Series solution of second-order linear DE
(교과서 6장 급수해)
1. Review on power series (멱급수)

Power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

(1) Radius of convergence
=> Apply ratio test to obtain a number R >0 such that the series converges on $|x-x_0| < R$

(2) Algebra

$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$

$(a)\ (f \pm g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-x_0)^n$

$(b)\ kf(x) = \sum_{n=0}^{\infty} k a_n (x-x_0)^n$

$(c)\ f(x)g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)(x-x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x-x_0)^2 + \cdots$

(3) Derivatives

$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$

$f''(x) = \sum_{n=2} n(n-1) a_n (x-x_0)^{n-2}$

(4) Shifting index of summation

Example) $\sum_{n=0}^{\infty} a_n x^{n+2} + \sum_{n=0}^{\infty} b_n x^n$

$= b_0 + b_1 x + \sum_{n=2}^{\infty} (a_{n-2} + b_n) x^n$

2. Finding power series solution of 2^{nd} order DE around a given point.

Def) A function f is called analytic (해석적) at x=c if f has a power series representation around the point c.

Example) f(x)=sinx is analytic everywhere. $g(x) = \frac{1}{(1-x^2)}$ is analytic except x=1, -1.

Theorem (Case: leading coefficient is constant)
Suppose that the functions p(x), q(x), f(x) are real analytic around x=c.
Then IVP $y'' + p(x)y' + q(x)y = f(x),\ y(c) = A, y'(c) = B$ has an analytic solution around x=c.

Example. $y'' + x^2 y = 0$. Find a power series solution around x=0

Try $y = \sum_{n=0}^{\infty} a_n x^n$

Then $y'' + x^2 y = 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} + a_{n-2}]x^n = 0$

$$a_2 = a_3 = 0$$

$$a_{n+2} = -\frac{a_{n-2}}{(n+2)(n+1)}, \qquad n = 2, 3, \ldots$$

$$y = a_0 + a_1 x - a_0/12x^4 - a_1/20x^5 + \cdots$$

$$= a_0(1 - x^4/12 + x^8/(56 \times 12) + \cdots) + a_1(x - x^5/20 + x^9/(72 \times 20) + \cdots)$$

where a_0 and a_1 are determined by IC.

Exercise Find a power series solution around 0 for DE $y'' + x^2 y' + 4y = 1 - x^2$

3. Ordinary points (보통점) (교과서 6장2절)

Def) $P(x)y'' + Q(x)y' + R(x)y = 0$ (Case: leading coefficient is not constant)
We say x=c is an ordinary point if Q/P and R/P are analytic at x=c

Example
$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

Every point except x= 1, -1 are ordinary points.

Theorem
$P(x)y'' + Q(x)y' + R(x)y = 0$. Suppose that Q/P and R/P are analytic a x=c
=> General solution $y = \sum_{n=0}^{\infty} a_n (x-c)^n$

$$= a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary and y_1 and y_2 are analytic at x=c and form a fundamental set of solutions.

Example (Legendre equation) $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$
(It appears when we study electric potential on spherical object)

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad a_{n+2} = \frac{(n+\alpha+1)(n-\alpha)}{(n+2)(n+1)} a_n, \quad n = 0,1,\dots$$

(a) Consider the case $\alpha = 1$

(b) What is radius of convergence?