DE class 05-29 /2023

System of linear ODEs (Phase diagram을 이용한 해의 분석)

Phase portrait of linear system.

Cases are divided by type of eigenvalues of coefficient matrix

$\lambda < \mu < 0$	sink(nodal)
$\lambda > \mu > 0$	source(nodal)
$\lambda < 0 < \mu$	saddle
$\lambda = \mu$	improper node

$\lambda, \overline{\lambda}$:complex

$Re(\lambda) > 0$	unstable spiral
$Re(\lambda) < 0$	stable spiral
$Re(\lambda) = 0$	center

(오른쪽 칼럼의 이름은 원점을 지칭하는 명칭으로 해가 갖는 동역학적 성질 설명)

(1) Nodal sink.

$$A = \begin{bmatrix} -6 & -2 \\ 5 & 1 \end{bmatrix} \qquad \begin{aligned} &\lambda = -4, -1 \\ &\overrightarrow{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \end{aligned}$$

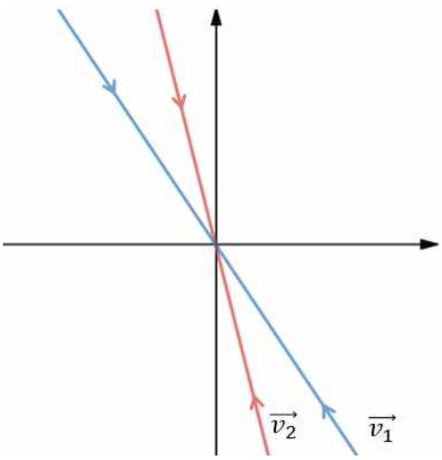

그림 1

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \frac{d}{dt}\vec{x}(0) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}$$

$$\text{at} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \quad \vec{x}' = \begin{bmatrix} -3-2 \\ \frac{5}{2}+1 \end{bmatrix} = \begin{bmatrix} -5 \\ \frac{7}{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} -6-2a \\ 5a+b \end{bmatrix}$$

$$5a + b = 0 \quad b = -5a$$
$$5a + b < 0 \quad b = 5a$$

$$
\begin{aligned}
\vec{x}(t) &= x_1 \boldsymbol{v_1} e^{-4t} + c_2 \boldsymbol{v_2} e^{-t} \\
&= e^{-t}\left(c_1 \boldsymbol{v_1} e^{-3t} + c_2 \boldsymbol{v_2}\right)
\end{aligned}
$$

$$\rightarrow c_2 \boldsymbol{v_2} e^{-t}$$

Two stores problem.

$x(t) =$ Daily profit of store A at time t.

$y(t) =$ Daily profit of store B at time t.

$x(t) > 0$ Making money.

$x(t) < 0$ Losing money.

$$
\begin{cases}
\dfrac{dx}{dt} = ax + by \\
\dfrac{dy}{dt} = cx + dy
\end{cases}
$$

$$
\begin{array}{ll}
x > 0 & a > 0 \\
y > 0 & b > 0
\end{array}
$$

$y > 0$ $\quad b < 0$ $\quad$ Store B steals customers from A.

(2) Source

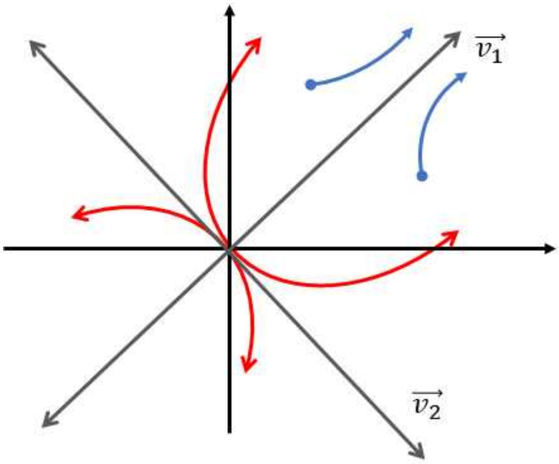

그림 2

$\lambda_1 > \lambda_2 > 0$

$\vec{x}(t) = c_1\overrightarrow{v_1}e^{\lambda_1 t} + c_2\overrightarrow{v_2}e^{\lambda_2 t}$

