DE NOTE : System of linear ODEs part 2

1) 선형연립미분방정식의 응용

Ex)

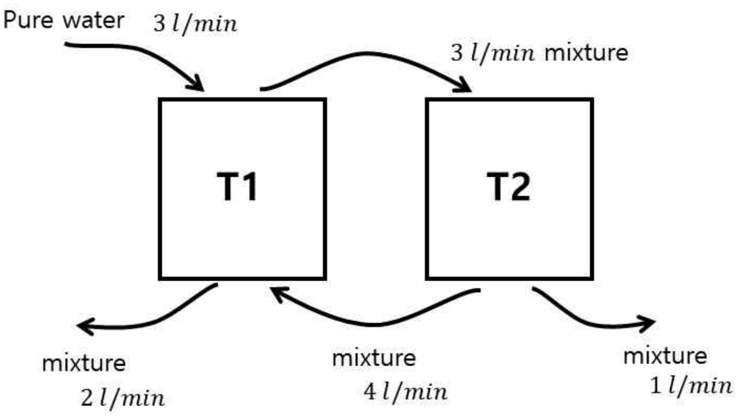

그림 1

$$\begin{aligned} \text{T1:}\ & Water(t=0)=20l \\ & Chlorine(t=0)=150g \end{aligned}$$

$$\begin{aligned} \text{T2:}\ & W(t=0)=10t \\ & C(t=0)=50g \end{aligned}$$

Determine the amount of chlorine in each tank at any time t>0.

Sol)

$$x_j(t) = amount\,of\,chlorine \in \tan k_j \,(\in y)$$

$$\begin{aligned} \varDelta x_1 &= rate\ \text{in} - \text{rate out} \\ &= (3l/\min)(0y/l)(\triangle t\min) + (3l/\min)(\frac{x_2}{10}g/l)(\triangle t\min) \\ &\quad -(2l/\min)(\frac{x_2}{10}g/l)(\triangle t\min) - (4l/\min)(\frac{x_2}{20}g/l)(\triangle t\min) \end{aligned}$$

$$x_1{}'(t) = \frac{3}{10}x_2 - \frac{6}{20}x_1$$

$$\begin{aligned} \triangle x_2(t) &= \in - out \\ &\approx (4l/\min)(\frac{x_1}{20}g/l)(\triangle t\min) \\ &\quad -(3l/\min)(\frac{x_2}{10}g/l)(\triangle t\min) \\ &\quad -(1l/\min)(\frac{x_2}{10}g/l)(\triangle t\min) \end{aligned}$$

$$x_2{}'(t) = \frac{1}{5}x_1 - \frac{4}{10}x_2$$

$$\boldsymbol{x}'(0) = A\boldsymbol{x} = \begin{bmatrix} -\frac{3}{10} & \frac{3}{10} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}\boldsymbol{x}$$

$$\boldsymbol{x}(0) = \begin{bmatrix} 150 \\ -50 \end{bmatrix}$$

$$\begin{vmatrix} -3/10-\lambda & 3/10 \\ 1/5 & -1/3-\lambda \end{vmatrix} = 0$$

$$\left(-\frac{3}{10}-\lambda\right)\left(-\frac{2}{5}-\lambda\right)-\frac{3}{50} = 0$$

$$\lambda = -\frac{1}{10}, -\frac{3}{10}$$

$$\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Omega = \begin{bmatrix} 3/2e^{-\frac{1}{10}t} & -e^{-\frac{3}{5}t} \\ e^{\frac{t}{10}} & e^{-\frac{3}{5}t} \end{bmatrix}$$

$$C_1 = \frac{20}{20} = 1$$

$$C_2 = \frac{30}{10} = 3$$

$$\frac{C_1}{C_2} = \frac{Q_1/20}{Q_2/10} = \frac{Q_1}{Q_2} \times \frac{1}{2}$$

2) Complex eigenvalues

$$\frac{d}{dt}\vec{x}(t)=\begin{bmatrix}-1/2 & 1\\ -1 & -1/2\end{bmatrix}\vec{x}(t)$$

$$\left(-\frac{1}{2}+\lambda\right)^2+1=0$$

$$\pm\frac{1}{2}+\lambda=\pm i$$

$$\lambda=-\frac{1}{2}\pm i$$

$$\Phi(t)=\vec{v}e^{\lambda t}$$

$$\lambda=-\frac{1}{2}+i$$

$$\begin{bmatrix}-\frac{1}{2}-(-\frac{1}{2}+i) & 1\\ -1 & -\frac{1}{2}-(-\frac{1}{2}+i)\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=0$$

$$\begin{bmatrix}-i & 1\\ -1 & -i\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=0$$

