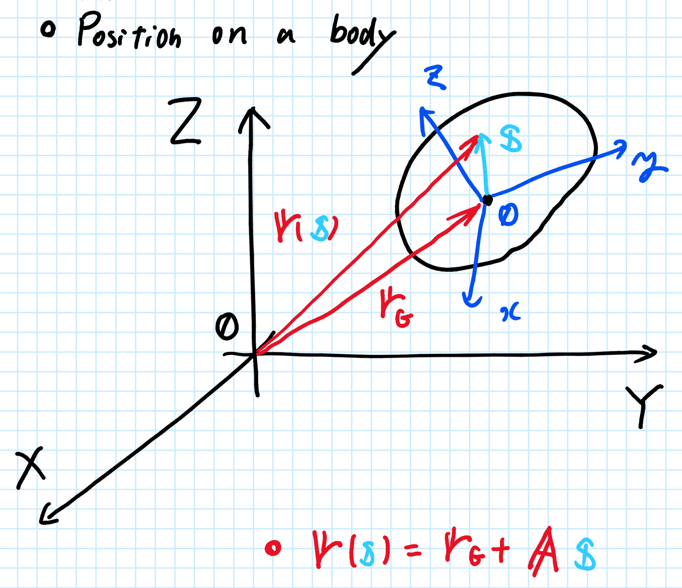
- Position on a body



- $r(s) = r_G + A s$
- $A := (2p_0^2 - 1)I + 2(p^T p + p_0 \tilde{p})$

$p_0 := \cos\frac{\Theta}{2}$,

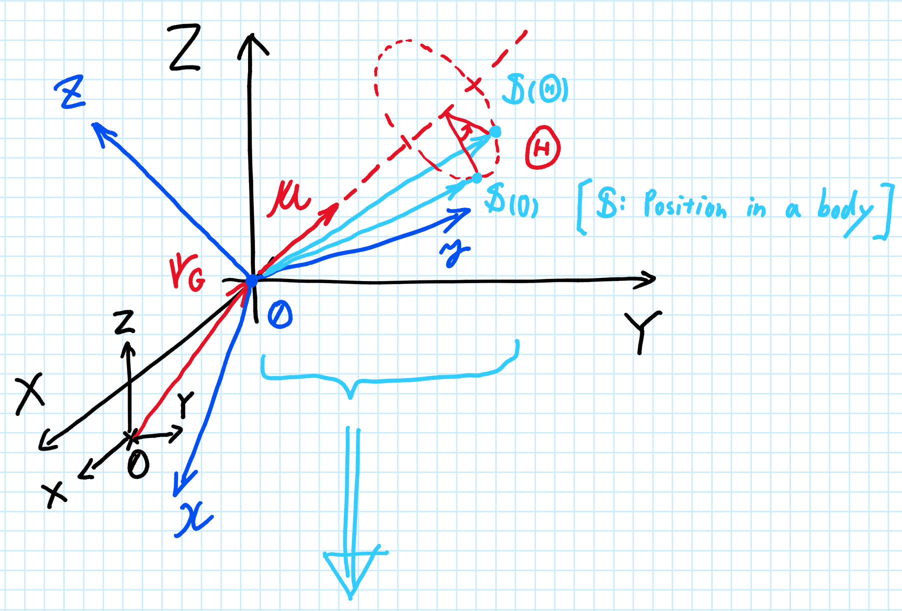
- $p = [p_1\ p_2\ p_3]^T := n \sin\frac{\Theta}{2}$, $\|n\| = 1$,

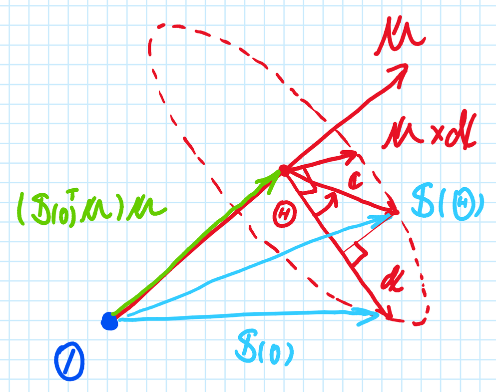
$\left[\Rightarrow p_0^2 + p_1^2 + p_2^2 + p_3^2 = \cos^2\frac{\Theta}{2} + \|n\|^2 \sin^2\frac{\Theta}{2} \equiv 1\right]$

} : Euler parameters

- $\tilde{x} := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \Rightarrow \left[\tilde{x} y = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -x_3 y_2 + x_2 y_3 \\ x_3 y_1 - x_1 y_3 \\ -x_2 y_1 + x_1 y_2 \end{bmatrix} = x \times y \right]$

- Euler parameters





- $s(\Theta) = (s(0)^T n)n + \underbrace{\cos\Theta\, d}_{} + \underbrace{\sin\Theta (n \times d)}_{}$

$\left[\|c\| \cos\Theta \frac{d}{\|d\|} = \|d\| \cos\Theta \frac{d}{\|d\|} \right]$ $\left[\|c\| \sin\Theta \frac{n \times d}{\|n \times d\|} = \|d\| \sin\Theta \frac{n \times d}{\|n\| \cdot \|d\| \sin\frac{\pi}{2}} = \sin\Theta (n \times d) \right]$ $\because\ \|c\| = \|d\|,\ \|n\| = 1.$

- $s(0) = (s(0)^T n)n + d$

$\Rightarrow$ • $d = s(0) - (s(0)^T n)n$

$$\Rightarrow s(\Theta) = (s(0)^T n)n + \cos\Theta\{s(0) - (s(0)^T n)n\} + \sin\Theta\{n \times s(0) - (s(0)^T n)(n \times n)\}$$
$$= (s(0)^T n)n + \{s(0) - (s(0)^T n)n\}\cos\Theta + \{n \times s(0)\}\sin\Theta$$

- $n \times (n \times d) = -\|n\| \cdot \|n \times d\| \frac{d}{\|d\|} = -d$

$\Leftrightarrow d = -n \times (n \times d) = -n \times (\underbrace{n \times s(0)}_{})$

$\left[n \times d = n \times \{s(0) - (s(0)^T n)n\} = n \times s(0) - (s(0)^T n)\, n \times n = n \times s(0) \right]$

