The Walk from Hamiltonian Mechanics to Quantum Mechanics

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1 Introduction

Hamiltonian theory -¿ Liouville theorem and Poisson brackets -¿ Operators instead of dynamical variables and Lie algebras / Lie brackets (Poisson bracket = Lie bracket, but now we have operators) -¿ KvN -¿ Difference between QM and CM -¿ To field theory

KvN mechanics is about the evolution of real, physical trajectories, with ψ only tracking probabilities. λ operators and the Liouvillian bridge this to a wavefunction language. But in QM, the wavefunction is itself the physical object, so the Hamiltonian acts on it directly, without mediation.

Momentum is the generator of spatial translations (this is true in classical mechanics). Now, we want to describe particles as waves because many experiments showed particles exhibiting wave behavior. If the theories to be useful at all, we would like the "momentum" of our wave to be the momentum of the particle we're modeling in the classical limit. Ok, so now we're treating a wave as our fundamental object, and we look at the operator that generates spatial translations in our wave. Ok, it's $\hat{p} = -i\hbar\partial_x$. In the classical limit (average), this value should look like the momentum in classical mechanics. Ehrenfest's theorem confirms this. Further, this makes the connection with KvN clean because the difference is that, in KvN, the λ operators don't generate spatial translation because the wavefunction "isn't the particle" (the λ operators were never meant to be measurements; I just write this because of how similar $\hat{\lambda}$ looks to $\hat{p}$ in form). The wavefunction isn't the fundamental object; it's just a representation of our ignorance.

KvN evolves the dynamic variables x and p. Those are the physical quantities; the state of our "particle." Then, $p = mv$ generates spatial translations. When we push to QM, our "particle" is the wave; here, $\hat{p} = -i\hbar\partial_x$ generates spatial translations ($x(t) + dx = x(t) + p(t)/m\ dt$ in classical, but $\psi(x + dx) = \psi(x) + i\hbar/m\hat{p}\psi(x)\ dx$

Hamilton-Jacobi theory. $\psi(x,t) = A(x,t)\exp\left(\frac{-i}{\hbar}S\right)$. S captures the classical information. S follows the HJ theory (plus corrections). A encodes probabilistic information. It's the added structure that makes QM $\neq$ HJ + corrections.