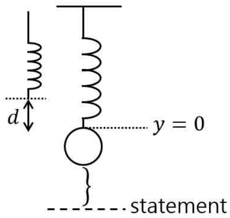
Section 2-3 Application of Second order ODE

Application of 2nd order DE

Analyze motion of object attached spring.

damper 마찰이나 저항을 일으키는 것.



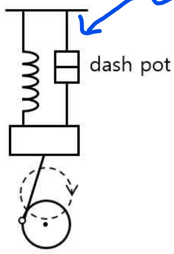


그림 1 그림 2

$$y(t) = displacement$$

시간에 따른 평면에 있는 평행운동을 하는 물체의 위치

$$\begin{cases} air\, rsistance \\ or\, viscosity\, of\, the\, medium \end{cases} \Rightarrow retarding\ \text{force}$$

$$\Downarrow$$

$$damping\ term$$

마찰력.

$$F = -ky - cy'$$

k는 용수철 상수, c는 변위.

Deriving force of magnitude $f(t)$.

복원력: ◦변위에 비례한다.
◦방향은 외부힘 방향의 반대이다.

$$F = -ky - cy' + f(t)$$

$$my'' = -ky - cy' + f(t)$$

$$y'' + \frac{k}{m}y' + \frac{y}{m}y = f(t)$$

저항력: 속도에 비례한다
: 방향은 속도의 방향의 반대이다.

1) Unforced motion. (자유운동)

$$f = 0$$

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$$

Case 1. $c^2 - 4km > 0$, λ_1, λ_2 (과잉 감쇠)

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$\lambda_2 < 0 \quad \lambda_1 = -\frac{c}{2m} + \frac{1}{2m}\sqrt{c^2 - 4km}$$

$$\sqrt{c^2 - 4km} < c$$

$$\Rightarrow -c+\sqrt{c^2-4km}<0$$

$$\lambda_1<0 \quad \lim_{t\to\infty} y(t)=0$$

Called "Over-damping."

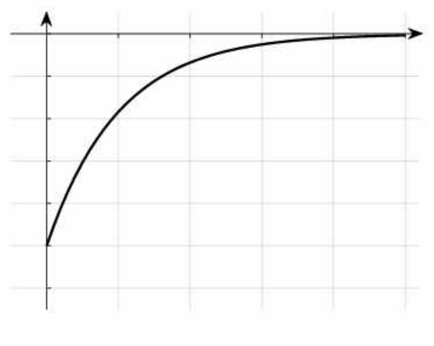


그림 3

Case 2. $c^2-4km=0$ (Critical damping), (임계 감쇠)

$$\lambda=-\frac{c}{2m}$$

$$y=e^{-\frac{c}{2m}t},\ te^{-\frac{c}{2m}t}$$

$$y=(C_1+C_2t)e^{-\frac{c}{2m}t}$$

$$\text{IC: } y(0)=-4,\ y'(0)=5$$

$$-4=C_1$$

$$5=(C_2-\frac{c}{2m}(-C_1))$$

$$=C_2+\frac{c}{2m}C_1$$

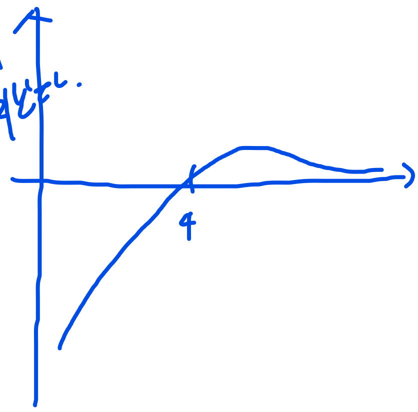
$$c=2, m=1, k=1$$

$$C_2=5-4=1$$

$$y=(-4+t)e^{-t}$$

$$\lim_{t\to 0}(-4+t)e^{-t}=0$$

t가 4일때 평형점을 지난다.



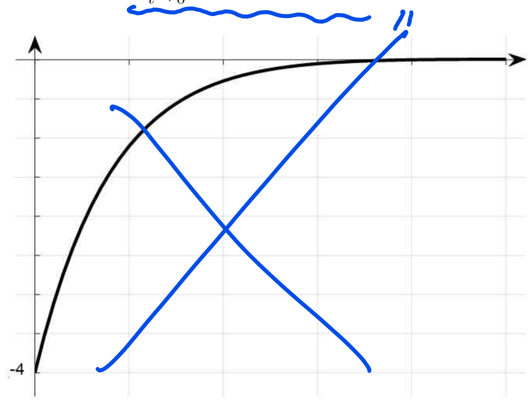


그림 4

The object passes through the equilibrium point exactly once.

