Chapter 3
Growth of Functions

Algorithm Analysis

School of CSEE

Growth of Function

- In this chapter we will study growth of function.
- The order of growth of the running time of an algorithm gives a simple characterization of the algorithm's efficiency and also allows us to compare the relative performance of alternative algorithms.

(input의 클래시)

precise X, simple O

time complexity. (주로 matter).
space.

Growth of Functions

- Although we can sometimes determine the exact running time of an algorithm, as we did for insertion sort in Chapter 2, the extra precision is not usually worth the effort of computing it.
- ***Asymptotic*** efficiency - how the running time increases with the size of the input *in the limit*.

  n이 클 때만 생각하겠다.
- Focus on what’s important by abstracting away *low-order terms* and *constant factors*.

Notations

when $n \to \infty$.

Theta	$f(n) = \theta(g(n))$	$f(n) \approx c\ g(n)$ 비슷하다.
BigOh	$f(n) = O(g(n))$	$f(n) \leq c\ g(n)$ → upper bound
Omega	$f(n) = \Omega(g(n))$	$f(n) \geq c\ g(n)$ → lower bound.
Little Oh	$f(n) = o(g(n))$	$f(n) < c\ g(n)$
Little Omega	$f(n) = \omega(g(n))$	$f(n) > c\ g(n)$

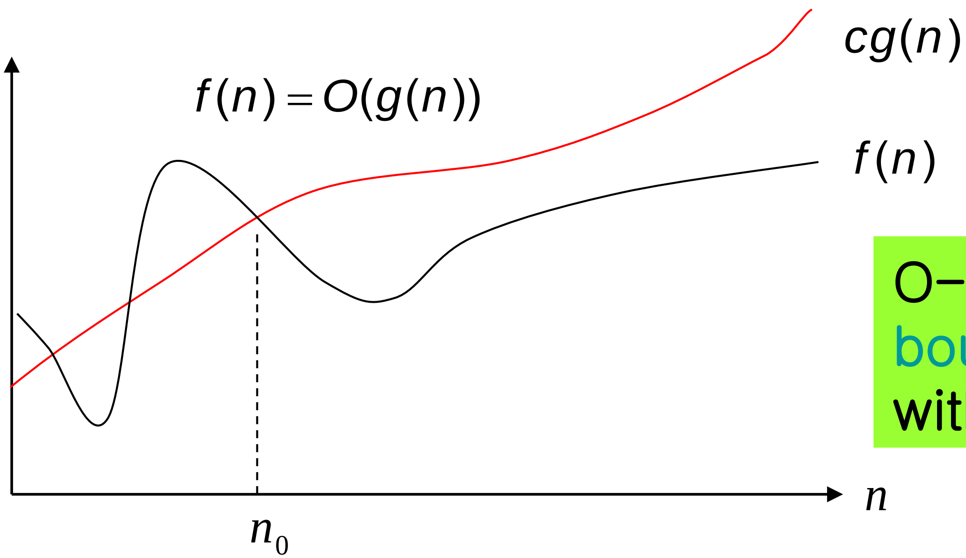
O-notation – upper bound

- $O(g(n)) = \{ f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le c\, g(n)$ for all $n \ge n_0 \}$

  If $f(n) \in O(g(n))$, we write $f(n) = O(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \to c < \infty \quad \Rightarrow \quad f(n) = O(g(n))$$

  Example : $2n^2 + 1 = O(n^2)$, with c =? and n_0 = ?



