To Compute The Result Of A Matrix Raised To A Power

Project for Advance Control System (EE 3302)

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-: Project topic :-

Write an algorithm to compute A^k, $where\ A \in \mathbb{R}^{n \times n}$ (n and k are arbitrary positive real numbers).

Approach

1. Finding eigen-values and eigen-vectors of the given matrix.
2. Diagonalizing the matrix
3. Finding formula for the power of a matrix

Let’s understand the above steps through an example

Example matrix

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Compute A^n for any $n \in \mathbb{N}$.

Step 1 : Finding eigen-values and eigen-vectors of matrix

We solve

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) = 0$$

and obtain the eigenvalues $\lambda = -1, 3$.

To find an eigenvector $\mathbf{x}$ corresponding to $\lambda = -1$, we solve $(A+I)\mathbf{x} = \mathbf{0}$ or

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We obtain an eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to $\lambda = -1$.

Similarly, solving $(A-3I)\mathbf{y} = \mathbf{0}$, we obtain an eigenvector $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to $\lambda = 3$.

Step 2 : Diagonalising the matrix

We can find an invertible matrix S (by cascading the eigen-vectors) which would diagonalize the given matrix

$$S = [\mathbf{x}\,\mathbf{y}] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ diagonalizes the matrix } A \text{, that is,}$$

$$S^{-1}AS = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \text{ or equivalently } A = S \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} S^{-1}.$$

Step 3 : Finding a formula for power of a matrix

To find the final formula, we use 2 identities:

1. If $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, then $A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$ for any $n \in \mathbb{N}$.

2. If $B = S^{-1}AS$, then $B^n = S^{-1}A^nS$ for any $n \in \mathbb{N}$

Please note: Both the above relations can be proved by principle of mathematical induction

Step 3 continued...

Previously, we had

$S = [\mathbf{x}\,\mathbf{y}] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ diagonalizes the matrix A, that is,

$$S^{-1}AS = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \text{ or equivalently } A = S\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}S^{-1}.$$

By using the identities, we can reach to a formula as mentioned below

Then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} A^n &= \left(S\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}S^{-1}\right)^n = S\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}^n S^{-1} = S\begin{bmatrix} (-1)^n & 0 \\ 0 & 3^n \end{bmatrix}S^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 3^n \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (-1)^n + 3^n & (-1)^{n+1} + 3^n \\ (-1)^{n+1} + 3^n & (-1)^n + 3^n \end{bmatrix}. \end{aligned}$$

Final formula

So, for $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, we could derive a formula:

$$A^n = \frac{1}{2}\begin{bmatrix} (-1)^n + 3^n & (-1)^{n+1} + 3^n \\ (-1)^{n+1} + 3^n & (-1)^n + 3^n \end{bmatrix}$$

In a similar manner, such formulas can be found out for any square matrix. And by using this formula we can calculate the result of a matrix raised to some power

Output for A^3 using the formula

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A^3 = \frac{1}{2}\begin{bmatrix} (-1)^3 + (3)^3 & (-1)^{3+1} + (3)^3 \\ (-1)^{3+1} + (3)^3 & (-1)^3 + (3)^3 \end{bmatrix}$$

$$\Rightarrow A^3 = \frac{1}{2}\begin{bmatrix} -1 + 27 & 1 + 27 \\ 1 + 27 & -1 + 27 \end{bmatrix}$$