$\quad\quad = e^{\lambda_1 t}\left[c_1\overrightarrow{v_1} + c_2\overrightarrow{v_2}e^{(\lambda_2 - \lambda_1)t}\right]$

$-(\lambda_1 - \lambda_2) < 0$

$\rightarrow c_1\overrightarrow{v_1}e^{\lambda_1 t}$

(3) Saddle

$$A=\begin{bmatrix}-2 & -3 \\ -3 & -2\end{bmatrix} \qquad \begin{aligned}&\lambda = 1,\ -5 \\ &v_1 = \begin{bmatrix}1 \\ -1\end{bmatrix}\ v_2 = \begin{bmatrix}1 \\ 1\end{bmatrix}\end{aligned}$$

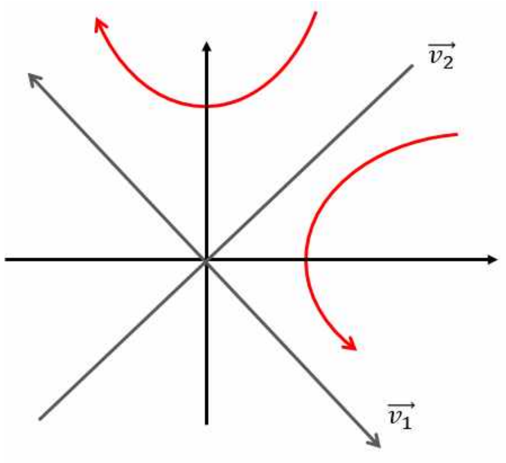

그림 3

Overcrowding effect.

$x_1(0) = x_2(0)$

$x_1(0) > x_2(0)$

$x_1(0) < x_2(0)$

(4) improper node

Ex)

$$A=\begin{bmatrix}-10 & 6\\ -6 & 2\end{bmatrix}\qquad \lambda=-4\quad v=\begin{bmatrix}1\\1\end{bmatrix}$$

$$(A+41)w=v$$

$$w=\begin{bmatrix}1\\ \frac{7}{6}\end{bmatrix}$$

$$\vec{x}(t)=c_1\vec{v_1}e^{-4t}+c_2(t\vec{v}+\vec{w})e^{-4t}$$

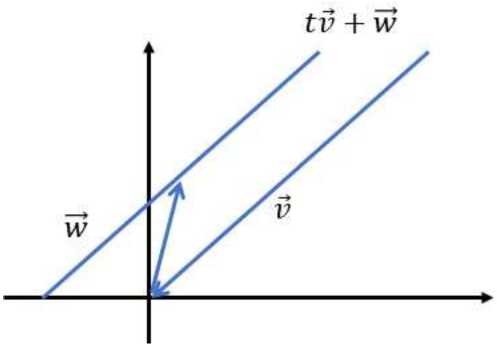

그림 4

$(\vec{v},\vec{w})>0\ \rightarrow$ Clockwise

$(\vec{v},\vec{w})<0\ \rightarrow$ Counter clockwise

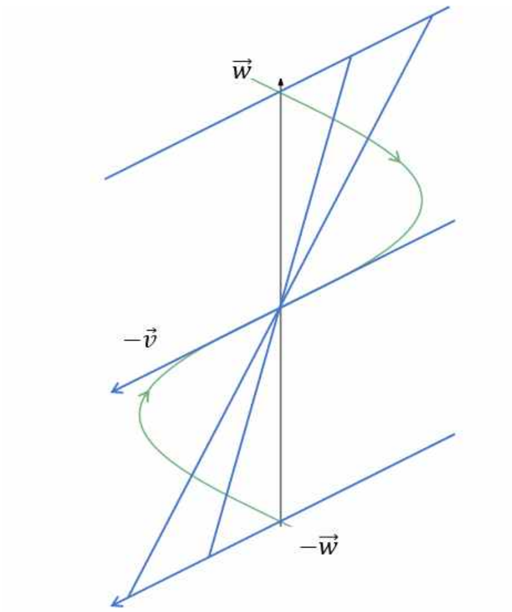

그림 5

$$(t\vec{v}+\vec{w})e^{-4t}$$

Curve transverses line.

$$\left[\left(c_1\vec{v_1}+c_2\vec{w}\right)+c_2\vec{v}t\right]e^{-4t}$$