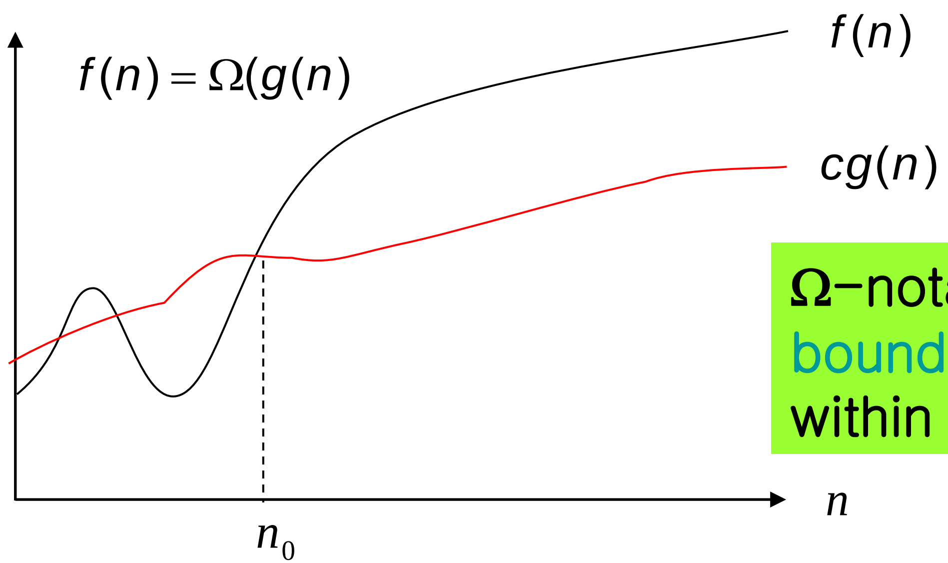
O-notation gives an upper bound on a function, to within a constant factor.

Ω-notation – lower bound

- $\Omega(g(n)) = \{ f(n)$: there exist positive constants c and n_0 such that $0 \le c\, g(n) \le f(n)$ for all $n \ge n_0 \}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \to c > 0 \quad \Rightarrow \quad f(n) = \Omega(g(n))$$

Example : $\sqrt{n} = \Omega(\lg n)$, with $c = 1$ and $n_0 = 16$



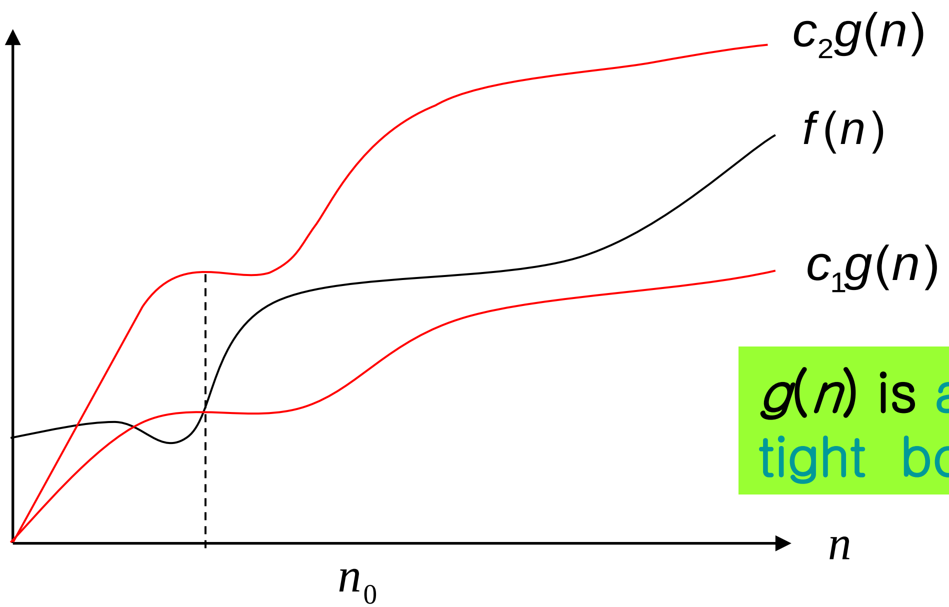
Ω−notation gives an lower bound on a function, to within a constant factor.

Θ-notation – tight (exact) bound

- $\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2 and n_0 such that $0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$ for all $n \ge n_0 \}$

$$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} \to c < \infty \quad \Rightarrow \quad f(n) = \Theta(g(n))$$

Example : $n^2/2 - 2n = \Theta(n^2)$, with $c_1 = 1/4$, $c_2 = 1/2$, $n_0 = 8$



$g(n)$ is an asymptotically tight bound for $f(n)$.

