미분방정식 05-04 수업

piecewise continuous function f s.t. $f(t+T)=f(t)$

$$\mathcal{L}[f](s)=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$$

Ex)

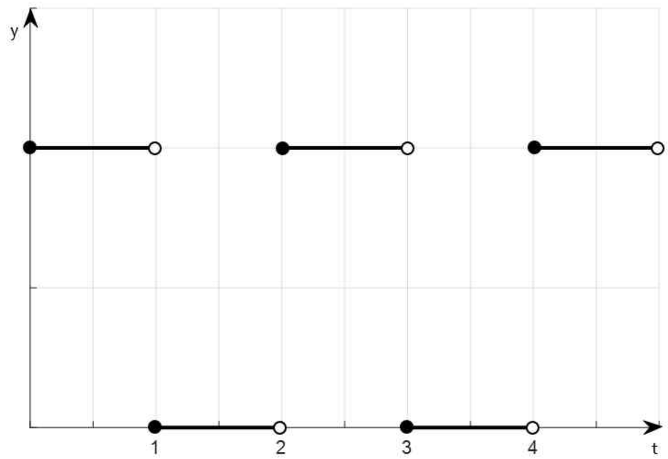

그림 1

E(t)= $\begin{cases}1, & 0\le t<1\\ 0, & 1\le t<2\end{cases}$ =>

Ex)

$$L\frac{di}{dt}+Ri=E(t),\ i(0)=0,$$ E(t)는 위의 예제의 함수.

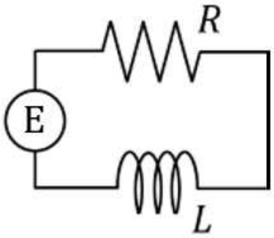

그림 2

$$LsI+RI=\frac{1}{s(1+e^{-s})}$$

$$\begin{aligned} I&=\frac{1}{(Ls+R)(s(1+e^{-s}))} \\ &=\frac{1}{s(Ls+R)}\sum_{k=0}^{\infty}(-1)^k e^{-ks} \end{aligned}$$

$$F(s)=\frac{1}{s(Ls+R)}$$

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s(L^2+R)}\right)&=\mathcal{L}^{-1}\left[\frac{L}{R}\left(\frac{1/L}{s}-\frac{1}{Ls+R}\right)\right] \\ &=\frac{L}{R}(\frac{1}{L}-\frac{1}{L}e^{-R/Lt}) \\ &=\frac{1}{R}(1-e^{-R/Lt})=f(t) \end{aligned}$$

$$I(s) = \sum_{k=0}^{\infty} (-1)^k e^{-ks} F(s)$$

$$\begin{aligned} i(t) &= \sum_{k=0}^{\infty} (-1)^k \mathcal{L}^{-1}\left[e^{-ks} F(s)\right] \\ &= \sum_{k=0}^{\infty} (-1)^k u_k(t) f(t-k) \end{aligned}$$

$$0 < t \le 1 \quad i(t) = f(t) = \frac{1}{R}\left(1 - e^{-R/Lt}\right)$$

$$1 \le t < 2 \quad i(t) = f(t) - u_1(t) f(t-1)$$

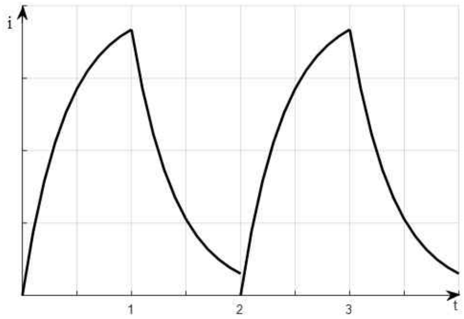

그림 3

$$\begin{aligned} i(t) &= \left(1 - e^{-R/Lt}\right) = \left(1 - e^{-R/L(t-1)}\right) \\ &= e^{-R/Lt}\left(e^{R/L} - 1\right) > 0 \end{aligned}$$

2) Impulse function. (충격 함수)

$$ay'' + by' + cy = g(t)$$

$$g(t) = \begin{cases} large - \end{cases}$$

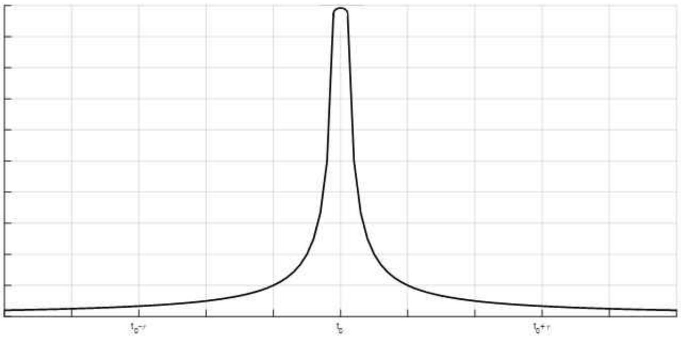

그림 4

$$I(\tau) = \int_{t_0 - \tau}^{t_0 + \tau} g(t)dt = \int_{-\infty}^{\infty} g(t)dt$$

(Strength of forcing function.)