$\Leftrightarrow s(0) - (s(0)^T n)n = -n \times (n \times s(0))$

$\Leftrightarrow (s(0)^T n)n = s(0) + n \times (n \times s(0))$

$$\Rightarrow \bullet\ s(\Theta) = (s(0)^T n)n + \{s(0) - (s(0)^T n)n\}\cos\Theta + \{n \times s(0)\}\sin\Theta$$
$$= s(0) + n \times (n \times s(0)) - n \times (n \times s(0))\cos\Theta + \{n \times s(0)\}\sin\Theta$$
$$= s(0) + n \times (n \times s(0))(1 - \cos\Theta) + \{n \times s(0)\}\sin\Theta$$
$$= s(0) + 2\, n \times (n \times s(0)) \sin^2\frac{\Theta}{2} + \{n \times s(0)\}\sin\Theta$$

$\left[1 - \cos\Theta = 1 - \left(\cos^2\frac{\Theta}{2} - \sin^2\frac{\Theta}{2}\right) = \sin^2\frac{\Theta}{2} + \cos^2\frac{\Theta}{2} - \left(\cos^2\frac{\Theta}{2} - \sin^2\frac{\Theta}{2}\right) = 2\sin^2\frac{\Theta}{2} \right]$

$$= s(0) + 2\tilde{n}(n \times s(0))\sin^2\frac{\Theta}{2} + \tilde{n} s(0) \sin\Theta \quad [a \times b = \tilde{a} b]$$
$$= s(0) + 2\tilde{n}\tilde{n} s(0) \sin^2\frac{\Theta}{2} + \tilde{n} s(0) \sin\Theta$$
$$= \left\{I + 2\tilde{n}\tilde{n}\sin^2\frac{\Theta}{2} + \tilde{n}\sin\Theta\right\} s(0)$$
$$= \underbrace{A}_{} s(0),$$

$$A := \left\{I + 2\tilde{n}\tilde{n}\sin^2\frac{\Theta}{2} + \tilde{n}\sin\Theta\right\}.$$
$$= I + 2\tilde{n}\tilde{n}\sin^2\frac{\Theta}{2} + 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}\tilde{n}$$
$$= \boxed{I + 2\sin\frac{\Theta}{2}\tilde{n}\left\{\sin\frac{\Theta}{2}\tilde{n} + \cos\frac{\Theta}{2} I\right\}} \quad \text{— ①}$$

- $$s(\Theta) = (s(0)^T n)n + \{s(0) - (s(0)^T n)n\}\cos\Theta + \{n \times s(0)\}\sin\Theta$$
$$= (n^T s(0))n + \{s(0) - (n^T s(0))n\}\cos\Theta + \tilde{n} s(0)\sin\Theta$$
$$= n(n^T s(0)) + \{s(0) - n(n^T s(0))\}\cos\Theta + \tilde{n} s(0)\sin\Theta$$
$$= \{n n^T + (I - n n^T)\cos\Theta + \tilde{n}\sin\Theta\} s(0)$$

$= A$

$$= \underset{\sim}{\mathbb{A}}\, \mathbb{s}(0),$$

$$\mathbb{A} := \mathbf{n}\mathbf{n}^T + (\mathbb{I} - \mathbf{n}\mathbf{n}^T)\cos\Theta + \tilde{\mathbf{n}}\sin\Theta$$

$$= \cos\Theta\, \mathbb{I} + (1 - \cos\Theta)\mathbf{n}\mathbf{n}^T + \sin\Theta\, \tilde{\mathbf{n}}$$

$$= \underbrace{\left(2\cos^2\frac{\Theta}{2} - 1\right)}_{\left[\begin{aligned}\cos\Theta &= \cos^2\frac{\Theta}{2} - \sin^2\frac{\Theta}{2}\\ &= 2\cos^2\frac{\Theta}{2} - 1\end{aligned}\right]}\mathbb{I} + \underbrace{2\sin^2\frac{\Theta}{2}}_{\left[\begin{aligned}\cos\Theta &= \cos^2\frac{\Theta}{2} - \sin^2\frac{\Theta}{2}\\ &= 1 - 2\sin^2\frac{\Theta}{2}\end{aligned}\right]}\mathbf{n}\mathbf{n}^T + 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}\tilde{\mathbf{n}}$$

$$= \boxed{\left(2\cos^2\frac{\Theta}{2} - 1\right)\mathbb{I} + 2\left\{\sin^2\frac{\Theta}{2}\, \mathbf{n}\mathbf{n}^T + \cos\frac{\Theta}{2}\sin\frac{\Theta}{2}\tilde{\mathbf{n}}\right\}} \quad \text{— ②}$$

- $p_0 := \cos\dfrac{\Theta}{2}, \quad \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \mathbf{n}\sin\dfrac{\Theta}{2}$

$$② \Rightarrow \mathbb{A} = (2p_0^2 - 1)\mathbb{I} + 2\left(\mathbf{p}\mathbf{p}^T + p_0 \underset{\sim}{\tilde{\mathbf{p}}}\right)$$

$$\left[\begin{aligned}\sin\frac{\Theta}{2}\tilde{\mathbf{n}} &= \sin\frac{\Theta}{2}\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}\\ &= \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} = \tilde{\mathbf{p}}\end{aligned}\right]$$

$$① \Rightarrow \mathbb{A} = \mathbb{I} + 2\tilde{\mathbf{p}}(\tilde{\mathbf{p}} + p_0\mathbb{I})$$

- Decomposition matrix for rotational matrix $\mathbb{A}$

$$\mathbb{A} = \mathbb{I} + 2\tilde{\mathbf{p}}(\tilde{\mathbf{p}} + p_0\mathbb{I})$$

$$= (1 - p_0^2)\mathbb{I} + p_0^2\mathbb{I} + 2\tilde{\mathbf{p}}(\tilde{\mathbf{p}} + p_0\mathbb{I})$$

$$= (p_1^2 + p_2^2 + p_3^2)\mathbb{I} + p_0^2\mathbb{I} + 2\tilde{\mathbf{p}}(\tilde{\mathbf{p}} + p_0\mathbb{I})$$

$$= (p_1^2 + p_2^2 + p_3^2)\mathbb{I} + p_0^2\mathbb{I} + 2p_0\tilde{\mathbf{p}} + 2\tilde{\mathbf{p}}\tilde{\mathbf{p}}$$

$$= \mathbf{p}\mathbf{p}^T + \begin{bmatrix} p_2^2 + p_3^2 & -p_1p_2 & -p_1p_3 \\ & p_1^2 + p_3^2 & -p_2p_3 \\ * & & p_1^2 + p_2^2 \end{bmatrix} + \left(p_0^2\mathbb{I} + p_0\tilde{\mathbf{p}} - p_0\tilde{\mathbf{p}}^T - \tilde{\mathbf{p}}\tilde{\mathbf{p}}^T\right) + \tilde{\mathbf{p}}\tilde{\mathbf{p}}$$

