Answer Set 1

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Problem 1

Base Step

To prove $f(n) \in \Theta(n^2)$ we need there to exist three positive constants $n_o \in \mathbb{N}$ and $c_1, c_2 \in \mathbb{R}$ such that $\forall n \geq n_o$ we have $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ where $g(n) = n^2$

If we choose $n_0 = 40, c_1 = \frac{133}{40000}$ and $c_2 = 1$ the inequality becomes

$$\frac{133}{40000} \cdot 40^2 \leq \frac{40^2}{5} - 8 \cdot 40 + \log 40 \leq 1 \cdot 40^2$$

$$5.32 \leq 5.32 \leq 40^2$$

Inductive Step

Right Inequality

$$\begin{aligned}
f(n+1) &= \frac{(n+1)^2}{5} - 8(n+1) + log(n+1) \\
&\leq \frac{n^2+2n+1}{5} - 8n - 8 + log2n \\
&\leq \frac{n^2}{5} - 8n + logn + \frac{2n+1}{5} - 8 \\
&\leq \frac{n^2}{5} - 8n + logn + \frac{2n+1}{5} - 8 \\
&\leq c_2 \cdot n^2 + \frac{2n+1}{5} - 8 \\
&\leq c_2 \cdot n^2 + 2n + 1 \\
&\leq n^2 + 2n + 1 \\
&\leq c_2 \cdot (n+1)^2
\end{aligned}$$

Left Inequality

$$f(n+1) = \frac{(n+1)^2}{5} - 8(n+1) + log(n+1)$$
$$\geq \frac{n^2+2n+1}{5} - 8n - 8 + logn$$
$$\geq \frac{n^2}{5} - 8n + logn + \frac{2n+1}{5} - 8$$
$$\geq \frac{n^2}{5} - 8n + logn + \frac{2n+1}{5} - 8$$
$$\geq c_1 \cdot n^2 + \frac{2n+1}{5} - 8$$

$\frac{2n+1}{5} - 8 \geq c_1 \cdot (2n+1)$

$$\geq c_1 \cdot n^2 + c_1 \cdot (2n+1)$$
$$\geq c_1 \cdot (n^2+2n+1)$$
$$\geq c_1 \cdot (n+1)^2$$

So we have found three positive constants $n_o \in \mathbb{N}$ and $c_1, c_2 \in \mathbb{R}$ such that $\forall n \geq n_o$ we have $c_1.g(n) \leq f(n) \leq c_2.g(n)$ where $g(n) = n^2$. So we can conclude that $f(n) \in \Theta(n^2)$

Problem 2

Proof of Part 1

m and w are partners in M but not in M'. Lets assume m's partner in M' is w' and w's partner in M' is m'.

We will prove this by contradiction. There can be two contradictions

i m and w both prefers their partner in M to their partner in M'.

ii m and w both prefers their partner in M' to their partner in M.

For (i) m and w prefers their partner in M than their partner in M'', that is they prefer each other. But m's partner in M' is w' and w's parttner in M' is m'. This contradicts the construction that M'' is a stable matching. Because as m and w prefers each other than their current partner in M'', they create a blocking pair. This gives a contradiction. So m and w can't prefer each other than their current matching in M''

For (ii) m and w prefers their partner in M' than their partner in M, that is they don't prefer each other. But m's partner in M is w and w's parttner in M is m. This contradicts the construction that m and w are partners in M. Because as m and w doesn't prefer each other than their current partner in M', they would never be partners in M. This gives a contradiction.

With the contradictions, we can conclude that one of m and w prefers its partner in M to its partner in M', and the other prefers its partner in M' to its partner in M

Proof of Part 2

Lets assume that man m and m' received the same partner w, this is possible if (m,w) is a pair in M and (m',w) is a pair in M'. Then $M >_m M'$ and $M' >'_m M$. Part 1 applied to the pair (m,w) implies that $m' >_w m$ and applied to the pair (m',w) implies that $m >_w m'$, which gives us a contradiction.

Proof of Part 3

Lets assume that M'' is not stable, then it exists a blocking pair (m,w). This is, $w >_m M''(m)$, from the definition of M'', $w >_m M'(m)$ and $w >_m M'(m)$. On the other hand $m >_w M(w)$. If $M''(w) = M(w)$ then (m,w) is a blocking pair in M, if $M''(w) = M'(w)$ then (m,w) is a blocking pair in M'. Since M and M' are both stable in each case we get a contradiction.

Problem 3

Part 1

Let $M_1 ... M_n$ be any sequence of stable matchings of size m, and let M be any stable matching of size n. The total number of choices available for $M_1 ... M_n$ and M is yx^n. We claim that the mapping

$$(a_i, b_j) \longleftrightarrow (M_j(a_i), M(b_j))$$

is a stable matching.

As M_j and M are matchings, this mapping is actually a matching. Lets assume this matching is blocked by the pair $((a,b),(c,d))$. Then, of the following conditions we must have either (i) or (ii) together with either (iii) or (iv).

i b prefers d to $M(b)$.

ii $d = M(b)$ and a prefers c to $M_j(a)$.

iii d prefers b to $M(d)$.

iv $b = M(d)$ and c prefers a to $M_j(c)$.

Of the four possibilities, the combination of (i) with (iii) is precluded by the stability of M, (ii) with (iv) by the stability of M_j and the others by simple incompatibility.

This justifies the claim, and we demonstrate it with an instance of at least yx^n stable matchings. Same can be shown for xy^m.

Part 2

Proof by induction:

Base case

For $n = 2^0$ the instance of size 1 admits a single stable matching. For $n = 2^1$ there exist at least one stable matching.

Inductive Step

Lets assume it true for $n = 2^k$. We apply repeatedly the construction of part 1 with $m = 2$ and the instance of size 2 shown in tables below. For this instance, both possible matchings are stable, so that $x = 2$ and, by the inductive hypothesis, $y = 2^{2^k-1}$. Hence, by part 1, there exists an instance of size $2 \cdot 2^k = 2^{k+1}$ with at least $max(2 \cdot (2^{2^k-1})^2, 2^{2^k-1} \cdot 2^{2^k}) = 2^{2^{k+1}-1}$ stable matchings.

$P(m_1)$	w_1	w_2
$P(m_2)$	w_2	w_1

Table 1: Men's Preferences

$P(w_1)$	m_2	m_1
$P(w_2)$	m_1	m_2

Table 2: Women's Preferences