CS 580: Stable Matchings

1.1 Stable Matching

College Admissions and the Stability of Marriage

Author(s): D. Gale and L. S. Shapley

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Textbook slides by Kevin Wayne

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.
- Participants rate members of the other type.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Perfect matching: everyone is matched 1-1.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
  - Unstable pair m-w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	favorite ↓		least favorite ↓
	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
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Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?
A. No. Bertha and Xavier will prefer each other to their current partners.

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ … least favorite ↓

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Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?
A. Yes.

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Women's Preference Profile

Gale-Shapley Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

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```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = most preferred woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Proof of Correctness: Termination

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Observation 1. Men propose to women in decreasing order of preference.

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Can a woman become unmatched at some point?

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Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched.

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Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched.

Claim. Algorithm terminates after at most n^2 iterations of while loop.
Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪

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Tightness: n(n-1) + 1 proposals required

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Question: Is there a better asymptotic analysis?

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Claim. Algorithm terminates after at most n^2 iterations of while loop.
Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪
Question: Is there a better asymptotic analysis? A: No. Input below requires $n(n-1) + 1$ proposals [stable matching is unique for worst case instance].

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	1st	2nd	3rd	4th	5th
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Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

Tightness: $n(n-1) + 1$ proposals required

Proof of Correctness: Perfection

How many people remain unmatched in the worst case?

Proof of Correctness: Perfection

Claim. All men and women get matched.

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Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.

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- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.

[Recall Observation 2: Once a woman is matched, she never becomes unmatched.]

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▪

Proof of Correctness: Stability

Claim. No unstable pairs.

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- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.

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S*
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Bertha-Zeus
. . .

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  men propose in decreasing order of preference

  $\Rightarrow$ Z prefers his GS partner to A.

  $\Rightarrow$ A-Z is stable.

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  - $\Rightarrow$ Z prefers his GS partner to A. ← men propose in decreasing order of preference
  - $\Rightarrow$ A-Z is stable.

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  - $\Rightarrow$ A prefers her GS partner to Z. ← women can only improve during the algorithm

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- Case 2: Z proposed to A.
  - $\Rightarrow$ A rejected Z (right away or later)
  - $\Rightarrow$ A prefers her GS partner to Z. ← women can only improve during the algorithm
  - $\Rightarrow$ A-Z is stable.

- In either case A-Z is stable, a contradiction. ▪

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

- Assume men are named 1, …, n.
- Assume women are named 1', …, n'.

Engagements.

- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - set entry to `0` if unmatched
  - if m matched to w then `wife[m]=w` and `husband[w]=m`

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man `m`.

Efficient Implementation

Women rejecting/accepting.

- Does woman `w` prefer man `m` to man `m'`?
- For each woman, create inverse of preference list of men.
- Constant time access for each query after O(n) preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man 3 to 6
since `inverse[3]` < `inverse[6]`

2 7

Understanding the Solution

Q. Is the stable matching unique?

Understanding the Solution

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An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
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Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

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Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

Claim. GS matching S* is man-optimal.

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- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference $\Rightarrow$ some man is rejected by valid partner.

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- Let Y be first such man, and let A be first valid woman that rejects him.

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- Let B be Z's partner in S.

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- Let B be Z's partner in S.
- Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B (since he proposes to A before proposing to B).

since this is first rejection by a valid partner

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- Also A prefers Z to Y.

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- Let B be Z's partner in S.
- Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B (since he proposes to A before proposing to B).

  since this is first rejection by a valid partner
- Also A prefers Z to Y.
- Thus A-Z is unstable in S. ▪

S

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Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S*.

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- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.

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- Let B be Z's partner in S.
- Z prefers A to B. ← man-optimality in Gale-Shapley algorithm

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- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in S.
- Z prefers A to B. ← man-optimality
- Thus, A-Z is an unstable in S. ▪

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An LP Formulation

For each man m and woman w, let $x_{mw} = 1$ if m is matched with w, and zero otherwise.

Notation: $x \succ_i y$ means that agent i ranks x above y.

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$$\sum_{j \prec_m w} x_{mj} + \sum_{i \prec_w m} x_{iw} + x_{mw} \leq 1 \qquad \forall m \in M,\, w \in W$$

$$x_{mw} \geq 0 \qquad \forall m \in M,\, w \in W$$

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x_{mw} &\geq 0 && \forall m \in M, w \in W
\end{aligned}
$$

Let P be the polyhedron defined by these inequalities.

What do the constraints mean?

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$$x_{mw} \geq 0 \qquad \forall m \in M, w \in W$$

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What do the constraints mean?

- Constraints 1-2 ensure each agent is matched with exactly one other agent.

An LP Formulation

For each man m and woman w, let $x_{mw} = 1$ if m is matched with w, and zero otherwise.
Notation: $x \succ_i y$ means that agent i ranks x above y.

