HW7 DE 21900136 김신후

1.
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix}.$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 2 & 1-\lambda & -(2-\lambda) \\ -3 & 2 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & 1 & 0 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 1 \end{vmatrix} = (2-\lambda)\left\{1\cdot\begin{vmatrix} 1-\lambda & 1 \\ -3 & 2 \end{vmatrix} + 1\begin{vmatrix} 1-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix}\right\}$$

$$= (2-\lambda)(2-2\lambda-(1\cdot -3) + (1-\lambda)^2 - 2)$$

$$= -(\lambda-2)^3. \quad \therefore \lambda = 2. \quad \text{*(Repeated eigenvalue case)}$$

$$(A-2I)\vec{X} = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \Rightarrow \begin{cases} -a+b+c=0 \\ 2a-b-c=0 \\ -3a+2b+2c=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=-c. \end{cases}$$

$$\vec{X} = \begin{bmatrix} 0 \\ b \\ -b \end{bmatrix} \quad \text{chose } b=1 \rightarrow \vec{X} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

generalized eigenvector : $(A-\lambda I)\vec{X_1} = \vec{X} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \Rightarrow \begin{cases} -a_1+b_1+c_1=0 \\ 2a_1-b_1-c_1=1 \\ -3a_1+2b_1+2c_1=-1 \end{cases} \Rightarrow \begin{cases} a_1=1 \\ b_1=1-c_1 \end{cases}$$

$$\text{Chose } b_1 = 0 \rightarrow \vec{X_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$(A-\lambda)\vec{X_2} = \vec{X_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{bmatrix}\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} -a_2+b_2+c_2=1 \\ 2a_2-b_2-c_2=0 \\ -3a_2+2b_2+2c_2=1 \end{cases}$$

$$\Rightarrow \begin{cases} a_2=1 \\ b_2=2-c_2 \end{cases} \quad \text{Chose } b_2=0 \rightarrow \vec{X_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

fundamental set : $X = C_1\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{2t} + C_2\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}te^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{2t}\right) + C_3\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\frac{t^2}{2}e^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}te^{2t} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}e^{2t}\right)$

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2. $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \vec{x}$. $\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - (-2 \cdot 4) = \lambda^2 - 2\lambda + 5$$

$$\Rightarrow \lambda = 1 \pm 2i.$$

◦ when $\lambda = 1+2i$.

$$A - \lambda I = \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}.$$

$$(A-\lambda I)\vec{v} = \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Longrightarrow \begin{cases} (2-2i)v_1 - 2v_2 = 0 & \cdots ㉠ \\ 4v_1 + (-2-2i)v_2 = 0 & \cdots ㉡ \end{cases}$$

㉠과 ㉡을 살펴보자면 ㉠에 $(2+2i)$를 곱하면 ㉠과 ㉡은 같은 식임을 알 수 있다.

$$\therefore v_2 = (1-i)v_1 \longrightarrow \vec{v} = v_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix}.$$

◦ when $\lambda = 1-2i$.

$$(A-\lambda I)\vec{v} = \begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} (1+i)v_1 - 2v_2 = 0 \\ 2v_1 + (-1+i)v_2 = 0 \end{cases}$$

$$\therefore v_2 = (1+i)v_1 \longrightarrow \vec{v} = v_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$\therefore X = c_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(1+2i)t} + c_2 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(1-2i)t}.$$

$$\Rightarrow X = c_1 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^t \cdot e^{2ti} + c_2 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^t e^{-2ti}.$$

$$= c_1 e^t (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ 1-i \end{bmatrix} + c_2 e^t (\cos 2t - i \sin 2t) \begin{bmatrix} 1 \\ 1+i \end{bmatrix}.$$

$$(\because e^{i\theta} = \cos\theta + i\sin\theta).$$

$$= c_1 e^t \begin{bmatrix} \cos 2t + i \sin 2t \\ \cos 2t + i \sin 2t - i \cos 2t - i^2 \sin 2t \end{bmatrix}$$

$$+ c_2 e^t \begin{bmatrix} \cos 2t - i \sin 2t \\ \cos 2t - i \sin 2t + i \cos 2t - i^2 \sin 2t \end{bmatrix}.$$

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$$= c_1 e^t \left[\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix} \right].$$

$$+ c_2 e^t \left[\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} - i \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix} \right].$$

$$= (c_1 + c_2) e^t \begin{bmatrix} \overset{1}{\cancel{2}} \cos 2t \\ \overset{1}{\cancel{2}}(\cos 2t + \sin 2t) \end{bmatrix} + (c_1 i - c_2 i) e^t \begin{bmatrix} \overset{1}{\cancel{2}} \sin 2t \\ \overset{1}{\cancel{2}}(\sin 2t - \cos 2t) \end{bmatrix}$$

$$= \hat{c}_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + \hat{c}_2 e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}. \quad //$$

$$\vec{x}(0) = \hat{c}_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \hat{c}_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \hat{c}_1 \\ \hat{c}_1 - \hat{c}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$$\therefore \hat{c}_1 = 2 \quad \hat{c}_2 = -1.$$