$$= \mathbf{p}\mathbf{p}^T - \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix} + (p_0\mathbb{I} + \tilde{\mathbf{p}})(p_0\mathbb{I} - \tilde{\mathbf{p}}^T) + \tilde{\mathbf{p}}\tilde{\mathbf{p}}$$

$$= \mathbf{p}\mathbf{p}^T - \tilde{\mathbf{p}}\tilde{\mathbf{p}} + (p_0\mathbb{I} + \tilde{\mathbf{p}})(p_0\mathbb{I} - \tilde{\mathbf{p}}^T) + \tilde{\mathbf{p}}\tilde{\mathbf{p}}$$

$$= [-\mathbf{p} \quad p_0\mathbb{I} + \tilde{\mathbf{p}}]\begin{bmatrix} -\mathbf{p}^T \\ p_0\mathbb{I} - \tilde{\mathbf{p}}^T \end{bmatrix} =: \mathbb{E}\mathbb{L}^T,$$

$$\boxed{\begin{aligned} &\mathbb{A} = \mathbb{E}\mathbb{L}^T, \\ &\left.\begin{aligned} \mathbb{E} &:= [-\mathbf{p} \quad p_0\mathbb{I} + \tilde{\mathbf{p}}], \\ \mathbb{L} &:= [-\mathbf{p} \quad p_0\mathbb{I} - \tilde{\mathbf{p}}]. \end{aligned}\right\} \end{aligned}}$$

- $\mathbb{E}\mathfrak{e} = [-\mathbf{p} \quad p_0\mathbb{I} + \tilde{\mathbf{p}}]\begin{bmatrix} p_0 \\ \mathbf{p} \end{bmatrix} = -p_0\mathbf{p} + p_0\mathbf{p} + \tilde{\mathbf{p}}\mathbf{p}$

$$= \tilde{\mathbf{p}}\mathbf{p} = \mathbf{p} \times \mathbf{p} = \mathbf{0},$$

- $\mathbb{L}\mathfrak{e} = [-\mathbf{p} \quad p_0\mathbb{I} - \tilde{\mathbf{p}}]\begin{bmatrix} p_0 \\ \mathbf{p} \end{bmatrix} = -p_0\mathbf{p} + p_0\mathbf{p} - \tilde{\mathbf{p}}\mathbf{p}$

$$= -\mathbf{p} \times \mathbf{p} = \mathbf{0},$$

- $\mathbb{E}\mathbb{E}^T = [-\mathbf{p} \quad p_0\mathbb{I} + \tilde{\mathbf{p}}]\begin{bmatrix} -\mathbf{p}^T \\ p_0\mathbb{I} + \tilde{\mathbf{p}}^T \end{bmatrix} = \mathbf{p}\mathbf{p}^T + p_0^2\mathbb{I} + p_0\tilde{\mathbf{p}} + \underbrace{p_0\tilde{\mathbf{p}}^T}_{[= -p_0\tilde{\mathbf{p}}]} + \underbrace{\tilde{\mathbf{p}}\tilde{\mathbf{p}}^T}_{[= -\tilde{\mathbf{p}}\tilde{\mathbf{p}}]}$

$$= \mathbf{p}\mathbf{p}^T - \tilde{\mathbf{p}}\tilde{\mathbf{p}} + p_0^2\mathbb{I}$$

$$= (p_1^2 + p_2^2 + p_3^2)\mathbb{I} + p_0^2\mathbb{I} = \mathbb{I},$$

- $\mathbb{L}\mathbb{L}^T = [-\mathbf{p} \quad p_0\mathbb{I} - \tilde{\mathbf{p}}]\begin{bmatrix} -\mathbf{p}^T \\ p_0\mathbb{I} - \tilde{\mathbf{p}}^T \end{bmatrix} = \mathbf{p}\mathbf{p}^T + p_0^2\mathbb{I} - p_0\tilde{\mathbf{p}} - \underbrace{p_0\tilde{\mathbf{p}}^T}_{[= -p_0\tilde{\mathbf{p}}]} + \underbrace{\tilde{\mathbf{p}}\tilde{\mathbf{p}}^T}_{[= -\tilde{\mathbf{p}}\tilde{\mathbf{p}}]}$

$$= \mathbb{I},$$

$$\Big\} \quad \mathbb{E}\mathbb{E}^T = \mathbb{L}\mathbb{L}^T = \mathbb{I}.$$

- $dt(\mathbb{E}\mathfrak{e}) = (dt\,\mathbb{E})\mathfrak{e} + \mathbb{E}\, dt\,\mathfrak{e} = \mathbf{0}$

$$\Leftrightarrow (dt\,\mathbb{E})\mathfrak{e} = -\mathbb{E}\, dt\,\mathfrak{e},$$

- $dt(\mathbb{L}\mathfrak{e}) = \mathbf{0}$

$$\Leftrightarrow (dt\,\mathbb{L})\mathfrak{e} = -\mathbb{L}\, dt\,\mathfrak{e},$$

$$\circ\ \mathbb{E}^T\mathbb{E} = \begin{bmatrix} -\boldsymbol{\varepsilon}^T \\ \varepsilon_0\mathbb{I} + \tilde{\boldsymbol{\varepsilon}}^T \end{bmatrix} \begin{bmatrix} -\boldsymbol{\varepsilon} & \varepsilon_0\mathbb{I} + \tilde{\boldsymbol{\varepsilon}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} & -\varepsilon_0\boldsymbol{\varepsilon}^T - \boldsymbol{\varepsilon}^T\tilde{\boldsymbol{\varepsilon}} \\ -\varepsilon_0\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}^T\boldsymbol{\varepsilon} & \varepsilon_0^2\mathbb{I} + \varepsilon_0\tilde{\boldsymbol{\varepsilon}}^T + \varepsilon_0\tilde{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\varepsilon}}^T\tilde{\boldsymbol{\varepsilon}} \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2 & -\varepsilon_0\boldsymbol{\varepsilon}^T - \boldsymbol{\varepsilon}^T\tilde{\boldsymbol{\varepsilon}} \\ -\varepsilon_0\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}^T\boldsymbol{\varepsilon} & \varepsilon_0^2\mathbb{I} - \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}} \end{bmatrix} = \begin{bmatrix} (1-\varepsilon_0^2) & -\varepsilon_0\boldsymbol{\varepsilon}^T - \boldsymbol{\varepsilon}^T\tilde{\boldsymbol{\varepsilon}} \\ -\varepsilon_0\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}^T\boldsymbol{\varepsilon} & \varepsilon_0^2\mathbb{I} - \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}} \end{bmatrix}$$