$-iv_1+v_2=0$ $\qquad$ $v_2=iv_1$ $\qquad$ $\vec{v}=v_1\begin{bmatrix}1\\ i\end{bmatrix}$

$v_1+iv_2=0$

$$\left.\begin{aligned}A\begin{bmatrix}1\\ i\end{bmatrix}&=\left(-\frac{1}{2}+i\right)\begin{bmatrix}1\\ i\end{bmatrix}\\ A\begin{bmatrix}1\\ -i\end{bmatrix}&=\left(-\frac{1}{2}-i\right)\end{aligned}\right\}$$

$$\Phi_1=\begin{bmatrix}1\\ i\end{bmatrix}e^{(-1/2+i)t}$$

$$\Phi_2=\begin{bmatrix}1\\ -i\end{bmatrix}e^{(-1/2-i)t}$$

$$\begin{aligned}\Phi_1&=e^{-1/2t}\left(\begin{bmatrix}1\\ 0\end{bmatrix}+i\begin{bmatrix}0\\ 1\end{bmatrix}\right)(\cos t+i\sin t)\\ &=e^{-1/2t}\left[\left(\begin{bmatrix}1\\ 0\end{bmatrix}\cos t-\begin{bmatrix}0\\ 1\end{bmatrix}\sin t\right)+i\left(\begin{bmatrix}0\\ 1\end{bmatrix}\cos t+\begin{pmatrix}1\\ 0\end{pmatrix}\sin t\right)\right]\end{aligned}$$

$$\Phi_2=\Phi_1$$

$$\Phi_1=e^{-1/2t}(\Psi_1+i\Psi_2)$$
$$\Phi_2=e^{-1/2t}(\Psi_1-i\Psi_2)$$

$$e^{-1/2t}\Psi_1=RE\Phi_1=\frac{\Phi_1+\Phi_2}{2}=\frac{1}{2}(\Phi_1+\Phi_2)$$

$$e^{-1/2t}\Psi_2=IM\Phi_1=\frac{\Phi_1-\Phi_2}{2i}=\frac{1}{2i}(\Phi_1-\Phi_2)$$

Fundamental set: $e^{-1/2t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix}, e^{-1/2t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$

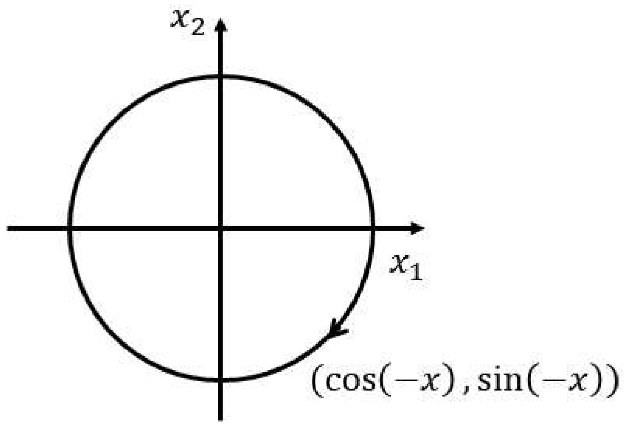

그림 2

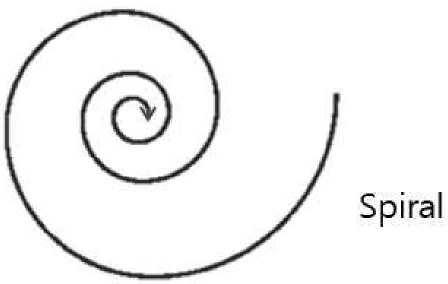

그림 3

$$x_1^2 + x_2^2 = e^{-t}$$
$$r(t) = e^{-\frac{1}{2}t}$$

Ex)

