NP-completeness, Cook-Levin, and additional reductions

CS 580

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What we need to show: Given as input a Clique instance, i.e. an undirected graph G = (V,E) and a number k, we can construct in polynomial time a 3SAT formula ϕ such that the graph G has a k-clique if and only if ϕ has a satisfying assignment.

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1. For each i, there is an i-th vertex in the clique: $\bigvee_{v \in V} x_{i,v}$.
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2. For all i, j, the i-th vertex is different from the j-th vertex: for each $v \in V$, add $\bar{x}_{i,v} \vee \bar{x}_{j,v}$.
3. For each non-edge $(u, v) \notin E$, u and v cannot both belong to the clique: for each i, j, we have $\bar{x}_{i,u} \vee \bar{x}_{j,v}$.

NP-completeness

Definition (NP-complete): A language L is NP-complete if:

- $L \in NP$
- Every language $A \in NP$ is poly-time reducible to L.

Recall: A language $A \subseteq \Sigma^*$ is polynomial time (mapping) reducible to a language $B \subseteq \Sigma^*$ if there exists a polynomial time computable function $f : \Sigma^* \to \Sigma^*$ such that for every $x \in \Sigma^*$, we have:

$$x \in A \text{ if and only if } f(x) \in B.$$

The function f is the **reduction**.

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Proof: Let $L \in NP$ be a language.

Since A is NP-complete, there exists a poly-time reduction f such that for every $x \in \Sigma^*$, we have: $x \in L$ if and only if $f(x) \in A$. Since $A \in P$, there is a poly-time TM M to decide A.

Then we can get a poly-time algorithm for L. On each input $x \in \Sigma^*$:

- Compute $f(x) = y$
- Run the TM M on y and output the same answer as M.

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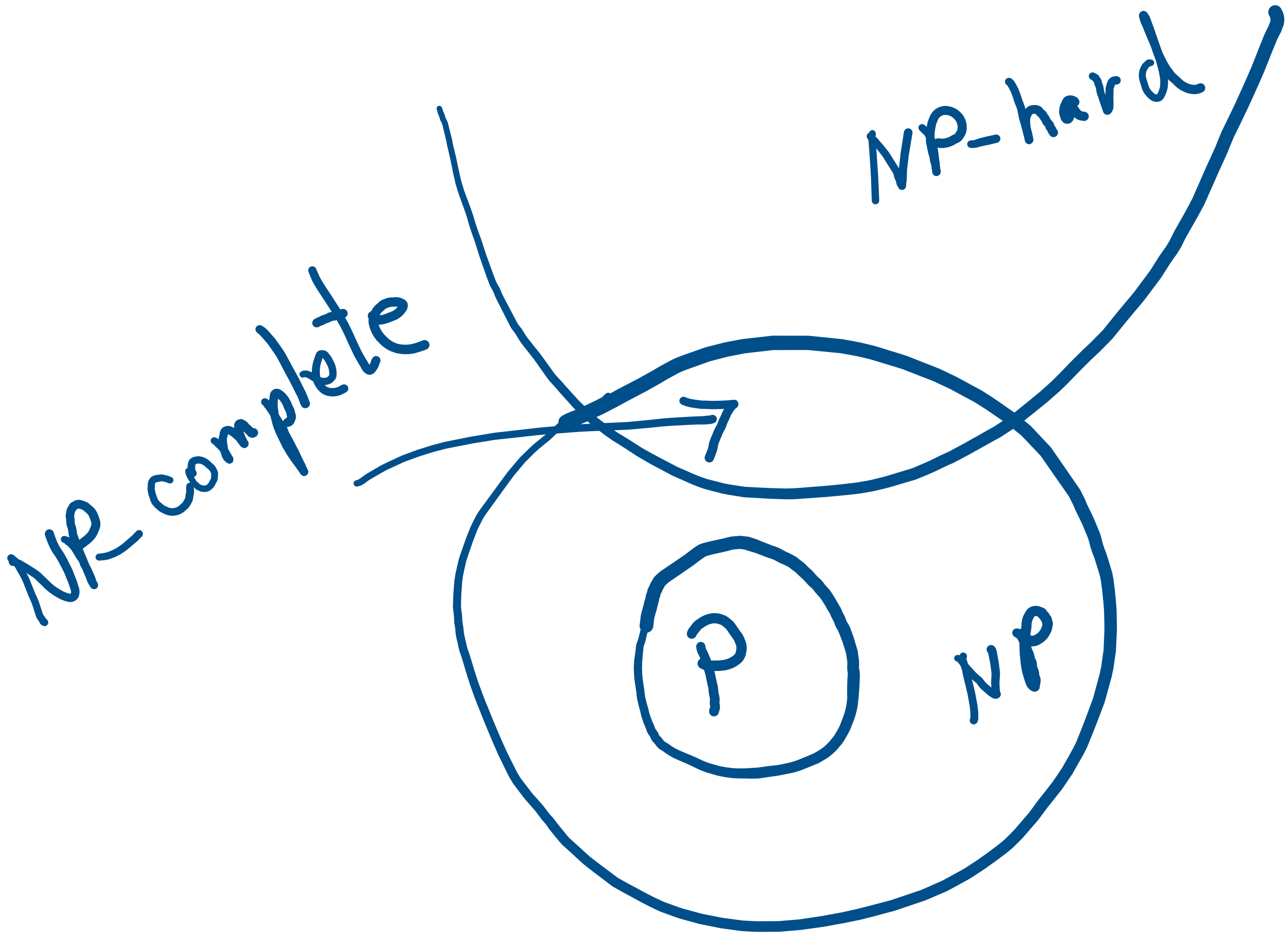
$$\begin{tikzcd} L \arrow[r, "g"] \arrow[rr, bend right] & A \arrow[r, "h"] & B \end{tikzcd}$$