“Total impulse” of $g(t)$ on $|t - t_0| < \tau$.

$$d_\tau(t) = \begin{cases} \dfrac{1}{2\tau} & -\tau < t < \tau \\ 0 & otherwise \end{cases}$$

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)dt = 1$$

Def)

Unit impulse function δ. (Dirac delta)

$$\delta(t) := \lim_{\tau \to 0+} d_\tau(t)$$

$$\delta(t - t_0) = \lim_{\tau \to 0+} d_\tau(t - t_0)$$

$$\mathcal{L}\left[\delta(t - t_0)\right] := \lim_{\tau \to 0} \mathcal{L}\left[d_\tau(t - t_0)\right]$$

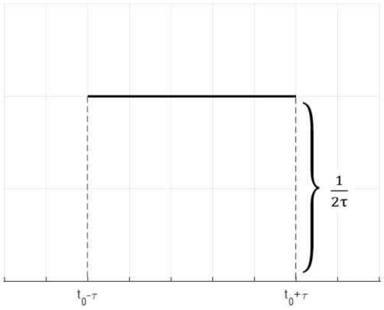

그림 5

$$\Rightarrow \quad d_\tau(t-t_0)=\frac{1}{\tau}(u_{t_0-\tau}(t)-u_{t_0+\tau}(t))$$

$$\begin{aligned}\mathcal{L}\left[d_\tau(t-t_0)\right] &= \frac{1}{2\tau}\left(\frac{e^{-(t_0-\tau)s}}{s}-\frac{e^{-(t_0+\tau)s}}{s}\right)\\ &= e^{-t_0 s}\left(\frac{e^{\tau s}-e^{-\tau s}}{2\tau s}\right)\end{aligned}$$

$$\begin{aligned}\lim_{\tau\to 0}\mathcal{L}\left[d_\tau(t-t_0)\right] &= e^{-t_0 s}\lim_{\tau\to 0}\frac{e^{\tau s}-e^{-\tau s}}{2\tau s}\\ &= e^{-t_0 s}\lim_{\tau\to 0}\frac{se^{\tau s}+se^{-\tau s}}{2s}\\ &= e^{-t_0 s}\frac{1}{2}(e^0+e^0)\\ &= e^{-st_0}\end{aligned}$$

Claim $\mathcal{L}\left[\delta(t-t_0)\right]=e^{-st_0}$.

Ex)

$$y'' + 2y' + 2y = \delta(t-\pi), \qquad y(0) = y'(0) = 0 \quad \text{Solve IVP.}$$

$$y = u_{\pi}(t) f(t-\pi)$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+2s+2}\right] = ?$$

$$\frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$$

$$\mathcal{L}[e^{-t}\sin t] = \frac{1}{(s+1)^2+1}$$

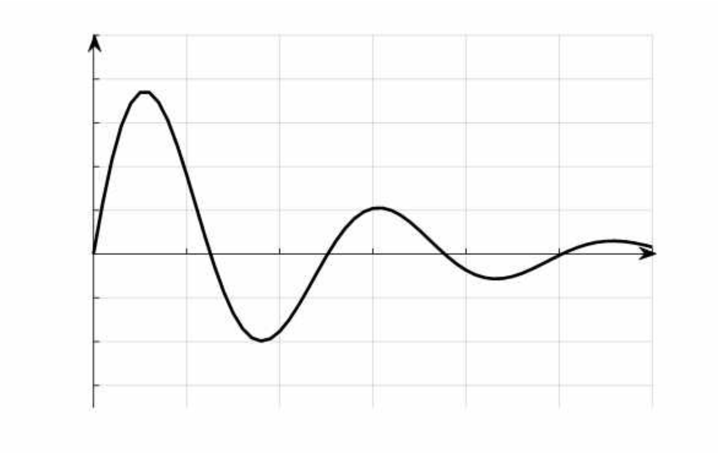

그림 6

연립 선형 미분방정식

System of Linear Differential equations

1. Review on matrix (Linear algebra)

$A=(a_{ij})$ n×m matrix

i th row $\begin{bmatrix} a_{i1} \cdots a_{im} \end{bmatrix}$ $\qquad$ ith column $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$

