科学计算作业 4

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1. 证明

$$(1)\Delta(f_k g_k) = f_k \Delta g_k + g_{k+1} \Delta f_k$$

，

$$(2)\sum_{k=0}^{n-1} f_k \Delta g_k = f_n g_n - f_0 g_0 - \sum_{k=0}^{n-1} g_{k+1} \Delta f_k$$

证 (1)：由定义可知 $\Delta = E - I$，则：

$$\Delta(f_k g_k) = f_{k+1} g_{k+1} - f_k g_k$$

$$f_k \Delta g_k + g_{k+1} \Delta f_k = f_k(g_{k+1} - g_k) + g_{k+1}(f_{k+1} - f_k) = f_{k+1} g_{k+1} - f_k g_k$$

证毕。

证 (2)：同样利用定义计算：

$$\sum_{k=0}^{n-1} f_k \Delta g_k = \sum_{k=0}^{n-1} f_k(g_{k+1} - g_k) = \sum_{k=0}^{n-1} f_k g_{k+1} - \sum_{k=0}^{n-1} f_k g_k$$

$$f_n g_n - f_0 g_0 - \sum_{k=0}^{n-1} g_{k+1} \Delta f_k = f_n g_n - f_0 g_0 - \sum_{k=0}^{n-1} g_{k+1} f_{k+1} + \sum_{k=0}^{n-1} g_{k+1} f_k = -\sum_{k=0}^{n-1} f_k g_k + \sum_{k=0}^{n-1} f_k g_{k+1}$$

证毕。

2. 利用差分算子，计算：

$$S_n = \sum_{k=1}^{n} k^3$$

容易知道：

$$S_n = \sum_{k=1}^{n} k^3 = \sum_{k=0}^{n} k^3$$

且由定义知 $\Delta = E - I$，则 $E = \Delta + I$，利用二项式定理展开：

$$E^n = (\Delta + I)^n = \sum_{k=0}^{n} C_n^k I^{n-k} \Delta^k$$

将算子作用于 S_k，得到：

$$S_{n+k} = \sum_{k=0}^{n} C_n^k I^{n-k} \Delta^k S_k = \sum_{k=0}^{n} C_n^k \Delta^k S_k$$

$$S_k = \sum_{k=1}^{n} k^3$$

$$\Delta S_k = (k+1)^3$$

$$\Delta^2 S_k = 3k^2 + 9k + 7$$

$$\Delta^3 S_k = 6k + 12$$

$$\Delta^4 S_k = 6$$

此时令 $k = 0$，则得到：

$$S_n = C_n^0 \times 0 + C_n 1 \times 1 + C_n^2 \times 7 + C_n^3 \times 12 + C_n^4 \times 6$$

$$S_n = n + \frac{7n(n-1)}{2!} + \frac{12n(n-1)(n-2)}{3!} + \frac{6n(n-1)(n-2)(n-3)}{4!}$$

$$S_n = n\frac{12 + 42(n-1) + 24(n-1)(n-2) + 3(n-1)(n-2)(n-3)}{12}$$

$$S_n = \frac{n^2(3n^2 - 6n + 3)}{12} = \frac{n^2(n-1)^2}{4}$$

3. 求多项式 $p(x) \in \mathbb{P}_4$，满足：

$$p(x_0) = f(x_0), p'(x_0) = f'(x_0), p''(x_0) = f''(x_0),$$

$$p'''(x_0) = f'''(x_0), p(x_1) = f(x_1).$$

利用基函数方法求解：

规定 $f(x_0), f'(x_0), f''(x_0), f'''(x_0), f_(x_1)$ 的插值基函数分别为

$$\alpha_1(x), \alpha_2(x), \alpha_3(x), \alpha_4(x), \alpha_5(x) \in \mathbb{P}_4$$

对于 $\alpha_1(x)$，其需要满足得条件有：

$$\begin{cases} \alpha_1(x_0) = 1 \qquad \alpha_1(x_1) = 0; \\ \alpha_1'(x_0) = 0 \\ \alpha_1''(x_0) = 0 \\ \alpha_1'''(x_0) = 0 \end{cases}$$