Case 3. $c^2-4km<0$ (under damping), (부족감쇠)

$$\lambda=-\frac{c}{2m}\pm\frac{i}{2m}\sqrt{4km-c^2}$$

$$=-\frac{2}{2m}\pm i\beta$$

$$y=e^{-\frac{c}{2m}t}\cos(\beta t),\ e^{-\frac{c}{2m}t}\sin(\beta t)$$

$$y = e^{-\frac{c}{2m}t}\left[C_1\cos(\beta t) + C_2\sin(\beta t)\right]$$

$$\lim_{t\to\infty} y(t) = 0$$

Ex)

$$c = k = 2,\ m = 1$$

$$\beta = \frac{1}{2}\sqrt{8-4} = 1$$

$$y(0) = -3,\ \ y'(0) = 2$$

$$y(t) = -e^{-t}(3\cos t + \sin t)$$

진동.

진폭이 줄어듭니다.

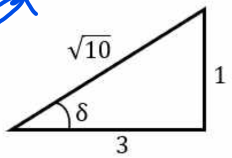


그림 5

진폭을 알아내기 위해 3cost + sint를 합친다.

$$\cos\delta = \frac{3}{\sqrt{10}},\ \sin\delta = \frac{3}{\sqrt{10}}$$

$$\begin{aligned} 3\cos t + \sin t &= \sqrt{10}(\cos\delta\cos t + \sin t\sin\delta) \\ &= \sqrt{10}\cos(t-\delta) \end{aligned}$$

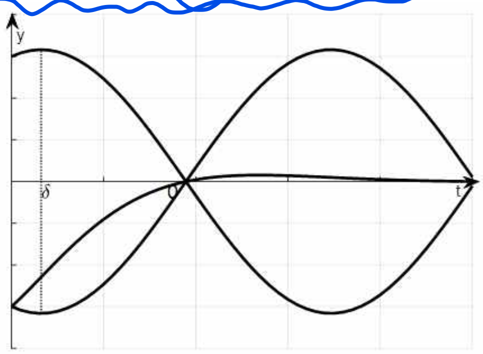


그림 6

$$\arctan\frac{1}{3} = \delta$$

$$y = -\sqrt{10}\,e^{-t}\cos(t-\delta)$$

envelope

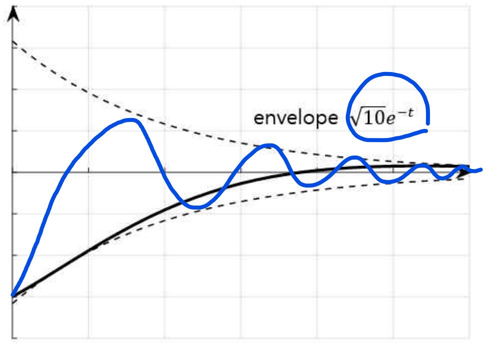


그림 7

Harmonic Oscillation (조화진동)

$$my'' + ky = 0, \quad c = 0 \quad \text{(no damping)}$$

$$mr^2 + k = 0 \qquad r^2 + \frac{k}{m} = 0 \qquad r = \pm\sqrt{\frac{k}{m}}i$$

$$y = C_1 \cos\sqrt{\frac{k}{m}}t + C_2 \sin\sqrt{\frac{k}{m}}t$$

Set $\omega = \sqrt{\frac{k}{m}}$ natural frequency

Period $\omega T = 2\pi \quad T = \frac{2\pi}{\omega}$

$$A = \sqrt{C_1^2 + C_2^2}$$

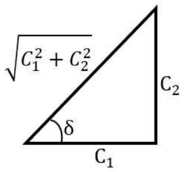


그림 8

$$y_1 = A(\cos\delta\cos(\omega t) + \sin\delta\sin(\omega t))$$
$$= A\cos(\omega t - \delta)$$

A, B Determined by initial conditions.