Let f be given by $f(x) = h(g(x))$

Definition (NP-hard): A language L is NP-hard if:

- Every language $A \in NP$ is poly-time reducible to L.



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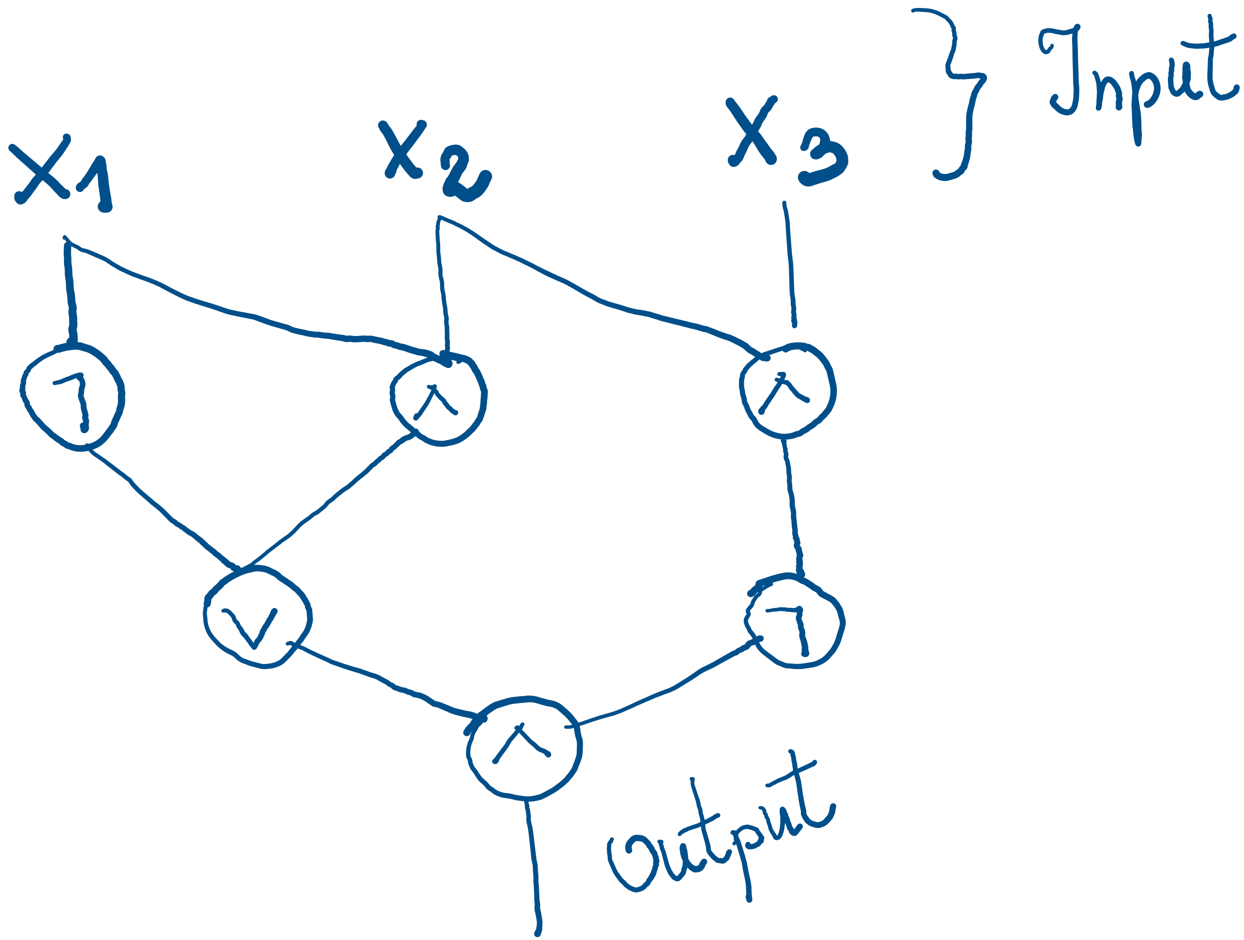

