# Max-flow min-cut theorem

**Augmenting path theorem.** A flow $f$ is a max-flow iff no augmenting paths.

**Max-flow min-cut theorem.** Value of the max-flow = capacity of min-cut.

Pf. The following three conditions are equivalent for any flow $f$:

i. There exists a cut $(A, B)$ such that $cap(A, B) = val(f)$.

*ii.* $f$ is a max-flow.

iii. There is no augmenting path with respect to $f$.

The value of a flow $f$ is: $val(f) = \sum_{e \text{ out of } s} f(e)$ .

Def. An $st$-cut (cut) is a partition $(A, B)$ of the nodes with $s \in A$ and $t \in B$.

Def. Its capacity is the sum of the capacities of the edges from $A$ to $B$.

***Recall:***

$$cap(A, B) = \sum_{e \text{ out of } A} c(e)$$

Weak duality. Let $f$ be any flow and $(A, B)$ be any cut. Then, $v(f) \leq cap(A, B)$.

Flow value lemma. Let $f$ be any flow and let $(A, B)$ be any cut. Then, the net flow across $(A, B)$ equals the value of $f$.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

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- Suppose that $(A, B)$ is a cut such that $cap(A, B) = val(f)$.
- Then, for any flow $f'$, $\underbrace{val(f') \leq cap(A, B)}_{\text{weak duality}} \underbrace{=}_{\text{by assumption}} val(f)$.
- Thus, $f$ is a max-flow. ▪

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[ ii ⇒ iii ] We prove contrapositive: ~iii ⇒ ~ii.

- Suppose that there is an augmenting path with respect to $f$.
- Can improve flow $f$ by sending flow along this path.
- Thus, $f$ is not a max-flow. ▪

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[ iii $\Rightarrow$ i ]

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[ iii ⇒ i ]

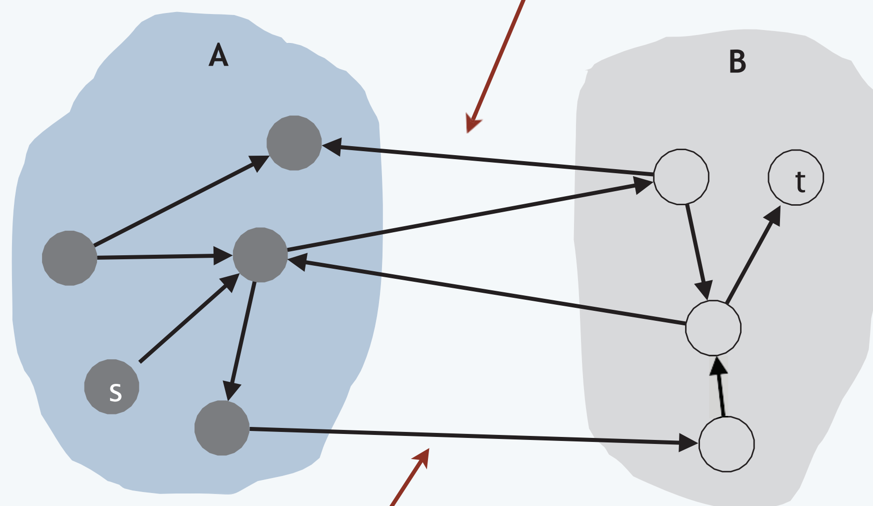
- Let $f$ be a flow with no augmenting paths.
- Let $A$ be set of nodes reachable from $s$ in residual graph $G_f$.
- By definition of cut $A$, $s \in A$.
- By definition of flow $f$, $t \notin A$.

edge e = (v, w) with v ∈B, w ∈A
must have f(e) = 0

original network G

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\
&= \sum_{e \text{ out of } A} c(e) \\
&= cap(A, B) \quad ▪
\end{aligned}
$$

flow-value lemma

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must have f(e) = c(e)