The Walk from Classical Electromagnetism to Special Relativity

1. Structure of Electromagnetism

Coulomb's Law,

$$\mathbf{F} = \frac{kqQ}{r^2}\hat{r},$$

was derived empirically using Newton's framework. Looking at how charged pith balls interact with one another allowed this law to be derived. This law is Galilean covariant.

Partly for convenience and partly to remedy spooky action at a distance, it's helpful to introduce fields like $\mathbf{E}$ which propagate the force. So now, the force on a charge isn't immediately caused by the position of another charge; it's mediated by a field $\mathbf{E}$: $\mathbf{F} = q\mathbf{E}$ (note that, as of this moment, the electric field is only masking the locality problem since it itself doesn't propagate yet).

Long before the electric field was proposed, people talked of magnetic fields. Little pieces of metal (magnets) seemed to push on each other, and Faraday could map out field lines using iron filings. People noticed that current-carrying wires also seemed to produce a magnetic field. They noticed that the tip of a metal compass seemed to rotate around a current carrying wire (but, compasses only reacted to magnetic forces). If one imagines the current carrying wire creating a field $\mathbf{B}$ around it, the compass seemed to interact with this field. Experiments found that $\mathbf{F} = \hat{I}(\mathbf{I} \times \mathbf{B})$. It was surmised that the charges making up the current themselves feel the force, so $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$. This is a profound change from the electric force law. This law is not Galilean invariant. If you shift frames, the form of $\mathbf{v} \times \mathbf{B}$ changes. This implies that the force law prefers a frame (both will see different force laws; one must be the right one in form so that they both see the same acceleration. The other frame has to rewrite their force law in terms of the other person's frame.[1]). This already breaks the structure that Newton's laws work under.

2. The Transition to Fields

It became useful to focus entirely on the fields. Helmholtz theorem tells us that vector fields are uniquely described by their divergence and curl, so coming up with dynamical equation of the divergence and curl of the electric and magnetic fields fully describe electromagnetic theory. Further work showed that $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho$, $\boldsymbol{\nabla} \cdot \mathbf{B} = \mathbf{0}$, $\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J}$, and $\boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}$. In the form written, all of these equations are derivable from experimental laws like Coulomb's law and Biot-Savart law. The divergence equations represent how the fields are produced by "sources": Imagine integrating over a small volume. If there are no charges ($\rho = 0$), then all field lines entering the volume must exit; if a field did start inside the volume, it would need to loop back around. In other words, if there is no charge, there can't be a source or terminus for electric fields. Further, if $\rho > 0$, then there must be a net outward flow of electric field lines, implying that the charge created an electric field line. For magnetism, $\boldsymbol{\nabla} \cdot \mathbf{B} = \mathbf{0}$ implies that there can't be any sources of magnetic fields (there can't be magnetic monopoles). The curl equations don't represent how sources or sinks produce these fields; they represent how the fields circulate around in space (the divergence equations represent how they spread out). Fields can rotate in space without beginning or ending; i.e. the fields can swirl without

[1]Technically, the form could be preserved if $\mathbf{B}$ transforms in such a way to keep $\mathbf{v}' \times \mathbf{B}' = \mathbf{v} \times \mathbf{B}$. When the relative velocity between the two frames is $v << c$, this looks like it's the case, but once v becomes significant, it's clear that $\mathbf{B}$ isn't transforming nicely, and the force law is not covariant. One frame is "right"; the other frame is "wrong".

having a source or sink (just look at the divergence and curl of the magnetic field).