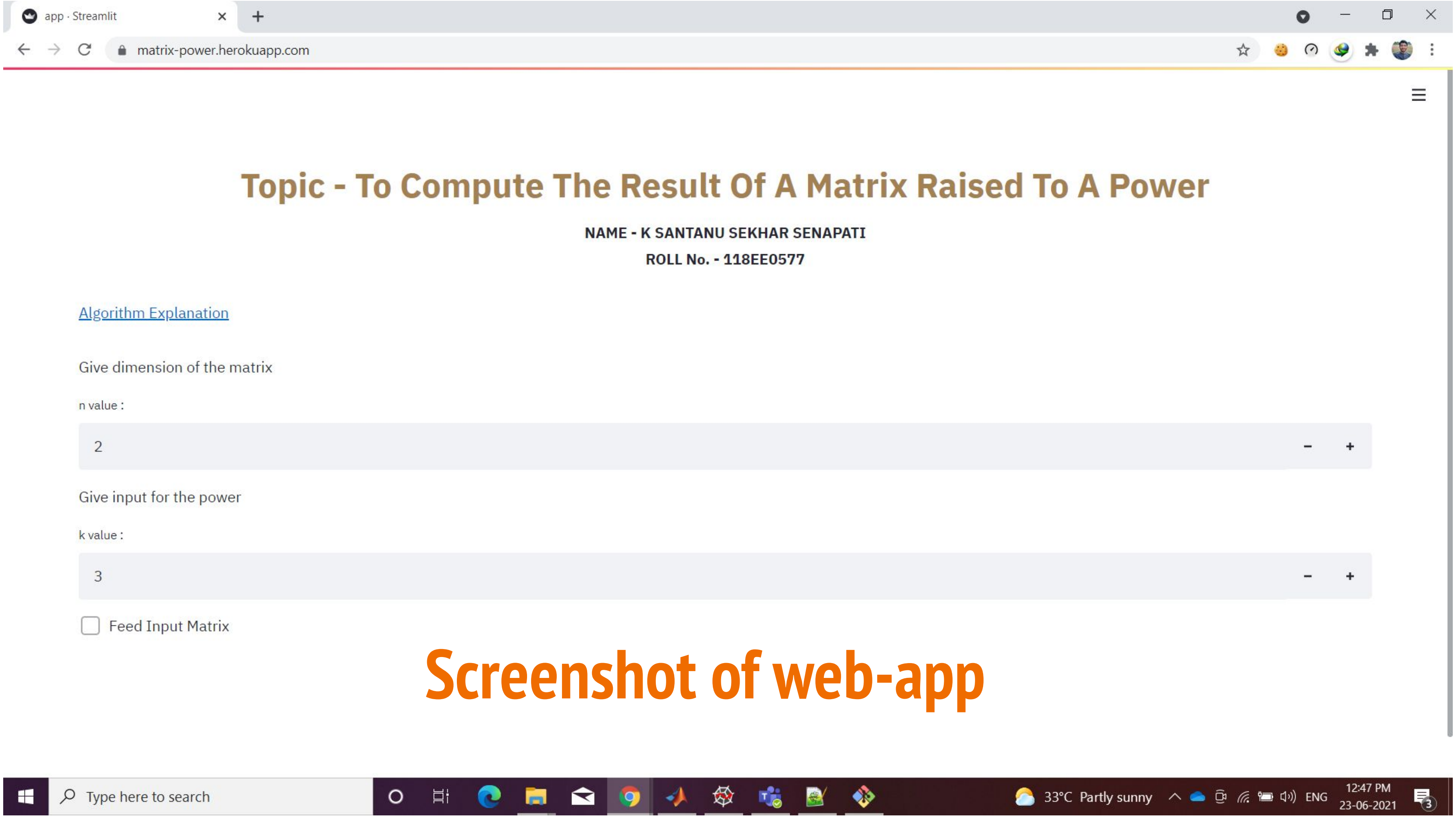
$$\Rightarrow A^3 = \frac{1}{2}\begin{bmatrix} 26 & 28 \\ 28 & 26 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} 13 & 14 \\ 14 & 13 \end{bmatrix}$$

Web-App

[Link to the webapp]

1. A matlab code is written to perform all the 3 steps explained earlier
2. Frontend of web-app is designed in streamlit (python library used to make custom web apps)
3. Then, the website is hosted in heroku (a cloud platform for hosting webapps)



Screenshot of web-app

app · Streamlit

matrix-power.herokuapp.com

Feed Input Matrix

```
* While entering the values press tab to go to the next input field and press enter after giving the inputs

* If you want to change the dimension of the input matrix then uncheck the checkbox before changing the input dimension

* The page may take some time to load the grid for providing input to the matrix, kindly wait
```

column-1	column-2
1	2
2	1

Matrix A (2x2) is given as input in the web-app

Given Input Matrix

	0	1
0	1	2
1	2	1

Calculate Result

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Given Input Matrix

	0	1
0	1	2
1	2	1

Matrix A

Calculate Result

Output Matrix

	0	1
0	13	14
1	14	13

Matrix A^3

Output from the web-app

Made with Streamlit

Type here to search

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Matlab code:

```
% Taking input for the dimension of matrix
n=input('Enter dimension of matrix : ');

% Taking input for the power
k=input('Enter power : ');

% Taking input for the matrix elements
for i=1:n
   for j=1:n
       fprintf('Enter A[%d][%d]',i,j);
       A(i,j)=input(' element');
   end
end
A=reshape(A,n,n);

% Printing the matrix
disp('Matrix A : ');
disp(A);

% Finding eigenvalues and eigenvectors
[S,D] = eig(A); % S is eigenvector matrix and D is a diagonal matrix with eigenvalues as the diagonal elements

% Computing the result A^k
A_k = S*D.^k/S;

% Printing the result
disp('Matrix A^k : ');
disp(A_k);
```

Program output:

```
Enter dimension of matrix :
   2
Enter power :
   3
Enter A[1][1] element
   1
Enter A[1][2] element
   2
Enter A[2][1] element
   2
Enter A[2][2] element
   1
Matrix A :
   1    2
   2    1
Matrix A^k :
  13.0000   14.0000
  14.0000   13.0000
```