

EE23BTECH11209 - K S Ballvardhan*

Question: A series of natural numbers $F_1, F_2, F_3, F_4, F_5, F_6, F_7, \dots$ obeys $F_{n+1} = F_n + F_{n-1}$ for all integers $n \geq 2$. If $F_6 = 37$, and $F_7 = 60$, then what is F_1 ? [GATE CS 2023]

Solution:

Parameter	Description	Value
$x(6)$	Seventh term of the sequence	60
$x(5)$	Sixth term of the sequence	37
$x(1)$	Second term of the sequence	?
$x(0)$	First term of the sequence	?

TABLE I
INPUT VALUES

On simplifying we get,

$$x(n) = (x(1) - x(0)) \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right) + x(0) \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{1 - \sqrt{5}}{2} \right)^n \right) \quad (9)$$

From the values in Table I:

$$5(x(1) - x(0)) + 8x(0) = 37 \quad (10)$$

$$8(x(1) - x(0)) + 13x(0) = 60 \quad (11)$$

$$\Rightarrow x(1) = 5, x(0) = 4 \quad (12)$$

$$\therefore x(0) = 4 \quad (13)$$

Taking z-transform of $X(z)$:

$$X(z) = x(0) + z^{-1}x(1) + \sum_{n=2}^{\infty} x(n) z^{-n} \quad (1)$$

$$= x(0) + z^{-1}x(1) + z^{-1} \sum_{n=1}^{\infty} x(n+1) z^{-n} \quad (2)$$

$$= x(0) + z^{-1}x(1) + z^{-1} \sum_{n=1}^{\infty} (x(n) + x(n-1)) \quad (3)$$

$$= x(0) + z^{-1}x(1) + z^{-1} (X(z) - x(1) + z^{-1}X(z)) \quad (4)$$

$$\Rightarrow X(z) = \frac{x(0) + (x(1) - x(0))z^{-1}}{1 - z^{-1} - z^{-2}} \quad (5)$$

Using Contour Integration to find the inverse Z-transform,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \quad (6)$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^n (x(1) - x(0) + x(0)z)}{z^2 - z - 1} dz \quad (7)$$

By residue theorem:

$$x(n) = \frac{1}{(0)!} \lim_{z \rightarrow \frac{1+\sqrt{5}}{2}} \frac{d}{dz} \left(\left(z + \frac{1+\sqrt{5}}{2} \right) X(z) \right) + \frac{1}{(0)!} \lim_{z \rightarrow \frac{1-\sqrt{5}}{2}} \frac{d}{dz} \left(\left(z + \frac{1-\sqrt{5}}{2} \right) X(z) \right) \quad (8)$$

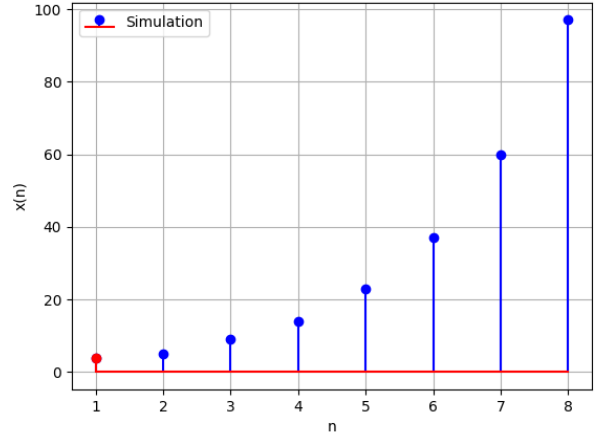


Fig. 1. Terms of the given sequence