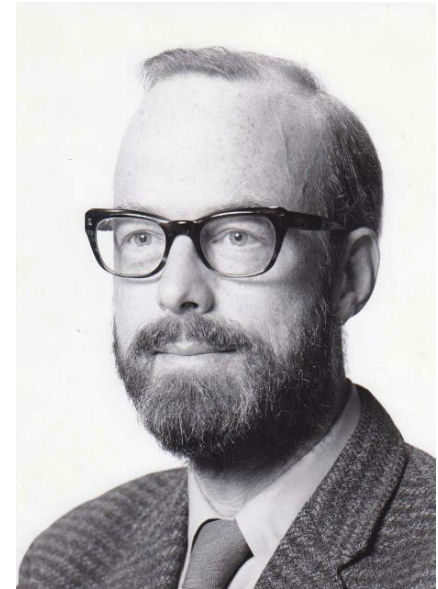


CS 580

Review: Quicksort

- Proposed by Tony Hoare in 1962.
- Divide-and-conquer algorithm.
- Sorts "in place" (like insertion sort, but not like merge sort).
- Very practical (with tuning).



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3. Recursively apply the above steps to the sub-array of elements with smaller values and separately to the sub-array of elements with greater values.

Quicksort an n -element array

Popular choice of pivot: x is a random element of the array

Exercise: write down recurrence for this choice of pivot and prove the expected run time is $O(n \log n)$.

- Assume all input elements are distinct.

(following analysis by Daniel Gildea).



The Algorithm

Quicksort(A, n)

1: Quicksort'($A, 1, n$)

Quicksort'(A, p, r)

1: **if** $p \geq r$ **then return**

2: $q = \text{Partition}(A, p, r)$

3: Quicksort'($A, p, q - 1$)

4: Quicksort'($A, q + 1, r$)

Partition(A, p, r)

1: $x = A[r]$

2: $i \leftarrow p - 1$

3: **for** $j \leftarrow p$ **to** $r - 1$ **do**

4: **if** $A[j] \leq x$ **then** {

5: $i \leftarrow i + 1$

6: Exchange $A[i]$ and $A[j]$ }

7: Exchange $A[i + 1]$ and $A[r]$

8: **return** $i + 1$

Worst Case Runtime

Suppose the pivot is chosen uniformly at random from the input array.

Question: What is the worst case number of comparisons for Partition on input of length n ?

Let $T(n)$ be the worst case number of comparisons. Write recurrence (with equality or inequality) for $T(n)$.

Worst-case analysis

Let T be the worst-case running time of Quicksort. Then

$$T(n) = T(1) + T(n - 1) + \Omega(n).$$

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Thus, we have:

Theorem A The worst-case running time of Quicksort is $\Omega(n^2)$.

Since each element belongs to a region in which **Partition** is carried out at most n times, we have:

Theorem B The worst-case running time of Quicksort is $O(n^2)$.

The Best Cases

The best cases are when the array is split half and half. Then each element belongs to a region in which **Partition** is carried out at most $\lceil \log n \rceil$ times, so it's $O(n \log n)$.

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Recurrence:

$$T(n) = 2 T(n/2) + n$$

$$T(0) = T(1) = 0 \text{ (best case)}$$

Randomized-Quicksort

The idea is to **turn pessimistic cases into good cases by picking up the pivot randomly.**

Quicksort(A, n)

1: **Quicksort'**(A, 1, n)

Quicksort'(A, p, r)

-1: **Pick t uniformly at random from $\{p, p + 1, \dots, r\}$**

0: **Exchange A[r] and A[t]**

1: **if $p \geq r$ then return**

2: **q = Partition(A, p, r)**

3: **Quicksort'**(A, p, q-1)

4: **Quicksort'**(A, q+1, r)

Expected Running Time of Randomized-Quicksort

Let n be the size of the input array. Suppose that the elements are pairwise distinct.*

Let $T(n)$ be the expected running time of Randomized-Quicksort on inputs of size n . By convention, let $T(0) = 0$.

Let x be the pivot. Note that the size of the left subarray after partitioning is the rank of x minus 1.

*A more involved analysis is required if this condition is removed.

Making a hypothesis

We claim that the expected running time is at most $cn \log n$ for all $n \geq 1$. We prove this by induction on n . Let a be a constant such that partitioning of a size n subarray requires at most an steps.

For the base case, we can choose a value of c so that the claim hold.

For the induction step, let $n \geq 3$ and suppose that the claim holds for all values of n less than the current one.

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Exercise: Prove by induction that $T(n)$ is at most $cn \log(n)$.

The expected running time satisfies the following:

$$\begin{aligned} T(n) &\leq an + \frac{\sum_{k=0}^{n-1} (T(k) + T(n-1-k))}{n} \\ &= an + \frac{2}{n} \sum_{k=1}^{n-1} T(k). \end{aligned}$$

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By our induction hypothesis, this is at most

$$an + \frac{2c}{n} \sum_{k=1}^{n-1} k \log k.$$

Note that

$$\sum_{k=1}^{n-1} k \lg k \leq \int_1^n x \lg x dx.$$

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The integration is equal to

$$\frac{1}{2}n^2 \lg n - \frac{n^2}{4} + \frac{1}{4}.$$

This is at most

$$\frac{1}{2}n^2 \lg n - \frac{n^2}{8}.$$

Note that $\int x \log(x) = \frac{1}{2}x^2 \log(x) - \frac{x^2}{4} + D$ for any constant D

By plugging this bound, we have

$$T(n) \leq cn \lg n + (a - \frac{c}{4})n.$$

Choose c so that $c > 4a$. Then,

$$T(n) \leq cn \lg n.$$

Thus, we have proven:

Theorem C Randomized Quicksort has the expected running time of $\Theta(n \lg n)$.