On the warping sum of knots

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Abstract

The warping sum $e(K)$ of a knot K is the minimal value of the sum of the warping degrees of a minimal diagram of K with both orientations. In this paper, knots K with $e(K) \leq 3$ are characterized, and some knots K with $e(K) = 4$ are given.

1 Introduction

In this paper, knot diagrams are oriented and on S^2, and they are considered up to ambient isotopy of S^2. An oriented knot diagram D is *monotone* if one can travel along D from a point on D so that one meets each crossing as an over-crossing first. The *warping degree* of a knot diagram D, denoted by $d(D)$, is the smallest number of crossing changes required to obtain a monotone diagram from D (defined by Kawauchi in [7]). The warping degree of an oriented knot diagram is dependent on both the diagram and the orientation, so it is not a knot invariant. On the other hand, this degree can be used to study a knot's orientation, its alternating behavior, its crossing number, and other properties ([9, 17, 18]). The warping degree has been also defined and studied for links and spatial graphs ([7, 8]), nanowords ([2]), virtual knot diagrams ([14]) and so on. Similar concepts have been studied from various view points (see, for example, [4, 11]). In particular, the warping degree relates to a knot invariant called the *ascending number* of a knot ([15], see

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also [1, 3, 5, 13]); the minimal warping degree $d(D)$ for all diagrams D with an orientation of an unoriented knot K is the ascending number $a(K)$ of K.

For a knot diagram D, let $-D$ denote the diagram D with its orientation reversed. The *warping sum* $e(D)$ of a knot diagram D is the sum $d(D) + d(-D)$ of warping degrees of D and $-D$. By definition, we have $e(-D) = e(D)$ and hence $e(D)$ is not orientation-dependent. We remark that $e(D)$ relates to the *span*, $\mathrm{spn}(D)$, of the "warping polynomial" of a knot diagram D. We have the equality $\mathrm{spn}(D) = c(D) - e(D)$ ([19]), where $c(D)$ is the crossing number of D. (For the knot invariant $\mathrm{spn}(K)$, see also [10, 12].) For an (oriented or unoriented) knot K, the *warping sum* of the knot K, denoted by $e(K)$, is defined to be the minimal value of $e(D)$ for all *minimal diagrams* D of K. In [17], the inequality $e(K) \leq c(K) - 1$ is proven, giving an upper bound on $e(K)$. Furthermore, it is shown that equality holds if and only if K is a prime alternating knot. In this paper, we provide a lower bound for $e(K)$. In particular, we show that any knot K which is neither the trivial knot, 3_1 nor 4_1 has $e(K) \geq 4$ even if K is not prime alternating (Theorem 2.9). From this theorem, we determine some knots K with $e(K) = 4$. The rest of the paper is organized as follows: In Section 2, we investigate the warping sum $e(K)$, and we determine which knots have $e = 0, 1, 2$ or 3 by considering another new invariant $md(K)$, called the minimal warping degree. Then we give some knots K with $e(K) = 4$. In Section 3, we generalize the warping sum $e(K)$ to define a related invariant $\hat{e}(K)$ by considering all diagrams of K, not only minimal diagrams and we determine which knots have $\hat{e} = 0, 1, 2$ or 3.

2 The warping sum $e(K)$

In this section, we study the warping sum $e(K)$ and we give some knots K with $e(K) = 4$. In [17], the following inequality is shown:

Theorem 2.1 ([17]). *For a nontrivial knot K, we have*

$$e(K) \leq c(K) - 1,$$

where the equality holds if and only if K is a prime alternating knot.

Since any minimal diagram of a prime alternating knot is a reduced alternating diagram, and the equality $e(D) = c(D) + 1$ holds for any nontrivial alternating diagram of a knot ([17]), any minimal diagram D of a nontrivial prime alternating knot K has $e(D) = c(K) - 1$. However, different minimal diagrams of the same prime alternating knot can give different breakdown of the warping degree $d(D)$ and $d(-D)$ as shown in Examples 2.2 and 2.3.

