CS 580: Algorithm Design and Analysis

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems

Algorithm?

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Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

1-D version. **Algorithm?**

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↖ fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

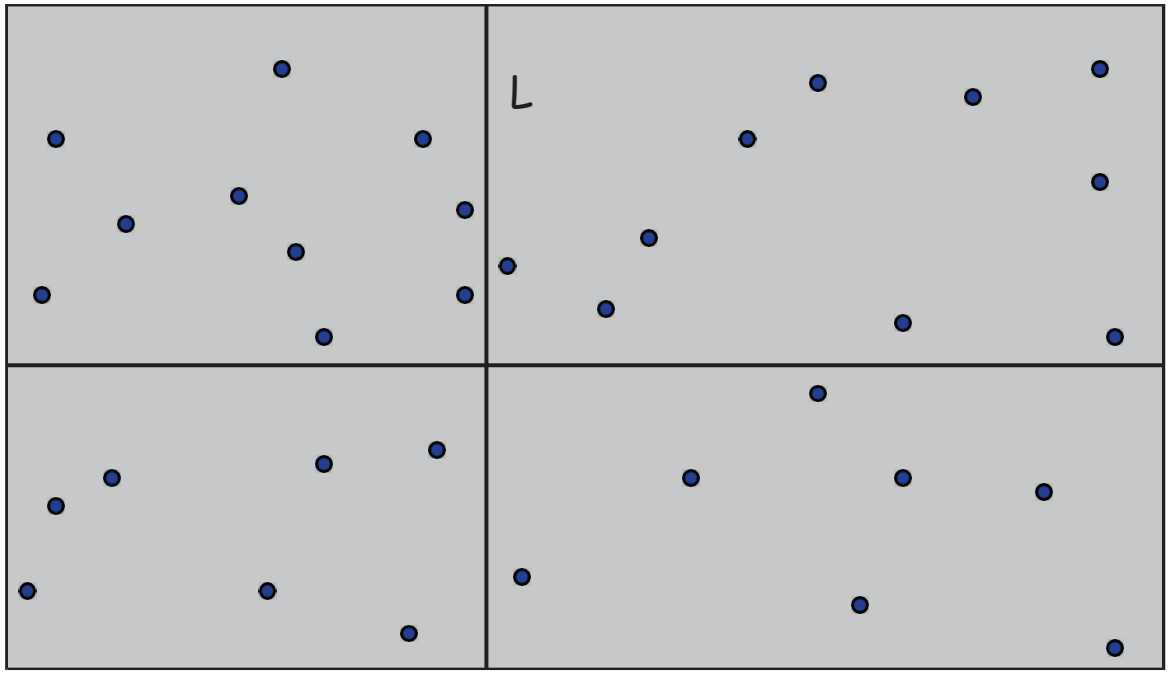
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

↑ to make presentation cleaner

Closest Pair of Points

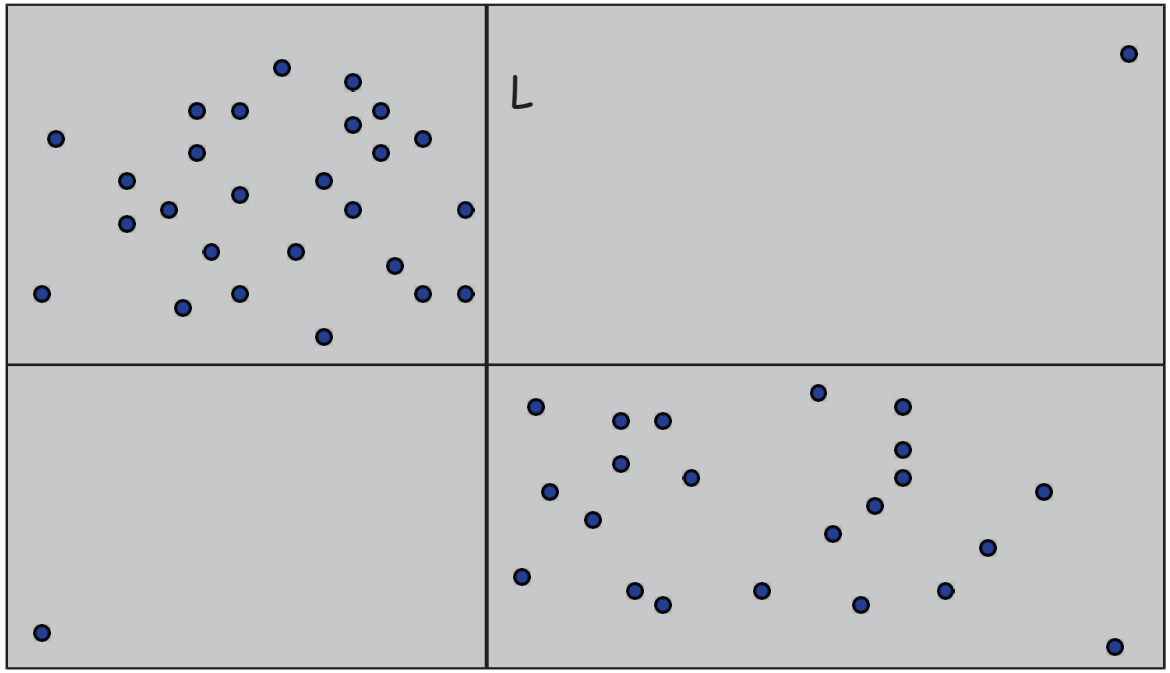
Divide. Sub-divide region into 4 quadrants.



Closest Pair of Points

Divide. Sub-divide region into 4 quadrants.