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Function computed by the circuit: To each circuit C we can associate a function $f_C : \{0,1\}^n \to \{0,1\}$, where if C outputs b on input $(x_1, \dots, x_n)$, then $f_C(x_1, \dots, x_n) = b$.

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Theorem: Let $t : N \to N$ be a function, where $t(n) \geq n$ [i.e. the TM reads the input]. If A ∈ $DTIME\big(t(n)\big)$, then for each $\mathrm{n} \in N$ there is a circuit $\mathrm{C_n}$ of size $\mathrm{O}\big(t(n)^2\big)$ such that for every $x = x_1 \dots x_n \in \{0,1\}^n$, we have $x \in A$ if and only if $C_n(x) = 1$.

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We will make some composite symbols to capture the state and symbol under the tape head together:

$$\underbrace{0010}_{\text{tape contents to the left of the head.}}\boxed{q\,1}\,110101$$

composite symbol

Let A ∈ $DTIME\big(t(n)\big)$. Then there is a TM M that runs in $O(t(n))$ that decides A; suppose it is C * t(n) for a constant C.

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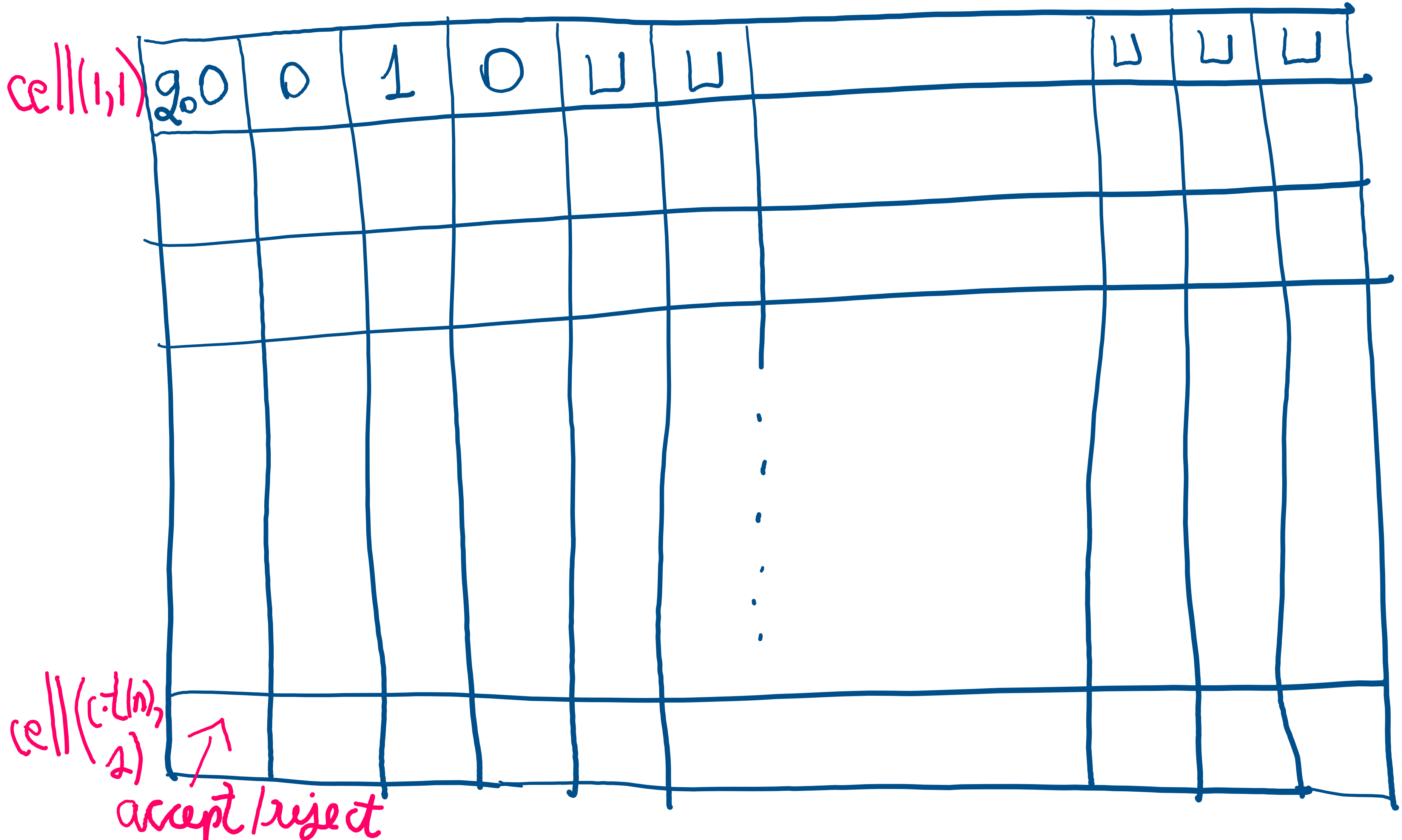