Damped Oscillation (감쇠진동)

$$my'' + cy' + ky = 0, \quad c \neq 0$$

$$mr'' + cr + k = 0 \quad k = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$$c^2 - 4km < 0 \qquad r = -\frac{c}{2m} \pm i\mu$$

$$\mu = \frac{\sqrt{4km - c^2}}{2m}$$

$$y = e^{-\frac{c}{2m}t}(C_1\cos(\mu t) + C_2\sin(\mu t))$$

$$
\begin{aligned}
\mu &= \frac{1}{2m}\sqrt{4km-c^2} \\
&= \sqrt{\frac{4km}{4m^2}-\frac{c^2}{4m^2}} \\
&= \sqrt{\frac{k}{m}-\frac{1}{4}\left(\frac{c}{m}\right)^2} \qquad \frac{k}{m}=\omega_0^2 \\
&= \left[\omega_0^2-\left(\frac{c}{2m}\right)^2\right]^{\frac{1}{2}} \\
&= \omega_0\left[1-\left(\frac{c}{2m\omega_0}\right)^2\right]^{\frac{1}{2}}
\end{aligned}
$$

Use $(1+t)^{\frac{1}{2}} \approx 1+\frac{1}{2}t$ when $t \ll 1$

$$\mu \approx \omega_0\left[1-\frac{1}{2}\left(\frac{c}{2m\omega_0}\right)^2\right]$$

$\hookrightarrow$ quasi frequency (준진동수)

$T_d = \frac{2\pi}{\mu}$ Quasi period (준주기)

$T_d > T$

Ex) **Overdamping case.**

$$c=6, k=5, m=1$$

$$c^2-4km=36-20>0$$

$$\lambda^2+6\lambda+5=(\lambda+5)(\lambda+1)=0 \quad \lambda=-1,-5$$

$$y=C_1e^{-5t}+C_2e^{-t}$$

$$y(0)=-4$$

$$y'(0)=0$$

$$-4=C_1+C_2$$

$$0=-5C_1-C_2$$

$$-4=-4C_1 \qquad C_1=1, \quad C_2=-5$$

$$
\begin{aligned}
y &= e^{-5t}-5e^{-t} \\
&= e^{-t}(-5+e^{-4t})<0
\end{aligned}
$$

The object remarks below the equilibrium point.

Ex)

$$\frac{c}{2m}=1$$

$$c^2-4km=0 \ \Rightarrow \ m=k$$

$$y=(C_1+C_2t)e^{-t}$$

Case $\frac{C_1}{C_2}$ has positive sign.

$$y=-(1+t)e^{-t}$$

$$y=(t+1)e^{-(t+1)}=e^{-1}(1+t)e^{-t}$$

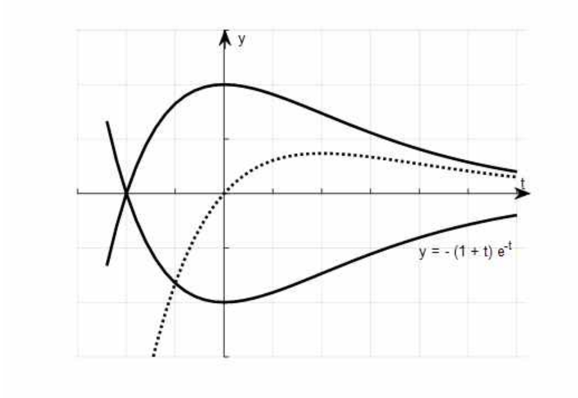


그림 9

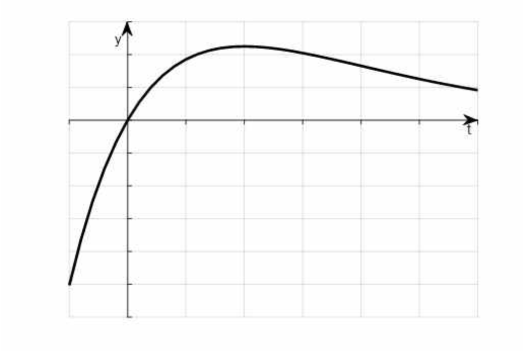


그림 10

$$y(0)=C_1$$

$$y'(0)=C_2-C_1 \quad C_2=y(0)+y'(0)$$

$$y=(y(0)+(y(0)+y'(0))t)e^{-t}$$

$y(0)=-1$ Not pass rest position
$y'(0)=-1$

2) Forced motion. (강제운동)