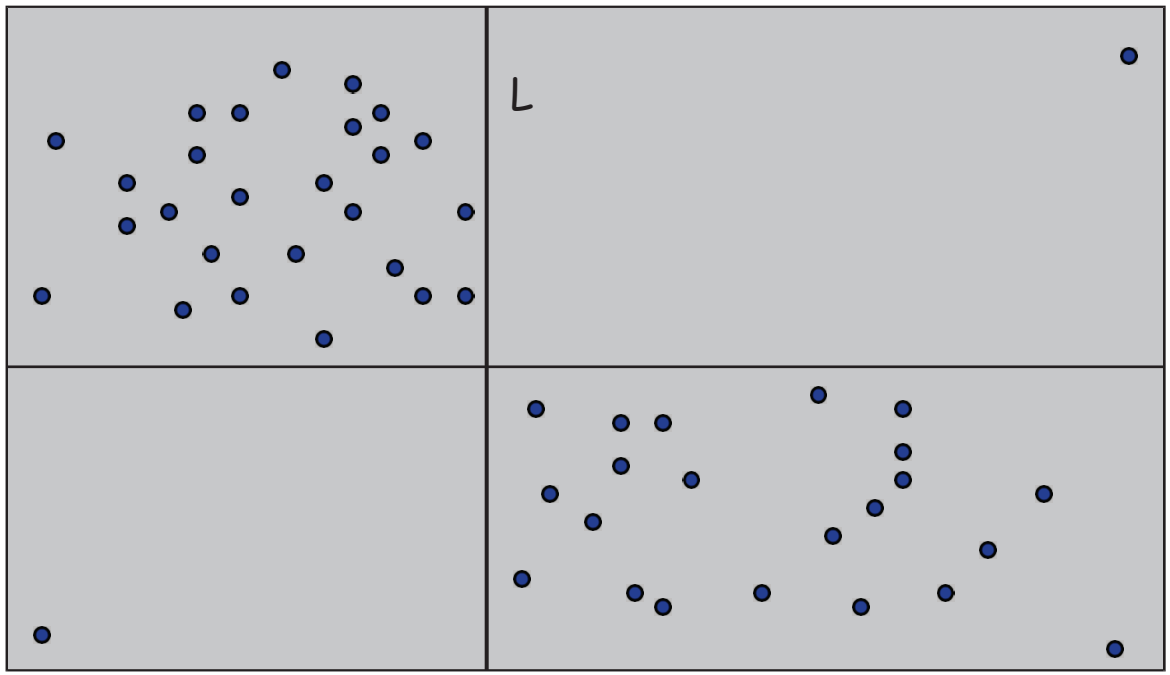
Q: **What should we do next?**



Closest Pair of Points

Divide. Sub-divide region into 4 quadrants.

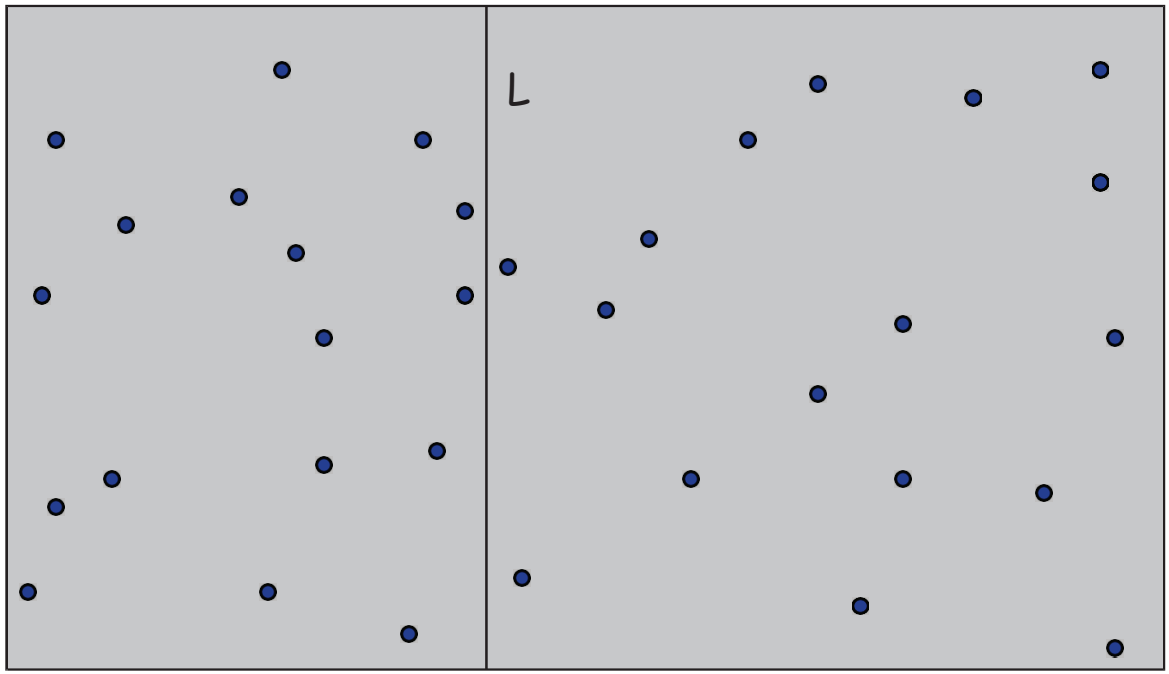
Obstacle. Impossible to ensure n/4 points in each piece.



Closest Pair of Points

Algorithm.

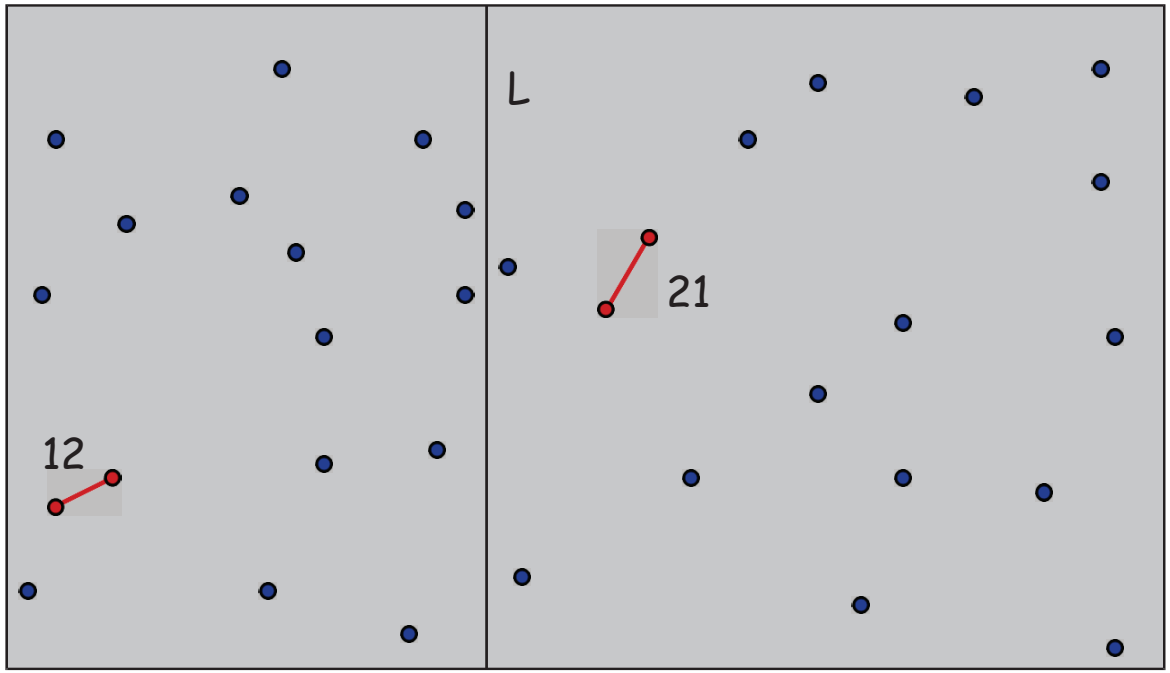
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



