미분방정식 05-04 수업

T = 주기

piecewise continuous function f s.t. $f(t+T)=f(t)$

$$\mathcal{L}[f](s)=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt = \sum_{k=0}^{\infty} e^{-kTs}\int_0^T e^{-st}f(t)\,dt$$

Ex)

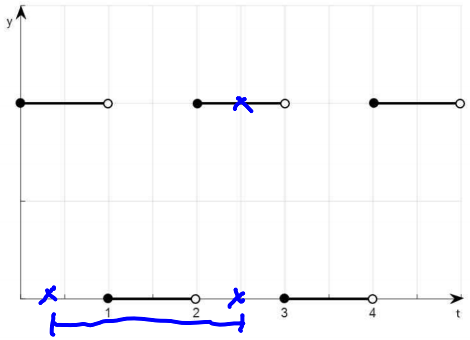


그림 1

$E(t)=\begin{cases}1, & 0\le t<1\\ 0, & 1\le t<2\end{cases}$ => $E(t+\underline{\underline{2}})=E(t)$

$$\mathcal{L}[E(t)] = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}E(t)\,dt = \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}\,dt$$

$$= \frac{1}{1-e^{-2s}}\left[-\frac{1}{s}\left(e^{-s}-1\right)\right]$$

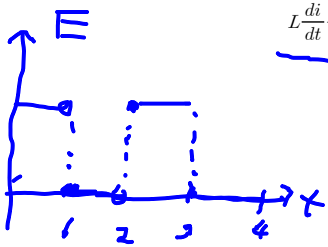
$$= \frac{1}{s}\,\frac{1-e^{-s}}{1-e^{-2s}} = \frac{1}{s}\,\frac{\cancel{1-e^{-s}}}{\cancel{(1-e^{-s})}(1+e^{-s})} = \frac{1}{s(1+e^{-s})}$$

$$\frac{1}{1+a} = \sum_{k=0}^{\infty}(-1)^k a^k \qquad = \frac{1}{s}\sum_{k=0}^{\infty}(-1)^k e^{-ks}$$

$|a|<1$

Ex)

$L\frac{di}{dt}+Ri=E(t)$, $i(0)=0$, E(t)는 위의 예제의 함수.



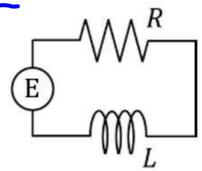


그림 2

$i(t)$?

$\mathcal{L}[i]=I$

$\mathcal{L}[Li'+Ri]=\mathcal{L}[E]$

$L(sI-i(0))+RI=\frac{1}{s(1+e^{-s})}$

L, R 상수 >0

$LsI+RI=\frac{1}{s(1+e^{-s})}$

$i(t)=\mathcal{L}^{-1}[I]$

$\mathcal{L}^{-1}\left[\frac{1}{1+e^{-s}}\right]$

$$I=\frac{1}{(Ls+R)(s(1+e^{-s}))}$$

$$=\frac{1}{s(Ls+R)}\sum_{k=0}^{\infty}(-1)^k e^{-ks}$$

$\mathcal{L}^{-1}[I]=\sum_{k=0}^{\infty}(-1)^k\mathcal{L}^{-1}\left[e^{-ks}\frac{1}{s(Ls+R)}\right]$

$$F(s)=\frac{1}{s(Ls+R)}$$

$\sum_{k=0}^{\infty}(-1)^k u_k(t)f(t-k)$

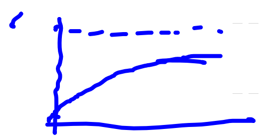
$$\mathcal{L}^{-1}\left(\frac{1}{s(Ls+R)}\right)=\mathcal{L}^{-1}\left[\frac{L}{R}\left(\frac{1/L}{s}-\frac{1}{Ls+R}\right)\right]$$

$$=\frac{L}{R}\left(\frac{1}{L}-\frac{1}{L}e^{-R/Lt}\right)$$

$$=\frac{1}{R}(1-e^{-R/Lt})=f(t)$$

$\frac{A}{s}+\frac{B}{Ls+R}$

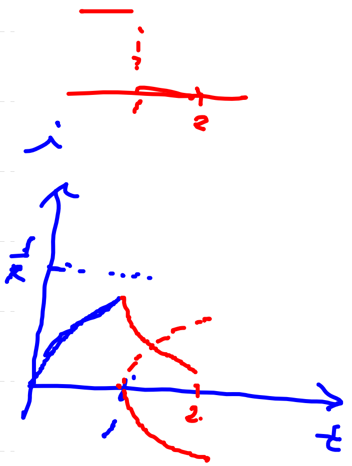
$\frac{1}{L}\quad\frac{1}{s+R/L}\quad\frac{1}{s}$