Periodic driving force $f=\cos(\omega t) \Rightarrow$ resonance & beats $\Rightarrow$ analogy with an electric circuit

$$y''+\frac{c}{m}y'+\frac{k}{m}y=\frac{A}{m}\cos(\omega t)$$

Periodic driving force.

$$y_p=a\cos(\omega t)+b\sin(\omega t)$$

Ex)

$$c=6, k=5, m=1 \qquad A=6\sqrt{5}, \omega=\sqrt{5}$$

$$c^2-4km=35-4 \bullet 5>0$$

Over damped.

$$y'' + 6y' + 5y = 6\sqrt{5}\cos(\sqrt{5}t)$$

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$\lambda^2 + 6\lambda + 5 = (\lambda+5)(\lambda+1)$$

특수해의 후보는 $y_p = a\cos(\sqrt{5}t) + b\sin(\sqrt{5}t)$

$$y_p' = \sqrt{5}(-a\sin(\sqrt{5}t) + b\cos(\sqrt{5}t))$$

$$y_p'' = 5(-a\cos(\sqrt{5}t) - b\sin(\sqrt{5}t))$$

특수해의 후보를 미분방정식에 대입하면

$$(\sqrt{5}b - 5a)\cos(\sqrt{5}t) - (5b + \sqrt{5}a)\sin(\sqrt{5}t) \\ = 6\sqrt{5}\cos(\sqrt{5}t)$$

$$(-5a + 6\sqrt{5}b + 5a)\cos(\sqrt{5}t) + (-5b - 6\sqrt{5}a + 5b)\sin(\sqrt{5}t) = 6\sqrt{5}\cos(\sqrt{5}t)$$

$$6\sqrt{5}b = 6\sqrt{5} \qquad -6\sqrt{5}a = 0 \qquad b = 1, a = 0$$

$$y_p = \sin(\sqrt{5}t)$$

$$y = C_1e^{-5t} + C_2e^{-t} + \sin(\sqrt{5}t) \\ = (transient) + (steady\,state)$$

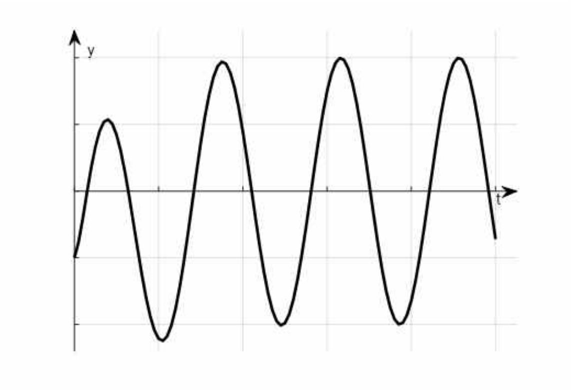


그림 11

Forced motion with damping.

Ex)

$$y'' + y' + 1.25y = 3\cos t$$

$$1 + \frac{1}{4} = \frac{5}{4}$$

$$m = 1, A = 3, k = \frac{5}{4}, c = 1$$

$$c^2 - 4km = 1 - 4 \bullet \frac{5}{4} = 1 - 5 < 0$$

$$\lambda^2 + \lambda + \frac{5}{4} = 0$$

$$(\lambda + \frac{1}{2})^2 = -1,\ \ \lambda + \frac{1}{2} = \pm i,\ \ \lambda = -\frac{1}{2} \pm i$$

$$y = e^{-\frac{1}{2}t}(C_1\cos t + C_2\sin t)$$

$$y_p = a\cos t + b\sin t$$

$$y_p{}' = -a\sin t + b\cos t$$

$$y_p{}'' = -a\cos t - b\sin t$$

$$y_p{}'' + y_p{}' + \frac{5}{4}y_p = (-a+b+\frac{5}{4}a)\cos t + (-b-a+\frac{5}{4}b)\sin t = 3\cos t$$

$$\frac{1}{4}a + b = 3 \qquad a + 4b = 12$$

$$-a + \frac{1}{4}b = 0 \qquad (4+\frac{1}{4})b = 12 \qquad b = \frac{48}{17}$$

$$a = \frac{1}{4}\frac{48}{17} = \frac{12}{17}$$

$$y_p = \frac{12}{17}\cos t + \frac{48}{17}\sin t$$

$$y = e^{-\frac{1}{2}t}(C_1\cos t + C_2\sin t) + y_p$$

$$\begin{cases} y(0) = 2 \\ y'(0) = 3 \end{cases} \Rightarrow C_1 = \frac{12}{17},\ C_2 = \frac{14}{17}$$