$$m, c, k \quad m = 1$$
$$c^2 - 4k < 0$$
$$k = 1, c = 1 \qquad 1 - 4 < 0$$
$$y'' - y' + y = 0$$
$$y_1 = y \qquad y_1' = y' = y_2$$
$$y_1 = y' \qquad y_2' = -y' - y = -y_2 - y_1$$
$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\vec{y}(t)$$
$$\begin{vmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-1-\lambda)+1=0$$
$$\lambda^2+\lambda+1=\left(\lambda+\frac{1}{2}\right)^2+\frac{3}{4}=0$$
$$\lambda+\frac{1}{2}=\pm i\frac{\sqrt{3}}{2}$$
$$\lambda=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$
$$\text{Eigenvector}\begin{bmatrix}\frac{1}{2}-i\frac{\sqrt{3}}{2} & 1 \\ -1 & -\frac{1}{2}-i\frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}v_1 \\ v_2\end{bmatrix}=0$$
$$\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)v_1+v_2=0$$
$$\begin{bmatrix}1 \\ -\frac{1}{2}+i\frac{\sqrt{3}}{2}\end{bmatrix}\text{ or }\begin{bmatrix}2 \\ -1+i\sqrt{3}\end{bmatrix}$$
$$\Phi_1=e^{-1/2t}\left(\begin{bmatrix}2 \\ -1\end{bmatrix}+i\begin{bmatrix}0 \\ \sqrt{3}\end{bmatrix}\right)\left(\cos\frac{\sqrt{3}}{2}t+i\sin\frac{\sqrt{3}}{2}t\right)$$
$$=e^{-1/2t}\left[\left(\begin{bmatrix}2 \\ -1\end{bmatrix}\cos\frac{\sqrt{3}}{2}t-\begin{bmatrix}0 \\ \sqrt{3}\end{bmatrix}\sin\frac{\sqrt{3}}{2}t\right)+i\left(\begin{bmatrix}0 \\ \sqrt{3}\end{bmatrix}\cos\frac{\sqrt{3}}{2}t+\begin{bmatrix}2 \\ -1\end{bmatrix}\sin\frac{\sqrt{3}}{2}t\right)\right]$$
$$\Psi_1=e^{-1/2t}\begin{bmatrix}2\cos\frac{\sqrt{3}}{2}t \\ -\cos\frac{\sqrt{3}}{2}t-\sqrt{3}\,sin\frac{\sqrt{3}}{2}t\end{bmatrix}$$
$$\Psi_2=e^{-1/2t}\begin{bmatrix}2\sin\frac{\sqrt{3}}{2}t \\ \sqrt{3}\,cos\frac{\sqrt{3}}{2}t-\sin\frac{\sqrt{3}}{2}t\end{bmatrix}$$
$$\begin{bmatrix}y \\ y'\end{bmatrix}=c_1\Psi_1+c_2\Psi_2$$
$$y=e^{-1/2t}\left(c_1\cos\frac{\sqrt{3}}{2}t+c_2\sin\frac{\sqrt{3}}{2}t\right)$$
$$\begin{bmatrix}y \\ y'\end{bmatrix}=\begin{bmatrix}2\cos\frac{\sqrt{3}}{2}t \\ -\cos\frac{\sqrt{3}}{2}t-3\sin\frac{\sqrt{3}}{2}t\end{bmatrix}$$
$$\left(\frac{y}{2}\right)^2+\left(\frac{y'+y/2}{\sqrt{3}}\right)^2=1$$
$$\begin{bmatrix}2 & 0 \\ -1 & -\sqrt{3}\end{bmatrix}\begin{bmatrix}\cos\frac{\sqrt{3}}{2}t \\ \sin\frac{\sqrt{3}}{2}t\end{bmatrix}$$
$$\begin{bmatrix}1 \\ 0\end{bmatrix}\rightarrow\begin{bmatrix}2 \\ -1\end{bmatrix}\quad\begin{bmatrix}0 \\ 1\end{bmatrix}\rightarrow\begin{bmatrix}0 \\ -\sqrt{3}\end{bmatrix}$$