Closest Pair of Points

Algorithm.

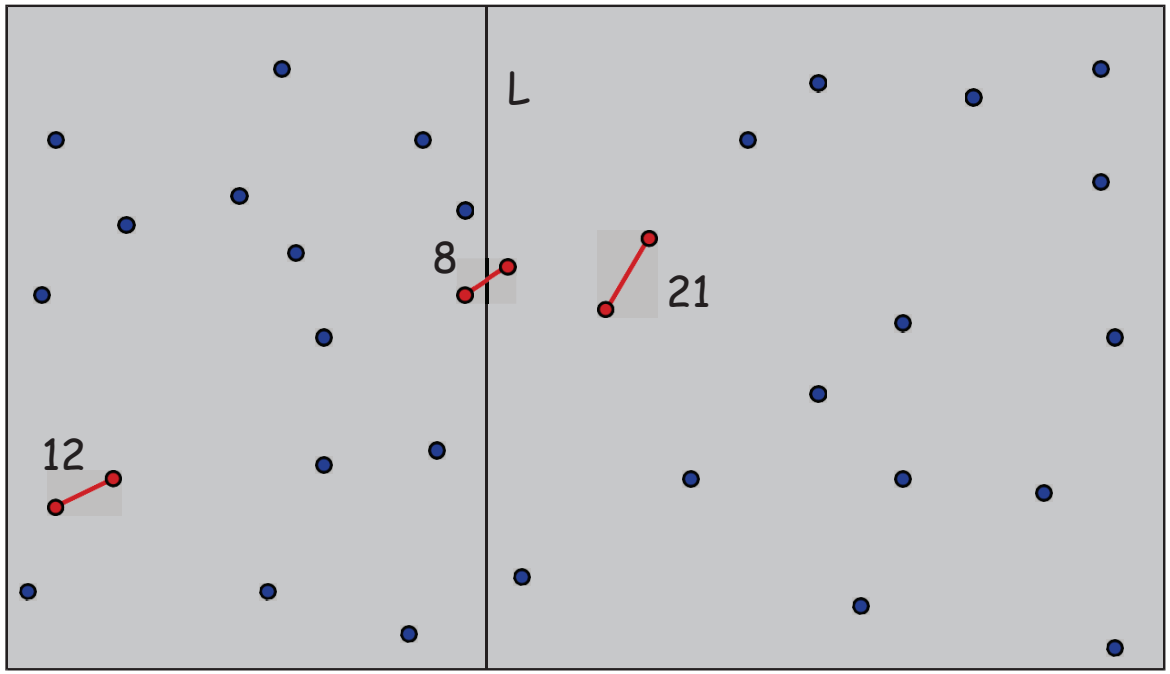
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

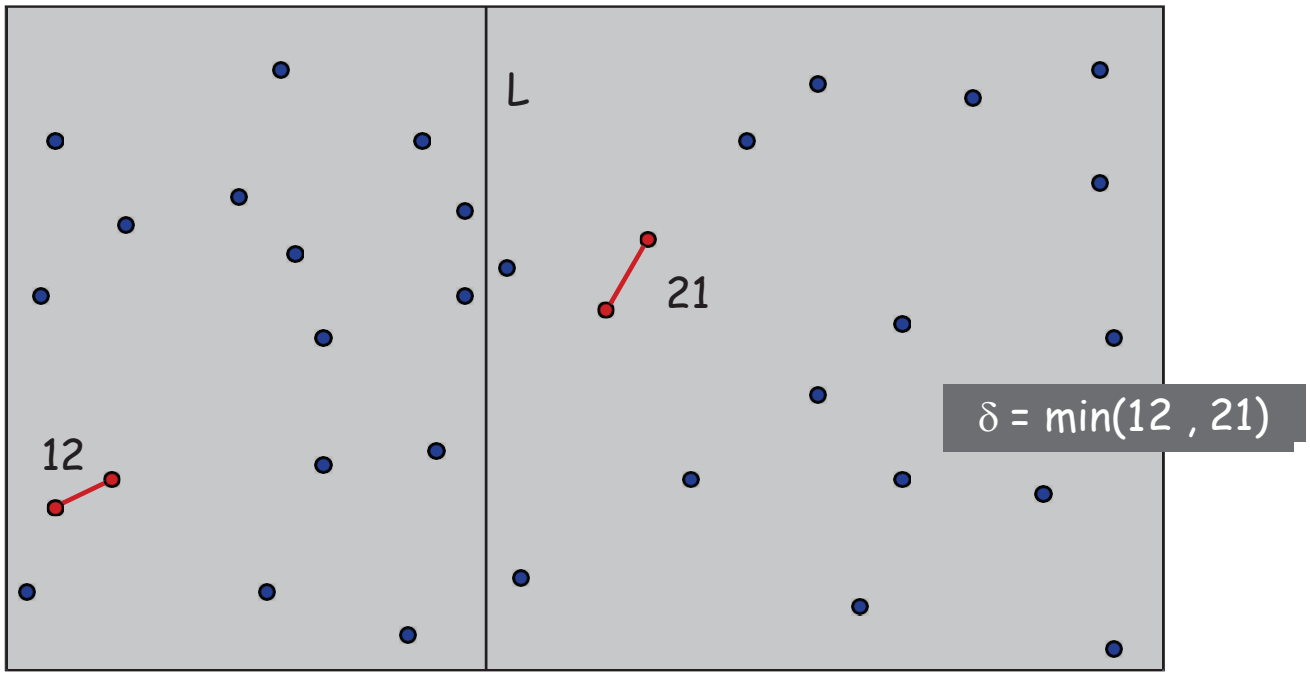
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.