$$= \mathbb{I} - \begin{bmatrix} \varepsilon_0^2 & \varepsilon_0\boldsymbol{\varepsilon}^T + \boldsymbol{\varepsilon}^T\tilde{\boldsymbol{\varepsilon}} \\ \varepsilon_0\boldsymbol{\varepsilon} + \tilde{\boldsymbol{\varepsilon}}^T\boldsymbol{\varepsilon} & (1-\varepsilon_0^2)\mathbb{I} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}} \end{bmatrix} = \mathbb{I} - \begin{bmatrix} \varepsilon_0^2 & \varepsilon_0\boldsymbol{\varepsilon}^T + (\tilde{\boldsymbol{\varepsilon}}^T\boldsymbol{\varepsilon})^T \\ \varepsilon_0\boldsymbol{\varepsilon} + \tilde{\boldsymbol{\varepsilon}}^T\boldsymbol{\varepsilon} & (1-\varepsilon_0^2)\mathbb{I} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}} \end{bmatrix}$$

$$= \mathbb{I} - \begin{bmatrix} \varepsilon_0^2 & \varepsilon_0\boldsymbol{\varepsilon}^T - (\tilde{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon})^T \\ \varepsilon_0\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon} & (1-\varepsilon_0^2)\mathbb{I} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}} \end{bmatrix} = \mathbb{I} - \begin{bmatrix} \varepsilon_0^2 & \varepsilon_0\boldsymbol{\varepsilon}^T - (\boldsymbol{\varepsilon}\times\boldsymbol{\varepsilon})^T \\ \varepsilon_0\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\times\boldsymbol{\varepsilon} & \underbrace{(1-\varepsilon_0^2)\mathbb{I} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}}}_{\left[\begin{array}{l}(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2)\mathbb{I} = \boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T - \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}} \\ \Leftrightarrow (1-\varepsilon_0^2)\mathbb{I} + \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T\end{array}\right]} \end{bmatrix} = \mathbb{I} - \begin{bmatrix} \varepsilon_0^2 & \varepsilon_0\boldsymbol{\varepsilon}^T \\ \varepsilon_0\boldsymbol{\varepsilon} & \boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^T \end{bmatrix}$$

$$= \mathbb{I} - \mathcal{E}\mathcal{E}^T,$$

$$\circ\ \mathbb{L}^T\mathbb{L} = \begin{bmatrix} -\boldsymbol{\varepsilon}^T \\ \varepsilon_0\mathbb{I} - \tilde{\boldsymbol{\varepsilon}}^T \end{bmatrix} \begin{bmatrix} -\boldsymbol{\varepsilon} & \varepsilon_0\mathbb{I} - \tilde{\boldsymbol{\varepsilon}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} & -\varepsilon_0\boldsymbol{\varepsilon}^T - \boldsymbol{\varepsilon}^T\tilde{\boldsymbol{\varepsilon}} \\ -\varepsilon_0\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}^T\boldsymbol{\varepsilon} & \varepsilon_0^2\mathbb{I} - \varepsilon_0\tilde{\boldsymbol{\varepsilon}}^T - \varepsilon_0\tilde{\boldsymbol{\varepsilon}} + \tilde{\boldsymbol{\varepsilon}}^T\tilde{\boldsymbol{\varepsilon}} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon} & -\varepsilon_0\boldsymbol{\varepsilon}^T - \boldsymbol{\varepsilon}^T\tilde{\boldsymbol{\varepsilon}} \\ -\varepsilon_0\boldsymbol{\varepsilon} - \tilde{\boldsymbol{\varepsilon}}^T\boldsymbol{\varepsilon} & \varepsilon_0^2\mathbb{I} - \tilde{\boldsymbol{\varepsilon}}\tilde{\boldsymbol{\varepsilon}} \end{bmatrix} = \mathbb{I} - \mathcal{E}\mathcal{E}^T,$$

$$\left.\right\} \quad \mathbb{E}^T\mathbb{E} = \mathbb{L}^T\mathbb{L} = \mathbb{I} - \mathcal{E}\mathcal{E}^T,$$

$$\circ\ \mathbb{E}\, d_t\mathbb{L}^T = \begin{bmatrix} -\boldsymbol{\varepsilon} & \varepsilon_0\mathbb{I} + \tilde{\boldsymbol{\varepsilon}} \end{bmatrix} \begin{bmatrix} -\dot{\boldsymbol{\varepsilon}}^T \\ \dot{\varepsilon}_0\mathbb{I} - \dot{\tilde{\boldsymbol{\varepsilon}}}^T \end{bmatrix} = \boldsymbol{\varepsilon}\dot{\boldsymbol{\varepsilon}}^T + \varepsilon_0\dot{\varepsilon}_0\mathbb{I} + \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}} - \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}}^T - \tilde{\boldsymbol{\varepsilon}}\dot{\tilde{\boldsymbol{\varepsilon}}}^T$$

$$= \boldsymbol{\varepsilon}\dot{\boldsymbol{\varepsilon}}^T + \tilde{\boldsymbol{\varepsilon}}\dot{\tilde{\boldsymbol{\varepsilon}}} + \dot{\varepsilon}_0\varepsilon_0\mathbb{I} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}}^T + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}}$$

$$= \begin{bmatrix} \varepsilon_1\dot{\varepsilon}_1 & \varepsilon_1\dot{\varepsilon}_2 & \varepsilon_1\dot{\varepsilon}_3 \\ \varepsilon_2\dot{\varepsilon}_1 & \varepsilon_2\dot{\varepsilon}_2 & \varepsilon_2\dot{\varepsilon}_3 \\ \varepsilon_3\dot{\varepsilon}_1 & \varepsilon_3\dot{\varepsilon}_2 & \varepsilon_3\dot{\varepsilon}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\dot{\varepsilon}_3 & \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 & 0 & -\dot{\varepsilon}_1 \\ -\dot{\varepsilon}_2 & \dot{\varepsilon}_1 & 0 \end{bmatrix} + \dot{\varepsilon}_0\varepsilon_0\mathbb{I} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}}^T + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}}$$