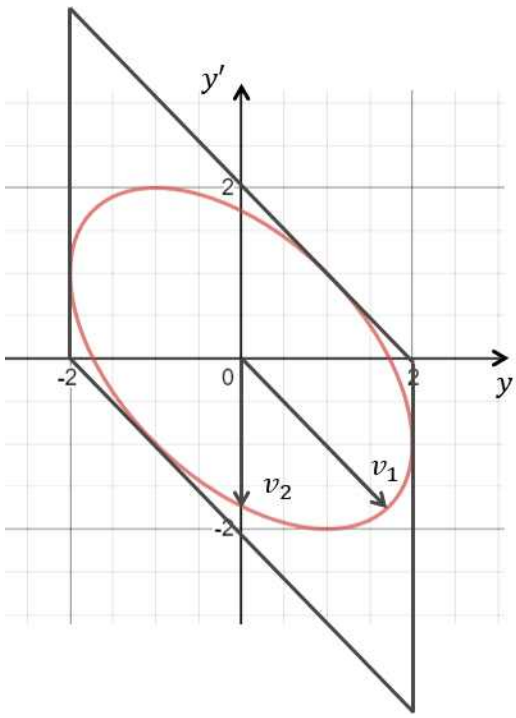

그림 **4**

3) Repeated Eigenvalues/ multiplicity

Case :

$X' = AX$ does not have n linearly independent eigenvectors.

$$X' = AX \qquad A = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix}$$

$$|A - \lambda I_2| = \begin{bmatrix} 1-\lambda & 3 \\ -3 & 7-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(7-\lambda) + 9 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$$

$$\begin{bmatrix} 1-4 & 3 \\ -3 & 3 \end{bmatrix} \qquad \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$\Phi_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \qquad E_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Try $\Phi_2(t) = X_1 te^{3t} + E_2 e^{4t}$ ($E_2 e^{4t}$: correction term)

$$\Phi_2{}' = E_1 e^{4t} + 4E_1 te^{4t} + 4E_2 e^{4t}$$

$$\Phi_2{}' - A(E_1 te^{4t} + E_2 e^{4t}) = (4E_1 - AE_1)te^{4t} + (E_1 + 4E_2 - AE_2)e^{4t}$$

$$E_1 + 4E_2 - AE_2 = 0$$
$$(A - 4I)E_2 = E_1$$

$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$-3a + 3b = 1$$

$$a = s, \; b = \frac{3s+1}{3}, \quad \begin{bmatrix} s \\ \frac{3s+1}{3} \end{bmatrix}$$

$$s = 1 \Rightarrow \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix}$$

$$\Phi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} 1 \\ \frac{4}{3} \end{bmatrix} e^{4t} = \begin{bmatrix} 1+t \\ \frac{4}{3}+t \end{bmatrix} e^{4t}$$

Ex)

$$A = \begin{bmatrix} -2 & -1 & -5 \\ 25 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{vmatrix} (-2-\lambda) & -1 & -5 \\ 25 & -7-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = \lambda^3 + 6\lambda^2 + 12\lambda + 8$$

$$(\lambda + 2)^3 = \lambda^3 + 6\lambda^2 + 3 \cdot 4\lambda + 8$$

Eigenvector $\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} = E_1$

$$\Phi_1 = E_1 e^{-2t}$$
$$\Phi_2 = E_1 te^{-2t} + E_2 e^{-2t}$$
$$\Phi_2{}' - A\Phi_2 = E_1 e^{-2t} - 2E_1 te^{-2t} - 2E_2 e^{-2t} - \left(AE_1 te^{-2t} + AE_2 e^{-2t}\right)$$

$$\begin{vmatrix} -2-\lambda & -1 & -5 \\ 25 & -7-\lambda & 0 \\ 0 & 1 & -3-\lambda \end{vmatrix} = -(2+\lambda)\begin{vmatrix} -(7+\lambda) & 0 \\ 1 & 3-\lambda \end{vmatrix} - 25\begin{vmatrix} -1 & -5 \\ 1 & 3-\lambda \end{vmatrix}$$
$$= (2+\lambda)(7+\lambda)(3-\lambda) - 25(\lambda-3+5)$$
$$= (\lambda+2)\{(21-25+(3-7)\lambda-\lambda^2)\}$$
$$= -(\lambda+2)(\lambda+2)^2$$
$$= -(\lambda+2)^3 \qquad \lambda = -2$$

$$\begin{bmatrix} 0 & -1 & -5 \\ 25 & -5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \qquad \begin{matrix} y+5z=0 \\ 5x=y \end{matrix} \qquad \begin{bmatrix} \frac{1}{5}y \\ y \\ -\frac{1}{5}y \end{bmatrix}$$

$$(A+2)E_2 = E_1$$

$$E_1 = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} \qquad \begin{matrix} y+5z=-1 \\ 5x-y=1 \end{matrix} \qquad y=0 \qquad E_2 = \begin{bmatrix} \frac{1}{5} \\ 0 \\ -\frac{1}{5} \end{bmatrix}$$