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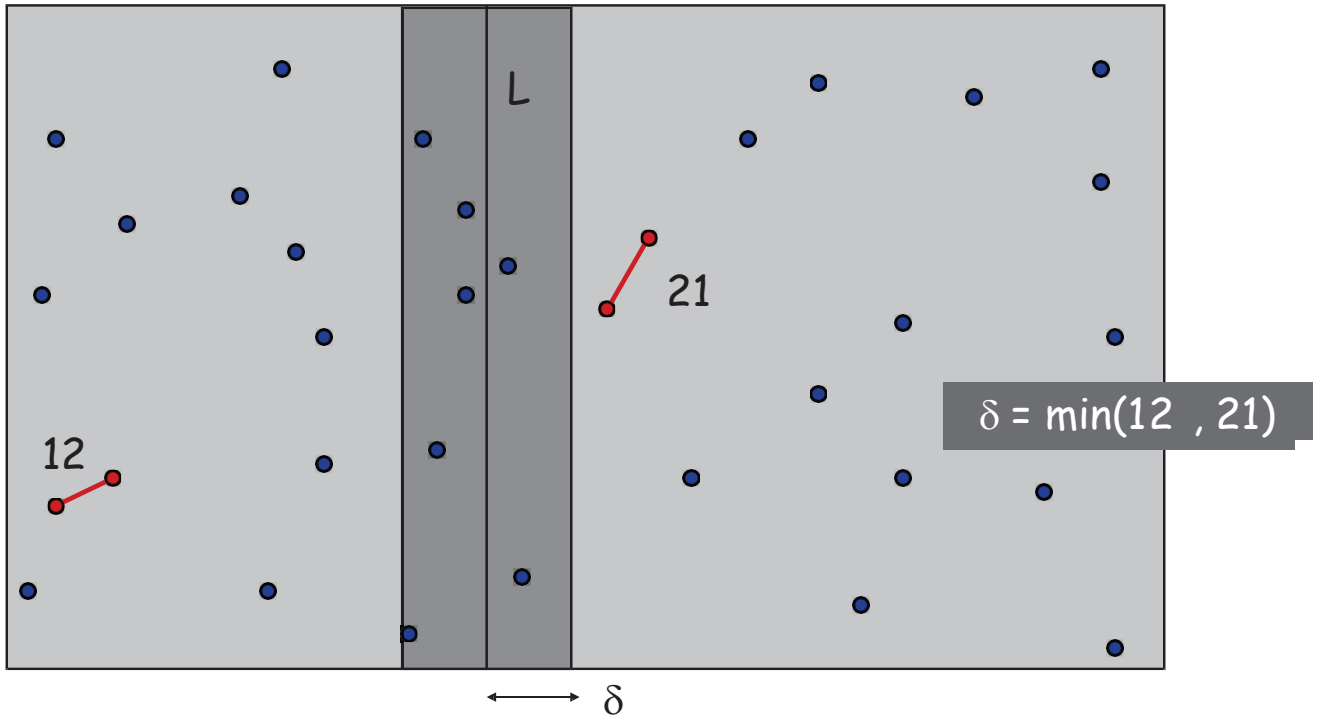
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