$\mathcal{L}[u_k(t)f(t-k)]$

$=e^{-ks}\mathcal{L}[f]=e^{-ks}\frac{1}{s(Ls+R)}$

$\mathcal{L}[e^{ct}f(t)]=F(s-c)$



$$I(s) = \sum_{k=0}^{\infty} (-1)^k e^{-ks} F(s)$$

$$f(t) = \frac{1}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\begin{aligned} i(t) &= \sum_{k=0}^{\infty} (-1)^k \mathcal{L}^{-1}\left[e^{-ks}F(s)\right] \\ &= \sum_{k=0}^{\infty} (-1)^k u_k(t) f(t-k) \end{aligned}$$

$$= u_0(t)f(t) \underbrace{- u_1(t)f(t-1)}_{x} + \cdots$$

$$0 < t \le 1 \quad i(t) = f(t) = \frac{1}{R}\left(1 - e^{-R/Lt}\right)$$

$$1 \le t < 2 \quad i(t) = f(t) - u_1(t) f(t-1)$$

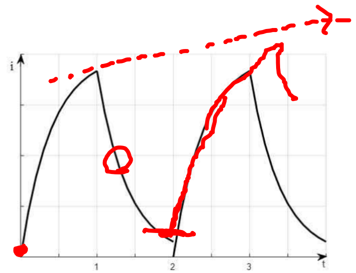


그림 3

R

$$\begin{aligned} i(t) &= \left(1 - e^{-R/Lt}\right) - \left(1 - e^{-R/L(t-1)}\right) \\ &= e^{-R/Lt}\left(e^{R/L} - 1\right) > 0 \end{aligned}$$

Laplace

2) Impulse function. (충격 함수)

$my'' + c'y' + ky = f(t)$

$y(0) = y'(0)$

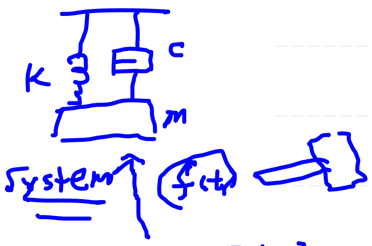
$(ms^2 + cs + k)Y = F(s)$

$Y = \frac{1}{ms^2 + cs + k} F$

$y = \mathcal{L}^{-1}[\cdot] * f$

mixed

Q: 유수하게 system 정보 분리?



system $f(t)$

$y(t) = (m, c, k) + f(t)$

response Laplace

$$ay'' + by' + cy = g(t)$$

$$g(t) = \{large -$$

$t = t_0$

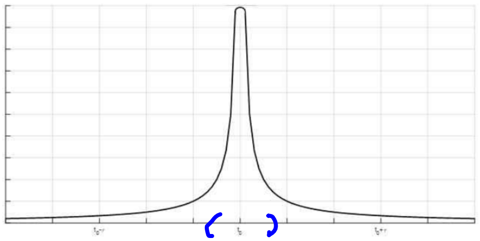


그림 4

$g(t) = \begin{cases} 0 & t \neq t_0 \\ +\infty & t = t_0 \end{cases}$

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)dt = \int_{-\infty}^{\infty} g(t)dt$$

= Total impulse = 충격량

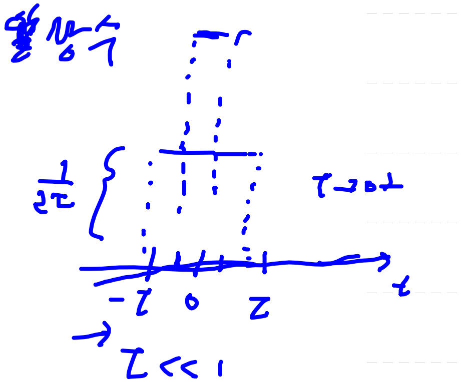
(Strength of forcing function.)