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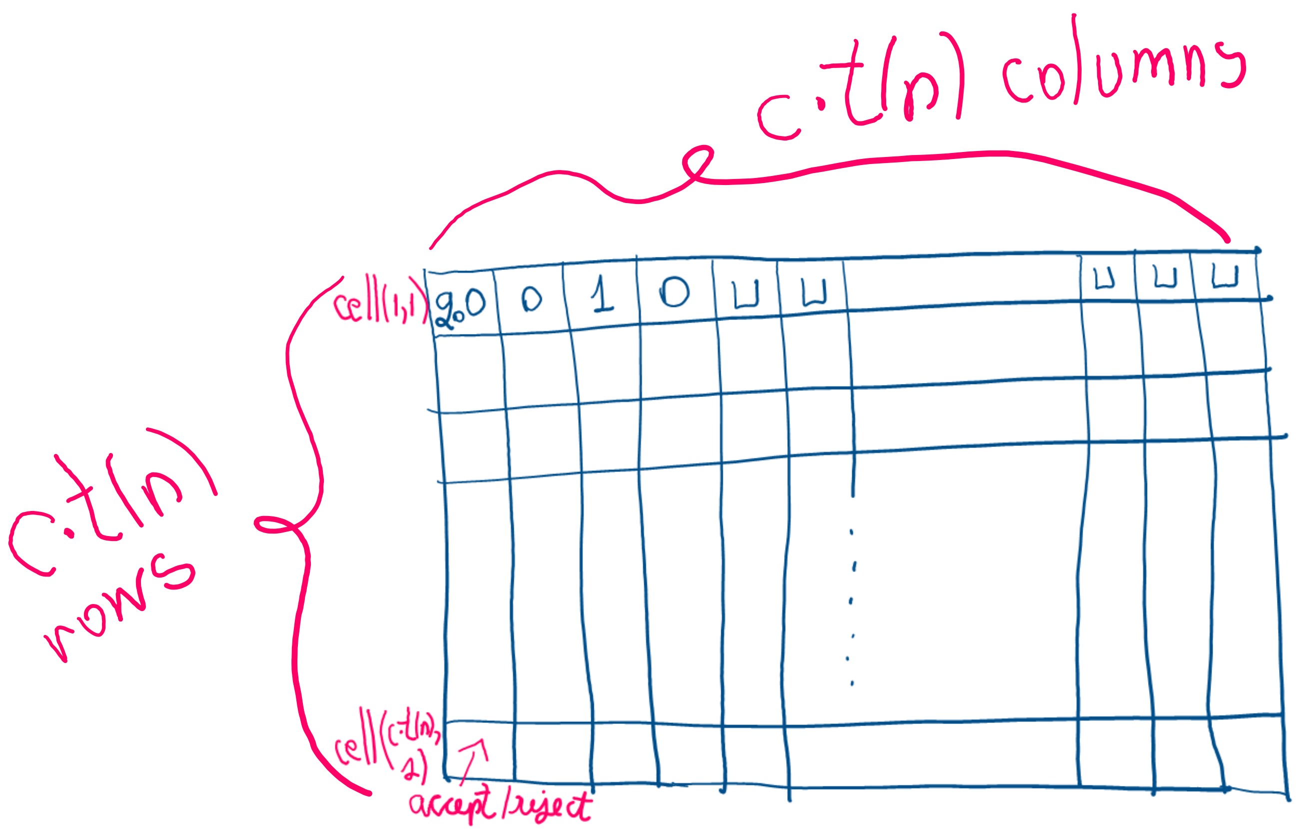

