Philosophical and Logical Structure of Classical Mechanics and Special Relativity

Classical Mechanics: Logical Structure and Motivation

1. Choose a Frame

- Postulate the existence of a class of **inertial frames**, related by Galilean transformations.
- All laws of physics take the same form in these frames.
- These frames agree on time t, position $\vec{x}$, and acceleration $\vec{a}$.

Conclusion: This shared notion of acceleration motivates why $\vec{a}$ is the natural quantity to relate to force.

2. Definition of Force

- Define a real force as anything that causes a deviation from uniform motion in an inertial frame:

$$\vec{F} := m\vec{a}$$

- This is not purely empirical — it is a definition *within* the logical framework.

2.1 Mass

- Under seemingly the same "influence", different objects might respond differently: they might accelerate in the same direction but do so at different magnitudes
- Rather than creating a new force for each individual object, define mass m as an intrinsic property of each object. The force law is then allowed to remain the same, but each object will respond differently. Force laws can always be written to include m consistently (just multiply by a factor until the masses are consistent.

2.2 Galilean Invariance and Covariance

- As mentioned, inertial frames are related by Galilean transformations.
- Acceleration $\vec{a}$ is invariant under these transformations (its a Galilean invariant): all inertial frames agree on it. This is one of the biggest reasons $\vec{F} = m\vec{a}$ works as a framework: all inertial observers agree on the value of $\vec{a}$.
- In practice, force laws $\vec{F}$ are often derived empirically. To be physically meaningful, the equation $\vec{F} = m\vec{a}$ must be **Galilean covariant**: it should transform consistently across inertial frames. Otherwise, the universe has "picked" a frame, which doesn't feel philosophically reasonable.
- For example, Newtonian gravity $\vec{F} = -GmM\,\hat{r}/r^2$ is Galilean covariant: $\ddot{\vec{r}}$ is invariant, and all inertial observers agree on spatial distances $\vec{r}$.

3. Fictitious Forces

- A non-inertial frame observes particles accelerating even when no force acts in the inertial frame.
- To preserve the form $\vec{F} = m\vec{a}$, we invent fictitious forces (e.g., centrifugal, Coriolis).
- These vanish in an inertial frame and are "by definition" not real.

4. Linearity and Independence of Forces

- Newton's second law is linear in $\vec{F}$, allowing:

$$\vec{F}_{\text{total}} = \vec{F}_1 + \vec{F}_2 + \cdots$$

- Forces in different directions act independently because of the Euclidean vector structure of space. No assumptions or defining characteristics of forces are required to explain this. It simply follows from the structure we impart onto space.

4.1 Linearity and the Possibility of Interacting Forces

Philosophically, it is possible to imagine a universe in which forces do not act independently — that is, where the presence of one force can modify the effect of another. For example, applying a force $\vec{F}_1$ alone might yield an acceleration $\vec{a}_1$, but applying $\vec{F}_1 + \vec{F}_2$ could yield an acceleration different from $\vec{a}_1 + \vec{a}_2$.

Newtonian mechanics explicitly rules out this possibility. It does so not by experimental necessity, but by definition: force is defined in such a way that its effects are additive. The law $\vec{F} := m\vec{a}$ enforces linearity, and any observed deviation from additivity is attributed to a different net force; i.e. if $F_{\text{net}} \neq F_1 + F_2$, then $F_{\text{new}} = F_{\text{net}}$ (where F_{new} is thought of as independent from $F_1 + F_2$).

Thus, the linearity and independence of forces are not empirically discovered facts, but part of the logical structure of the theory. Only those interactions whose effects combine linearly are permitted to be called "forces" in this framework.

5. In Practice

Look at how different objects respond to what is seemingly the "same" influence, and empirically derive some force law F. Assuming that the laws of physics stay static over time, this force law can be used to find the acceleration of an object. The acceleration of an object under the influence of multiple different forces can be derived

In Newtonian mechanics, mass can be defined operationally by comparing the accelerations of different objects under the same influence. If two bodies experience the same force and respond with accelerations $\vec{a}_1$ and $\vec{a}_2$, we define their mass ratio by

$$\frac{m_1}{m_2} := \frac{a_2}{a_1}.$$

This allows us to define force via $\vec{F} = m\vec{a}$ in a way that is consistent across many objects.