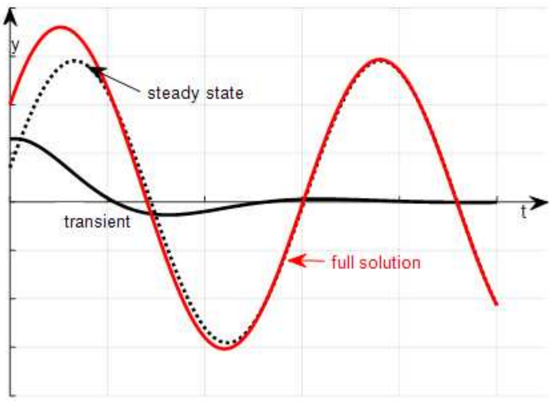


그림 12

Amplitude of steady state.

$$\frac{12}{17}(\cos t + 4\sin t) = \frac{12}{\sqrt{17}}\cos(t-\delta)$$

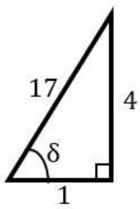


그림 13

$$\tan\delta = 4$$

$$\frac{12}{\sqrt{17}} = \frac{12}{\sqrt{16+1}} = \frac{12}{4\sqrt{1+\frac{1}{16}}} = 3\left(1+\frac{1}{16}\right)^{-\frac{1}{2}}$$
$$\fallingdotseq 3(1-\frac{1}{32})$$
$$\fallingdotseq 3\times\frac{31}{32}$$

3 = Amplitude of external force

$$my'' + cy' + ky = F_0\cos(\omega t)$$
$$y = C_1 y_1 + C_2 y_2 + A\cos(\omega t) + B\sin(\omega t)$$
$$S(t) = F_0(A'\cos(\omega t) + B'\sin(\omega t)) = R\cos(\omega t - \delta)$$
$$R = F_0\sqrt{A^2 + B^2}$$
$$\frac{R}{F_0} = \frac{1}{\sqrt{m^2(\omega_0^2 - \omega^2) + c^2\omega^2}}$$
$$= \frac{1}{k\left[\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \frac{c^2}{mk}\frac{\omega^2}{\omega_0^2}\right]^{\frac{1}{2}}}$$
$$\frac{Rk}{F_0} = \frac{1}{\left[\left(1-\frac{\omega^2}{\omega_0^2}\right) + \frac{c^2}{mk}\frac{\omega^2}{\omega_0^2}\right]^{\frac{1}{2}}}$$
$$\Gamma = \frac{c^2}{mk} \ll 1,\quad \frac{\omega}{\omega_0} \to 1 \quad \text{blow up}$$

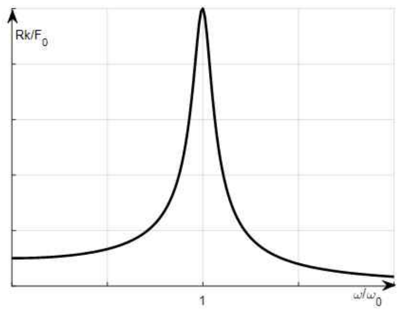


그림 14

F_0에 곱해지는 Amplitude ⇒ Resonance
Γ는 c가 작을 때 작아진다.

* Analogy with an electric circuit.
R: Resistance
L: Inductance

C: Capacitance

$E(t)$: Eletromotive force

$$E(t) = Li'(t) + Ri(t) + \frac{1}{C}q(t)$$

$$i = q'$$

$$E(t) = Lq''(t) + Rq'(t) + \frac{1}{C}q(t)$$

$$q'' + \frac{R}{L}q' + \frac{1}{LC}q = \frac{1}{L}E$$

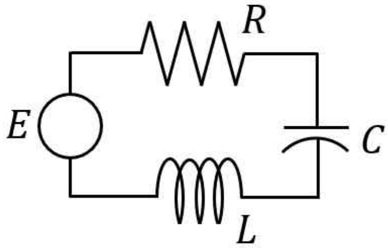


그림 15

Mass $\leftrightarrow L$

Damping constant $\leftrightarrow R$

Spring constant $\leftrightarrow \frac{1}{C}$

Driving force $\leftrightarrow E$

$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{1}{m}f$

Ex)

