phase portrait

DE class 05-29 /2023

System of linear ODEs (Phase diagram을 이용한 해의 분석)

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \text{homogeneous}$$

$\vec{x}(0) = \vec{x}_0$

$\Rightarrow \vec{x}(t) \Rightarrow ?$

= trajectory

Phase portrait of linear system.

Cases are divided by type of eigenvalues of coefficient matrix

$\lambda < \mu < 0$	sink(nodal)
$\lambda > \mu > 0$	source(nodal)
$\lambda < 0 < \mu$	saddle
$\lambda = \mu$	improper node
$\lambda, \overline{\lambda}$:complex	
$Re(\lambda) > 0$	unstable spiral
$Re(\lambda) < 0$	stable spiral
$Re(\lambda) = 0$	center

$\vec{x}(t) \to \vec{0}$

$\vec{x}(t) \to +\infty$

(오른쪽 칼럼의 이름은 원점을 지칭하는 명칭으로 해가 갖는 동역학적 성질 설명)

(1) Nodal sink.

$A = \begin{bmatrix} -6 & -2 \\ 5 & 1 \end{bmatrix}$ $\quad \lambda = -4, -1$

$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$\vec{x}(t) = e^{-t}(c_2\vec{v}_2 + c_1\vec{v}_1 e^{-3t})$

$t \to \infty$

$\approx c_2\vec{v}_2 e^{-t}$

$\vec{x}(t) = c_1\vec{v}_1 e^{-4t} + c_2\vec{v}_2 e^{-t}$

$\vec{x}(0) \Rightarrow c_1, c_2$

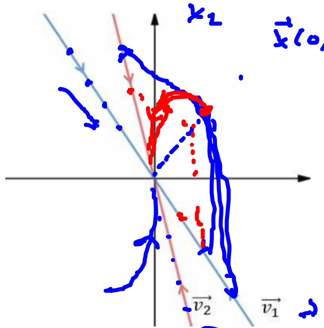
$\vec{x}(0) = \vec{v}_1$

$\vec{x}(t) = \vec{v}_1 e^{-4t}$

$\vec{x}(0) = \vec{v}_2$

$\vec{x}(t) = \vec{v}_2 e^{-t}$

$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$\lambda = -4$ 고유벡터

그림 1 (-1)

$\Rightarrow \vec{v}_2$ (-1 > -4) 점근선

$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \frac{d}{dt}\vec{x}(0) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \end{bmatrix}$

at $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \quad \vec{x}' = \begin{bmatrix} -3-2 \\ \frac{5}{2}+1 \end{bmatrix} = \begin{bmatrix} -5 \\ \frac{7}{2} \end{bmatrix}$

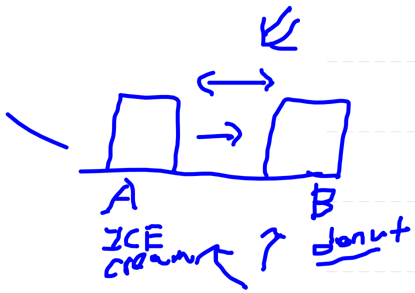
$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} -6-2a \\ 5a+b \end{bmatrix}$

$5a + b = 0 \quad b = -5a$

$5a + b < 0 \quad b = 5a$

$$\begin{aligned}\vec{x}(t) &= x_1\boldsymbol{v_1}e^{-4t}+c_2\boldsymbol{v_2}e^{-t}\\ &= e^{-t}\left(c_1\boldsymbol{v_1}e^{-3t}+c_2\boldsymbol{v_2}\right)\\ &\to c_2\boldsymbol{v_2}e^{-t}\end{aligned}$$



Two stores problem.

- $x(t) =$ Daily profit of store A at time t.
- $y(t) =$ Daily profit of store B at time t.
- $x(t) > 0$ Making money.
- $x(t) < 0$ Losing money.

