EE23BTECH11209 - K S Ballvardhan*

Question: A series of natural numbers F_1, F_2, F_3, F_4, F_5, F_6, F_7,....obeys $F_{n+1} = F_n + F_{n-1}$ for all integers $n \geq 2$. If $F_6 = 37$, and $F_7 = 60$, then what is F_1? [GATE CS 2023]

Solution:

Parameter	**Description**	**Value**
$x(6)$	Seventh term of the sequence	60
$x(5)$	Sixth term of the sequence	37
$x(1)$	Second term of the sequence	?
$x(0)$	First term of the sequence	?

TABLE I
INPUT VALUES

Taking z-transform of $X(z)$:

$$X(z) = x(0) + z^{-1}x(1) + \sum_{n=2}^{\infty} x(n) z^{-n} \tag{1}$$

$$= x(0) + z^{-1}x(1) + z^{-1}\sum_{n=1}^{\infty} x(n+1) z^{-n} \tag{2}$$

$$= x(0) + z^{-1}x(1) + z^{-1}\sum_{n=1}^{\infty} (x(n) + x(n-1)) z^{-n} \tag{3}$$

$$= x(0) + z^{-1}x(1) + z^{-1}\left(X(z) - x(1) + z^{-1}X(z)\right) \tag{4}$$

$$\implies X(z) = \frac{x(0) + (x(1) - x(0)) z^{-1}}{1 - z^{-1} - z^{-2}} \tag{5}$$

Using Contour Integration to find the inverse Z-transform,

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)\ z^{n-1}\ dz \tag{6}$$

$$= \frac{1}{2\pi j}\oint_C \frac{z^n (x(1) - x(0) + x(0) z)}{z^2 - z - 1}\ dz \tag{7}$$

By residue theorem:

$$x(n) = \frac{1}{(0)!}\lim_{z\to\frac{1+\sqrt{5}}{2}} \frac{d}{dz}\left((z + \frac{1+\sqrt{5}}{2})X(z)\right) + \frac{1}{(0)!}\lim_{z\to\frac{1-\sqrt{5}}{2}} \frac{d}{dz}\left(\left(z + \frac{1-\sqrt{5}}{2}\right)X(z)\right) \tag{8}$$

On simplifying we get,

$$x(n) = (x(1) - x(0))\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) + (x(0))\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) \tag{9}$$

From the values in Table I:

$$5(x(1) - x(0)) + 8x(0) = 37 \tag{10}$$

$$8(x(1) - x(0)) + 13x(0) = 60 \tag{11}$$

$$\implies x(1) = 5, x(0) = 4 \tag{12}$$

$$\therefore x(0) = 4 \tag{13}$$

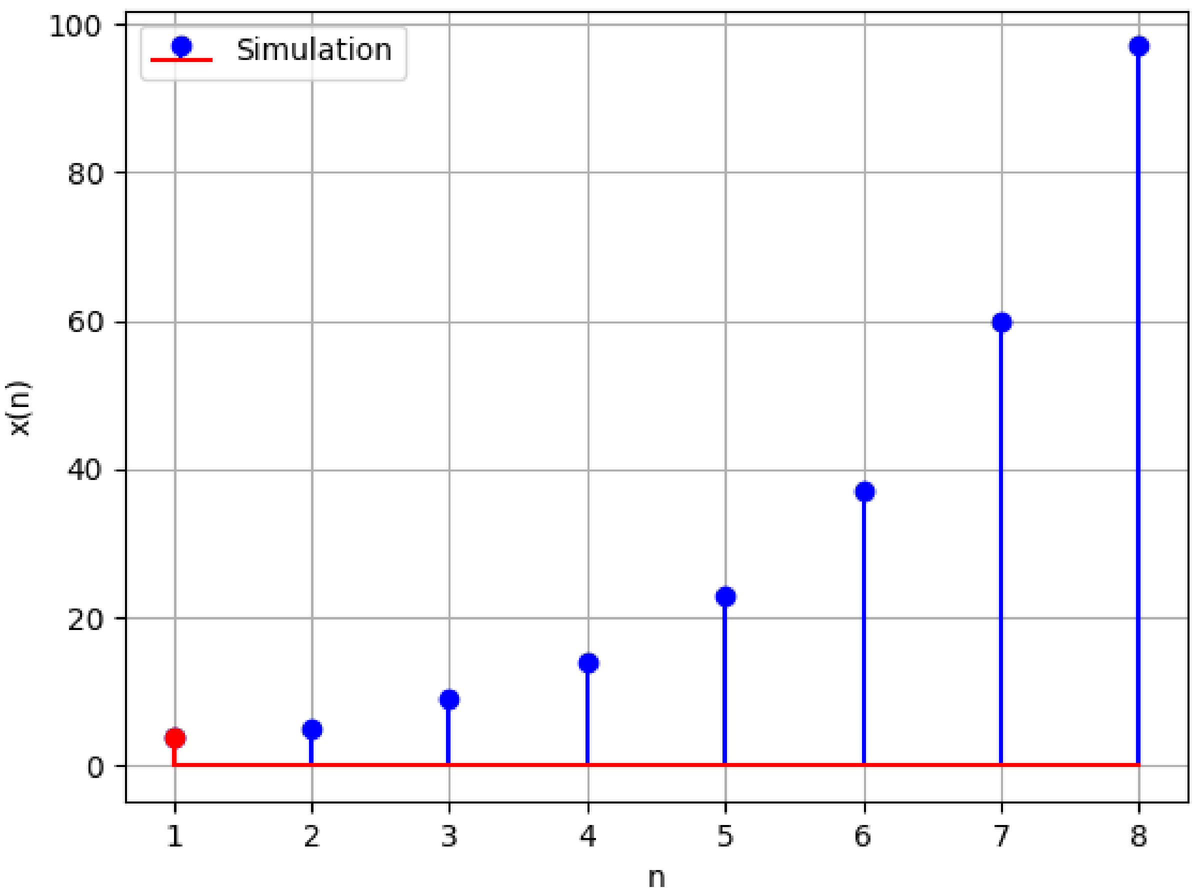

Fig. 1. Terms of the given sequence