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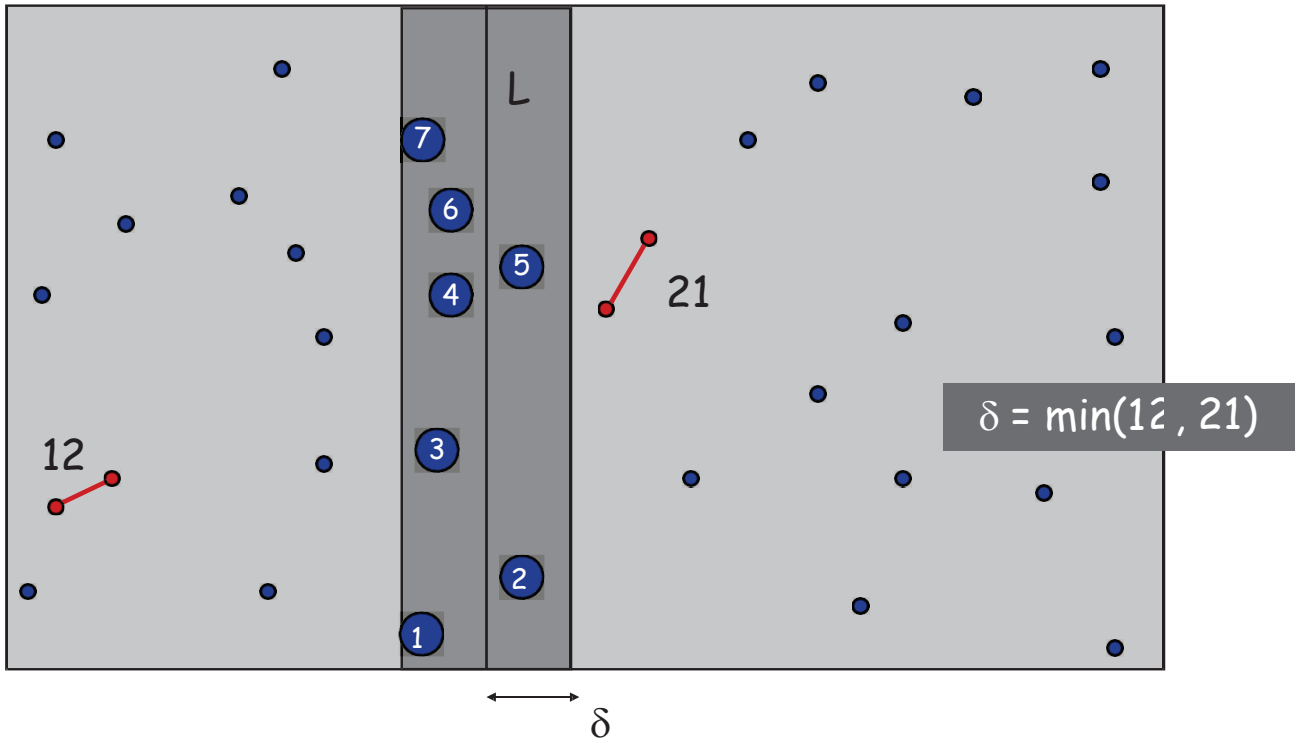
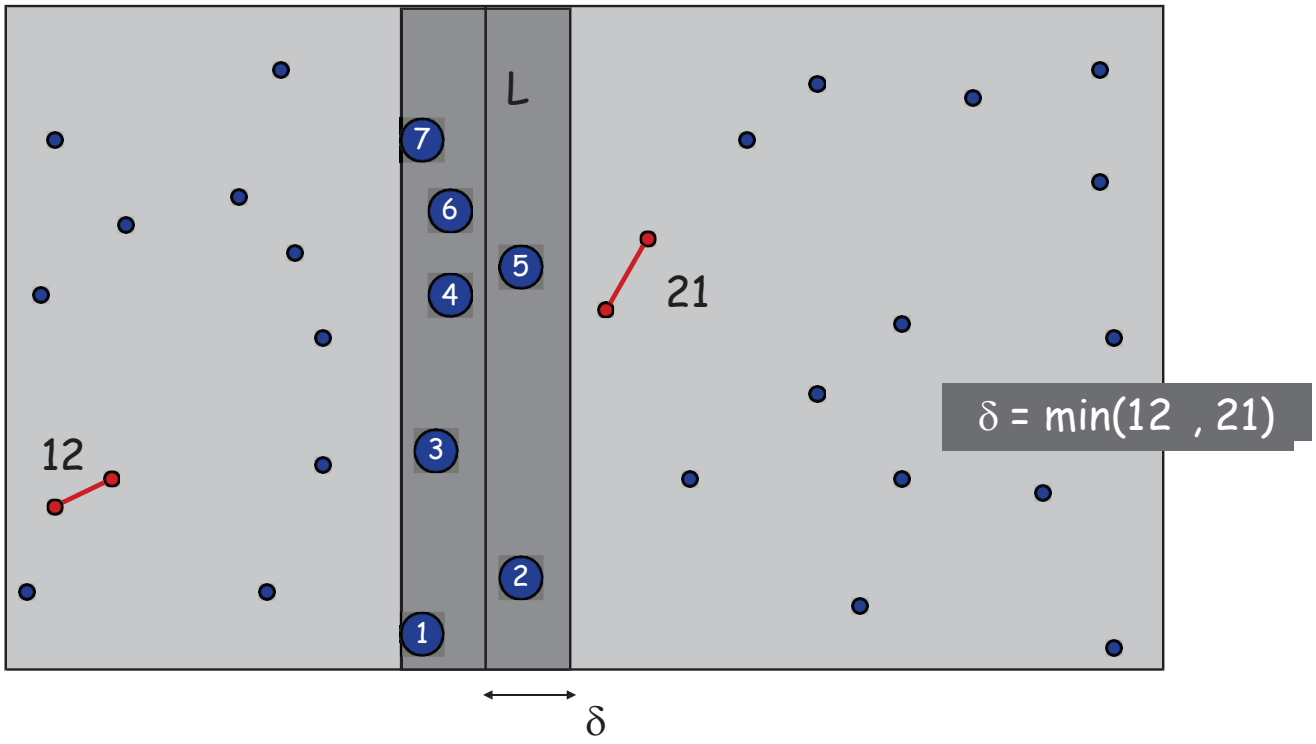
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Would like to check distances to points that are "close" in sorted list.

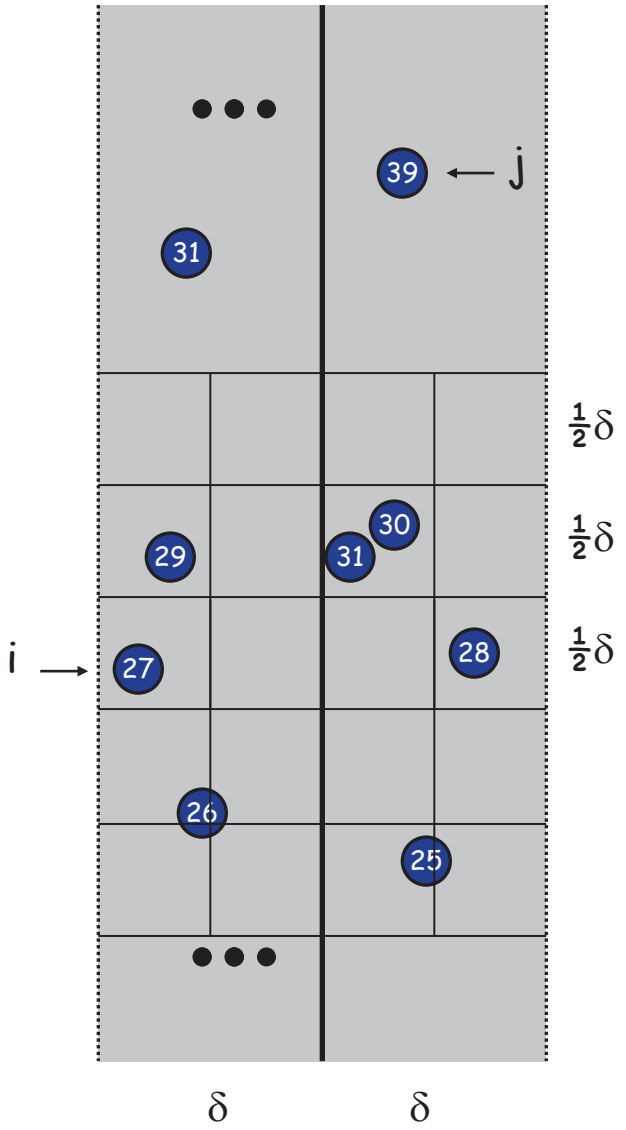


Closest Pair of Points

Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Q: **Can we find x such that $\forall i, j$:**