=

$$\begin{cases}\dfrac{dx}{dt} = ax+by\\[4pt] \dfrac{dy}{dt} = cx+dy\end{cases}$$

$$\frac{dx}{dt} = ax + by \qquad b<0$$

$$\frac{dy}{dt} = cx + dy$$

$b, c > 0$?

$x>0 \quad a>0$
$y>0 \quad b>0$

$y>0 \quad b<0$ Store B steals customers from A.

(2) Source

$\vec{v}_1$ $(\lambda_1 = 3)$

점근선

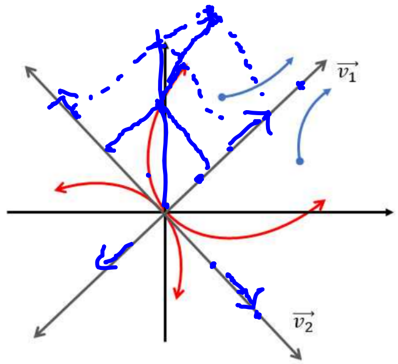


그림 2

$\lambda_1 > \lambda_2 > 0$

$\vec{x}(t) = c_1\vec{v_1}e^{\lambda_1 t} + c_2\vec{v_2}e^{\lambda_2 t}$

$\quad = e^{\lambda_1 t}\left[c_1\vec{v_1} + c_2\vec{v_2}e^{(\lambda_2-\lambda_1)t}\right]$

$-(\lambda_1 - \lambda_2) < 0$

$\rightarrow c_1\vec{v_1}e^{\lambda_1 t}$

$\frac{d\vec{x}}{dt} = A\vec{x}$

$A \quad \lambda_1 = 3$

$\quad\ \ \lambda_2 = 1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $(\lambda_1 = 3)$

$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $(\lambda_2 = 1)$

$c_1\vec{v}_1 e^{3t} + c_2\vec{v}_1 e^{t}$

$= e^{3t}\left(c_1\vec{v}_1 + \underbrace{c_2\vec{v}_2 e^{-2t}}_{0}\right)$

(3) Saddle

$$A=\begin{bmatrix}-2 & -3\\ -3 & -2\end{bmatrix} \qquad \lambda = 1,\ -5$$

$$\boldsymbol{v}_1=\begin{bmatrix}1\\ -1\end{bmatrix} \quad \boldsymbol{v}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\frac{dx}{dt} = -2x - 3y$$

$$\frac{dy}{dt} = -3x - 2y$$

$$\vec{x}(t) = c_1 \vec{v}_1 e^{t} + c_2 \vec{v}_2 e^{-5t}$$

$$\vec{x}(0) = \vec{v}_2$$

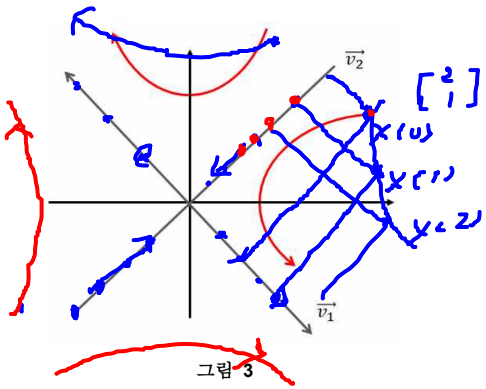
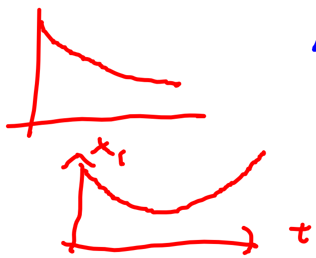
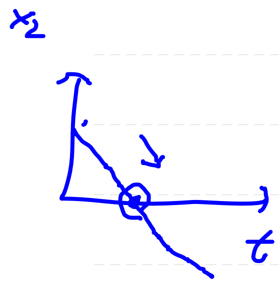