Example 2.2. Two minimal diagrams of 7_6, D_1 and D_2 depicted in Figure 1 have $d(D_1, -D_1) = (3, 3)$ and $d(D_2, -D_2) = (2, 4)$, where we denote the pair $(d(D), d(-D))$ by $d(D, -D)$. Hence $d(D)$ is not preserved by some flype moves although $e(D)$ is preserved by flypes on reduced alternating diagrams of a prime alternating knot; $e(D_1) = e(D_2) = 6$.

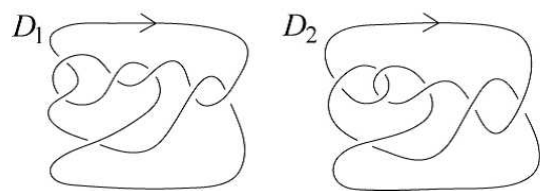

Figure 1: Minimal diagrams D_1 and D_2 of 7_6.

Example 2.3. Two minimal diagrams D_1 and D_2 of $K = 8_{12}$ depicted in Figure 2 have $d(D_1, -D_1) = (3, 4)$ and $d(D_2, -D_2) = (2, 5)$.

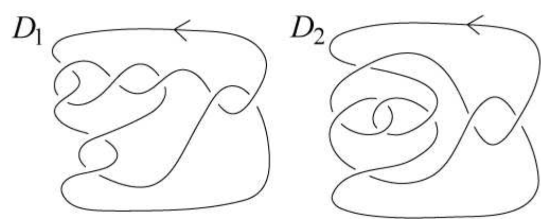

Figure 2: Minimal diagrams D_1 and D_2 of 8_{12}.

We define the *minimal warping degree*, $md(K)$, of a knot K to be the minimal value of the warping degree, $d(D)$, for all minimal diagrams D of K with all possible orientations. Note that the minimal warping degree $md(K)$ and the warping sum $e(K)$ are computable for prime knots K with up to 12 crossings by checking all the diagrams with up to 12 crossings using *LinKnot* ([6]). By definition, we have the inequality $u(K) \leq a(K) \leq md(K)$ for the unknotting number $u(K)$ and the ascending number $a(K)$ of a knot K since $a(K)$ is the minimal value of the warping degree for all diagrams of K, not only minimal diagrams. Knots with ascending number one are determined as follows:

Theorem 2.4 (Ozawa, [15])**.** *A knot K has ascending number one if and only if K is a twist knot.*

A *twist knot* is a knot whose Conway notation is "2 n" or "-2 $-n$" (where n is a positive integer). In this paper, we consider twist knots to be only those twist knots with described by the positive integers "2 n" without loss of generality. We have the following:

Theorem 2.5. *A knot K has minimal warping degree one if and only if $K = 3_1$ or 4_1.*

To prove Theorem 2.5, we prepare the following lemmas:

Lemma 2.6. *Each twist knot has a unique minimal diagram.*

Proof. It is known that links with Conway notation "p q" (p and q are positive integers) have only one minimal diagram on S^2 (see, for example, [16]). Hence any twist knot has only one minimal diagram. □

Lemma 2.7. *Let n be a positive integer, and let D be a knot diagram with Conway notation "2 n". We have $d(D, -D) = \left(\frac{n+1}{2}, \frac{n+1}{2}\right)$ if n is odd, and $d(D, -D) = \left(\frac{n}{2}, \frac{n+2}{2}\right)$ or $\left(\frac{n+2}{2}, \frac{n}{2}\right)$ if n is even.*

Proof. Since D is alternating, we can obtain the warping degree with an orientation by starting at any point just before an over-crossing and counting the number of crossings that we pass under the first time we meet them as we travel around D ([17]). For example, see Figure 3. □

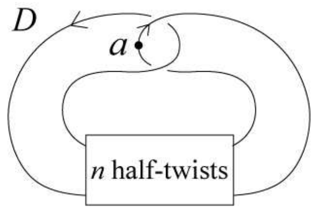

Figure 3: For the case n is odd and the orientation shown by the arrow, by taking the start point a we meet $\frac{n+1}{2}$ crossing points as an under-crossing first in the n half-twists.