Philosophically, things could change when moving from statics to dynamics. There's no reason a priori that $\nabla \cdot \mathbf{E} = \rho$ once we allow ρ to change in time, for example (as we can no longer use Coulomb's law to prove $\nabla \cdot \mathbf{E} = \rho$). Inevitably, experiment is what confirms the dynamical form of the above four laws, but there are some philosophical hints as to what might change. For instance, the divergence equations represent how electric and magnetic fields are produced by sources. Even if these sources are allowed to move, there's strong reason to suspect the equations don't change: just look at each instant in time. **E** field is produced by the charge configuration (the sources) in the volume at that time. If the sources themselves aren't changing, there's no reason to suspect that the fields would suddenly change (why would the source's velocity matter). If there were a correction term added to Gauss's law, for instance, $\nabla \cdot \mathbf{E} = \rho + \mathbf{f}(\partial_\mathbf{t}\mathbf{E}, \partial_\mathbf{t}\mathbf{B})$, then the electric field lines could begin or terminate on something other than charge (field lines could still be produced even when $\rho = 0$); i.e. there would be another source. This isn't wrong a priori, but it just doesn't *feel* right. As mentioned earlier, the curl equations really have nothing to do with sources or sinks[2], so they'd be ideal candidates to capture the dynamical motion of the fields. This is what's backed up by experiment, as we will now discuss. Imagine moving a loop of charge being dragged through a magnetic field. Faraday found that, as you might expect, there's a symmetry here: why should anything change if, instead of moving the loop, you moved the magnet with the opposite velocity? In experiment, moving the loop with velocity $\mathbf{v}$, the magnet with velocity $-\mathbf{v}$ or decreasing / increasing the magnetic field strength all produced a current in the wire. The charges aren't moving, yet a force still acts on them (as you'd expect due to symmetry). The force on them must be electrical in nature since the charges aren't moving; i.e. a changing magnetic field produces an electric field. Specifically, it was found that $\nabla \times \mathbf{E} = -\partial_\mathbf{t}\mathbf{B}$. This addition is profound because it shows a mixing of electric and magnetic fields. It would seem reasonable that, due to symmetry, there's probably a $\partial_t\mathbf{E}$ term in the $\nabla \times \mathbf{B}$. Maxwell realized, from a purely mathematical point of view, that this must be true. Charge conservation is violated if there isn't such a term:

$$\nabla \cdot \mathbf{J} = \nabla \cdot \nabla \times \mathbf{B} = \mathbf{0}.$$

But, this violates charge conservation: $\nabla \cdot \mathbf{J} + \partial_\mathbf{t}\rho = \mathbf{0}$. If no term is added, charge can be created and destroyed. Let's add a $\partial_t\mathbf{E}$ term:

$$\nabla \cdot \mathbf{J} = \nabla \cdot (\nabla \times \mathbf{B} - \partial_\mathbf{t}\mathbf{E} = -\partial_\mathbf{t}(\nabla \cdot \mathbf{E} = -\partial_\mathbf{t}\rho.$$

Note that this derivation is further evidence that $\nabla \cdot \mathbf{E} = \rho$ should remain the same in dynamics (looking at it another way, you can prove Gauss's law in dynamics by assuming the other Maxwell's equations hold along with charge conservation). Thus, the final form of Maxwell's equations are

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho \\ \nabla \times \mathbf{E} &= -\partial_t\mathbf{B} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} &= \mathbf{J} + \partial_\mathbf{t}\mathbf{E}. \end{aligned}$$

3. The Wave Equation

Take the curl of the Faraday and Maxwell-Ampere equation in vacuum:

$$\begin{aligned} &\nabla \times \nabla \times \mathbf{E} = -\partial_\mathbf{t}(\nabla \times \mathbf{B}) \\ \to &\nabla(\nabla \cdot \mathbf{E}) - \nabla^\mathbf{2}\mathbf{E} = -\partial_\mathbf{t}\mathbf{J} - \partial_\mathbf{t}^\mathbf{2}\mathbf{E} \\ \to &\nabla\rho - \nabla^2\mathbf{E} = -\partial_\mathbf{t}\mathbf{J} - \partial_\mathbf{t}^\mathbf{2}\mathbf{E} \\ \to &\nabla^2\mathbf{E} = \partial_\mathbf{t}^\mathbf{2}\mathbf{E}. \end{aligned}$$

[2]You may ask: isn't **J** a "source" for **B**? Yes, but not in the way I mean here. Here, I mean source as something that creates starts or ends to field lines. **J** doesn't create starts or ends to field lines; it only creates magnetic field lines that rotate around the current; i.e. it tells us how **B** rotates, but it doesn't create a start or end to magnetic field lines. **E** emits from ρ, but **B** doesn't emit from **J**. A dynamic term added to Gauss's law would imply another source of field lines, but a dynamic term added to the curl equations doesn't imply another source of field lines; it just tells us that the fields rotate a little differently.

$$
\begin{aligned}
&\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{J} + \partial_{\mathbf{t}}(\boldsymbol{\nabla} \times \mathbf{E}) \\
\to &\nabla(\boldsymbol{\nabla} \cdot \mathbf{B}) - \nabla^{\mathbf{2}}\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{J} - \partial_{\mathbf{t}}^{\mathbf{2}}\mathbf{B} \\
\to &\nabla^2\mathbf{B} = \partial_{\mathbf{t}}^{\mathbf{2}}\mathbf{B}.
\end{aligned}
$$