$$(A+2)E_3 = E_2 \quad y = 5z = -\frac{1}{5}$$

$$25x-5y=0 \quad x=y=0 \quad z=-\frac{1}{25}$$

$$E_3 = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{25} \end{bmatrix}$$

$$\Phi_1 = e^{-2t}\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

$$\Phi_2 = te^{-2t}\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} + e^{-2t}\begin{bmatrix} \frac{1}{5} \\ 0 \\ -\frac{1}{5} \end{bmatrix}$$

$$\Phi_3 = \frac{1}{2}t^2e^{-2t}\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} + te^{-2t}\begin{bmatrix} \frac{1}{5} \\ 0 \\ -\frac{1}{5} \end{bmatrix} + e^{-2t}\begin{bmatrix} 0 \\ 0 \\ -\frac{1}{25} \end{bmatrix}$$
$$= e^{-2t}\begin{bmatrix} \frac{1}{2}t^2 + \frac{1}{5}t \\ \frac{5}{2}t^2 \\ -\frac{1}{2}t^2 - \frac{1}{25} \end{bmatrix}$$
$$= te^{-2t}(-2E_1 - AE_1) + (E_1 - 2E_1 - AE_2)e^{-2t}$$
$$(A+2I)E_2 = E_1$$
$$E_2 = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$$

$$\Phi_2 = E_1 te^{-2t} + E_2 e^{-2t} = \begin{bmatrix} -1-t \\ -4-5t \\ 1+t \end{bmatrix} e^{-2t}$$

$$\Phi_3 = \frac{1}{2} E_1 t^2 e^{-2t} + E_2 te^{-2t} + E_3 e^{-2t}$$

$$\Phi_3{}' = \left[E_1 re^{-2t} - E_2 t^2 e^{-2t} + E_2 e^{-2t} - 2E_2 te^{-2t} - 3E_3 3^{-2t} \right]$$

$$\Phi_3{}' = A\Phi_3 = A\left(\frac{1}{2}E_1 t^2 + E_2 t + E_3\right) e^{-2t}$$

$$\begin{cases} -E_1 = \frac{1}{2}AE_1 \\ E_1 - 2E_2 = AE_2 \end{cases}$$

$$E_2 - 2E_3 = AE_3$$

$$\Rightarrow (A+2)E_3 = E_2 \qquad E_3 = \begin{bmatrix} -\frac{24}{25} \\ -4 \\ 1 \end{bmatrix}$$

$$\Phi_3 = \frac{1}{2}\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} t^2 e^{-2t} + \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -\frac{24}{25} \\ -4 \\ 1 \end{bmatrix} e^{-2t}$$

$$= \begin{bmatrix} -\frac{24}{25} - t - \frac{1}{2}t^2 \\ -4 - 4t - \frac{3}{2}t^2 \\ 1 + t + \frac{1}{2}t^2 \end{bmatrix} e^{-2t}$$

Question) How can we diagonalize A?

$$AP = PD$$

$$P = [v_1 \cdots v_n]$$

$$Av_j = P\begin{bmatrix} 0 \\ \vdots \\ \lambda_j \\ \vdots \\ 0 \end{bmatrix} = \lambda_j v_j$$

$$\Rightarrow D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad \lambda_j : eigenvalues\, of\, A$$

$P = [v_1^{-1} \cdots v_n^{-1}]$ $\qquad$ v_j^{-1}: Eigenvectors associated λ_j

Ex)

$$\boldsymbol{x}' = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} \boldsymbol{x}$$

$$\begin{vmatrix} 3-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda) - 3 = 0$$

$$\lambda^2-8\lambda+12=(\lambda-6)(\lambda-2)=0$$
$$\lambda=2,6$$
$$\begin{bmatrix}-3\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}$$

Ex)

