r}}{(r^2+\delta^2)^{3/2}} \times \frac{\partial\,\mathbf{x}}{\partial\alpha}\,d\alpha$$

Discretization gives:

$$\mathbf{u}_j = \frac{-\Gamma}{4\pi}\sum_{l=1}^{N}\frac{\mathbf{x}_j - \mathbf{x}_{l+1/2}}{(|\,\mathbf{x}_j - \mathbf{x}_{l+1/2}|^2 + \delta^2)^{3/2}} \times (\mathbf{x}_{l+1} - \mathbf{x}_l)$$



Vorton method

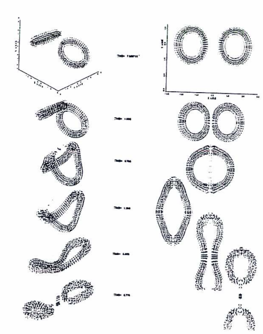
$$\Omega_j = \bar{\omega}_j \cdot \mathrm{Vol}_j$$

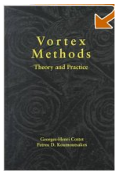
$$\frac{d\mathbf{x}_i}{dt} = \frac{-1}{4\pi}\sum_{j=1}^{N}\frac{|\,\mathbf{x}_i - \mathbf{x}_j|^2 + (5/2)\epsilon^2}{(|\,\mathbf{x}_i - \mathbf{x}_j|^2 + \epsilon^2)^{5/2}}(\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{\Omega}_j$$

$$\frac{d\mathbf{\Omega}_i}{dt} = \frac{1}{4\pi}\sum_{j=1}^{N}\left\{\frac{|\,\mathbf{x}_i - \mathbf{x}_j|^2 + (5/2)\epsilon^2}{(|\,\mathbf{x}_i - \mathbf{x}_j|^2 + \epsilon^2)^{5/2}}\mathbf{\Omega}_i \times \mathbf{\Omega}_j\right.$$
$$+ 3\frac{|\,\mathbf{x}_i - \mathbf{x}_j|^2 + (5/2)\epsilon^2}{(|\,\mathbf{x}_i - \mathbf{x}_j|^2 + \epsilon^2)^{5/2}}\Big(\mathbf{\Omega}_i \cdot ((\mathbf{x}_i - \mathbf{x}_j) \times \mathbf{\Omega}_j)\Big)(\mathbf{x}_i - \mathbf{x}_j)$$
$$\left. + \frac{105\epsilon^4}{\mathrm{Re}}\frac{\mathrm{Vol}_i\mathbf{\Omega}_j - \mathrm{Vol}_j\mathbf{\Omega}_i}{|\,\mathbf{x}_i - \mathbf{x}_j|^2 + \epsilon^2)^{9/2}}\right\}$$



A “vorton” simulation of the collision and reconnection of two vortex rings






For more information:

Vortex Methods: Theory and Applications by
Georges-Henri Cottet & Petros D. Koumoutsakos



# Vortex Sheets



Often the vorticity is concentrated in a thin vortex sheet. Its strength is given by the integral of the vorticity across the sheet:

$$\gamma = \int \omega\, dn = \left(\mathbf{u}_1 - \mathbf{u}_2\right)\cdot \mathbf{s}$$

Its velocity is given by

$$\boldsymbol{u}\left(\boldsymbol{x}\right) = \frac{1}{2\pi}\int_A \frac{\mathbf{k}\times\mathbf{r}}{r^2}\omega(\boldsymbol{x}')\,da' = \frac{1}{2\pi}P\int_S \frac{\mathbf{k}\times\mathbf{r}}{r^2}\gamma(s')\,ds'$$

Where P stands for the “principal value” of the integral.



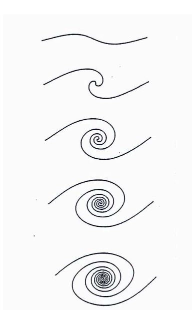
To eliminate the need to consider a principal value integral and to stabilize short waves, the integral is usually regularized by adding a small numerical parameter

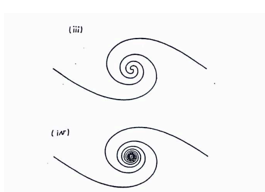
$$\boldsymbol{u}\left(\boldsymbol{x}\right) = \frac{1}{2\pi}\int_S \frac{\mathbf{k}\times\mathbf{r}}{r^2+\delta^2}\gamma(s')\,ds'$$

The sheet is usually discretized by replacing it with a row of point vortices

$$\frac{d}{dt}(x_i, y_i) = \frac{1}{2\pi}\sum_{j=1}^{N}\frac{\Gamma_j(-(y_i - y_j),(x_i - x_j))}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \delta^2}}$$



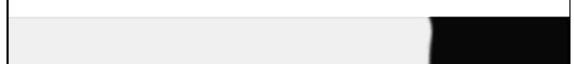





Computation of the rollup of a vortex sheet by a regularized point vortex method. Left: evolution in time. Top: effect of reducing delta.



