Section 3-1 Laplace Transform - Part 1

1. Definition and basic properties

미분방정식의 해를 구하기 위해서 사용하는 변환법의 아이디어는 다음과 같다.

$$\begin{tikzcd}
& \text{DE} \arrow[r, "T"] & \text{Algebraic equation} \arrow[d, "\text{solve}"] \\
\text{solution} & & \text{solution} \arrow[ll, "T^{-1}"']
\end{tikzcd}$$

그림 1

미분방정식을 변환을 통해 대수방정식으로 변화하여 대수방정식의 해를 구하고 대수방정식의 해를 다시 역변환을 취하면 본래의 미분방정식의 해를 구할 수 있다.

왜 라플라스 변환인가?

다음과 같음 mass-spring system에서 시스템에 입력되는 함수가 불연속일 때 기존의 비동차 방정식의 풀이법을 사용할 수 없음.

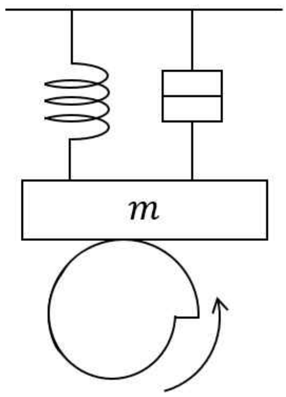

그림 2

라플라스변환의 정의

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)dt$$

Ex)

$$f = e^{at}$$

$$
\begin{aligned}
\mathcal{L}\left[e^{at}\right](s) &= \int_0^\infty e^{-st} e^{at} dt \\
&= \int_0^\infty e^{(a-s)t} dt \\
&= \lim_{k\to\infty} \int_0^k e^{(a-s)t} dt \\
&= \lim_{k\to\infty} \frac{1}{(a-s)} \left[e^{(a-s)k} - 1\right] \\
&= \frac{1}{s-a} \qquad (a-s<0)
\end{aligned}
$$

Ex)

$$
\mathcal{L}[\sin t](s) = \int_0^\infty e^{-st} \sin t \, dt
$$

$$
\begin{aligned}
\mathcal{L} &= -e^{-st}\cos t|_0^\infty - s\int_0^\infty e^{-st}\cos t \, dt \\
&= 1 - s(e^{-st}\sin t|_0^\infty + s\mathcal{L}) \\
\mathcal{L} &= 1 - s^2\mathcal{L} \\
(1+s^2)\mathcal{L} &= 1 \\
\mathcal{L} &= \frac{1}{1+s^2}
\end{aligned}
$$

Theorem (라플라스 변환의 선형성)

$$
\mathcal{L}[\alpha f + \beta g] = \alpha F(s) + \beta G(s)
$$
$$
F = \mathcal{L}[f],\ G = \mathcal{L}[g] \qquad s > a
$$

Ex)

$$
\begin{aligned}
\mathcal{L}[\cos(3t) - \sin(4t)](s) &= \mathcal{L}[\cos(3t)](s) - \mathcal{L}[\sin(4t)](s) \\
&= \frac{s}{s^2+3^2} - \frac{4}{s^2+4^2}
\end{aligned}
$$

Ex)

$$
\begin{aligned}
\mathcal{L}\left[2t^2e^{-3t} - 2t + 1\right] &= 2\mathcal{L}\left[t^2e^{-3t}\right] - 4\mathcal{L}[t] + \mathcal{L}[1] \\
&= 2\frac{2!}{(s+a)^3} - \frac{4}{s^2} + \frac{1}{s}
\end{aligned}
$$

Q: When $\mathcal{L}[f](s)$ exist?

$f \mapsto \int_0^\infty e^{-st} f(t)dt$ is defined for proper range of s.

Theorem.