Theorem and Rule

Theorem 3.1

For two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

L'Hôpital's Rule

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}$$

o-notation

- $o(g(n)) = \{ f(n)$: for any positive constant $c > 0$, there exist a constant $n_0 > 0$ such that $0 \le f(n) < c\, g(n)$ for all $n \ge n_0 \}$

  $f(n)$ is ***asymptotically smaller*** than $g(n)$ if $f(n) = o(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \to 0 \Rightarrow \quad f(n) = o(g(n))$$

$$n - 5,\ n,\ n^2,\ 10^{10}n^2 + 10^5 n + 10^9,\ n^2 - 9 \in o(n^3)$$

$o(f) = O(f) - \theta(f)$.

$o(f)$ is usually called "little oh of f".

$$\cancel{n^3 \in o(n^3)}$$

ω-notation

- $\omega(g(n))$ = { $f(n)$: for any positive constant $c > 0$, there exist a constant $n_0 > 0$ such that $0 \le c\, g(n) < f(n)$ for all $n \ge n_0$ }

  $f(n)$ is ***asymptotically larger*** than $g(n)$ if $f(n) = \omega(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} \to \infty \;\Rightarrow\; f(n) = \omega(g(n))$$

$$n^2, 10^{10}n^2 + 10^5 n + 10^9, n^3 - 9 \in \omega(n)$$

Note: $g(n) = o(f(n)) \Leftrightarrow f(n) = \omega(g(n))$

$\omega(f) = \Omega(f) - \theta(f)$.

$\omega(f)$ is usually called "little omega of f"

$$\cancel{n \in \omega(n)}$$

Example

Two functions *f* and *g* have the same order if and only if

$$\Theta(f(n)) = \Theta(g(n))$$

$$\Theta((8n^3 + n)^{1/2} + \log n)$$
$$= \Theta(2n^{3/2} + 6n\ln n)$$
$$= \Theta(n^{3/2})$$

simplest form

$$= \Theta(n^{3/2} + 23n^{1/3})$$
$$= \Theta(\frac{n^3 + n + 1}{4n^{3/2} + \ln n})$$

Also specify O or θ

Proposition

Θ determines an equivalence relation on F – i.e., for $f(n)$, $g(n) \in F$

1. **$f(n) \in \Theta(f(n))$ (reflexive)**
2. **$f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$ (symmetric)**
3. **$f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n)) \Rightarrow f(n) \in \Theta(h(n))$ (transitive)**

Two functions f and g have the same order if $f(n) \in \Theta(g(n))$

* Read page 51 & 52

Time Complexity of an *Algorithm*

The time complexity of an algorithm is the largest time required on any input of size n.

- **$O(n^2)$: Upper bound on the worst case running time of an algorithm.**
- **$\Omega(n^2)$: Lower bound on the best case of running time of an algorithm.**
- **$\theta(n^2)$: Do both.**

Optimality of algorithm

- Each problem has inherent complexity; that is, there is some minimum amount of work required to solve it.
- Lower bound of problem : minimal number of operation needed to solve the problem
- When the worst-case time complexity of an algorithm matches lower bound of the problem, the algorithm is said to be optimal.
- “Optimal” does not mean “the best known”; it means “the best possible”.