$$= \begin{bmatrix} \varepsilon_1\dot{\varepsilon}_1 & \varepsilon_1\dot{\varepsilon}_2 & \varepsilon_1\dot{\varepsilon}_3 \\ \varepsilon_2\dot{\varepsilon}_1 & \varepsilon_2\dot{\varepsilon}_2 & \varepsilon_2\dot{\varepsilon}_3 \\ \varepsilon_3\dot{\varepsilon}_1 & \varepsilon_3\dot{\varepsilon}_2 & \varepsilon_3\dot{\varepsilon}_3 \end{bmatrix} + \begin{bmatrix} -\varepsilon_3\dot{\varepsilon}_3 - \varepsilon_2\dot{\varepsilon}_2 & \dot{\varepsilon}_1\varepsilon_2 & \dot{\varepsilon}_1\varepsilon_3 \\ \dot{\varepsilon}_2\varepsilon_1 & -\varepsilon_1\dot{\varepsilon}_1 - \varepsilon_3\dot{\varepsilon}_3 & \dot{\varepsilon}_2\varepsilon_3 \\ \dot{\varepsilon}_3\varepsilon_1 & \dot{\varepsilon}_3\varepsilon_2 & -\varepsilon_1\dot{\varepsilon}_1 - \varepsilon_2\dot{\varepsilon}_2 \end{bmatrix} + \dot{\varepsilon}_0\varepsilon_0\mathbb{I} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}}^T + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}}$$

$$= \begin{bmatrix} \dot{\varepsilon}_1\varepsilon_1 & \dot{\varepsilon}_1\varepsilon_2 & \dot{\varepsilon}_1\varepsilon_3 \\ \dot{\varepsilon}_2\varepsilon_1 & \dot{\varepsilon}_2\varepsilon_2 & \dot{\varepsilon}_2\varepsilon_3 \\ \dot{\varepsilon}_3\varepsilon_1 & \dot{\varepsilon}_3\varepsilon_2 & \dot{\varepsilon}_3\varepsilon_3 \end{bmatrix} + \begin{bmatrix} -\varepsilon_3\dot{\varepsilon}_3 - \varepsilon_2\dot{\varepsilon}_2 & \varepsilon_1\dot{\varepsilon}_2 & \varepsilon_1\dot{\varepsilon}_3 \\ \varepsilon_2\dot{\varepsilon}_1 & -\varepsilon_1\dot{\varepsilon}_1 - \varepsilon_3\dot{\varepsilon}_3 & \varepsilon_2\dot{\varepsilon}_3 \\ \varepsilon_3\dot{\varepsilon}_1 & \varepsilon_3\dot{\varepsilon}_2 & -\varepsilon_1\dot{\varepsilon}_1 - \varepsilon_2\dot{\varepsilon}_2 \end{bmatrix} + \dot{\varepsilon}_0\varepsilon_0\mathbb{I} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}}^T + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}}$$

$$= \dot{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}^T + \dot{\varepsilon}_0\varepsilon_0\mathbb{I} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}}^T + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}} + \dot{\tilde{\boldsymbol{\varepsilon}}}\tilde{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}^T + \dot{\varepsilon}_0\varepsilon_0\mathbb{I} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}}^T + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}} - \dot{\tilde{\boldsymbol{\varepsilon}}}\tilde{\boldsymbol{\varepsilon}}^T$$

$$= \dot{\boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}^T + (\dot{\varepsilon}_0\mathbb{I} + \dot{\tilde{\boldsymbol{\varepsilon}}})(\varepsilon_0\mathbb{I} - \tilde{\boldsymbol{\varepsilon}}^T) = \begin{bmatrix} -\dot{\boldsymbol{\varepsilon}} & \dot{\varepsilon}_0\mathbb{I} + \dot{\tilde{\boldsymbol{\varepsilon}}} \end{bmatrix} \begin{bmatrix} -\boldsymbol{\varepsilon}^T \\ \varepsilon_0\mathbb{I} - \tilde{\boldsymbol{\varepsilon}}^T \end{bmatrix} = (d_t\mathbb{E})\mathbb{L}^T$$

$$\Leftrightarrow\ \mathbb{E}(d_t\mathbb{L})^T = (d_t\mathbb{E})\mathbb{L}^T,$$

$$\circ\ d_t\mathbb{A} = d_t(\mathbb{E}\mathbb{L}^T) = (d_t\mathbb{E})\mathbb{L}^T + \mathbb{E}(d_t\mathbb{L})^T$$

$$= 2\,\mathbb{E}(d_t\mathbb{L})^T,$$

$$\circ\ \mathcal{E}^T\mathcal{E} = \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = 1 \ \Leftrightarrow\ 2(d_\tau\mathcal{E})^T\mathcal{E} = 0$$

$$\Leftrightarrow\ (d_\tau\mathcal{E})^T\mathcal{E} = \mathcal{E}^T d_\tau\mathcal{E} = 0,$$

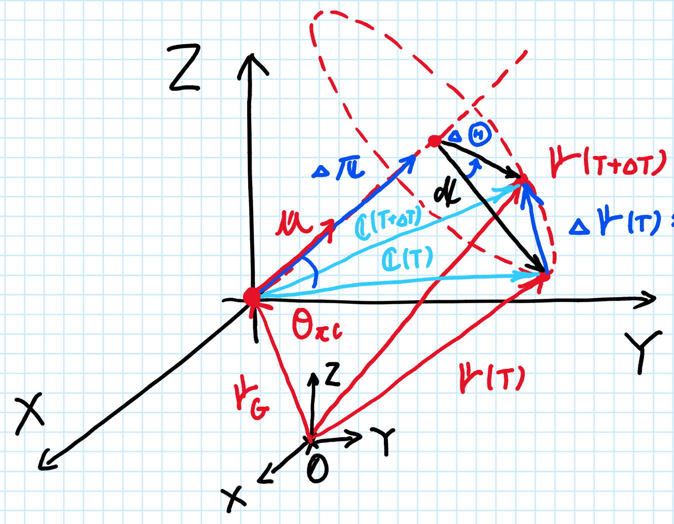
$$\circ\ \mathbb{L}(d_\tau\mathbb{L})^T = \begin{bmatrix} -\boldsymbol{\varepsilon} & \varepsilon_0\mathbb{I} - \tilde{\boldsymbol{\varepsilon}} \end{bmatrix} \begin{bmatrix} -\dot{\boldsymbol{\varepsilon}}^T \\ \dot{\varepsilon}_0\mathbb{I} - \dot{\tilde{\boldsymbol{\varepsilon}}}^T \end{bmatrix} = \boldsymbol{\varepsilon}\dot{\boldsymbol{\varepsilon}}^T + \varepsilon_0\dot{\varepsilon}_0\mathbb{I} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}} - \varepsilon_0\underbrace{\dot{\tilde{\boldsymbol{\varepsilon}}}^T}_{\left[\dot{\tilde{\boldsymbol{\varepsilon}}}^T = -\dot{\tilde{\boldsymbol{\varepsilon}}}\right]} + \tilde{\boldsymbol{\varepsilon}}\underbrace{\dot{\tilde{\boldsymbol{\varepsilon}}}^T} = \boldsymbol{\varepsilon}\dot{\boldsymbol{\varepsilon}}^T + \varepsilon_0\dot{\varepsilon}_0\mathbb{I} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}} + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}} - \tilde{\boldsymbol{\varepsilon}}\dot{\tilde{\boldsymbol{\varepsilon}}}$$