Suppose f is piecewise continuous in $[0, k]$, for every positive k suppose $\exists M, b$ such that

$$|f(t)| \le Me^{bt} \qquad t \ge 0$$

$$\Rightarrow \int_0^\infty e^{-st} f(t)dt \text{ Converges for } s > b$$

Proof)

$\int_0^\infty e^{-st} |f(t)| \le M \int_0^\infty e^{(b-s)t} dt$ Finite when $b - s < 0$

Example)

Even though $t^{-\frac{1}{2}}, t > 0$ does not satisfy growth condition.

$$\mathcal{L}\left[t^{-\frac{1}{2}}\right] = \int_0^\infty e^{-st} t^{-\frac{1}{2}} dt$$

$$\text{Let } x = t^{-\frac{1}{2}}, dx = \frac{1}{2} t^{-\frac{1}{2}} dt$$

$$\mathcal{L}[x] = 2\int_0^\infty e^{-sx^2} dx \qquad \sqrt{5}\,x = y$$

$$= 2\int_0^\infty e^{-y^2} \frac{1}{\sqrt{5}} dy$$

$$= \frac{2}{\sqrt{5}} \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{5}}$$

Definition)

$$\mathcal{L}[g] = G, g = \mathcal{L}^{-1}[G]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right](t) = e^{at}$$

2. Inverse Laplace Transform

Definition. (Inverse Laplace transform)

$$\mathcal{L}[g] = G(s),\ \ g(t) = \mathcal{L}^{-1}[G]$$

Ex)

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right](t) = e^{at}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t) = \sin t$$

Q: $\mathcal{L}^{-1}[g]$ is unique?

$$\mathcal{L}[e^{-t}](s) = \frac{1}{s+1},\ \ s>-1$$

$$h(t) = \begin{cases} e^{-t} & t \neq 3 \\ 0 & t = 3 \end{cases}$$

$$\mathcal{L}[h(t)](s) = \frac{1}{s+1}$$

$$\int_0^\infty e^{-st}h(t)dt = \int_0^3 e^{-st}h(t)dt + \int_3^\infty e^{-st}h(t)dt$$
$$= \int_0^\infty e^{-st}e^{-t}dt$$

ANS) If we demand $\mathcal{L}^{-1}[g]$ is continuous, then $\mathcal{L}^{-1}[g]$ is unique.

Ex)

$$\mathcal{L}^{-1}\left[\frac{2s-5}{s^2+16}\right] = 2\mathcal{L}^{-1}\left[\frac{s}{s^2+4^2}\right] - 5\mathcal{L}^{-1}\left[\frac{1}{s^2+4^2}\right]$$
$$= 2\cos(4t) - \frac{5}{4}\sin(4t)$$

3. Differentiation and Laplace transform.

Essential theorem to use Laplace transform in solving differential equation.

Theorem 1.

Suppose that f is continuous and f' is piecewise continuous on $[0, A]$ for $A > 0$.
Suppose that $\exists\ K, a, M > 0$ such that $|f(t)| \leq Ke^{at}$ for $t \geq M$
Then $\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$

Proof)

$$① \int_0^A e^{-st}f'(t)dt = \int_0^{t_1} e^{-st}f'(t)dt + \int_{t_1}^{t_2} + \cdots \int_{t_k}^{A} e^{-st}f'(t)dt$$

$$t_1, t_2, \cdots, t_k \in (0, A)$$

Points of discontinuity of f'

$$\int_{t_{i-1}}^{t_i} e^{-st} f'(t)dt = e^{-st} f(t)\Big|_{t_{i-1}}^{t_i} = -e^{-s0} f(o) + e^{-st} f(A)$$

$$\lim_{A\to\infty} \left| e^{-sA} f(A) \right| \le \lim_{A\to\infty} e^{-sA} k e^{aA} \quad e^{-(s-a)A} \to 0, \;\; s > a$$

Corollary 2.