As shown, the magnetic and electric fields can propagate as waves even in vacuum. This is huge, and a major hint that fields take on a life of their own and are themselves dynamic variables. The wave equations are odd for a few reasons:

- Waves usually propagate in something: ocean waves propagate in water, sound waves propagate in air, waves in a string propagate through the string, etc. But here, the waves seem to propagate through nothing.
- It predicts a wave speed c, but nothing seemed special about the frame we formulated electromagnetism in. It seems like we could have formulated electromagnetism in any other frame and gotten the same thing (this is the same problem that the force from the magnetic field hinted at earlier).

There are two ways one can resolve this conundrum:

- We conclude that our frame is special. We're measuring c because we're moving along with a preferred frame: the frame that the "luminous aether" moves in. This aether is the medium in which light travels.
- We're right to think that nothing is special about this frame. If we chose any other frame, we would have gotten the same result. c is the speed of propagation in all frames.

The first choice seems appealing. Galilean transformations are preserved, and there's a medium of propagation for light. Our wave equation isn't Galilean covariant (if I shift frames to one that isn't moving along with the aether, the equation has a different form), but at least the familiar structure of space and time is saved. The second is appealing for different reasons: we don't have to introduce some arbitrary medium ad hoc, and, while the structure of space and time need to change to keep c constant in all frames, the wave equations would be covariant under this new structure. In the end, what put the nail in the coffin was the Michelson-Morley experiment, which more-or-less showed that the speed of light is constant regardless of your frame.

Lorentz Transformations and the Structure of Spacetime

Given the two postulates of relativity:

- The laws of physics are the same in all inertial frames
- The speed of light is constant in all inertial frames,

one can find a relationship between the space and time coordinates of two frames moving at a constant velocity v relative to one another in the x direction:

$$
\begin{aligned}
t' &= \gamma(t - \beta/cx) \\
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z.
\end{aligned}
$$

This is the **Lorentz transformation** for a "boost" in the x direction. One can derive similar expressions for boosts in the y and z directions. These transformations tell us that the structure of spacetime is no longer Galilean: space and time warp to keep the speed of light constant in all frames. All rotational transformations (3 in total; one for each axis) and spacetime translations (4 in total; one for each spacetime coordinate) also keep the speed of light constant and the physics the same. These give us 10 total, independent transformations. Any general transformation that keeps the speed of light constant and keeps the physics the same will be some combination of these 10 transformations:

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu}\, x^{\nu} + a^{\mu}.$$

The 3 rotations and 3 boosts are linear transformations and so can both be wrapped under Λ. Spacetime translations are affine transformations (they move the origin) and so have to separated as a^{μ}. Λ are called the Lorentz transformations, and the entire combination $\Lambda + a$ are called the Poincare transformations. Consider spacetime points (x, t) and $(x + \Delta x, t + \Delta t)$. Under all 10 Poincare transformations, the interval

$$\Delta s^2 = |(x + \Delta x, t + \Delta t) - (x, t)|^2 = |(\Delta x, \Delta t)|^2 = \Delta t^2 - \Delta x^2$$

is invariant. Taking the differential limit, we obtain an invariant line element

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

This can define a geometry over Minkowski spacetime: the spacetime structure that relativistic physics lives in[3]. Notice that ds^2 almost looks like an inner product in 4D, but the spatial components receive a negative:

$$ds^2 = dx^{\mu}\eta_{\mu\nu}dx^{\nu} = \langle dx, dx\rangle_{\eta},$$

where, in matrix form,

$$[\eta] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

η is called the Minkowski metric. Notice that η is a fixed object and doesn't transform under Lorentz transformations (if it did, the form for ds^2 would change in different frames).

With this in mind, let's see if we can add some mathematical and geometrical structure to Minkowski spacetime. The vectors in Minkowski spacetime live on $\mathbb{R}^4$, and comes defined with a symmetric, bilinear form $\eta : V \times V \to \mathbb{R}$ defined by $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The inner product defined by this metric is (as with all metrics) given by $\langle A, B\rangle_{\eta} = A^{\mu}\eta_{\mu\nu}B^{\nu}$ for $A, B \in \mathbb{R}^4$[4]. The Lorentz group is the transformation group that preserves the line element ds^2 or, equivalently, preserves η. This group acts on V by: $v^{\mu} = \Lambda^{\mu}_{\ \nu}v^{\nu}$ (this is what defines a four vector. Technically, the metric η can act on all $\mathbb{R}^4$ vectors, but it only really has meaning it acts on vectors with the extra physical meaning of transforming under Lorentz transformations). Points in spacetime live in an affine space modeled on V (this allows us to talk of vectors but also spacetime translations). The Poincare group is the group of affine transformations that preserve the Minkowski metric on V.

