EE23BTECH11209 - K S Ballvardhan*

Question: A series of natural numbers F_1 , F_2 , F_3 , F_4 , F_5 , F_6 , F_7 ,....obeys $F_{n+1} = F_n + F_{n-1}$ for all integers $n \ge 2$. If $F_6 = 37$, and $F_7 = 60$, then what is F_1 ? [GATE CS 2023]

Solution:

Parameter	Description	Value
F_5	Sixth term of the sequence	37
F_6	Seventh term of the sequence	60
F_1	First term of the sequence	?
F_2	Second term of the sequence	?

TABLE I INPUT VALUES

Taking z-transform of F_n :

$$F(z) = F_1 + z^{-1}F_2 + \sum_{n=2}^{\infty} F_n z^{-n}$$

$$= F_1 + z^{-1}F_2 + z^{-1} \sum_{n=1}^{\infty} F_{n+1} z^{-n}$$

$$= F_1 + z^{-1}F_2 + z^{-1} \sum_{n=1}^{\infty} (F_n + F_{n-1}) z^{-n}$$
(3)

$$= F_1 + z^{-1}F_2 + z^{-1} \left(F(z) - F_1 + z^{-1}F(z) \right)$$

$$= F_1 + z^{-1}F_2 + z^{-1}\left(F(z) - F_1 + z^{-1}F(z)\right)$$
(4)

$$\implies F(z) = \frac{F_1 + (F_2 - F_1)z^{-1}}{1 - z^{-1} - z^{-2}}$$
 (5)

Using Contour Integration to find the inverse Ztransform,

$$F_n = \frac{1}{2\pi i} \oint_C F(z) \ z^{n-1} \ dz$$

$$= \frac{1}{2\pi i} \oint_C \frac{z^n (F_2 - F_1 + F_1 z)}{z^2 - z - 1} \ dz$$
(6)

By residue theorem:

$$F_{n} = \frac{1}{(0)!} \lim_{z \to \frac{1+\sqrt{5}}{2}} \frac{d}{dz} \left((z + \frac{1+\sqrt{5}}{2}) F(z) \right) + \frac{1}{(0)!} \lim_{z \to \frac{1-\sqrt{5}}{2}} \frac{d}{dz} \left(\left(z + \frac{1-\sqrt{5}}{2} \right) F(z) \right)$$
(8)

On simplifying we get,

$$F_{n} = (F_{2} - F_{1}) \left(\left(\frac{1 + \sqrt{5}}{2} \right)^{n} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n} \right) + (F_{1}) \left(\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right)$$
(9)

From the values in Table I:

$$5(F_2 - F_1) + 8F_1 = 37 \tag{10}$$

$$8(F_2 - F_1) + 13F_1 = 60 (11)$$

$$\implies F_2 = 5, F_1 = 4$$
 (12)

$$\therefore F_1 = 4 \tag{13}$$

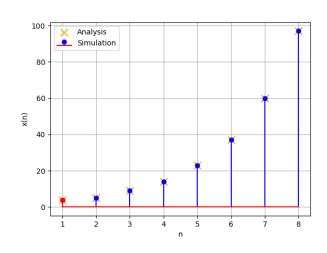


Fig. 1. Terms of the given sequence