"Total impulse" of $g(t)$ on $|t - t_0| < \tau$.

$\{d_\tau\}$

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & otherwise \end{cases}$$

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)dt = 1$$



$\tau \to 0+$

$\tau \ll 1$

Def)

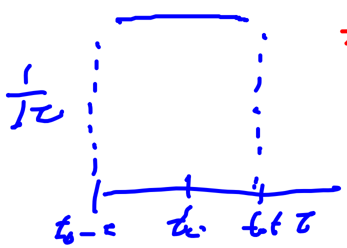
Unit impulse function δ. (Dirac delta)

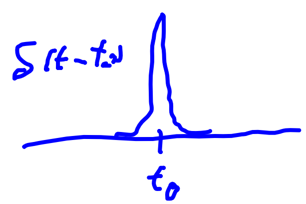
$$\delta(t) := \lim_{\tau \to 0+} d_\tau(t)$$

$= \begin{cases} +\infty & t = 0 \\ 0 & t \neq 0 \end{cases}$

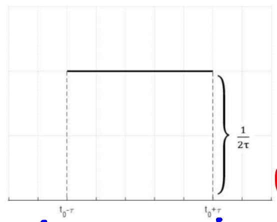
$$\delta(t - t_0) = \lim_{\tau \to 0+} d_\tau(t - t_0)$$

$$\mathcal{L}[\delta(t - t_0)] := \lim_{\tau \to 0} \mathcal{L}[d_\tau(t - t_0)]$$





square pulse $\Rightarrow$



$t_0-\tau$ $\qquad$ $t_0+\tau$

그림 5

$$\mathcal{L}[\delta(t-t_0)] = \lim_{\tau\to 0+}\mathcal{L}[d_\tau(t-t_0)]$$

$$\Rightarrow \quad d_\tau(t-t_0)=\frac{1}{2\tau}\left(u_{t_0-\tau}(t)-u_{t_0+\tau}(t)\right)$$

$$\mathcal{L}[u_c(t)] = e^{-cs}\frac{1}{s}$$

$$\mathcal{L}\left[d_\tau(t-t_0)\right]=\frac{1}{2\tau}\left(\frac{e^{-(t_0-\tau)s}}{s}-\frac{e^{-(t_0+\tau)s}}{s}\right)$$
$$=e^{-t_0 s}\left(\frac{e^{\tau s}-e^{-\tau s}}{2\tau s}\right)$$

$$\frac{e^x-1}{x}\to 1 \quad x\to 0$$

$\tau\to 0+$

L'Hôpital

$$\lim_{\tau\to 0+}\frac{f(\tau)}{g(\tau)}=\frac{0}{0}$$
$$=\lim_{\tau\to 0+}\frac{f'(\tau)}{g'(\tau)}$$

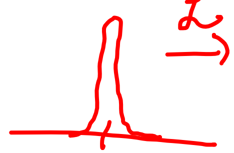
$$\lim_{\tau\to 0}\frac{e^{\tau s}-1}{\tau s}=1$$

$$\lim_{\tau\to 0}\mathcal{L}\left[d_\tau(t-t_0)\right]=e^{-t_0 s}\lim_{\tau\to 0}\frac{e^{\tau s}-e^{-\tau s}}{2\tau s}$$
$$=e^{-t_0 s}\lim_{\tau\to 0}\frac{se^{\tau s}+se^{-\tau s}}{2s}$$
$$=e^{-t_0 s}\frac{1}{2}(e^0+e^0)$$
$$=e^{-st_0}\cdot 1$$

$$\frac{e^{\tau s}-1}{2\tau s}+\frac{1-e^{-\tau s}}{2\tau s}$$

$$u_c(t)f(t-c) \xrightarrow{\mathcal{L}} e^{-cs}F(s)$$

Claim $\mathcal{L}\left[\delta(t-t_0)\right]=e^{-st_0}$.