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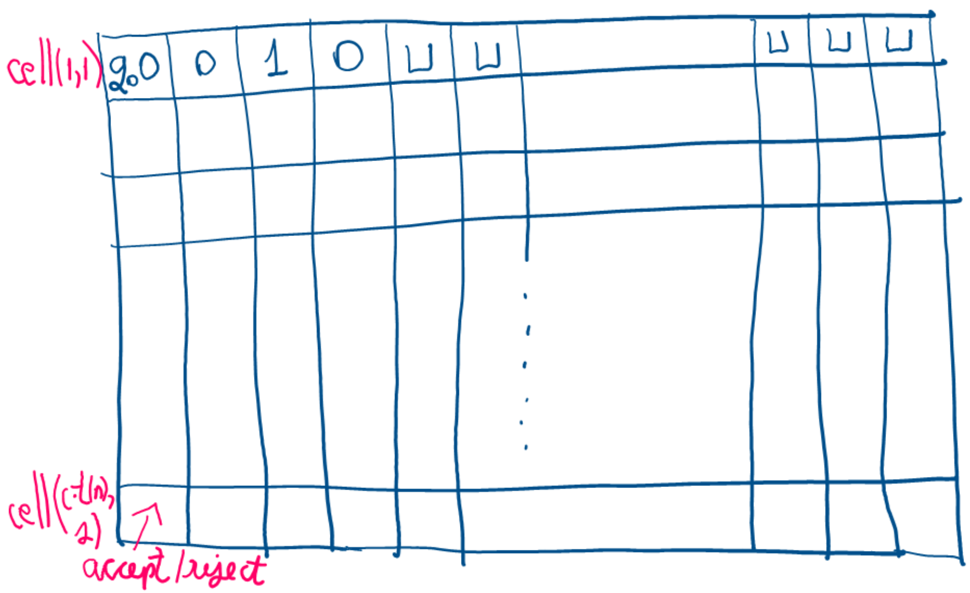

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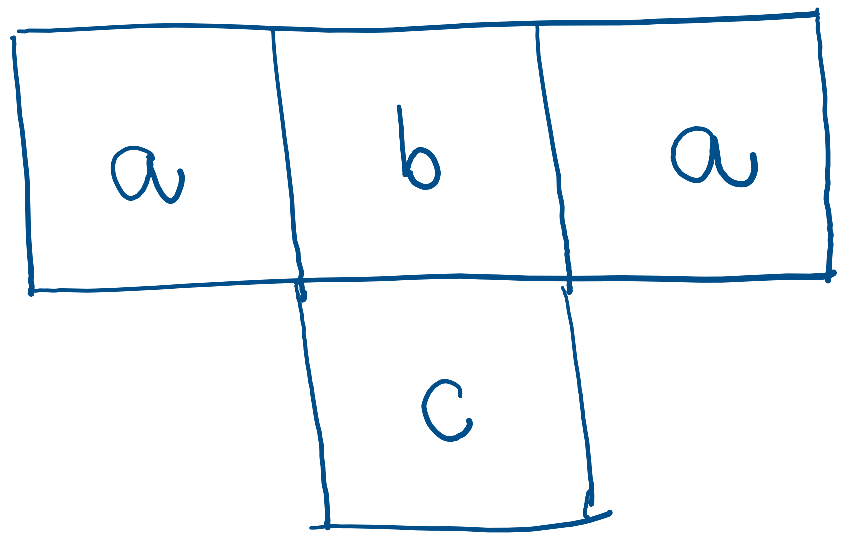
Observation: if we know cell(i-1,j-1), cell(i-1,j), and cell (i-1,j+1), we also know cell(i,j) by the way the transition function δ of the machine works.