$$A^T=(a_{ji})$$

(1) Addition

$$A+B=(a_{ij})+(b_{ij})=(a_{ij}+b_{ij})$$
$$A+B=B+A$$
$$A+(B+C)=(A+B)+C$$

(2) Multiply by scalar

$$\alpha A=\alpha(a_{ij})=(\alpha a_{ij})$$

(3) Multiplication

$$AB=\underset{n\times m}{(a_{ij})}\underset{m\times p}{(b_{ij})}=(\sum a_{ik}b_{kj})$$

(4) Identity matrix $I_n=\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

$$AI_n=I_nA=A \qquad A_{n\times m}$$

(5) Invertible matrix

$$AB=BA=I$$

$B=A^{-1}$ Inverse of A

$$\Leftrightarrow \det A\neq 0$$

Linear system : When does it have a solution $A\vec{x}=\vec{b}$ $\quad A:n\times n$?

$$\begin{gathered} a_{11}x_1+\cdots a_{1n}x_n=b_1 \\ \vdots \\ a_{n1}x_1+\cdots a_{nn}x_n=b_n \end{gathered}$$

$$A=(a_{ij}) \quad \vec{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_2 \end{bmatrix} \quad \vec{b}=\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

A is invertible $\Rightarrow \vec{x}=A^{-1}\vec{b}$

$A\vec{x}=\vec{0}$ (homogeneous equation) has only trivial solution $\vec{x}=0$

2. Linear independence.

$$\boldsymbol{v}, \boldsymbol{w} \quad \boldsymbol{w} \neq c\boldsymbol{v} \text{ (c: scalar)}$$

$$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \qquad \text{no redundant vector}$$

$$\boldsymbol{v}_2 \neq c\boldsymbol{v}_1 \text{ and } \boldsymbol{v}_3 \neq \alpha_1 \boldsymbol{w}_1 + \beta \boldsymbol{v}_2$$

(characterize) $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ are linearly independent

$$\Leftrightarrow c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = 0$$
$$\Rightarrow c_1 = \cdots = c_n = 0$$

Example)

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \boldsymbol{v}_3 = \begin{bmatrix} -4 \\ 1 \\ -11 \end{bmatrix}$$

Determine whether they are linearly independent or not.
Search for non-zero c_1, c_2, c_3 .

$$\begin{bmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$c_1 + 2c_2 - 4c_3 = 0$$
$$c_2 - 3c_3 = 0$$

$$c_3 = t, \quad c_2 = 3t$$
$$c_1 = -2c_2 + 4c_3$$
$$= -6t + 4t = -2t$$

$$\begin{bmatrix} -2t \\ 3t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \Rightarrow -2\boldsymbol{v}_1 + 3\boldsymbol{v}_2 + \boldsymbol{v}_3 = 0$$

$$\det[\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \boldsymbol{v}_3] = 0$$

How to evaluate determinant ?

$$\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = a_1\begin{vmatrix} c_2 & c_3 \\ b_2 & b_3 \end{vmatrix} - a_2\begin{vmatrix} c_1 & c_3 \\ b_1 & b_3 \end{vmatrix} + a_3\begin{vmatrix} c_1 & c_2 \\ b_1 & b_2 \end{vmatrix}$$

$\vec{v}, \cdots, \overrightarrow{v_n}$ lineally independent $\Rightarrow \det(v_1 \cdots v_n) \neq 0$

3. Introduction to linear system

Homogeneous Linear system with constant coefficients (연립미분방정식)

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$x_1 = c_1 e^{2t}$$

$$x_2 = c_2 e^{-3t}$$

$$\vec{x}(t) = c_1\begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{-3t} \end{bmatrix}$$

Ex)

$$x_1{}'(t) = 3x_1 + 3x_2 + 8$$

$$x_2{}'(t) = x_1 + 5x_2 + 4e^{3t}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$$

In general $\boldsymbol{X}' = A\boldsymbol{X} + \boldsymbol{G},\ \boldsymbol{X}(t_0) = \boldsymbol{X}^0$

Theorem. $I \ni t_0$

Suppose $a_{ij}(t), g_j(t)$ are continuous on I.

Then Initial Value Problem $\boldsymbol{X}' = A\boldsymbol{X} + \boldsymbol{G},\ \boldsymbol{X}(t_0) = \boldsymbol{X}^0$ has an unique solution defined at all $t \in I$.