可以利用 $\alpha_1(x)$ 在 x_0 的 Taylor 展开：

$$\alpha_1(x)=\alpha_1(x_0)+\alpha_1'(x_0)(x-x_0)+\frac{\alpha_1''(x_0)}{2!}(x-x_0)^2+\frac{\alpha_1'''(x_0)}{3!}(x-x_0)^3+\frac{\alpha_1''''(x_0)}{4!}(x-x_0)^4$$

代入得：

$$\alpha_1(x)=1+\frac{\alpha_1''''(x_0)}{4!}(x-x_0)^4$$

再利用条件 $\alpha_1(x_1)=0$ 可求得：

$$\frac{\alpha_1''''(x_0)}{4!}=-\frac{1}{(x_1-x_0)^4}$$

所以求得：

$$\alpha_1(x)=1-\frac{(x-x_0)^4}{(x_1-x_0)^4}$$

利用类似的方式，可以求得：

$$\begin{cases}\alpha_1(x)=1-\frac{(x-x_0)^4}{(x_1-x_0)^4}\\ \alpha_2(x)=(x-x_0)-\frac{(x-x_0)^4}{(x_1-x_0)^3}\\ \alpha_3(x)=\frac{(x-x_0)^2}{2}-\frac{(x-x_0)^4}{2(x_1-x_0)^2}\\ \alpha_4(x)=\frac{(x-x_0)^3}{6}-\frac{(x-x_0)^4}{6(x_1-x_0)}\\ \alpha_5(x)=\frac{(x-x_0)^4}{(x_1-x_0)^4}\end{cases}$$

所以可得多项式为：

$$p(x)=\left[1-\frac{(x-x_0)^4}{(x_1-x_0)^4}\right]f(x_0)+\left[(x-x_0)-\frac{(x-x_0)^4}{(x_1-x_0)^3}\right]f'(x_0)+\left[\frac{(x-x_0)^2}{2}-\frac{(x-x_0)^4}{2(x_1-x_0)^2}\right]f''(x_0)$$
$$+\left[\frac{(x-x_0)^3}{6}-\frac{(x-x_0)^4}{6(x_1-x_0)}\right]f'''(x_0)+\left[\frac{(x-x_0)^4}{(x_1-x_0)^4}\right]f(x_1)$$

设原函数可以表示为：

$$f(x)=p(x)+K(x)(x-x_0)^4(x-x_1)$$

构造函数 $\phi(t)$，其中 $K(x)$ 为待定函数：

$$\phi(t)=f(t)-p(t)-K(x)(t-x_0)^4(t-x_1)$$

计入重根，则 $\phi(t)$ 有 6 个根，反复使用 Rolle 中值定理，则 $[x_0,x_1]$ 间存在 ξ，使得：

$$\phi^{(5)}(\xi)=f^{(5)}(\xi)-K(x)5!=0$$

由此推出：

$$K(x)=\frac{f^{(5)}(\xi)}{5!}$$

所以余项为：

$$R_n(x)=\frac{f^{(5)}(\xi)}{5!}(x-x_0)^4(x-x_1)$$

4. 设 $\alpha_k, \alpha_{k+1}, \beta_k, \beta_{k+1}$ 是在 $e_k = [x_k, x_{k+1}]$ 上的 3 次 Hermite 插值基函数，求证：

(1)α_k, α_{k+1} 均是非负函数，且 $\alpha_k + \alpha_{k+1} = 1$;

(2)$|\beta_k(x)|, |\beta_{k+1}(x)| \leq \frac{4}{27}h_k$，其中 $h_k = x_{k+1} - x_k$;

(3) 设 $f \in C^1[x_k, x_{k+1}]$，根据 (1) 和 (2) 的结果，求证 $||f - H_3||_\infty \leq \frac{35}{27}||f'||_\infty h_k$