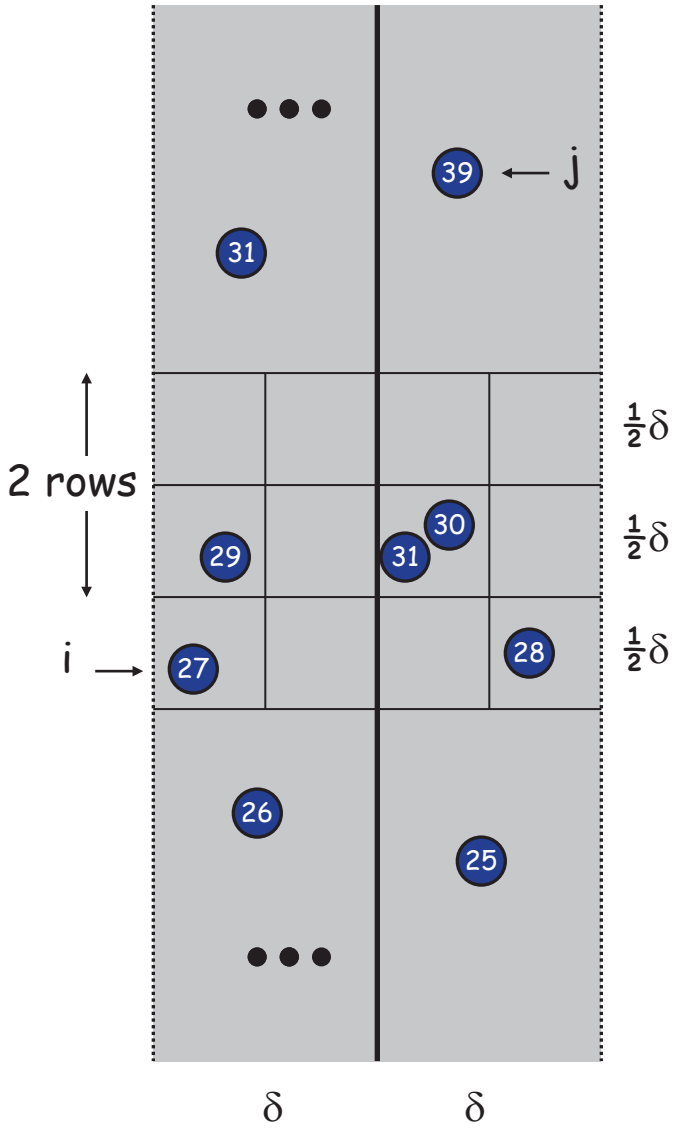
- If $|i - j| \geq x$, then the distance between s_i and s_j is at least δ. {For points in the δ strips}



Closest Pair of Points

Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.



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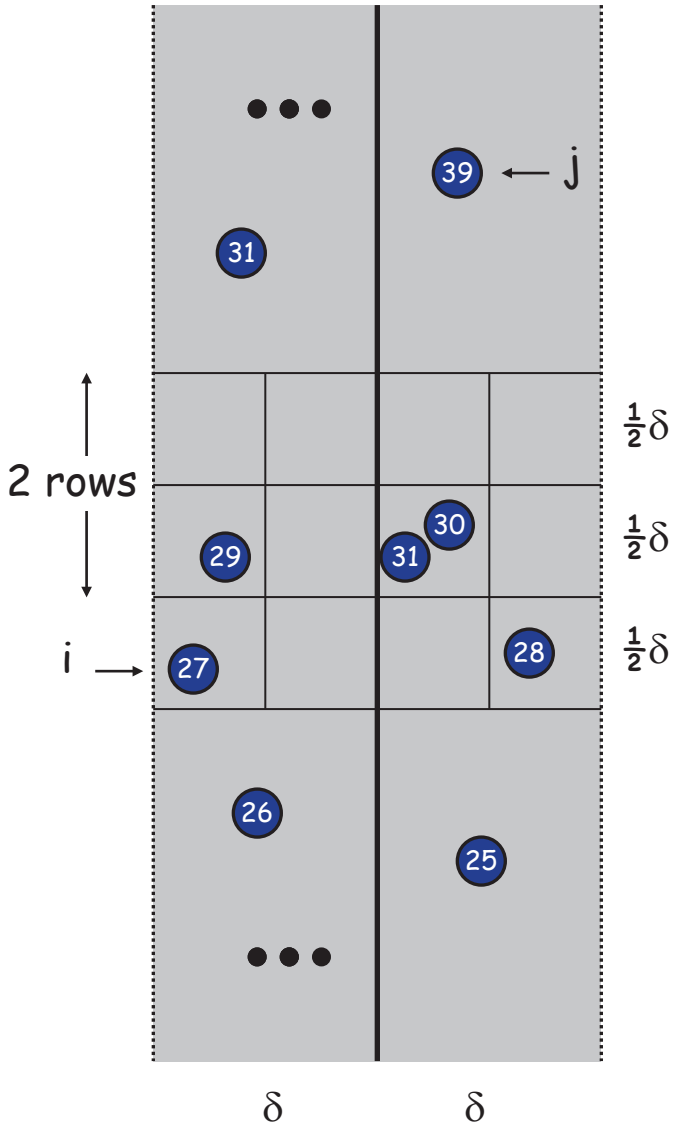
Pf.

- How many points in the same box?



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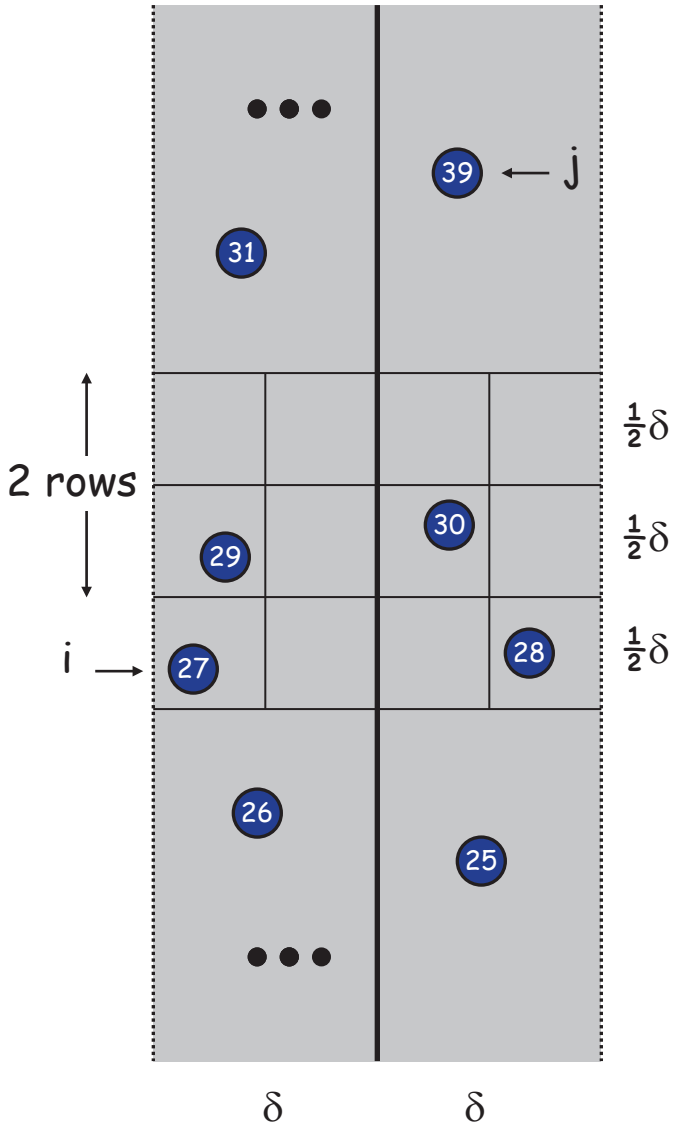
- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.