$$= \begin{bmatrix} \varepsilon_1\dot{\varepsilon}_1 + \varepsilon_0\dot{\varepsilon}_0 & \varepsilon_1\dot{\varepsilon}_2 & \varepsilon_1\dot{\varepsilon}_3 \\ \varepsilon_2\dot{\varepsilon}_1 & \varepsilon_2\dot{\varepsilon}_2 + \varepsilon_0\dot{\varepsilon}_0 & \varepsilon_2\dot{\varepsilon}_3 \\ \varepsilon_3\dot{\varepsilon}_1 & \varepsilon_3\dot{\varepsilon}_2 & \varepsilon_3\dot{\varepsilon}_3 + \varepsilon_0\dot{\varepsilon}_0 \end{bmatrix} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}} + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}} - \tilde{\boldsymbol{\varepsilon}}\dot{\tilde{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \underbrace{-\varepsilon_2\dot{\varepsilon}_2 - \varepsilon_3\dot{\varepsilon}_3} & \varepsilon_1\dot{\varepsilon}_2 & \varepsilon_1\dot{\varepsilon}_3 \\ \varepsilon_2\dot{\varepsilon}_1 & \underbrace{-\varepsilon_1\dot{\varepsilon}_1 - \varepsilon_3\dot{\varepsilon}_3} & \varepsilon_2\dot{\varepsilon}_3 \\ \varepsilon_3\dot{\varepsilon}_1 & \varepsilon_3\dot{\varepsilon}_2 & \underbrace{-\varepsilon_1\dot{\varepsilon}_1 - \varepsilon_2\dot{\varepsilon}_2}_{\left[(d_t\mathcal{E})^T\mathcal{E} = \dot{\varepsilon}_0\varepsilon_0 + \dot{\varepsilon}_1\varepsilon_1 + \dot{\varepsilon}_2\varepsilon_2 + \dot{\varepsilon}_3\varepsilon_3 = 0\right]} \end{bmatrix} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}} + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}} - \tilde{\boldsymbol{\varepsilon}}\dot{\tilde{\boldsymbol{\varepsilon}}}$$

$$= \begin{bmatrix} 0 & -\dot{\varepsilon}_3 & \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 & 0 & -\dot{\varepsilon}_1 \\ -\dot{\varepsilon}_2 & \dot{\varepsilon}_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{bmatrix} - \dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}} + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}} - \tilde{\boldsymbol{\varepsilon}}\dot{\tilde{\boldsymbol{\varepsilon}}} = -\dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}} + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}} - (\tilde{\boldsymbol{\varepsilon}}\dot{\tilde{\boldsymbol{\varepsilon}}} - \dot{\tilde{\boldsymbol{\varepsilon}}}\tilde{\boldsymbol{\varepsilon}}) = -\dot{\varepsilon}_0\tilde{\boldsymbol{\varepsilon}} + \varepsilon_0\dot{\tilde{\boldsymbol{\varepsilon}}} - \underbrace{(\tilde{\boldsymbol{\varepsilon}}\dot{\boldsymbol{\varepsilon}})^\sim}_{\left[\begin{array}{l}(\tilde{\mathbf{a}}\mathbf{b})^\sim\mathbf{c} = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} \\ \quad = (\mathbf{a}^T\mathbf{c})\mathbf{b} - (\mathbf{b}^T\mathbf{c})\mathbf{a} \\ \quad = (\mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T)\mathbf{c} \\ \Leftrightarrow (\tilde{\mathbf{a}}\mathbf{b})^\sim = \mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T \\ \quad = \begin{bmatrix} 0 & b_1a_2 - a_1b_2 & b_1a_3 - a_1b_3 \\ b_2a_1 - a_2b_1 & 0 & b_2a_3 - a_2b_3 \\ b_3a_1 - a_3b_1 & b_3a_2 - a_3b_2 & 0 \end{bmatrix} \\ \quad = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} = \tilde{\mathbf{a}}\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\tilde{\mathbf{a}}\end{array}\right]} = (-\dot{\varepsilon}_0\boldsymbol{\varepsilon} + \varepsilon_0\dot{\boldsymbol{\varepsilon}} - \tilde{\boldsymbol{\varepsilon}}\dot{\boldsymbol{\varepsilon}})^\sim$$

$$= \left(\begin{bmatrix} -\boldsymbol{\varepsilon} & \varepsilon_0\mathbb{I} - \tilde{\boldsymbol{\varepsilon}} \end{bmatrix}\begin{bmatrix} \dot{\varepsilon}_0 \\ \dot{\boldsymbol{\varepsilon}} \end{bmatrix}\right)^\sim = (\mathbb{L}\, d_\tau\mathcal{E})^\sim.$$

