

EE23BTECH11209 - K S Ballvardhan*

Question: A series of natural numbers $F_1, F_2, F_3, F_4, F_5, F_6, F_7, \dots$ obeys $F_{n+1} = F_n + F_{n-1}$ for all integers $n \geq 2$. If $F_6 = 37$, and $F_7 = 60$, then what is F_1 ? [GATE CS 2023]

Solution:

Parameter	Description	Value
F_5	Sixth term of the sequence	37
F_6	Seventh term of the sequence	60
F_0	First term of the sequence	?
F_1	Second term of the sequence	?
$F(z)$	Generating function of the given sequence	?

TABLE I
INPUT VALUES

Let the sequence be:

$$F_1, F_2, F_3 \dots \quad (1)$$

Given recurrence relation is

$$F_{n+1} = F_n + F_{n-1} \quad (2)$$

Taking z-transform of F_n :

$$F(z) = F_1 + z^{-1}F_2 + \sum_{n=2}^{\infty} F_n z^{-n} \quad (3)$$

$$\Rightarrow F(z) = F_1 + z^{-1}F_2 + z^{-1} \sum_{n=1}^{\infty} F_{n+1} z^{-n} \quad (4)$$

$$\Rightarrow F(z) = F_1 + z^{-1}F_2 + z^{-1} \sum_{n=1}^{\infty} (F_n + F_{n-1}) z^{-n} \quad (5)$$

$$\Rightarrow F(z) = F_1 + z^{-1}F_2 + z^{-1} (F(z) - F_1 + z^{-1}F(z)) \quad (6)$$

$$\therefore F(z) = \frac{F_1 + (F_2 - F_1)z^{-1}}{1 - z^{-1} - z^{-2}} \quad (7)$$

Using Contour Integration to find the inverse Z-transform,

$$F_n = \frac{1}{2\pi j} \oint_C F(z) z^{n-1} dz \quad (8)$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^n (F_2 - F_1 + F_1 z)}{z^2 - z - 1} dz \quad (9)$$

By residue theorem:

$$F_n = \frac{1}{(0)!} \lim_{z \rightarrow \frac{1+\sqrt{5}}{2}} \frac{d}{dz} \left(\left(z + \frac{1+\sqrt{5}}{2} \right) F(z) \right) + \frac{1}{(0)!} \lim_{z \rightarrow \frac{1-\sqrt{5}}{2}} \frac{d}{dz} \left(\left(z + \frac{1-\sqrt{5}}{2} \right) F(z) \right) \quad (10)$$

$$(11)$$

On simplifying we get,

$$F_n = (F_2 - F_1) \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) + (F_1) \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right) \quad (12)$$

From the values in Table I:

$$5(F_2 - F_1) + 8F_1 = 37 \quad (13)$$

$$8(F_2 - F_1) + 13F_1 = 60 \quad (14)$$

$$\Rightarrow F_2 = 5, F_1 = 4 \quad (15)$$

$$\therefore \boxed{F_1 = 4} \quad (16)$$

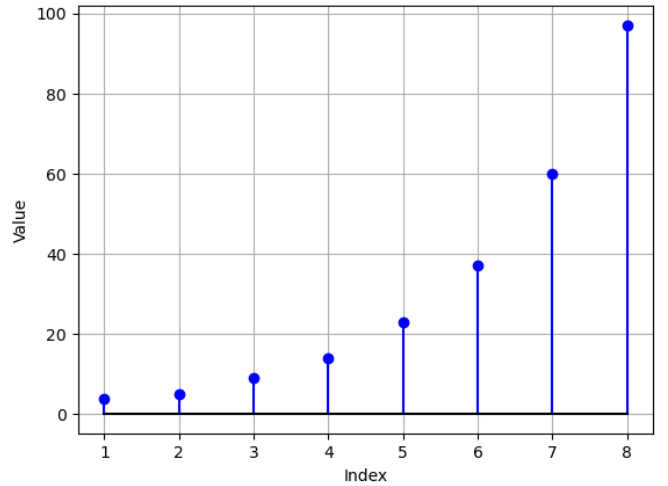


Fig. 1. Terms of the given sequence