$f, \cdots, f^{(n-1)}$ continuous $f^{(n)}$ piecewise continuous on $[0, A]$

$\exists\, K, a, M > 0$ Such that $\left| f^{(i)}(t) \right| \le ke^{at}$ for $t \ge M$, $i = 0, 1, 2, \cdots, n-1$

$$\Rightarrow \mathcal{L}\left[f^{(n)}\right] = s^n \mathcal{L}[f] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)})(0)$$

Table 1.

f	$\mathcal{L}[f](s)$
1	$\frac{1}{s} \quad s > 0$
t^n $n = 1, 2, \cdots$	$\frac{n!}{s^{n+1}} \quad s > 0$
$\cos(kt)$	$\frac{s}{s^2 + k^2} \quad s > 0$
$\sin(kt)$	$\frac{k}{s^2 + k^2} \quad s > 0$
$\vdots$	$\vdots$
e^{at}	$\frac{1}{s-a} \quad s > 0$

1계 미분에 대한 정리를 2계 미분에 반복적으로 적용하면

$$\begin{aligned} \mathcal{L}[f''](s) &= s\mathcal{L}[f'] - f'(0) \\ &= s[sF(s) - f(0)] - f'(0) \\ &= s^2 f(s) - sf(0) - f'(0) \end{aligned}$$

Example) 다음의 초깃값 문제의 해를 라플라스변환을 이용하여 구하여라.

$$y' - 4y = 1, \;\; y(0) = 1$$

(풀이) 방정식에 라플라스 변환을 적용

$$\mathcal{L}[y' - 4y] = \mathcal{L}[1]$$

If we set $Y(s) = \mathcal{L}[y]$

$$sY(s) - y(0) - 4Y(s) = \frac{1}{s}$$

$$
\begin{aligned}
&(s-4)Y(s) = 1+\frac{1}{s} \\
&\Rightarrow Y(s) = \frac{s+1}{s(s-4)} \\
&\qquad\quad = \frac{1}{s(s-4)}+\frac{1}{s-4}
\end{aligned}
$$

(here we express the rational function as sum of partial fraction, which is crucial step)

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}[\frac{1}{s(s-4)}+\frac{1}{s-4}] \\
&= \frac{1}{4-0}(e^{4t}-1)+e^{4t} \\
&= \frac{5}{4}e^{4t}-\frac{1}{4}
\end{aligned}
$$

Example) 2계미분방정식의 초깃값 문제 (라플라스 변환 이용)

$$y''+4y'+3y=e^t \qquad y(0)=0, y'(0)=2$$

(풀이)

$$\mathcal{L}[y''+4y'+3y]=\mathcal{L}[e^t]$$

$$s^2Y(s)-sy(0)-y'(0)+4(sY(s)-y(0))+3Y(s)=\frac{1}{s-a}$$

$$s^2Y-2+4sY+3Y=\frac{1}{s-1}$$

$$(s^2+4s+3)Y=2+\frac{1}{s-1}=\frac{2s-1}{s-1}$$

$$Y=\frac{2}{s^2+4s+3}+\frac{1}{(s-1)(s^2+4s+3)}$$

$$
\begin{aligned}
Y &= \frac{2s-1}{(s-1)(s+1)(s+3)} \qquad \text{(부분합으로 표현)} \\
&= \frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+3}
\end{aligned}
$$

$$2s-1=A(s+1)(s+3)+B(s-1)(s+3)+C(s-1)(s+1)$$

$$
\begin{cases}
s=-1 & -3=(-2)(2)B & B=3/4 \\
s=1 & 1=(2)(4)A & A=1/8 \\
s=-3 & -7=(-4)(-2)C & C\equiv 7/8
\end{cases}
$$

$$
\begin{aligned}
&Y=\frac{1/8}{s-1}+\frac{3/4}{s+1}-\frac{7/8}{s+3} \\
&y=\mathcal{L}^{-1}[Y] \\
&\ = \frac{1}{8}e^t+\frac{3}{4}e^{-t}-\frac{7}{8}e^{-3t}
\end{aligned}
$$