그림 3





Overcrowding effect. $\Rightarrow$ $x_1(1) = x_2(1)$ $\Rightarrow$

$x_1(0) = x_2(0)$

$x_1(0) > x_2(0)$ $x_1 \searrow,\ x_2 \searrow$ $x_1 \nearrow$

$x_1(0) < x_2(0)$

(4) improper node

Ex)

$$A=\begin{bmatrix}-10 & 6\\ -6 & 2\end{bmatrix}\qquad \lambda=-4\quad \boldsymbol{v}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$(A+41)\boldsymbol{w}=\boldsymbol{v}$$

$$\boldsymbol{w}=\begin{bmatrix}1\\ \frac{7}{6}\end{bmatrix}$$

$$\vec{x}(t)=c_1\vec{v}\,e^{-4t}+c_2(t\vec{v}+\vec{w})e^{-4t}$$

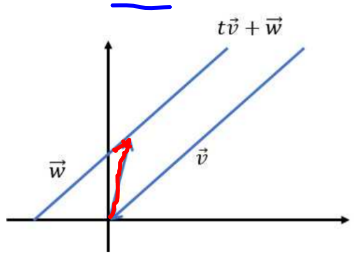


그림 4

$(\vec{v},\vec{w})>0 \rightarrow$ Clockwise

$(\vec{v},\vec{w})<0 \rightarrow$ Counter clockwise

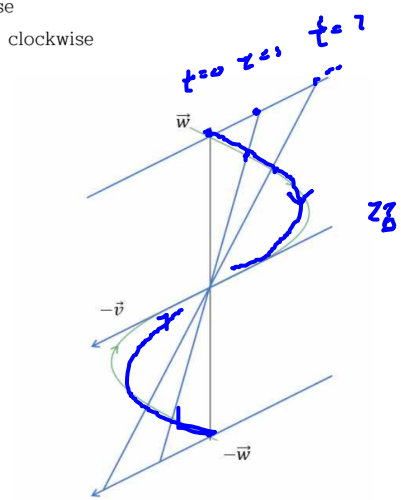
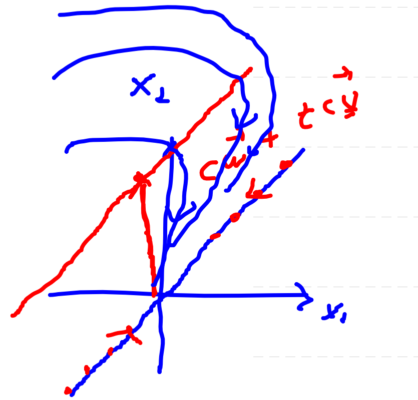


그림 5



$\vec{x}(0)=c\vec{w}$

$\vec{x}(t)=c(t\vec{v}+\vec{w})e^{-4t}$

$t\vec{v}+\vec{w}$

$$(t\vec{v}+\vec{w})e^{-4t}$$

Curve transverses line.

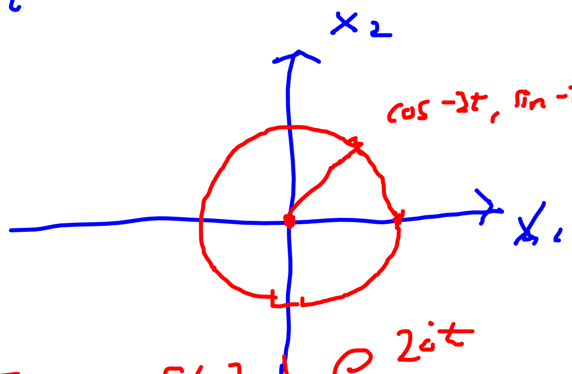
$$\left[(c_1\vec{v_1}+c_2\vec{w})+c_2\vec{v}t\right]e^{-4t}$$

$\frac{d\vec{x}}{dt} = A\vec{x}$

$(c=1)$ $\pm 2i$ $\begin{bmatrix}1\\ i\end{bmatrix}$ 고유벡터

$$\begin{bmatrix}1\\ i\end{bmatrix} e^{2it} = \left(\begin{bmatrix}1\\ 0\end{bmatrix} + i\begin{bmatrix}0\\ 1\end{bmatrix}\right)(\cos 2t + i\sin 2t)$$

$$Re(\,''\,) = \begin{bmatrix}1\\ 0\end{bmatrix}\cos 2t - \begin{bmatrix}0\\ 1\end{bmatrix}\sin 2t = \begin{bmatrix}\cos 2t\\ -\sin 2t\end{bmatrix} = \begin{bmatrix}\cos -2t\\ \sin -2t\end{bmatrix}$$