t_0

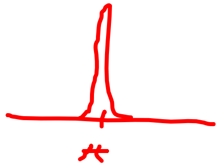
$$\mathcal{L}[\delta(t)]=1$$

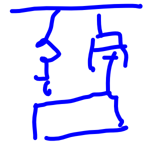
$$f*\delta=f$$

$$f*g=f$$

$$\mathcal{L}[f*g]=\mathcal{L}\{f\}\mathcal{L}\{g\}=\mathcal{L}[f]$$

$Y = \mathcal{L}[y]$ set





given

over critical under

system

Laplace 변환

Ex)

$$y'' + 2y' + 2y = \delta(t-\pi), \quad y(0) = y'(0) = 0 \quad \text{Solve IVP.}$$

$\mathcal{L}[y'' + 2y' + 2y] = e^{-\pi s}$

$s^2Y + 2sY + 2Y \Rightarrow Y = \dfrac{e^{-\pi s}}{s^2+2s+2}$

$\mathcal{L}^{-1}[Y]$

$$y = u_\pi(t) f(t-\pi)$$

$y = u_\pi(t) \boxed{f(t-\pi)}$

$e^{-\pi s}$

$\mathcal{L}[f] = \dfrac{1}{s^2+2s+2}$

system 정의 $\Rightarrow$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+2s+2}\right] = ?$$

$e^{-t}\sin t$

$\mathcal{L}[u_c(t) f(t-c)] = e^{-cs}\mathcal{L}[f]$

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$$

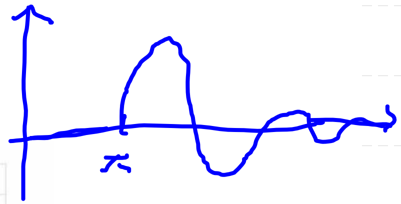
$$\mathcal{L}[e^{-t}\sin t] = \frac{1}{(s+1)^2+1}$$

underdamping (system)

$\Rightarrow y = u_\pi(t)\, e^{-(t-\pi)} \sin(t-\pi)$

$y = u_0 e^{-t} \sin t$

f



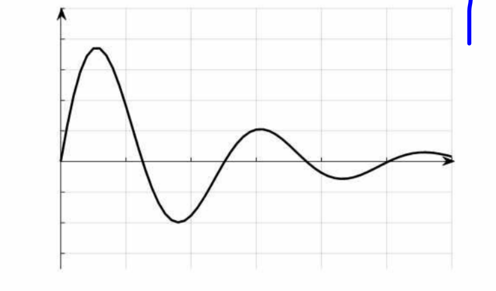


그림 6

$\vec{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$y' + 3y = e^t$ $y = y(t)$

ICE   Juice

daily profit

$\begin{cases} x_1(t) > 0 \\ x_2(t) < 0 \end{cases}$

연립 선형 미분방정식

System of Linear Differential equations

1. Review on matrix (Linear algebra)

$A=(a_{ij})$ n×m matrix

연벡터

i th row $[a_{i1} \cdots a_{im}]$ j th column $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$

행벡터

~~$A^T=(a_{ji})$~~

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1(t) - 2x_2(t) \\ \dfrac{dx_2}{dt} = -x_1(t) + 1.5x_2(t) \end{cases}$$

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 3 & -2 \\ -1 & 1.5 \end{bmatrix}\vec{x}$$

(1) Addition

$$A+B=(a_{ij})+(b_{ij})=(a_{ij}+b_{ij})$$
$$A+B=B+A$$
$$A+(B+C)=(A+B)+C$$

(2) Multiply by scalar

$$\alpha A = \alpha(a_{ij}) = (\alpha a_{ij})$$

(3) Multiplication

$$AB=\underbrace{(a_{ij})}_{n\times m}\underbrace{(b_{ij})}_{m\times p}=\left(\sum a_{ik}b_{kj}\right)$$

$$3\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 12 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ & \end{bmatrix}$$

(4) Identity matrix $I_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

$$AI_n = I_n A = A \quad A_{n\times m}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

연립방정식   (5) Invertible matrix   n×n 행렬

$$AB=BA=I$$

$B=A^{-1}$ Inverse of A

$\Leftrightarrow \det A \neq 0$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear system : When does it have a solution $A\vec{x}=\vec{b}$ $A: n\times n$?

$$a_{11}x_1 + \cdots a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{n1}x_1 + \cdots a_{nn}x_n = b_n$$

$ax+by=c$

$a'x+b'y=c'$

$$\begin{bmatrix} a & b \\ a' & b' \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ c' \end{bmatrix}$$