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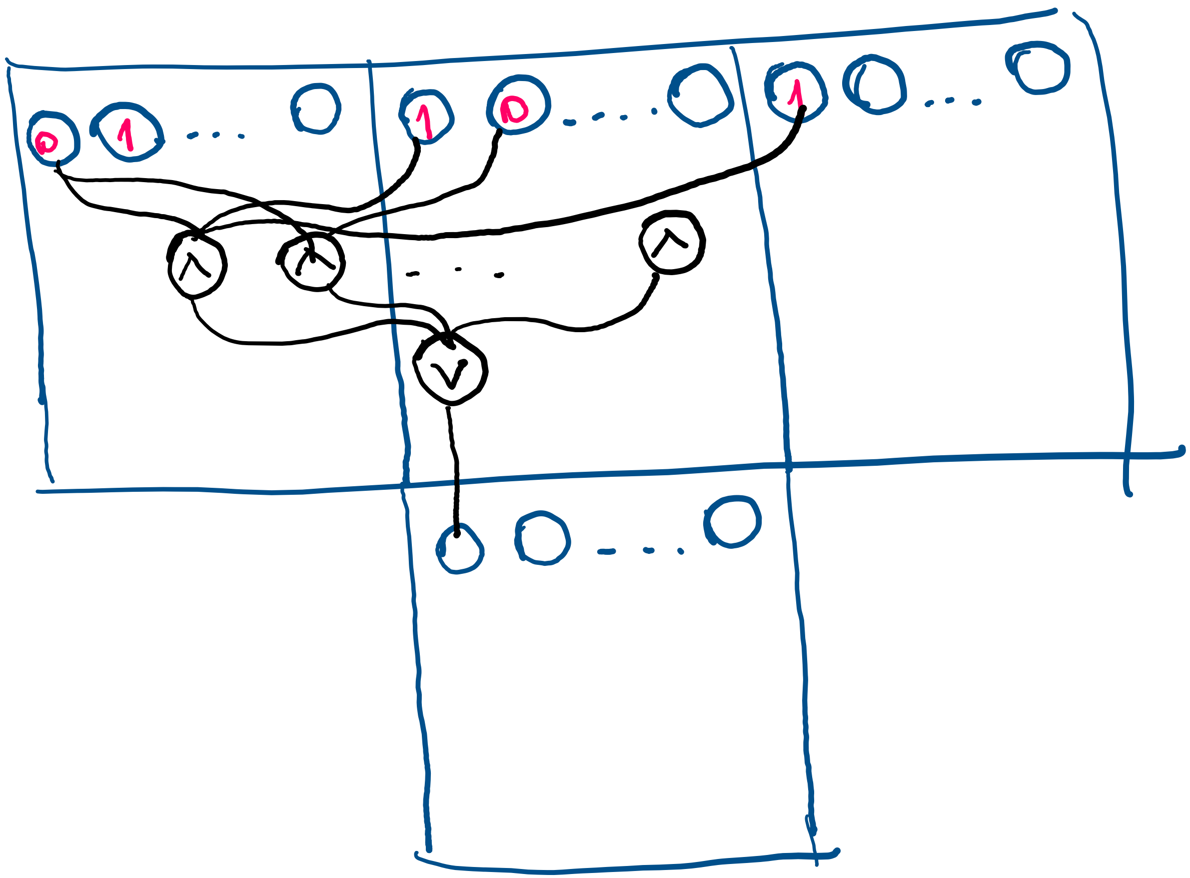

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The cells in row 1 have no predecessors and are handled in a special way:

- light[$1,1,q_0 1$] is connected to input x_1. Since the start state is q_0 and the head starts at x_1.
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- In particular light$[1,n+1,$⊔], … , light$[1,C\ * t(n),$⊔] are on.

What is the output gate?