$$E(t) = 17\sin(2t)$$

$$i(t) = ?$$

$$\begin{cases} q(0) = \frac{1}{2000} \\ q'(0) = i(0) = 0 \end{cases}$$

$$Lq'' + Rq' + \frac{q}{C} = E$$

$$10q'' + 120q' + 10^3 1 = 17\sin(2t)$$

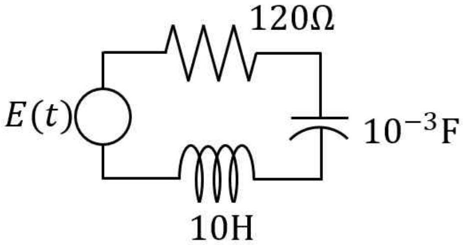


그림 16

$$q'' + 12q' + 10^3 q = \frac{17}{10}\sin(2t)$$

$$\lambda^2 + 12\lambda + 100 = 0$$

$$\begin{aligned} \lambda &= \frac{-12 \pm \sqrt{12^2 - 4 \cdot 100}}{2} \\ &= -6 \pm \sqrt{36 - 100} \\ &= -6 \pm 8i \end{aligned}$$

$$q(t) = e^{-6t}(C_1\cos(8t) + C_2\sin(8t)) + q_0$$

$$q_0 = a\cos(2t) + b\sin(2t)$$

$$-4a\cos(2t) - 4b\sin(2t) + 12(-2a\sin(2t) + 2b\cos(2t)) + 100(a\cos(2t) + b\sin(2t)) = 1.7\sin(2t)$$

$$-4a + 24b + 100a = 0 \qquad 96a + 24b = 0$$

$$-4b - 24a + 100b = 1.7 \qquad -24a + 96b = 1.7$$

$$4a + b = 0$$

$$-a + 4b = \frac{1.7}{24}$$

$$17b = \frac{1.7}{6} \qquad b = \frac{1}{60}$$

$$a = -\frac{1}{240}$$

$$q_0 = -\frac{1}{240}\cos(2t) + \frac{1}{60}\sin(2t)$$

$$q(0) = C_1 - \frac{1}{240} = \frac{1}{2000}$$

$$q'(0) = -6C_1 + 8C_2 + \frac{1}{30} = 0$$

$$C_1 = \frac{1}{2400} + \frac{1}{2000} = \left(\frac{1}{24} + \frac{1}{200}\right)\frac{1}{10}$$

$$C_1 = \frac{14}{3000} = \frac{7}{1500}, \quad C_2 = -\frac{1}{1500}$$

$$i(t) = q'(t) = -\frac{\sqrt{2}}{30}e^{-6t}\sin\left(8t + \frac{\pi}{4}\right) + \frac{\sqrt{17}}{120}\sin(2t + \alpha)$$

$$q(t) = \frac{1}{1500}e^{-6t}[7\cos(8t) - \sin(8t)] + \frac{1}{240}[-\cos(2t) + 4\sin(2t)]$$

$$i(t) = q'(t) = -\frac{1}{30}e^{-6t}[\cos(8t) + \sin(8t)] + \frac{1}{120}[4\cos(2t) + \sin(2t)]$$

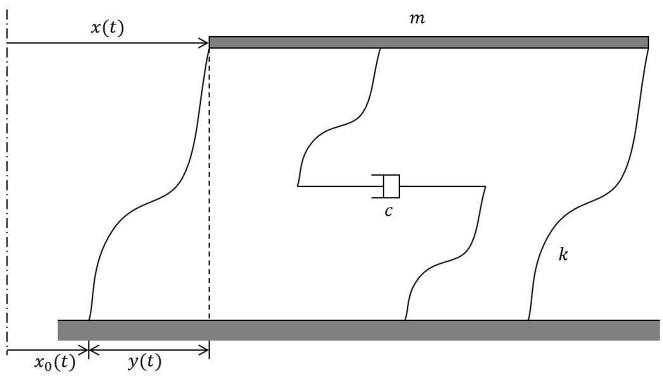


그림 17