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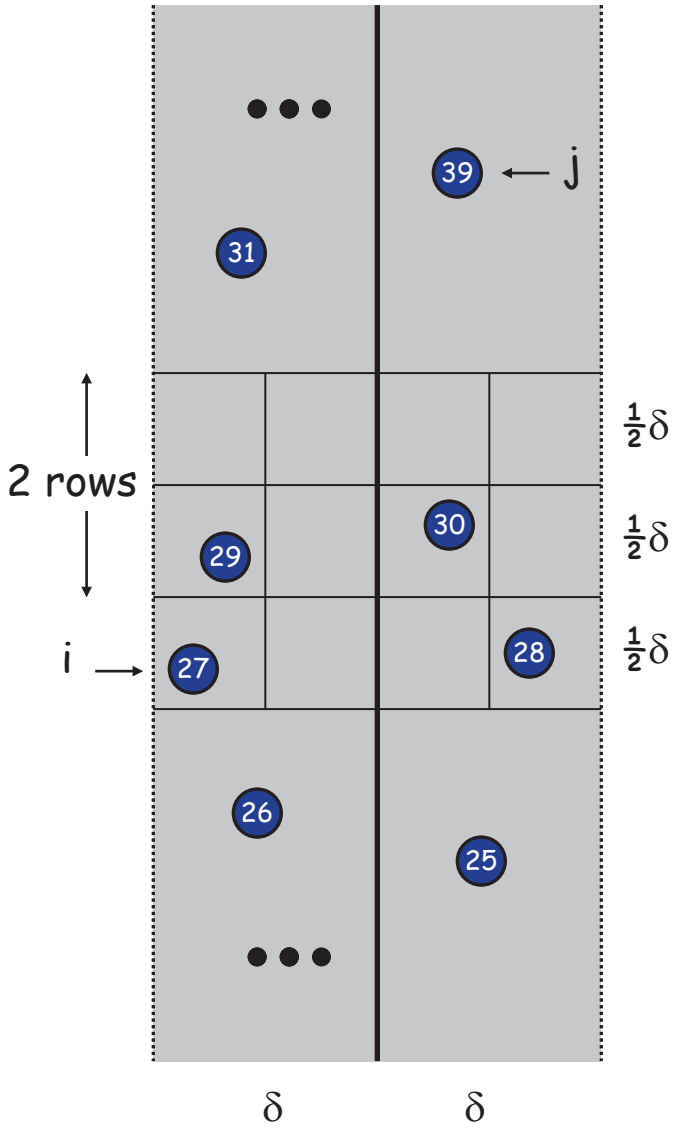
Why?



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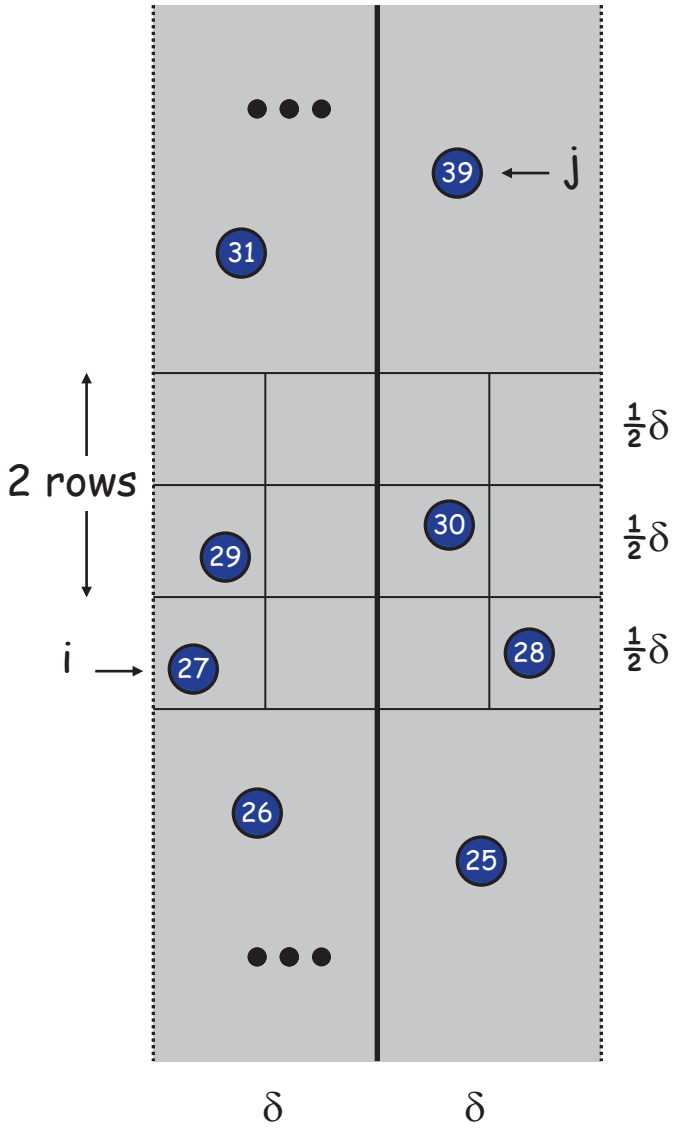
- How can we use this to prove the claim?