Example

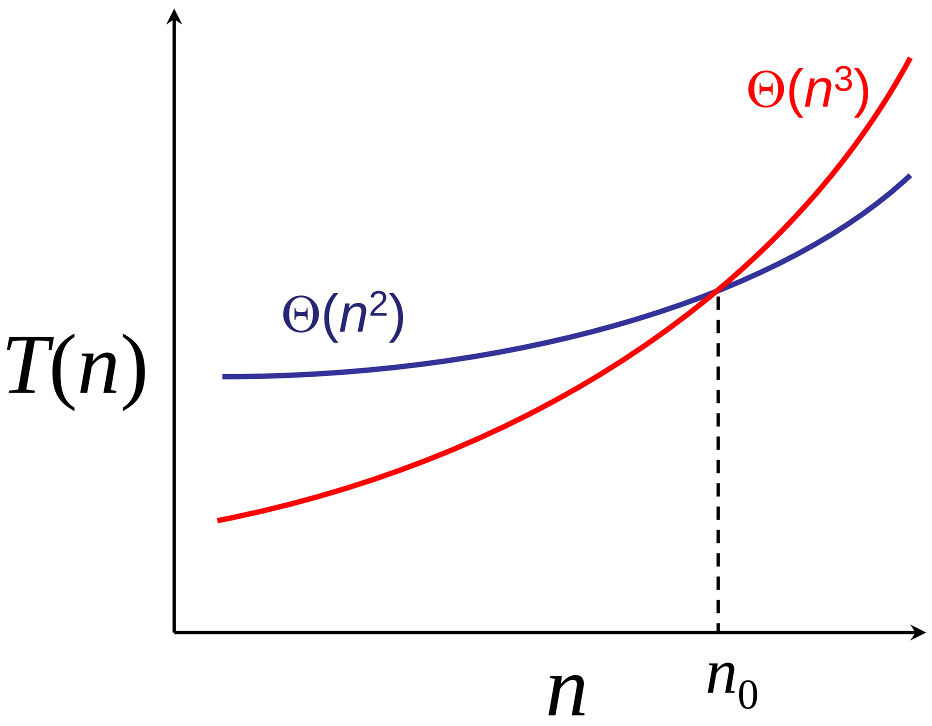
Time for several algorithms for a given problem

- ✓ Algorithm1 : $\theta(n^2)$
- ✓ Algorithm2 : $\theta(n^2 \lg n)$
- ✓ Algorithm3 : $\theta(n^3)$