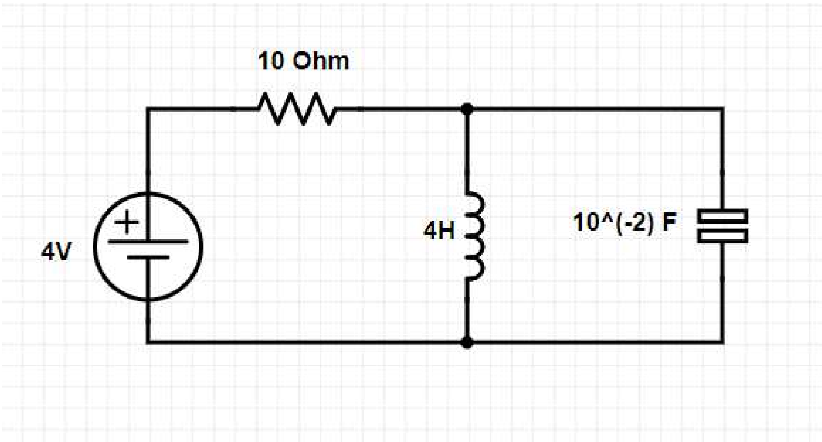

그림 5

Two internal loop and one external loop.

$$\begin{cases}10i_1+4(i_1{}'-i_2{}')=4\\10i_1+100q_2=4\end{cases}$$
$$\Rightarrow\begin{cases}i_1{}'=-10i_2\\2(i_1{}'-i_2{}')=-5i_1+2\end{cases}$$
$$\begin{bmatrix}1&0\\2&-2\end{bmatrix}\begin{bmatrix}i_1{}'\\i_2{}'\end{bmatrix}=\begin{bmatrix}0&-10\\-5&0\end{bmatrix}\begin{bmatrix}i_1\\i_2\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}$$
$$\begin{bmatrix}i_1{}'\\i_2{}'\end{bmatrix}=\begin{bmatrix}1&0\\1&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}0&-10\\-5&0\end{bmatrix}\begin{bmatrix}i_1\\i_2\end{bmatrix}+\begin{bmatrix}0\\-1\end{bmatrix}$$
$$\boldsymbol{i}=A\boldsymbol{i}+G$$
$$A=\begin{bmatrix}0&-10\\\frac{5}{2}&-10\end{bmatrix}\quad G=\begin{bmatrix}0\\-1\end{bmatrix}$$
$$|A-\lambda I|=-\lambda(-10-\lambda)+25=0$$
$$\lambda^2+10\lambda+25=(\lambda+5)^2=0$$
$$\Phi_1=\begin{bmatrix}2\\1\end{bmatrix}e^{-5t}$$
$$\Phi_2=E_1te^{-5t}+E_2e^{-5t}$$
$$\Phi_2{}'=A\Phi_2\quad\Rightarrow\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}5&-10\\5/2&-5\end{bmatrix}E_2$$
$$\Phi_2=\begin{bmatrix}2\\1\end{bmatrix}te^{-2t}+\begin{pmatrix}1\\3/10\end{pmatrix}e^{-5t}$$

$$\Omega(t) = \begin{bmatrix} 2e^{-5t} & (1+2t)e^{-5t} \\ e^{-5t} & (\frac{3}{10}+t)e^{-5t} \end{bmatrix}$$

$$\Omega^{-1} = e^{5t}\begin{bmatrix} -\frac{1}{4}(3+10t) & \frac{3}{2}(1+2t) \\ \frac{5}{2} & -5 \end{bmatrix}$$

$$\begin{aligned} U(t) &= \int \Omega^{-1}(t)G(t)dt \\ &= \begin{bmatrix} \int -\frac{5}{2}(1+2t)e^{5t}dt \\ \int 5e^{5t}dt \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{10}e^{3t} - e^{3t}t \\ e^{5t} \end{bmatrix} \end{aligned}$$

$$\Psi_1 = \Omega(t)U(t) = \begin{bmatrix} \frac{2}{5} \\ 0 \end{bmatrix}$$

$$\boldsymbol{i}(t) = \Omega C + \begin{bmatrix} 2/5 \\ 0 \end{bmatrix}$$

$$\boldsymbol{i}(0) = \begin{pmatrix} 2/5 \\ 2/5 \end{pmatrix}$$

Initial condition $i_1 - i_2 = 0$ on L.

$$10i_1(0) + 100q_2(0) = 4 \quad i_1(0) = \frac{2}{5}$$

$$-\begin{bmatrix} 2/5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2/5 \\ 2/5 \end{bmatrix} = \Omega(0)C \qquad C = \Omega^{-1}\begin{bmatrix} 0 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\boldsymbol{i}(t) = \Omega\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2/5 \\ 0 \end{bmatrix}$$