It may look like we've added a ton of information here, but we haven't. All we've done is the following:

- Find the transformations that are consistent with the two physical postulates of special relativity
- We find that the quantity $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ remains invariant under these transformations
- We note that the structural form of ds^2 looks like an inner product, so we define a metric η to reproduce ds^2 in inner product language
- Since all transformations preserve ds^2, and since ds^2 is defined via η, the transformations preserve η
- Thus, η can be used to find many other invariant quantities like "lengths" and "angles"
- The geometrical interpretation of these quantities comes from the geometrical interpretation of η as made precise in the mathematics of differential geometry.

[3]Alternatively, you can demand that the spacetime interval is invariant, and define Lorentz transformations as the set of transformations that keep the spacetime interval invariant.

[4]If one checks the inner product axioms, one will find that this symmetric, bilinear form obeys all but positive-definiteness. As a result, it's technically not an inner product, but a psuedo-inner product or a Minkowski inner product.

The above paragraph is just making all of the background mathematical structure precise. If I give you two four-vectors, η can give us a Lorentz invariant *notion* of an angle and distance between them. In a way, we're defining a geometry over our physical setup, and it's the unique geometry we can define that is Lorentz invariant.

With this metric, the defining property of the Lorentz transformations is given by

$$\begin{aligned} ds^2 = \langle dx^\mu, dx^\nu \rangle_\eta &= dx^\mu \eta_{\mu\nu} dx^\nu \\ = ds'^2 = \langle dx'^\mu, dx'^\nu \rangle &= dx'^\mu \eta_{\mu\nu} dx'^\nu \\ &= \Lambda^\rho{}_\mu dx^\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu dx^\nu \\ \rightarrow \eta_{\mu\nu} &= \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu, \end{aligned}$$

and some invariant quantities are:

$$\begin{aligned} &s_A^2 = \langle A, A \rangle_\eta && \text{giving us an invariant notion of distance and the line element} \\ &\langle A, B \rangle_\eta && \text{giving us an invariant notion of angles} \\ &d^4x = dt\; dx\; dy\; dz && \text{giving us an invariant notion of volume} \end{aligned}$$

All observers agree on these quantities, so integrals over d^4x or ds will give a Lorentz invariant notion of volume and distance.

As we mentioned in the beginning of this summary, electromagnetism is naturally Lorentz invariant and therefore naturally compatible with special relativity: all observers agree on the form of Maxwell's equations (and, consequently, on the form of the wave equation). Further, all observers agree that the speed of light is constant in their frame. With the Lorentz transformation connecting any two frames, there's no need for an aether and no need to treat one frame as preferential. The equations all transform covariantly. However, Maxwell's equations as they're written above are not manifestly Lorentz invariant. That is, while they are Lorentz invariant, it's not immediately obvious since the equations aren't written in terms of Lorentz invariant and covariant quantities. All that we have to do is find a way to write Maxwell's equations in terms of covariant quantities, and we'll be golden. This isn't too hard, and I'll spare you most of the details. As a thought experiment, imagine an observer sits at rest with an electron, observing it moving under the action of an electromagnetic field. In this frame, there can't be any magnetic forces since the electron's at rest. Now, if we shift to a frame moving at a constant velocity relative to this one, we'll see the electron as moving, and suddenly a magnetic force is acting on it. Different frames see a different mixture of electric and magnetic fields. A more precise treatment finds that

$$\partial_\mu F^{\mu\nu} = j^\nu, \text{ where}$$

$$\begin{aligned} j &= \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix} \\ F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\ A &= \begin{pmatrix} \phi \\ \vec{A} \end{pmatrix}, \end{aligned}$$

where A^μ is a four vector comprising the scalar potential ϕ and the vector potential $\vec{A}$, j^μ is a four vector comprising the charge density ρ and the current density $\vec{J}$, and $F^{\mu\nu}$ is the electromagnetic field tensor (it transforms as a tensor under the Lorentz transformation) which mixes the electric and magnetic fields.