We have the following corollary:

Corollary 2.8. *Let K be a twist knot with Conway notation "2 n". Then the minimal warping degree $md(K)$ equals $\lfloor \frac{n+1}{2} \rfloor$.*

Note that the warping sum $e(K)$ equals $c(K) - 1 = n + 1$ for twist knots K with Conway notation "2 n" because K is a prime alternating knot. Now we prove Theorem 2.5.

Proof of Theorem 2.5. Let K be a knot with $md(K) = 1$. K is not the trivial knot because $md(K)$ is not zero. Then we also have $a(K) = 1$ from $a(K) \leq md(K)$. By Theorem 2.4, K must be a twist knot. A twist knot K with Conway notation "2 n" has $md(K) = 1$ if and only if $n = 1$ or 2 by Corollary 2.8, that is, $K = 3_1$ or 4_1. $\square$

By Theorem 2.5, we can determine that the minimal warping degree $md(K)$ equals 2 for some knots K. For example, we have $md(7_6) = md(8_{12}) = 2$ by Examples 2.2 and 2.3. We show the following theorem:

Theorem 2.9. *Let K be a knot. Then the small values for the warping sum $e(K)$ are determined as follows.*
(0): $e(K) = 0$ if and only if K is the trivial knot.
(1): There are no knots K with $e(K) = 1$.

(2): $e(K) = 2$ if and only if K is the 3_1 knot.
(3): $e(K) = 3$ if and only if K is the 4_1 knot.

Proof. (0): If a knot K has a diagram D with the warping degree $d(D)$ equals 0, i.e., K has a monotone diagram, then K is the trivial knot. (1): If a knot diagram D has the warping sum $e(D) = 1$, then $d(D, -D) = (0, 1)$ or $(1, 0)$. This means D is a diagram of the trivial knot, which has $e(K) = 0$. (2): If $e(D) = 2$, then $d(D, -D) = (0, 2), (1, 1)$ or $(2, 0)$. For the case $(0, 2)$ or $(2, 0)$, D represents the trivial knot, which has $e(K) = 0$. For the case $(1, 1)$, D represents 3_1 or 4_1 if D is a minimal diagram by Theorem 2.5. For the minimal diagram D of 3_1, we have $d(D, -D) = (1, 1)$. For the minimal diagram D of 4_1, we have $d(D, -D) = (1, 2)$ or $(2, 1)$. Hence only 3_1 has $e(K) = 2$. (3): If $e(D) = 3$, then $d(D, -D) = (0, 3), (1, 2), (2, 1)$ or $(3, 0)$. Similarly to (2), we can see that only 4_1 has $e(K) = 3$. □

We have the following corollary:

Corollary 2.10. *Let K be a knot. If the value of the warping sum, $e(K)$, is 4 or 5, then the minimal warping degree, $md(K)$, equals 2.*

Proof. Let D be a minimal diagram of a knot K with $e(D) = e(K) = 4$ or 5. Then K must be neither the trivial knot, 3_1 nor 4_1 by Theorem 2.9, and the warping degree of any minimal diagram of K must be neither 0 nor 1 by Theorem 2.5. Hence $d(D, -D)$ is $(2, 2)$, $(2, 3)$ or $(3, 2)$, and therefore $md(K) = 2$. □

Since a prime alternating knot K has always the relation between the warping sum $e(K)$ and the crossing number $c(K)$ that $e(K) = c(K) - 1$, we have $e(5_1) = e(5_2) = 4$. In the following example, we give two knots which are non-alternating or non-prime with $e(K) = 4$.