Consider the following linear program:

$$\sum_{w \in W} x_{mw} = 1 \qquad \forall m \in M$$

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$$\sum_{j \prec_m w} x_{mj} + \sum_{i \prec_w m} x_{iw} + x_{mw} \leq 1 \qquad \forall m \in M, w \in W$$

$$x_{mw} \geq 0 \qquad \forall m \in M, w \in W$$

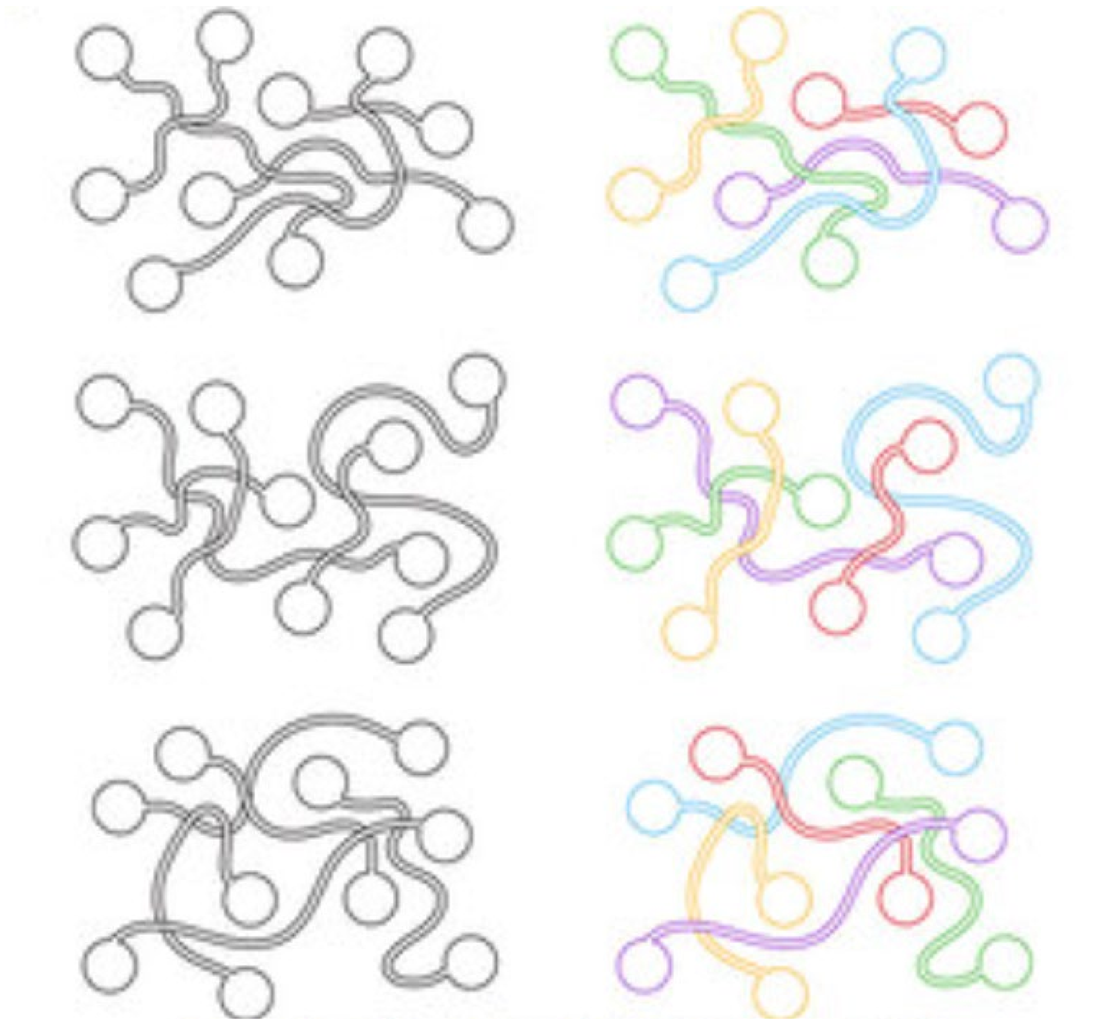
Let P be the polyhedron defined by these inequalities.

What do the constraints mean?

- Constraints 1-2 ensure each agent is matched with exactly one other agent.
- Constraint 3 ensures stability. Why? Suppose $\sum_{j \prec_m w} x_{mj} = 1$ and $\sum_{i \prec_w m} x_{iw} = 1$

Then m is matched to a woman j whom he ranks below w. Similarly, w is matched to a man she ranks below m. This would make the pair (m,w) blocking.

Matchings as a random process



Matchings as a random process

Consider arbitrary initial configuration (e.g. everyone is unmatched, or there is an arbitrary initial matching).

At each time step t = 1, 2, 3, …:

- An agent (say A) is selected uniformly at random and is matched with a random uniform agent (say B) from the other side.

- If A and B prefer each other to their current matches, they form a match, breaking any existing matches. Otherwise, nothing changes.

Question: What happens in the long term?