Answer: $\vec{x}(t) = 2e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} - e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}.$

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3. $A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$. $\vec{g}(t) = \begin{bmatrix} t+2 \\ 1+e^{6t} \end{bmatrix}$. $\vec{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

분리.

$$\vec{g}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{6t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 16 = (\lambda-5)(\lambda+3) = 0$$

$$\Rightarrow \lambda = 5 \text{ or } \lambda = -3.$$

○ $(A - 5I) = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $\Rightarrow \begin{cases} -v_1 + v_2 = 0 \\ v_1 - v_2 = 0 \end{cases} \Rightarrow \{ v_1 = v_2$.

Chose $v_1 = 1 \rightarrow \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

○ $(A + 3I) = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \{ v_1 = -v_2$.

Chose $v_1 = 1 \longrightarrow \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} = \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix},\quad X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} = \begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix}.$$

$\Phi(t) = \begin{bmatrix} e^{5t} & e^{-3t} \\ e^{5t} & -e^{-3t} \end{bmatrix}$. $\det(\Phi(t)) = -e^{5t} \cdot e^{-3t} - e^{5t} \cdot e^{-3t} = -2e^{2t}$.

$$\operatorname{adj}(\Phi(t)) = \begin{bmatrix} -e^{-3t} & -e^{-3t} \\ -e^{5t} & e^{5t} \end{bmatrix}.$$

$$\Phi^{-1}(t) = \frac{1}{\det(\Phi(t))} \operatorname{adj}(\Phi(t)) = -\frac{1}{2} e^{-2t} \begin{bmatrix} -e^{-3t} & -e^{-3t} \\ -e^{5t} & e^{5t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{-5t} & \frac{1}{2}e^{-5t} \\ \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \end{bmatrix}.$$

$$X_p = \Phi(t) \int \Phi^{-1}(t) \vec{g}(t)\, dt = \begin{bmatrix} e^{5t} & e^{-3t} \\ e^{5t} & -e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{1}{2}e^{-5t} & \frac{1}{2}e^{-5t} \\ \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \end{bmatrix} \begin{bmatrix} t+2 \\ 1+e^{6t} \end{bmatrix} dt.$$

$$= \begin{bmatrix} e^{5t} & e^{-3t} \\ e^{5t} & -e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{1}{2}te^{-5t} + e^{-5t} + \frac{1}{2}e^{-5t} + \frac{1}{2}e^{t} \\ \frac{1}{2}te^{3t} + e^{3t} - \frac{1}{2}e^{3t} - \frac{1}{2}e^{9t} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^{5t} & e^{-3t} \\ e^{5t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} -\frac{1}{10}te^{-5t} - \frac{1}{50}e^{-5t} - \frac{3}{10}e^{-5t} + \frac{1}{2}e^{t} \\ \frac{1}{6}te^{3t} - \frac{1}{18}e^{3t} + \frac{1}{6}e^{3t} - \frac{1}{18}e^{9t} \end{bmatrix} = \begin{bmatrix} \frac{1}{15}t - \frac{47}{225} + \frac{4}{9}e^{6t} \\ -\frac{4}{15}t - \frac{97}{225} + \frac{5}{9}e^{6t} \end{bmatrix}.$$

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$$X = \begin{bmatrix} e^{5t} & e^{-3t} \\ e^{5t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{15}t - \frac{47}{255} + \frac{4}{9}e^{6t} \\ -\frac{4}{15}t - \frac{97}{225} + \frac{5}{9}e^{6t} \end{bmatrix}$$

$$\vec{X}(0) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{47}{225} + \frac{4}{9} \\ -\frac{97}{225} + \frac{5}{9} \end{bmatrix}.$$

$$\Rightarrow \begin{cases} c_1 + c_2 - \frac{47}{225} + \frac{4}{9} = -2 \\ c_1 - c_2 - \frac{97}{225} + \frac{5}{9} = 1. \end{cases}$$

$$\Rightarrow \begin{cases} c_1 + c_2 = -\frac{503}{225} \\ c_1 - c_2 = \frac{197}{225} \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{17}{25} \\ c_2 = -\frac{14}{9}. \end{cases}$$

$$\therefore X = \begin{bmatrix} e^{5t} & e^{-3t} \\ e^{5t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} -\frac{17}{25} \\ -\frac{14}{9} \end{bmatrix} + \begin{bmatrix} \frac{1}{15}t - \frac{47}{225} + \frac{4}{9}e^{6t} \\ -\frac{4}{15}t - \frac{47}{225} + \frac{5}{9}e^{6t} \end{bmatrix}.$$