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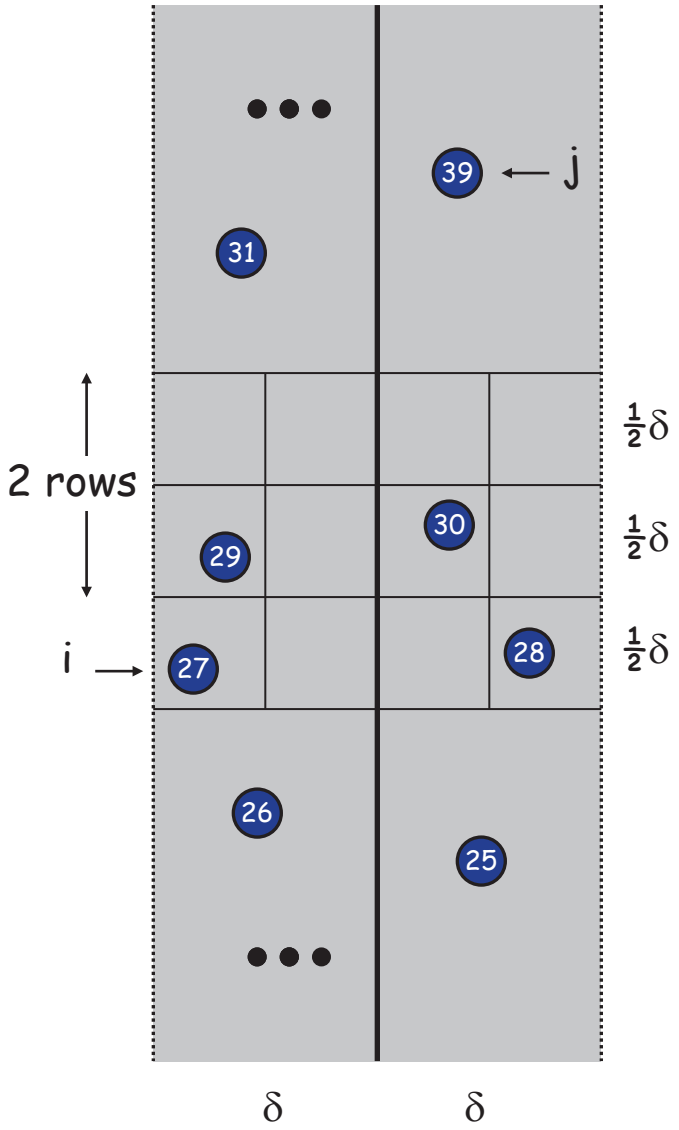
Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Still true if we replace 12 with 7.



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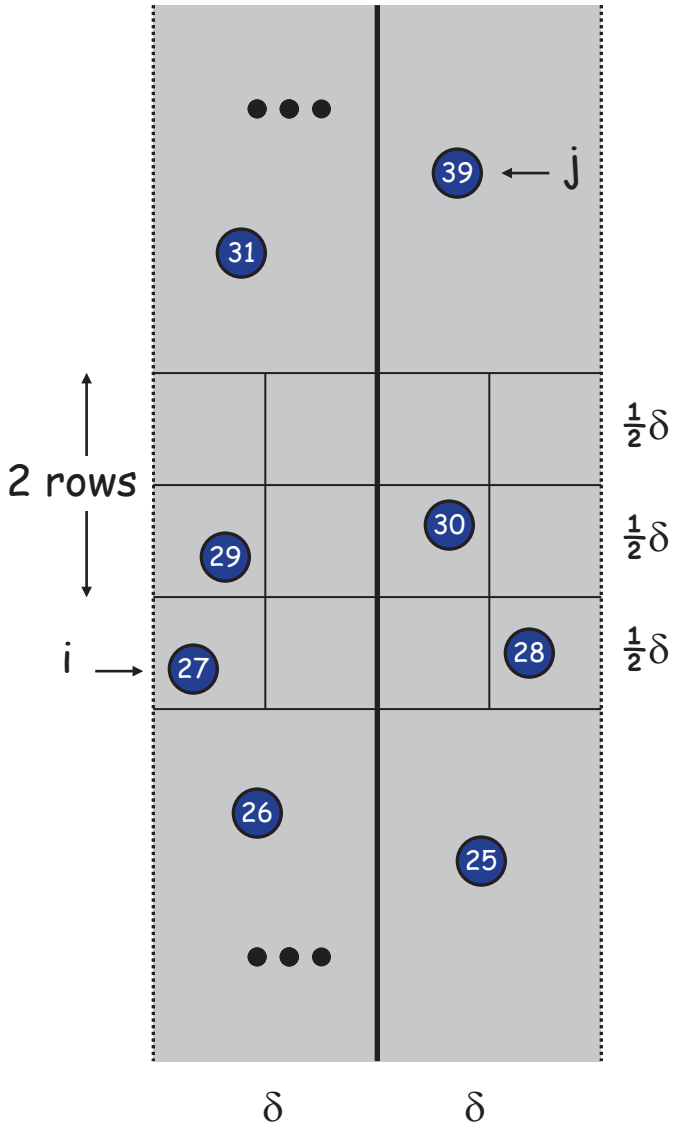
Q: How can we use this to find closest pair of points in the 2δ-strip?



Closest Pair Algorithm

```
Closest-Pair(p_1, …, p_n) {
   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ_1 = Closest-Pair(left half)
   δ_2 = Closest-Pair(right half)
   δ   = min(δ_1, δ_2)

   Delete all points further than δ from separation line L

   Sort remaining points by y-coordinate.

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these
   distances is less than δ, update δ.

   return δ.
}
```

Closest Pair Algorithm

Runtime

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Closest-Pair(p_1, …, p_n) {
   Compute separation line L such that half the points      ?
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Runtime

$O(n \log n)$

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O(n log n)

2T(n / 2)

O(n)

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O(n)

Closest Pair of Points: Analysis

Running time.

$$\mathrm{T}(n) \leq 2T(n/2) + O(n \log n) \quad \Rightarrow \quad \mathrm{T}(n) = O(n \log^2 n)$$

Closest Pair of Points: Analysis

Challenge. Can we achieve O(n log n)?