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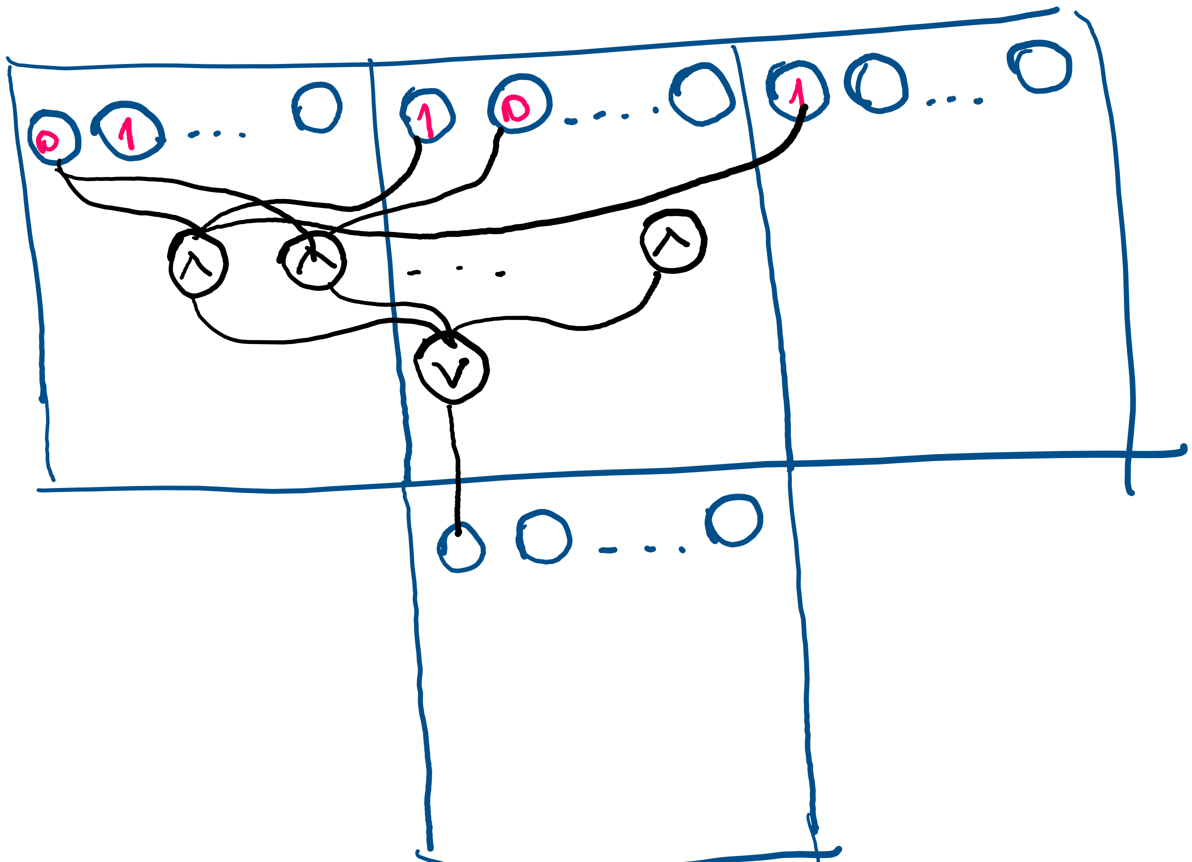
The output gate is the one attached to light[C $* \, t(n), 1, q_{accept}$].

Proof (cont): Let A ∈ $DTIME(t(n))$. Then there is a TM M that runs in $O(t(n))$ that decides A; suppose it is C * t(n) for a constant C.

How many gates do we use for each cell?



