CS 580, Fall 2022; Instructor: Simina Brânzei.
TAs: Shoaib Khan, Anuj Singh, Lu Yan, Zheng Zhong.
Due: September 18, 11:59PM EST. Submit to Gradescope

Problem Set 1

Reading material : Slides on stable matching. Additional reading in Kleinberg-Tardos Ch. 1.

Collaboration policy : Acknowledge your collaborators on the homework. You may discuss proof strategies, but the solution should be written individually in your own words.

Submission format : The solutions must be typed in Latex and submitted via Gradescope.

Problem 1. *(10 points) Consider the function* $f : \mathbb{N} \to \mathbb{R}$ *given by* $f(n) = \frac{n^2}{5} - 8n + \log n$*. Prove, using mathematical induction, that* $f(n) \in \Theta(n^2)$*.*

Problem 2. *(20 points) The stable matchings possess an elegant "lattice" structure, that is, given two different stable matchings,* M *and* M'*, if each men is given the better of his partners between* M *and* M'*, then the result* M'' *is still a stable matching.*

Show the lattice structure property by proving the following statements:

1. *(10 pts) Suppose* m *and* w *are partners in* M *but not in* M'*. Show that one of* m *and* w *prefers its partner in* M *to its partner in* M'*, and the other prefers its partner in* M' *to its partner in* M*.*

   Hint: It may be helpful to think about a bipartite graph formed by the matchings M *and* M'*. A path in this graph must alternate edges from* M *to* M'*.*

2. *(5 pts) Show that* M'' *is a perfect matching. That is, the partner of* m *and* m' *in* M'' *cannot be the same. You can directly use the result from 1.*

3. *(5 pts) Show that there is no unstable pair in* M''*.*

Problem 3. *(20 pts) (20 pts) Show that the number of stable matchings can grow exponentially with the size of the instance by proving the following statements:*

1. *(14 pts) Given an instance of* m *men and* m *women with* x *stable matchings and another instance of* n *men and* n *women with with* y *stable matchings, there is an instance of* mn *men and* mn *women with at least* $\max(xy^m, yx^n)$ *stable matchings.*

   Hint: Suppose the men are labeled $a_1, ..., a_m$ *and* $c_1, ..., c_n$*, and the women are labeled* $b_1, ..., b_m$ *and* $d_1, ..., d_n$*, consider the instance of size* mn *in which*

   (a) *the men are labeled* (a_i, b_j) *where* $i = 1, ..., m$ *and* $j = 1, ..., n$*;*
   (b) *the women are labeled* (c_i, d_j) *where* $i = 1, ..., m$ *and* $j = 1, ..., n$*;*
   (c) *man* (a_i, b_j) *prefers* (c_k, d_l) *to* $(c_{k'}, d_{l'})$ *if* b_j *prefers* d_l *to* $d_{l'}$*, or if* $l = l'$ *and* a_i *prefers* c_k *to* $c_{k'}$*;*

(d) *woman* (c_i, d_j) *prefers* (a_k, b_l) *to* $(a_{k'}, b_{l'})$ *if* d_j *prefers* b_l *to* $b_{l'}$*, or if* $l = l'$ *and* c_i *prefers* a_k *to* $a_{k'}$.

2. *(6 pts) For each* $n \geqslant 0$ *where* n *is a power of* 2*, there is an instance of* n *men and* n *women with at least* 2^{n-1} *stable matchings.*