$A=(a_{ij})$ $\vec{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_2 \end{bmatrix}$ $\vec{b}=\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

A is invertible $\Rightarrow$ $\vec{x}=A^{-1}\vec{b}$

$A\vec{x}=\vec{c}$

$(A^{-1}A)\vec{x} = A^{-1}\vec{c}$

$\vec{x} = A^{-1}\vec{c}$

$A\vec{x}=\vec{0}$ (homogeneous equation) has only trivial solution $\vec{x}=0$

2. Linear independence.

$\boldsymbol{v}, \boldsymbol{w} \quad \boldsymbol{w} \neq c\boldsymbol{v}$ (c: scalar)

$\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ no redundant vector

$\boldsymbol{v_2} \neq c\boldsymbol{v_1}$ and $\boldsymbol{v_3} \neq \alpha_1 \boldsymbol{w_1} + \beta \boldsymbol{v_2}$

(characterize) $\boldsymbol{v_1}, \cdots, \boldsymbol{v_n}$ are linearly independent

$$\Leftrightarrow c_1\boldsymbol{v_1} + \cdots + c_n\boldsymbol{v_n} = 0$$
$$\Rightarrow c_1 = \cdots = c_n = 0$$

Example)

$$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \boldsymbol{v_2} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \boldsymbol{v_3} = \begin{bmatrix} -4 \\ 1 \\ -11 \end{bmatrix}$$

Determine whether they are linearly independent or not.
Search for non-zero c_1, c_2, c_3.

$$\begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} c_1 + 2c_2 - 4c_3 &= 0 \\ c_2 - 3c_3 &= 0 \end{aligned}$$

$$\begin{aligned} &c_3 = t, \quad c_2 = 3t \\ &c_1 = -2c_2 + 4c_3 \\ &\quad = -6t + 4t = -2t \end{aligned}$$

$$\begin{bmatrix} -2t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \Rightarrow -2\boldsymbol{v_1} + 3\boldsymbol{v_2} + \boldsymbol{v_3} = 0$$

$$\det[\boldsymbol{v_1} \ \boldsymbol{v_2} \ \boldsymbol{v_3}] = 0$$

How to evaluate determinant ?

$$\det\begin{bmatrix} a_1\, a_2\, a_3 \\ b_1\, b_2\, b_3 \\ c_1\, c_2\, c_3 \end{bmatrix} = a\begin{vmatrix} b_2\, b_3 \\ c_2\, c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1\, b_3 \\ c_1\, c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1\, b_2 \\ c_1\, c_2 \end{vmatrix}$$

$$\begin{vmatrix} c_1\, c_2\, c_3 \\ b_1\, b_2\, b_3 \\ a_1\, a_2\, a_3 \end{vmatrix} = a_1\begin{vmatrix} c_2\, c_3 \\ b_2\, b_3 \end{vmatrix} - a_2\begin{vmatrix} c_1\, c_3 \\ b_1\, b_3 \end{vmatrix} + a_3\begin{vmatrix} c_1\, c_2 \\ b_1\, b_2 \end{vmatrix}$$

$\vec{v}, \cdots, \overrightarrow{v_n}$ lineally independent $\Rightarrow \det(v_1 \cdots v_n) \neq 0$

3. Introduction to linear system

Homogeneous Linear system with constant coefficients (연립미분방정식)

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$x_1 = c_1 e^{2t}$$

$$x_2 = c_2 e^{-3t}$$

$$\vec{x}(t) = c_1\begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$$

Ex)

$$x_1{}'(t) = 3x_1 + 3x_2 + 8$$

$$x_2{}'(t) = x_1 + 5x_2 + 4e^{3t}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 3\, 3 \\ 1\, 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$$

In general $\boldsymbol{X}' = A\boldsymbol{X} + \boldsymbol{G},\ \boldsymbol{X}(t_0) = \boldsymbol{X}^{\boldsymbol{0}}$

Theorem. $I \ni t_0$

Suppose $a_{ij}(t), g_j(t)$ are continuous on I.

Then Initial Value Problem $\boldsymbol{X}' = A\boldsymbol{X} + \boldsymbol{G},\ \boldsymbol{X}(t_0) = \boldsymbol{X}^{\boldsymbol{0}}$ has an unique solution defined at all $t \in I$.