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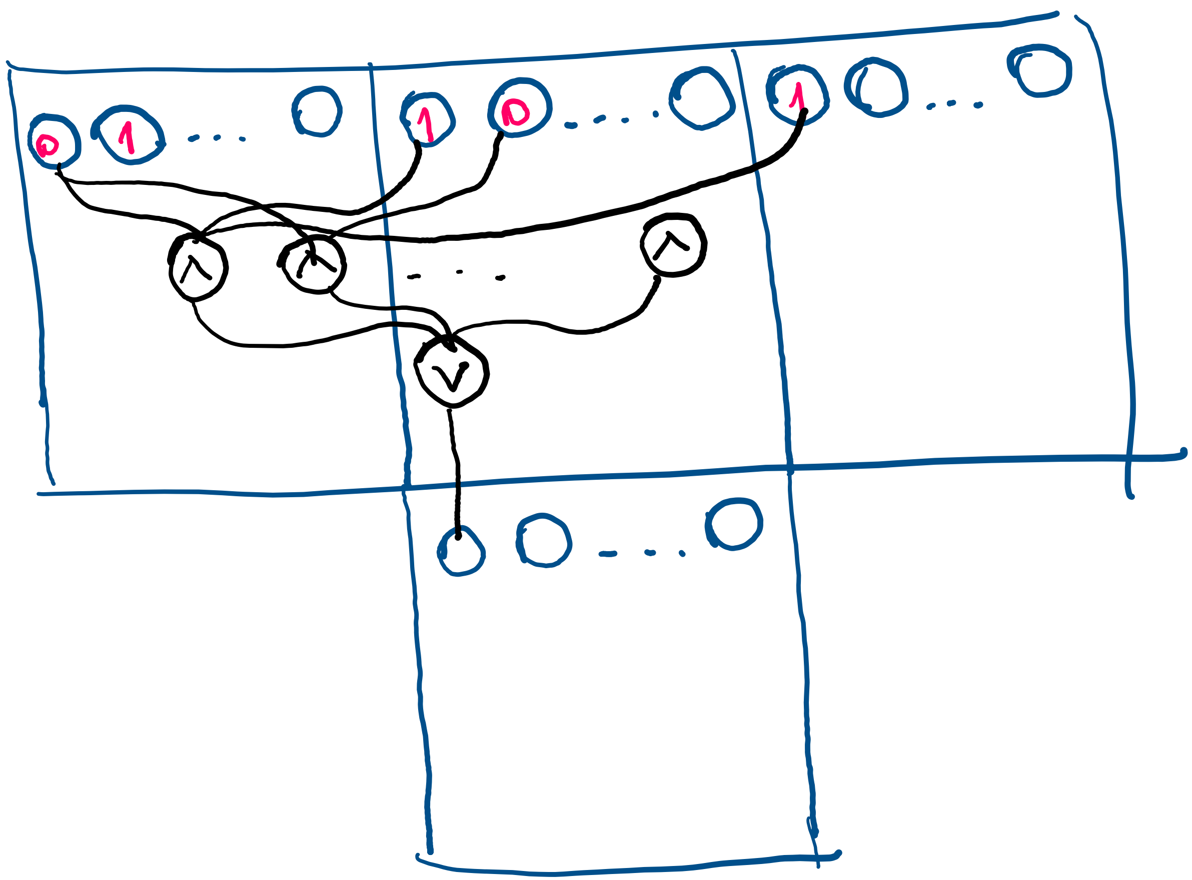
For each cell we use $O(f(k))$ gates to obtain its input from the three cells above. There are $C * t(n)^2$ cells, so overall the circuit has $O(t(n)^2)$ gates.



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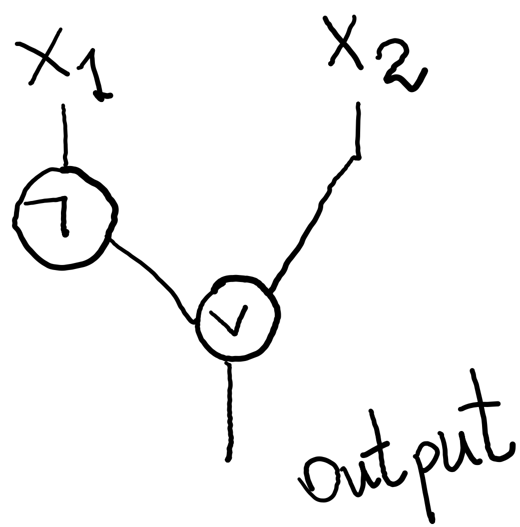
Claim: CircuitSAT ∈ NP.

Proof: Design a verifier for CircuitSAT. Given as input (C,x), where C is a circuit and $\mathrm{x} = x_1, \ldots, x_n$ is an input for C, check if C(x) = 1.

E.g: C



$x_1 = 0$
$x_2 = 1$

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Can we show that CircuitSAT is NP-complete? **Hint:** Use the previous theorem.

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