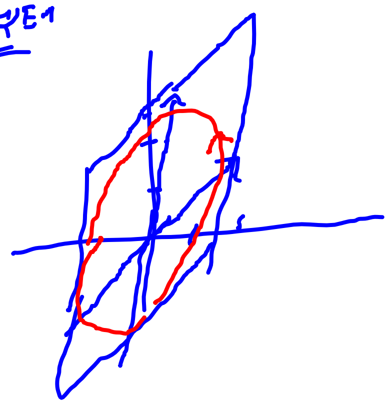
Center

$$\left(\begin{bmatrix}a_1\\ a_2\end{bmatrix} + i\begin{bmatrix}b_1\\ b_2\end{bmatrix}\right) e^{2it} =$$

$$Re \quad \begin{bmatrix}a_1\\ a_2\end{bmatrix}\cos 2t - \begin{bmatrix}b_1\\ b_2\end{bmatrix}\sin 2t = \begin{bmatrix}a_1 & -b_1\\ a_2 & -b_2\end{bmatrix}\begin{bmatrix}\cos 2t\\ \sin 2t\end{bmatrix}$$

$\begin{bmatrix}2-i\\ 1-3i\end{bmatrix}$ 고유벡터 $\quad ''\ \begin{bmatrix}2\\ 1\end{bmatrix} \quad \begin{matrix}-1\\ -3\end{matrix} \quad \begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$

$$\Rightarrow = \begin{bmatrix}\cos 2t + \sin 2t\\ \cos 2t + 3\sin 2t\end{bmatrix}$$



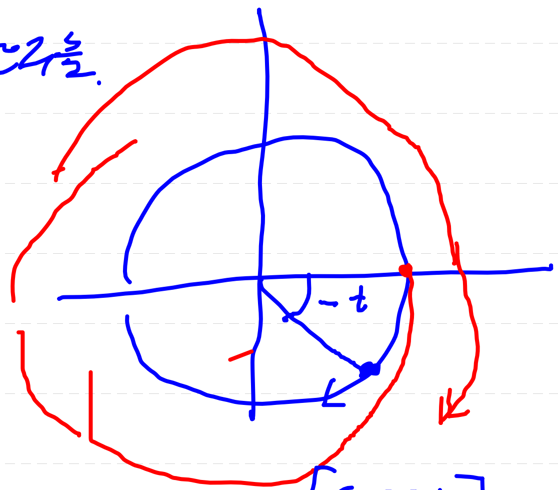
$\lambda = 1 \pm 2i \quad \mathrm{Re}\,\lambda > 0$

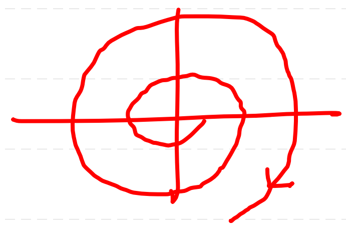
$\begin{bmatrix} 1 \\ i \end{bmatrix} e^{t} e^{2it}$

$\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$e^{t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix}$

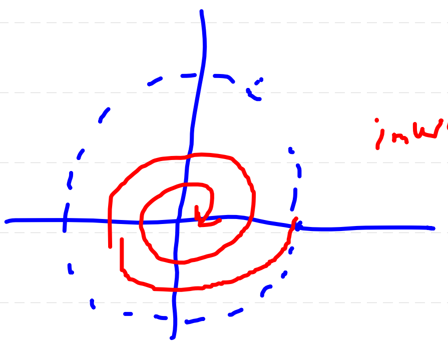
원의 반지름





unstable spiral

$\lambda = -1 + 2i \qquad e^{-t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix}$



inward spiral

stable