➔ If someone proves the given problem needs at least n^2 basic operations, then the algorithm1 is optimal.

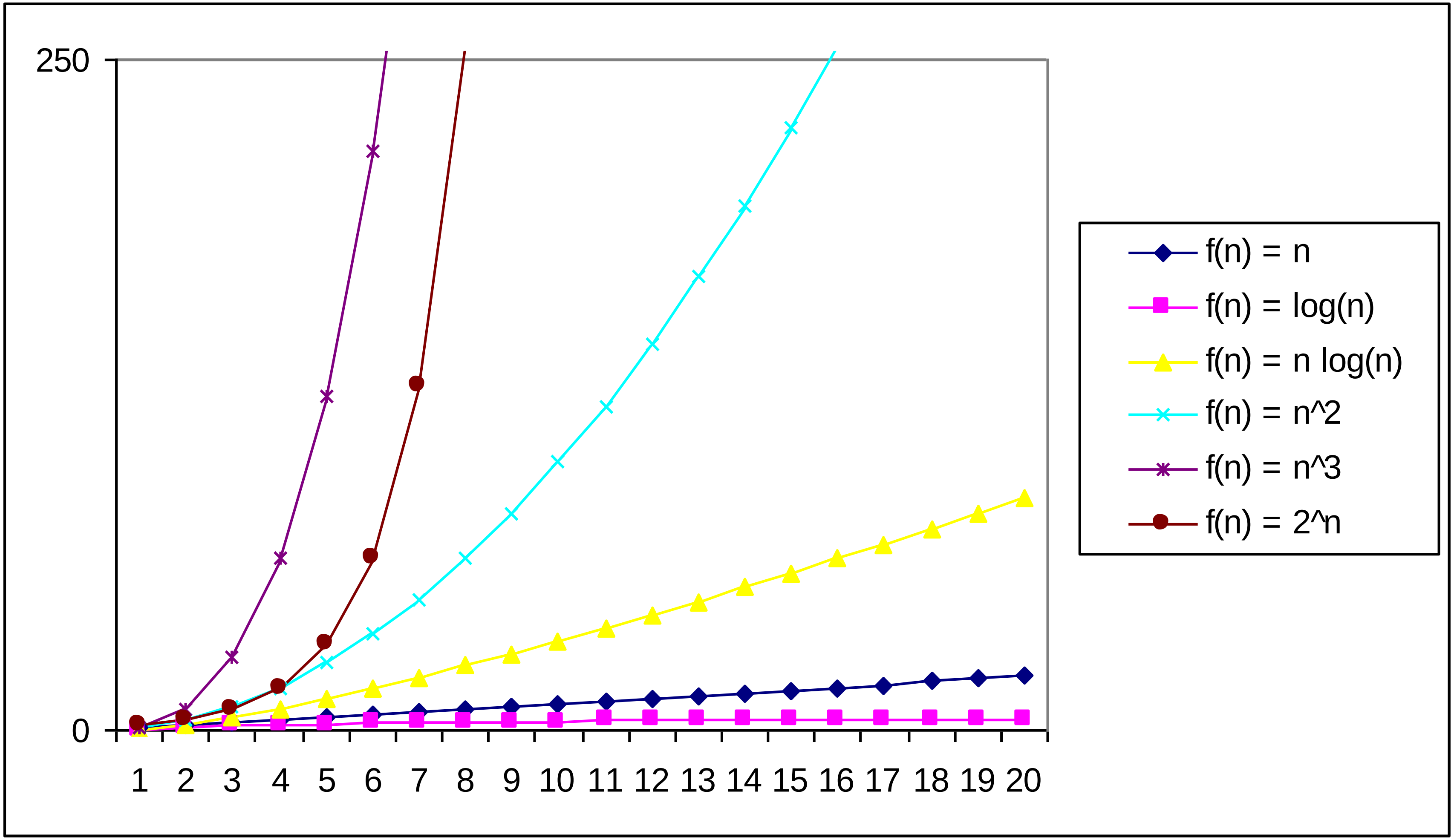
Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.