Example

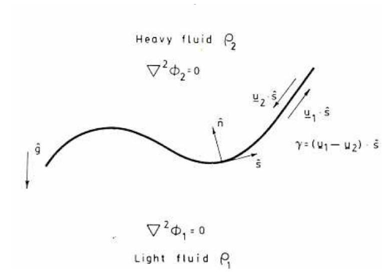


High Reynolds number density current

Computational Fluid Dynamics

# Vortex Methods for Stratified Flows

Computational Fluid Dynamics




Computational Fluid Dynamics

Define the vortex sheet strength, the average velocity and the acceleration following an interface point:

$$\gamma = (\mathbf{u}_1 - \mathbf{u}_2)\cdot\mathbf{s} \qquad \mathbf{U} = \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2) \qquad \frac{D\mathbf{u}_i}{Dt} = \frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{u}_i$$

By subtracting the tangential component of the inviscid Euler equations on either side of the interface

$$\rho_i\left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i\cdot\nabla\mathbf{u}_i\right) = -\nabla p - \rho_i g\mathbf{j}$$

We derive an equation for the evolution of the vortex sheet strength

$$A = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$$

$$\frac{D\gamma}{Dt} + \gamma\frac{\partial \mathbf{U}}{\partial s}\cdot\mathbf{s} = 2A\left\{\frac{D\mathbf{U}}{Dt}\cdot\mathbf{s} + \frac{1}{8}\frac{\partial\gamma^2}{\partial s}\right\} + 2Ag\mathbf{j}\cdot\mathbf{s} - \frac{2}{\rho_1+\rho_2}\frac{\partial(p_1 - p_2)}{\partial s}$$

Computational Fluid Dynamics

The evolution of the vortex sheet strength is found by

$$\frac{D\gamma}{Dt} + \gamma\frac{\partial \mathbf{U}}{\partial s}\cdot\mathbf{s} = 2A\left\{\frac{D\mathbf{U}}{Dt}\cdot\mathbf{s} + \frac{1}{8}\frac{\partial\gamma^2}{\partial s}\right\} + 2Ag\mathbf{j}\cdot\mathbf{s} - \frac{2}{\rho_1+\rho_2}\frac{\partial(p_1 - p_2)}{\partial s}$$

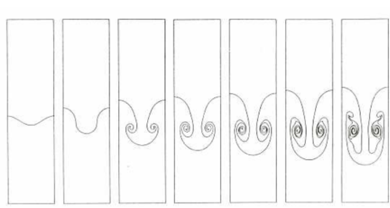
Once the vortex sheet strength is known, the velocity is found by integrating over the sheet

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{2\pi}\int_S \frac{\mathbf{k}\times\mathbf{r}}{r^2+\delta^2}\gamma(s')\,ds'$$

The integration is sometimes replaced by the Vortex in Cell method

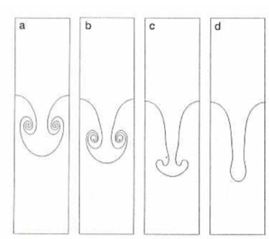
Computational Fluid Dynamics

The Rayleigh Taylor Instability

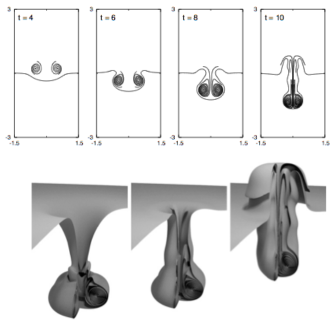


Computational Fluid Dynamics

The Rayleigh Taylor Instability for different density ratios



## Computational Fluid Dynamics




Simulation of a 3D vortex ring colliding with an initially flat interface

## Computational Fluid Dynamics
## Vortex Methods

Related Methods

- Panel and boundary integral method for flow over solid bodies
- Boundary Integral Methods for free surface flows
- Contour dynamics methods for “patches” of constant vorticity

## Computational Fluid Dynamics

There is no doubt that new solution strategies will continue to be developed. However, incremental advances of current approaches are likely to be the main vehicle for future advances in the computations of complex flows. The finite volume approach currently at the core of CFD has, in particular, proven to be exceedingly robust and versatile.