CS 580: Algorithm Design and Analysis

Order Statistics

The **selection problem** is the problem of computing, given a set A of n distinct numbers and a number i, $1 \leq i \leq n$, the i^{th}h **order statistics** (i.e., the i^{th} smallest number) of A.

We will consider some special cases of the order statistics problem:

- the **minimum**, i.e. the first,
- the **maximum**, i.e. the last, and
- the **median**, i.e. the "halfway point."

Order Statistics

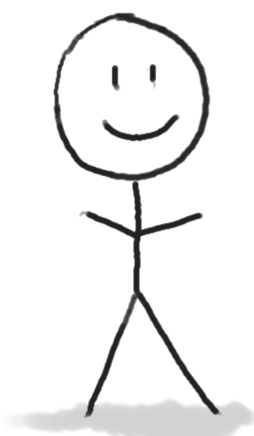
Medians occur at $i = \lfloor (n+1)/2 \rfloor$ and $i = \lceil (n+1)/2 \rceil$. If n is odd, the median is unique, and if n is even, there are two medians.

Recall

We work in the **comparison model**:

- Given input array $x = (x_1, \ldots, x_n)$, we can access the array via comparison queries: “Is $x_i < x_j$?”

Is  bigger than  ?



Algorithm

YES

There is a genie that knows what the right order is.

The genie can answer YES/NO questions of the form:
is [this] bigger than [that]?

Finding the Min

How many comparisons are necessary and sufficient for finding the minimum?

Algorithm:

Given vector $x = (x_1, \dots, x_n)$ *as input, consider the standard algorithm.*

Let $q = x_1$.

For each $i = 2, \dots, n$:

If $x_i < q$:

$q = x_i$

Return q

This algorithms makes $n - 1$ comparisons.

Finding the Min

Can we do better than $n - 1$ comparisons?

Finding the Min

Can we do better than $n - 1$ comparisons?

If not, then show a lower bound of the form:

- Every deterministic algorithm for finding the min (which is correct on every input) makes at least $n - 1$ comparisons.

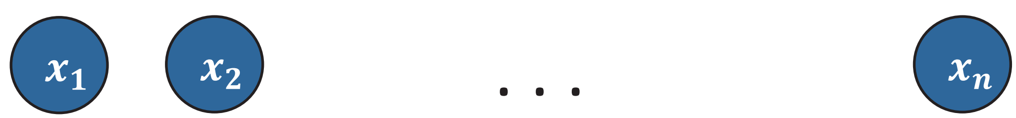
Lower Bound:

Consider any (correct) deterministic algorithm A for finding the min.

Construct a graph G with:

- vertices $x_1, \dots, x_n$.
- edge (x_i, x_j) if A compares elements x_i and x_j at some point during its execution.

What happens if the graph G is not connected at the end of A's execution?



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- edge (x_i, x_j) if A compares elements x_i and x_j at some point during its execution.

If G is not connected at the end of A's execution, then the algorithm can output the wrong answer on some inputs.

For G to be connected, it must have at least $n-1$ edges (if it has exactly $n-1$, it is a tree) $\Rightarrow$ A makes at least $n-1$ comparisons.

Selection *(Find s-th smallest element)*

Selection is a trivial problem **if the input numbers are sorted.** If we use a sorting algorithm having $O(n \lg n)$ worst-case running time, then the selection problem can be solved in in $O(n \lg n)$ time.

But using a sorting is more like using a cannon to shoot a fly since only one number needs to computed.

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Task: Design a selection algorithm inspired by QuickSort

but with fewer comparisons.

$O(n)$ expected-time selection using the randomized partition

Idea: In order to find the s-th order statistics in a region of size n, use the **randomized partition** to split the region into two subarrays. Let $\mathsf{k}-1$ and $n-\mathsf{k}$ be the size of the left subarray and the size of the right subarray. If $k=s$, the pivot is the key that's looked for. If $\mathsf{s} \leq \mathsf{k}-1$, look for the **?-th element in the left subarray**.

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Analysis

Define

- $T(n, s) =$ expected # comparisons for selection of s-th statistic
- $T(n) = \max_{s} T(n, s)$ is the expected # comparisons of selection for the worst case index s.

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Task: Write an inequality to upper bound $T(n)$ as a function of the amount of work done in Partition and in recursive selection calls.

Analysis

Recall: $T(n)$ is the expected # comparisons of selection for the worst case index s.

For each i, $0 \le i \le n-1$, the size of the left subarray is equal to i with probability $1/n$. Assuming that the larger interval is taken, for some $\alpha > 0$, $T(n)$ is at most

Work for partition

$$\alpha n + \frac{1}{n} \sum_{1 \le k \le n-1, k \ne s} T(\max(k, n-k)).$$

Expected work for recursive call

This is at most

$$\alpha n + \frac{2}{n} \left(\sum_{k=\lceil n/2 \rceil}^{n-1} T(k) \right).$$

Analysis (cont'd)

Assume that there is $c > 0$ such that $T(k) \leq ck$ for all $k < n$.

Then the sum $\sum_{k=\lceil n/2 \rceil}^{n-1} T(k)$ is at most $\sum_{k=\lceil n/2 \rceil}^{n-1} ck$. This is at most

$$
\begin{aligned}
& \sum_{k=1}^{n-1} ck - \sum_{k=1}^{\lceil n/2 \rceil - 1} ck \\
= \; & \frac{cn(n-1)}{2} - \frac{c}{2}\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\left\lceil \frac{n}{2} \right\rceil \\
\leq \; & \frac{cn(n-1)}{2} - \frac{c}{2}\left(\frac{n}{2} - 1\right)\frac{n}{2} \\
= \; & cn\left(\frac{3n}{8} - \frac{1}{4}\right).
\end{aligned}
$$

Analysis (cont'd)

So, if c is sufficiently large,

$$T(n) \leq \alpha n + c\left(\frac{3}{4}n - \frac{1}{2}\right).$$

By making the constant c at least 4α, we have that αn is at most $\frac{cn}{4}$. Then $\mathrm{T(n)} \leq \mathrm{c} \cdot n$.

Min and Max

How many comparisons are necessary and sufficient for computing both the minimum and the maximum?

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Well, to compute the maximum $n - 1$ comparisons are necessary and sufficient. The same is true for the minimum. So, the number should be $2n - 2$ for computing both.

Min and Max

Hint: *Actually you can do better by processing the input numbers in pairs*

Min and Max Algorithm

Simultaneous computation of max and min can be done in $\leq 3n/2$ comparisons

Assume n is even:

- Divide the numbers in pairs and find the larger and smaller one in each pair
- From the n/2 larger items, find the maximum
- From the n/2 smaller items, find the minimum

For n odd, compare the remaining (n-th item) with both the min and max.