Now, when introducing a new force law, we can always choose its form (i.e., scale it appropriately) so that the same set of mass values still satisfy $\vec{F} = m\vec{a}$. This ensures internal consistency.

However, if two objects with previously defined masses respond non-proportionally to the same new influence, we conclude that this force depends on some additional property (e.g., electric charge). In this way, the breakdown of proportionality becomes a signal of new physics.

- Luckily, all objects don't behave absolutely differently under seemingly the same influence; if they did, we'd need a new system

- The Newtonian system works off of the assumption that the universe won't change how it operates (forces will always influence objects in the same way).

Summary (Classical Mechanics)

Classical mechanics assumes the existence of inertial frames, where acceleration is objective and time is absolute. Forces are defined as causing acceleration in such frames. Fictitious forces arise only when we shift to non-inertial perspectives. Linearity and independence of forces arise naturally from the vector structure of space.

Special Relativity: Logical Structure and Motivation

Special relativity doesn't really alter the Newtonian framework from a systematic perspective. Newton's laws provide a defined system for practical physics applications. Special relativity alters the structure of spacetime beneath this structure. So, all we have to do is patch the holes in the framework left by this transition. The fundamental physics is in the new structure of spacetime.

1. No Absolute Time or Preferred Frame

- All inertial frames are equivalent, but now related by Lorentz (not Galilean) transformations.
- Time and simultaneity are relative.
- Observers disagree on t and x, but agree on the spacetime interval:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

2. Proper Time and Four-Velocity

- Since t is not invariant, acceleration defined as $d\vec{v}/dt$ is frame-dependent.
- Proper time τ is invariant across frames.
- Define the **four-velocity**:

$$u^\mu = \frac{dx^\mu}{d\tau}$$

3. Force as Deviation from Geodesic

- In flat spacetime, free particles move on straight worldlines:

$$\frac{du^\mu}{d\tau} = 0$$

- Define the **four-force**:

$$F^\mu := \frac{dp^\mu}{d\tau} = m\frac{du^\mu}{d\tau}$$

4. Why This Form?

- We require laws that:
  - Are covariant under Lorentz transformations.
  - Reduce to $\vec{F} = m\vec{a}$ in the Newtonian limit.
- This motivates and justifies the relativistic definition of force.

Summary (Special Relativity)

Special relativity replaces absolute time with invariant spacetime intervals and proper time. The concept of force is redefined via four-vectors, ensuring covariance and agreement with Newtonian mechanics in the low-speed limit. Free motion corresponds to geodesic worldlines, and force is whatever causes deviation from them.

Final Perspective

Both classical mechanics and special relativity define force relative to what counts as "free" motion:

- In classical mechanics: free motion = constant velocity in inertial frames.
- In special relativity: free motion = geodesic motion in spacetime.

In both cases, force is introduced to preserve the consistency of these free-motion definitions across frames. The laws are minimal and logical structures imposed to preserve symmetry and consistency of the underlying spacetime geometry.

In the relativistic framework, free motion is defined by the condition $a^\mu = 0$. Any object for which $a^\mu \neq 0$ is, by definition, experiencing a force. In classical mechanics, acceleration $\vec{a}$ was a Galilean invariant — all inertial frames agreed on its value. However, this was more of a convenient happenstance than a deep principle. What truly mattered was that the law $\vec{F} = m\vec{a}$ was Galilean covariant: all inertial frames agreed on the *do* of the equation and on whether or not $\vec{a} = 0$. They did not, for example, agree on whether $\vec{v} = 0$, which highlights the special role of acceleration.

In special relativity, the analogous Lorentz-covariant form is $f^\mu = ma^\mu$. Although inertial frames no longer agree on the value of a^μ, they *do* agree on whether $a^\mu = 0$, and they all agree on the form of this equation. This makes it a natural generalization of Newton's second law. Therefore, we take $a^\mu = 0$ as the defining condition for free (i.e., unaccelerated and force-free) motion in special relativity. In the classical limit, $f^\mu = ma^\mu \to \vec{F} = m\vec{a}$.