EE23BTECH11209 - K S Ballvardhan*

Question: A series of natural numbers F_1, F_2, F_3, F_4, F_5, F_6, F_7,....obeys $F_{n+1} = F_n + F_{n-1}$ for all integers $n \geq 2$. If $F_6 = 37$, and $F_7 = 60$, then what is F_1? [GATE CS 2023]

Solution:

Parameter	Description	Value
F_5	Sixth term of the sequence	37
F_6	Seventh term of the sequence	60
F_1	First term of the sequence	?
F_2	Second term of the sequence	?

TABLE I
INPUT VALUES

Let the sequence be:

$$1, F_2, F_3 \dots \tag{1}$$

Given recurrence relation is

$$F_{n+1} = F_n + F_{n-1} \tag{2}$$

Taking z-transform of F_n:

$$F(z) = F_1 + z^{-1}F_2 + \sum_{n=2}^{\infty} F_n z^{-n} \tag{3}$$

$$= F_1 + z^{-1}F_2 + z^{-1}\sum_{n=1}^{\infty} F_{n+1} z^{-n} \tag{4}$$

$$= F_1 + z^{-1}F_2 + z^{-1}\sum_{n=1}^{\infty} (F_n + F_{n-1}) z^{-n} \tag{5}$$

$$= F_1 + z^{-1}F_2 + z^{-1}\left(F(z) - F_1 + z^{-1}F(z)\right) \tag{6}$$

$$\implies F(z) = \frac{F_1 + (F_2 - F_1) z^{-1}}{1 - z^{-1} - z^{-2}} \tag{7}$$

Using Contour Integration to find the inverse Z-transform,

$$F_n = \frac{1}{2\pi j} \oint_C F(z)\ z^{n-1}\ dz \tag{8}$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^n (F_2 - F_1 + F_1 z)}{z^2 - z - 1}\ dz \tag{9}$$

By residue theorem:

$$F_n = \frac{1}{(0)!} \lim_{z \to \frac{1+\sqrt{5}}{2}} \frac{d}{dz}\left((z + \frac{1+\sqrt{5}}{2})F(z)\right) + \frac{1}{(0)!} \lim_{z \to \frac{1-\sqrt{5}}{2}} \frac{d}{dz}\left(\left(z + \frac{1-\sqrt{5}}{2}\right)F(z)\right) \tag{10}$$

On simplifying we get,

$$F_n = (F_2 - F_1)\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) + (F_1)\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) \tag{11}$$

From the values in Table I:

$$5(F_2 - F_1) + 8F_1 = 37 \tag{12}$$

$$8(F_2 - F_1) + 13F_1 = 60 \tag{13}$$

$$\implies F_2 = 5, F_1 = 4 \tag{14}$$

$$\therefore \boxed{F_1 = 4} \tag{15}$$

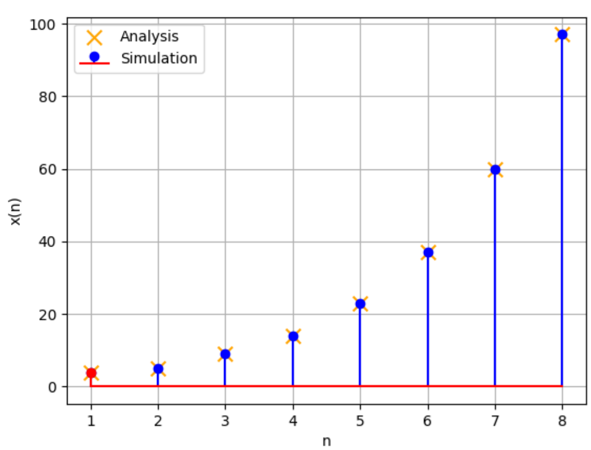

Fig. 1. Terms of the given sequence