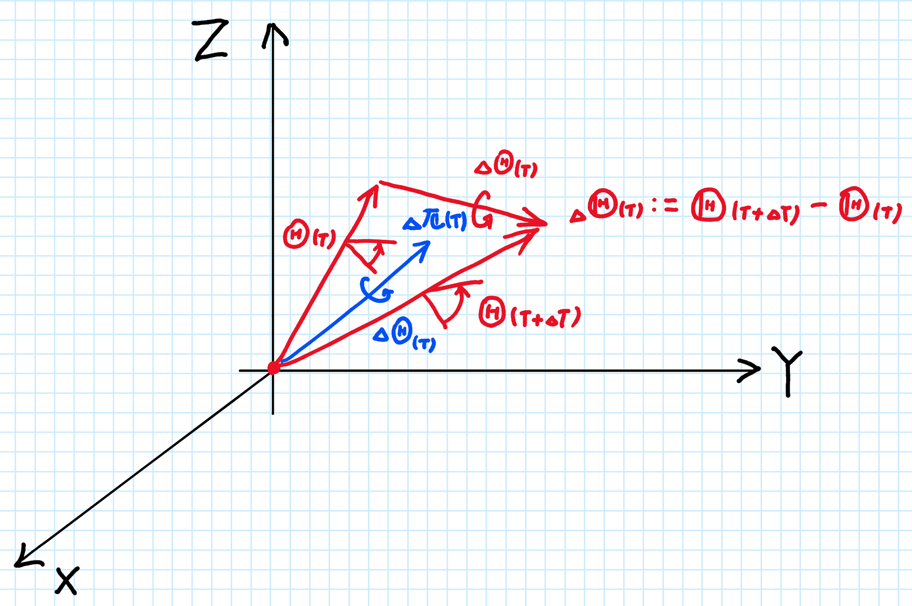
○ *Angular velocity*



$\Delta \mathbf{r}(T) := \mathbf{r}(T+\Delta T) - \mathbf{r}(T)$
$= \underbrace{\Delta \pi(T) \times \mathbf{c}(T)} + \Delta \mathbf{r}_G(T)$

$\left[\|\Delta \mathbf{r}\| = \Delta\Theta \, \|\mathbf{c}(T)\| \sin\theta_{\pi c} = \Delta\Theta \|\mathbf{d}\| \right]$

※ $\circ\ \Delta\Theta := \Theta(T+\Delta T) - \Theta(T)$
$\neq \Delta\pi(T)$



$\circ\ \mathbf{r}(T+\Delta T) = \mathbf{r}(T) + \Delta\mathbf{r}(T)$

$= \mathbf{r}(T) + \{\Delta\pi(T) \times \mathbf{c}(T) + \Delta\mathbf{r}_G(T)\}$

$$\Rightarrow \lim_{\Delta T\to 0} \frac{\Delta \mathbf{r}(T)}{\Delta T} = \lim_{\Delta T \to 0} \frac{\mathbf{r}(T+\Delta T) - \mathbf{r}(T)}{\Delta T}$$

$$= d_T \mathbf{r}(T)$$

$$\left[\text{※ } \mathbf{n}(T) = \text{const.} \Rightarrow \omega(T) := \lim_{\Delta T\to 0} \frac{\Delta\pi(T)}{\Delta T} = \lim_{\Delta T\to 0}\frac{\Delta\Theta(T)}{\Delta T}\mathbf{n} = \lim_{\Delta T\to 0} \frac{\Theta(T+\Delta T) - \Theta(T)}{\Delta T}\mathbf{n} = d_T\Theta(T)\,\mathbf{n} =: \omega(T)\mathbf{n}, \quad \omega := d_T\Theta. \right]$$

$$\Rightarrow d_T \mathbf{r} = \lim_{\Delta T\to 0} \frac{\Delta\mathbf{r}}{\Delta T} = \lim_{\Delta T\to 0}\left\{ \frac{\Delta\pi(T)\times\mathbf{c}(T)}{\Delta T} + \frac{\Delta\mathbf{r}_G(T)}{\Delta T} \right\} = \left(\lim_{\Delta T\to 0} \frac{\Delta\pi}{\Delta T}\right)\times\mathbf{c}(T) + \lim_{\Delta T \to 0}\frac{\mathbf{r}_G(T+\Delta T) - \mathbf{r}_G(T)}{\Delta T}$$

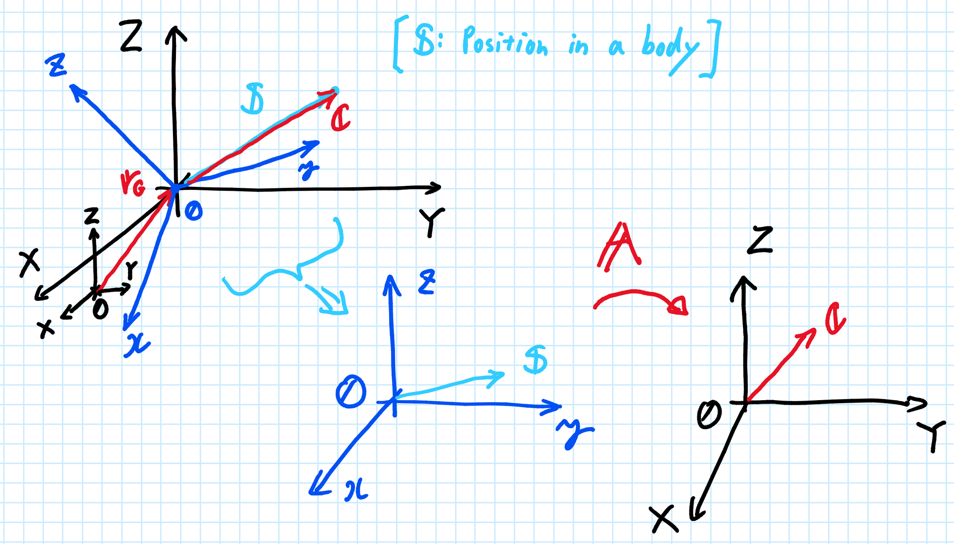
$$= \omega(T)\times\mathbf{c}(T) + d_T\mathbf{r}_G(T), \quad \left[\omega(T) := \lim_{\Delta T\to 0}\frac{\Delta\pi}{\Delta T} \right]$$

$$\Leftrightarrow d_T\mathbf{r} = d_T\mathbf{r}_G(T) + \omega(T)\times\mathbf{c}(T)$$
$$= d_T\mathbf{r}_G(T) + \tilde{\omega}(T)\,\mathbf{c}(T)$$
$$= d_T\mathbf{r}_G(T) + \tilde{\omega}(T)\,A\,\mathbf{s}.$$

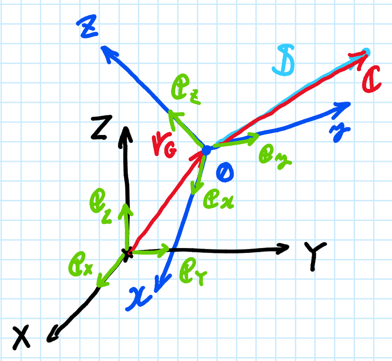
$$\circ\ d_T\mathbf{r}(\mathbf{s},T) = d_T\{\mathbf{r}_G + A\mathbf{s}\}$$
$$= d_T\mathbf{r}_G + (d_T A)\,\mathbf{s}.$$

$$\Rightarrow \boxed{d_T A = \tilde{\omega} A}$$

[$\mathbf{s}$: Position in a body]