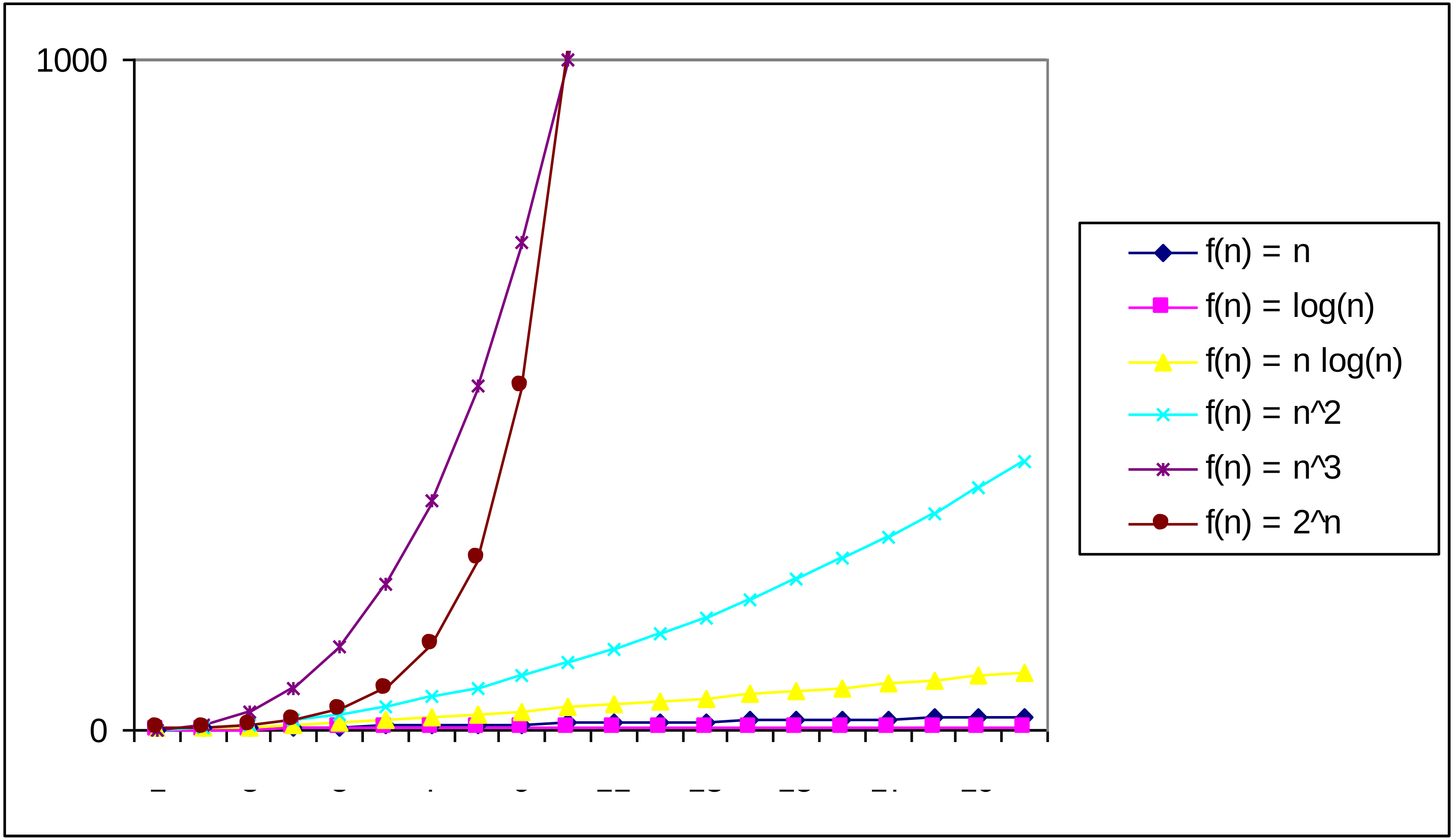


- Asymptotic analysis is a useful tool to help to structure our thinking toward better algorithm.
- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing

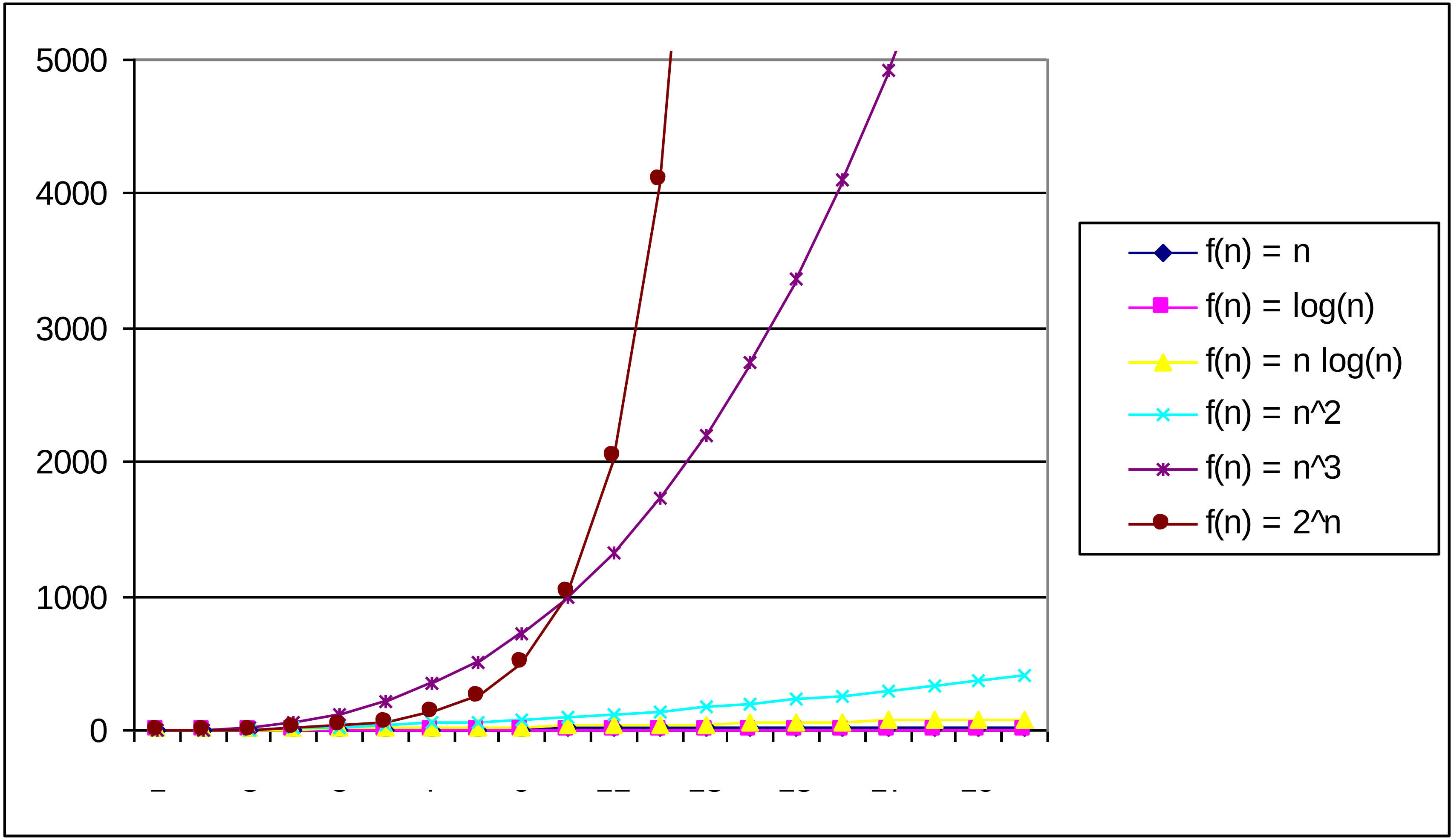
Practical Complexity



Practical Complexity



Practical Complexity



Hierarchy of Orders

Given function *f*(*n*) and *g*(*n*), we say that *f(n)* has smaller order than *g*(*n*), if O(*f*(*n*)) is strictly contained in O(*g*(*n*)).

,i.e., $O(f(n)) \subset O(g(n))$.

Example:

$$O(1) \subset O(\log n) \subset O(\sqrt{n}) \subset O(n) \subset O(n \log n)$$
$$\subset O(n^2) \subset O(n^3) \subset O(2^n) \subset O(n2^n) \subset O(n!)$$

Where does $O(n^{1/1000})$ belong?

Time complexity comparison

n	Cray-1 Fortran[a] $3n^3$ nanoseconds	TRS-80 Basic[b] 19,500,000n nanoseconds
10	3 microseconds	.2 seconds
100	3 milliseconds	2.0 seconds
1,000	3 seconds	20.0 seconds
2,500	50 seconds	50.0 seconds
10,000	49 minutes	3.2 minutes
1,000,000	95 years	5.4 hours

[a] Cray-1 is a trademark of Cray Research, Inc.

[B] TRS-80 is a trademark of Tandy Corporation.

Time complexity comparison

Algorithm	1	2	3	4	5
Time function (microsec.)	33n	46n lg n	$13n^2$	$3.4n^3$	2^n
Input size(n)			Solution time		
10	.0033 sec.	.0015 sec.	.0013 sec.	.0034 sec.	.001 sec.
100	.003 sec.	.03 sec.	.13 sec.	3.4 sec.	$4 \cdot 10^{16}$ yr.
1,000	.033 sec.	.45 sec.	13 sec.	.96 hr.	
10,000	.33 sec.	6.1 sec.	22 min.	39 days	
100,000	3.3 sec.	1.3 min.	1.5 days	108 yr.	
Time allowed			Maximum solvable input size (approx.)		
1 second	30,000	2,000	280	67	20
1 minute	1,800,000	82,000	2,200	260	26

Table is adapted from *Programming Peals* by John Bently.