미분방정식 강의노트 라플라스 변환 pt 2

Shifting theorem. (이동정리)

1) 왜 shifting theorem이 필요한가?

비동차 이계선형 미분방정식에 라플라스 변환을 적용했을 때 맞이하는 상황 =>

$$s^2Y(s)-sy(0)-y'(0)+A(sY(s)-y(0))+BY(s)=(s^2+As+B)Y(s)-y(0)s-y'(0)-Ay(0)$$

$$(s^2+As+B)Y(s)=y_1+F(s)$$

$$\begin{aligned}Y(s)&=\frac{y_1}{s^2+As+B}+F(s)\frac{1}{s^2+As+B}\\&=\frac{y_1}{(s-\alpha)(s-\beta)}\end{aligned}$$

$$\mathcal{L}^{-1}\left[\frac{1}{\alpha-\beta}\left(\frac{1}{s-\alpha}-\frac{1}{s-\beta}\right)\right]=\frac{1}{\alpha-\beta}\left(e^{\alpha t}-e^{\beta t}\right)$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-\alpha)^2}\right]=? \qquad \mathcal{L}[t]=\frac{1}{2^2}$$

$$\mathcal{L}[e^{\alpha t}t]=\frac{1}{(s-\alpha)^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+\alpha)^2+\beta^2}\right]=?$$

2) Step functions. (shifting theorems)

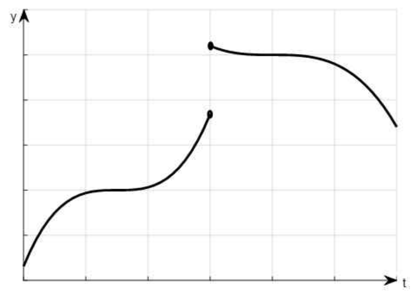

그림 1

불연속 함수 f의 라플라스 변환 $\mathcal{L}[f]=?$

Step function / Heaviside function

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

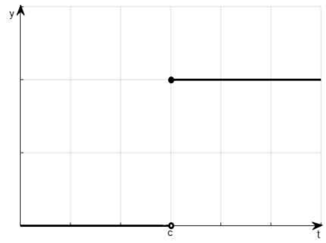

그림 2

Rectangular pulse.

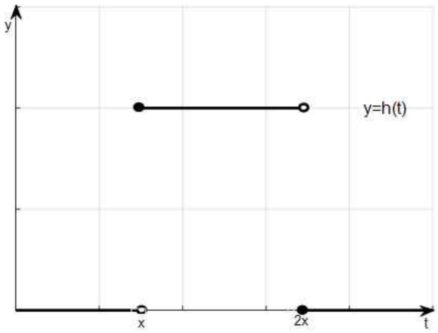

그림 3

$$h(t) = \begin{cases} 0 & t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$

$$h(t) = t_\pi(t) - u_{2\pi}(t)$$

(Laplace transform of step function)

$$\begin{aligned} \mathcal{L}[u_c(t)] &= \int_0^\infty e^{-st} u_c(t) dt \\ &= \int_c^\infty e^{-st} dt = -\frac{1}{s} e^{-st}|_c^\infty \\ &= \frac{1}{s} e^{-cs}, \quad s > 0 \end{aligned}$$

Application => When a signal function g starts from t=c

$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$$

표현하기 : $g = u_c(t) f(t-c)$

$$\mathcal{L}[g] = ?$$

Theorem. (이동정리)

$$\mathcal{L}\,[u_c(t)f(t-c)] = e^{-cs}\mathcal{L}\,[f]$$

Proof)

$$\begin{aligned}
&\int_0^\infty e^{-st}u_c(t)f(t-c)dt \\
&= \int_0^\infty e^{-st}f(t-c)dt \qquad \tau = t-c \\
&= \int_0^\infty e^{-s(\tau-c)}f(\tau)d\tau \\
&= e^{-cs}\int_0^\infty e^{-s\tau}f(\tau)d\tau = e^{-cs}\mathcal{L}\,[f](s)
\end{aligned}$$

Ex)

$$f(t) = \begin{cases} \sin t & 0 \le t < \dfrac{\pi}{4} \\ \sin t + \cos(t - \dfrac{\pi}{4}) & t \ge \dfrac{\pi}{4} \end{cases}$$

$$\mathcal{L}\,[f] = ?$$

(표현하기) $\quad f(t) = \sin t + u_{\pi/4}\cos(t - \dfrac{\pi}{4})$

Use linearity of Laplace transform and shifting theorem

$$\begin{aligned}
\mathcal{L}\,[f] &= \mathcal{L}\,[\sin t] + \mathcal{L}\,[u_{\pi/4}(t)\cos(t-\pi/4)] \\
&= \frac{1}{s^2+1} + e^{-\pi/4s}\mathcal{L}\,[\cos t] \\
&= \frac{1}{s^2+1} + e^{-\pi/4s}\frac{s}{s^2+1}
\end{aligned}$$

Ex)

$$F(s) = \frac{1-e^{-2s}}{s^2}$$

$$\mathcal{L}^{-1}[F] = ?$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}\left[e^{-2s}\frac{1}{s^2}\right]$$

$$\mathcal{L}\,[t] = \frac{1}{s^2}, \qquad \mathcal{L}\,[t^n] = \frac{n!}{s^{n+1}}(s>0)$$

Here the second term has an exponential function. So we have to use shifting theorem

$$\mathcal{L}\,[u_2(t)f(t-2)] = e^{-2s}\mathcal{L}\,[f](s)$$

Since $\mathcal{L}\,[f](s) = 1/s^2$, $f(t) = t$.

$$\mathcal{L}\,[u_2(t)(t-2)] = e^{-2s}\frac{1}{s^2}$$

ANS)

$$t - u_2(t)(t-2) = \begin{cases} t & t < 2 \\ 2 & t \geq 2 \end{cases}$$

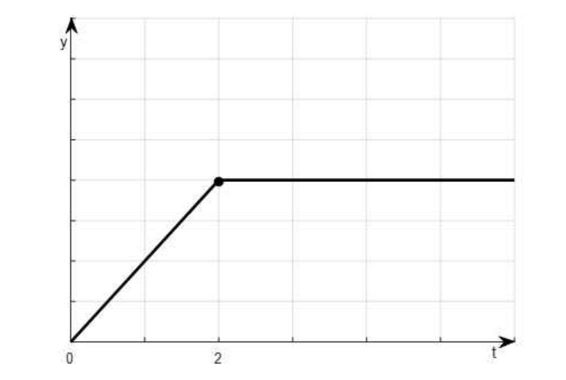

그림 4

Motivation: inverse Laplace transformation of shifted function

Theorem. (두번째 이동 정리)

$$\mathcal{L}\left[e^{ct}f(t)\right] = F(s-c), \quad s < a+c$$

$$\text{if } \mathcal{L}[f](s) = F(s)$$

Ex)

$$\begin{aligned} \mathcal{L}\left[te^{2t}\right] &= \mathcal{L}[t](s-2) \\ &= \frac{1}{(s-2)^2} \end{aligned}$$

Ex)

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-4s+5}\right]$$

$$s^2 - 4s + 5 = (s-2)^2 + 1$$

$$D = (-4)^2 - 4 \times 5 = 16 - 20 < 0$$

$$\frac{1}{(s-2)^2+1} \leftarrow \frac{1}{s^2+1}$$

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$$

$$\mathcal{L}\left[e^{ct}\sin t\right] = \frac{1}{(s-c)^2+1}$$

$$\Rightarrow c = 2$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2+1}\right] = e^{2t}\sin t$$