$\circ\ \mathbf{r} = r_{X_i}\mathbf{e}_{X_i}$
$= r_{x_i}^{\circ}\mathbf{e}_{x_i}$



$$\Rightarrow r_{X_i} = r_{x_j}^{\circ}\,\mathbf{e}_{x_j}^T\mathbf{e}_{X_i}$$
$$= A_{ij}\, r_{x_j}^{\circ}$$

$$\Leftrightarrow A = \begin{bmatrix} \mathbf{e}_X^T \\ \mathbf{e}_Y^T \\ \mathbf{e}_Z^T \end{bmatrix} \cdot [\mathbf{e}_x\ \mathbf{e}_y\ \mathbf{e}_z] = \mathbf{e}_{X_i}^T\mathbf{e}_{x_j}$$

$$\Rightarrow \dot{A} = \underbrace{\dot{\mathbf{e}}_{X_i}^T}_{[\,d\mathbf{e}_{X_i} \equiv 0:\ \text{Fixed}\,]}\mathbf{e}_{x_j} + \mathbf{e}_{X_i}^T\dot{\mathbf{e}}_{x_j} = \mathbf{e}_{X_i}^T\dot{\mathbf{e}}_{x_j}$$

$$= \mathbf{e}_{X_i}^T(\omega\times\mathbf{e}_{x_j})$$

$$= \mathbf{e}_{X_i}^T(\omega_x^{\circ}\,\mathbf{e}_x\times\mathbf{e}_{x_j} + \omega_y^{\circ}\,\mathbf{e}_y\times\mathbf{e}_{x_j} + \omega_z^{\circ}\,\mathbf{e}_z\times\mathbf{e}_{x_j}) \quad [\,\omega = \omega_{X_i}\mathbf{e}_{X_i} = \omega_{x_i}^{\circ}\mathbf{e}_{x_i}\,]$$

$$= \mathbf{e}_{X_i}^T[-\omega_y^{\circ}\mathbf{e}_z + \omega_z^{\circ}\mathbf{e}_y \quad \omega_x^{\circ}\mathbf{e}_z - \omega_z^{\circ}\mathbf{e}_x \quad -\omega_x^{\circ}\mathbf{e}_y + \omega_y^{\circ}\mathbf{e}_x]$$

$$= \mathbf{e}_{X_i}^T\,[\mathbf{e}_x\ \mathbf{e}_y\ \mathbf{e}_z]\begin{bmatrix} 0 & -\omega_z^{\circ} & \omega_y^{\circ} \\ \omega_z^{\circ} & 0 & -\omega_x^{\circ} \\ -\omega_y^{\circ} & \omega_x^{\circ} & 0 \end{bmatrix} = \mathbf{e}_{X_i}^T\mathbf{e}_{x_j}\,\tilde{\omega}^{\circ}$$

$$= A\tilde{\omega}^{\circ}.$$

$$\circ\ \boxed{d_T A = \tilde{\omega} A = A\tilde{\omega}^{\circ}}$$

$\circ\ \|\mathbf{c}\|^2 = \mathbf{c}^T\mathbf{c} = \mathbf{s}^T A^T A\mathbf{s}\ [\,\mathbf{c} = A\mathbf{s}\,]$

$\circ\ \|\mathbf{c}\|^2 = \|\mathbf{s}\|^2 = \mathbf{s}^T\mathbf{s}$, $\quad A$: Rotation transformation

$$\Rightarrow A^T A = I.$$

$$\Rightarrow \begin{cases} \cdot\ d_T A\cdot A^T = \tilde{\omega}\, A\cdot A^T = \tilde{\omega}, \\ \cdot\ A^T\cdot d_T A = A^T A\,\tilde{\omega}^{\circ} = \tilde{\omega}^{\circ} \end{cases}$$

$$\boxed{\begin{aligned} \tilde{\omega} &= (d_T A)A^T, \\ \tilde{\omega}^{\circ} &= A^T(d_T A). \end{aligned}}$$

◎ Angular velocity and Euler parameter

$$\circ\ \tilde{\omega}^{\circ} = A^T(d_T A) = A^T\tilde{\omega}A$$

$$\circ\ \tilde{\omega}^{\circ} = A^T(d_T A) = 2(E L^T)^T E(d_T L)^T = 2 L E^T E (d_T L)^T = 2L(I - \mathbf{e}\mathbf{e}^T)(d_T L)^T$$

$$= 2L(d_T L)^T - 2(\underbrace{L\mathbf{e}}_{[=0]})\mathbf{e}^T(d_T L)^T = 2L(d_T L)^T$$

$$= 2(L\, d_T\mathbf{e})^{\sim}$$

$$\Leftrightarrow \omega^{\circ} = 2L\, d_T\mathbf{e}$$

$$\Leftrightarrow \frac{1}{2}L^T\omega^{\circ} = L^T L\, d_T\mathbf{e} = (I - \mathbf{e}\mathbf{e}^T)\, d_T\mathbf{e} = d_T\mathbf{e} - \mathbf{e}(\underbrace{\mathbf{e}^T d_T\mathbf{e}}_{[=0]})$$

$$= d_T\mathbf{e}$$

$$\boxed{\begin{aligned} \omega^{\circ} &= 2L\, d_T\mathbf{e}, \\ d_T\mathbf{e} &= \frac{1}{2}L^T\omega^{\circ}. \end{aligned}}$$

- $d_T \mathbb{W}^{\Delta} = 2(d_T \mathbb{L}) d_T \mathcal{E} + 2 \mathbb{L} d_T^2 \mathcal{E}.$
- $(d_T \mathbb{L}) d_T \mathcal{E} = \begin{bmatrix} -\dot{\boldsymbol{k}} & \dot{k}_0 \mathbb{I} - \tilde{\dot{\boldsymbol{k}}} \end{bmatrix} \cdot \begin{bmatrix} \dot{k}_0 \\ \dot{\boldsymbol{k}} \end{bmatrix} = -\dot{k}_0 \dot{\boldsymbol{k}} + \dot{k}_0 \dot{\boldsymbol{k}} - \tilde{\dot{\boldsymbol{k}}} \dot{\boldsymbol{k}}$

$$= -\dot{\boldsymbol{k}} \times \dot{\boldsymbol{k}} = 0$$

$$\Rightarrow \boxed{d_T \mathbb{W}^{\Delta} = 2 \mathbb{L} d_T^2 \mathcal{E}}$$