Example 2.11. The non-alternating knot 8_{21} has $e(8_{21}) = 4$. The Granny knot G, a non-prime alternating knot, has $e(G) = 4$. These values of the warping sum are realized by the minimal diagrams shown in Figure 4.

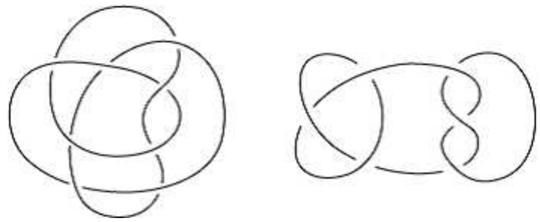

Figure 4: The minimal diagrams of 8_{21} (left) and Granny knot (right) with $e = 4$.

3 The reduced warping sum $\hat{e}(K)$

For a knot K, the warping sum $e(K)$ is defined to be the minimal value of the warping sum $e(D)$ for all minimal diagrams D of K. By considering all diagrams D including non-minimal diagrams, we might obtain a value of $e(D)$ (for a non-minimal diagram D of K) that is smaller than $e(K)$. For example, the knot 6_3 has $e(6_3) = 5$, and it has a non-minimal diagram D with $e(D) = 4$ (see Figure 5).

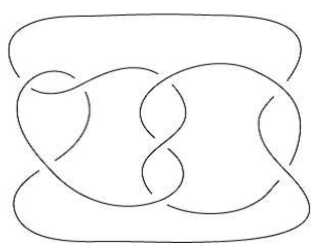

Figure 5: The knot 6_3 has a non-minimal diagram D with $e(D) = 4$.

We define the *reduced warping sum*, $\hat{e}(K)$, of a knot K to be the minimum value of warping sum $e(D)$ over all possible diagrams D of K. We show the following theorem:

Theorem 3.1. *Let K be a knot. Then the small values for the reduced warping sum $\hat{e}(K)$ are determined as follows.*
(0) $\hat{e}(K) = 0$ if and only if K is the trivial knot.
(1) There are no knots K with $\hat{e}(K) = 1$.

(2) $\hat{e}(K) = 2$ *if and only if* K *is a twist knot.*
(3) There are no knots K *with* $\hat{e}(K) = 3$.

Proof. (0) and (1): Similar to (0) and (1) of the proof of Theorem 2.9. (2): If a diagram D of a knot K realizes $e(D) = \hat{e}(K) = 2$, then $d(D, -D)$ should be $(1, 1)$ since K is not the trivial knot because $\hat{e}(K) \neq 0$. By Theorem 2.4, a non-trivial knot K which has an oriented diagram D with $d(D) = 1$ is a twist knot. Using Ozawa's method in [15], we can check that all twist knots have a diagram D with $e(D) = 2$ (see Figure 6). (3): Since $3 = 0+3$ or $1+2$,

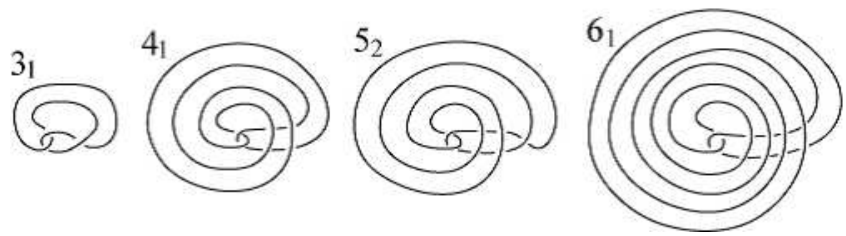

Figure 6: Every twist knot has a diagram with warping degree equal to one, regardless of which orientation is chosen (Ozawa's method in [15]).

a diagram D with $e(D) = 3$ represents the trivial knot or a twist knot. □

By Theorem 3.1, we can conclude that $\hat{e}(6_3) = 4$ (Figure 5).

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