(1) 由题意可设

$$\alpha_k(x) = (Ax + B)(x - x_{k+1})^2$$

并且由 $\alpha_k(x_k) = 1, \alpha_k'(x_k) = 0$，计算得：

$$\alpha_k(x) = \left(\frac{x_{k+1} - 3x_k + 2x}{x_{k+1} - x_k}\right)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2$$

并可以用类似方法计算得 $\alpha_{k+1}(x), \beta_k(x), \beta_{k+1}(x)$，得到：

$$\begin{cases} \alpha_{k+1}(x) = \left(\frac{x_k - 3x_{k+1} + 2x}{x_k - x_{k+1}}\right)\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 \\ \beta_k(x) = (x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 \\ \beta_{k+1}(x) = (x - x_{k+1})\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 \end{cases}$$

对于 $\alpha_k, \left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 \geq 0$，只需证 $\frac{x_k - 3x_{k+1} + 2x}{x_k - x_{k+1}} \geq 0$

$\frac{x_k - 3x_{k+1} + 2x}{x_k - x_{k+1}}$ 是单调递减函数，只需将 $x = x_2$ 代入，即得证。$\alpha_{k+1}(x) \geq 0$ 同理。

而后计算 $\alpha_k(x) + \alpha_{k+1}(x)$：

$$\alpha_k(x) + \alpha_{k+1}(x) = \frac{(x_{k+1} - 3x_k + 2x)(x - x_{k+1})^2 - (x_k - 3x_{k+1} + 2x)(x - x_k)^2}{(x_{k+1} - x_k)^3}$$

$$= \frac{x_{k+1}^3 - 3x_k x_{k+1}^2 + 3x_{k+1}x_k^2 - x_k^3}{(x_{k+1} - x_k)^3} = 1$$

得证。

(2) 已经求得：

$$\beta_k(x) = (x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2$$

则可以求导得到极值点：

$$\beta_k'(x) = 3x^2 - 2x(2x_{k+1} + x_k) + x_{k+1}(x_{k+1} + 2x_k) = 0$$

求解得两个解 $x_1 = x_{k+1}$（舍去），$x_2 = \frac{x_{k+1} + 2x_k}{3}$，将其代入得到：

$$\left(\frac{x_{k+1} - x_k}{3}\right)\left(\frac{2x_k - 2x_{k+1}}{x_{k+1} - x_k}\right)^2 = \frac{4}{27}h_k$$

$\beta_{k+1}(x)$ 同理可解。

由此得证，$|\beta_k(x)|, |\beta_{k+1}(x)| \leq \frac{4}{27}h_k$。

(3) 由题可知：

$$|f - H_3| = |\alpha_k(x)f(x_k) + \alpha_{k+1}(x)f(x_{k+1}) + \beta_k(x)f'(x_k) + \beta_{k+1}(x)f'(x_{k+1}) - f(x)|$$

缩放得：

$$\leq |\alpha_k(x)f(x_k) + \alpha_{k+1}(x)f(x_{k+1}) - f(x)| + ||f'||_\infty \frac{4}{27}h_k$$

利用之前结论做一些变换：

$$= |\alpha_k(x)f(x_k) + \alpha_{k+1}(x)f(x_{k+1}) - \alpha_k(x)f(x) - \alpha_{k+1}(x)f(x)| + ||f'||_\infty \frac{8}{27}h_k$$

$$= |\alpha_k(x)(f(x_k) - f(x)) + \alpha_{k+1}(x)(f(x_{k+1}) - f(x))| + ||f'||_\infty \frac{8}{27}h_k$$

对 $f(x)$ 分别在 x_k, x_{k+1} 处进行一阶 Taylor 展开，其中 $\xi_1 \in [x_k, x], \xi_2 \in [x, x_{k+1}]$：

$$|\alpha_k(x)f'(\xi_1)(x - x_k) + \alpha_{k+1}(x)f'(\xi_2)(x - x_{k+1})| + ||f'||_\infty \frac{8}{27}h_k$$

进一步缩放：

$$\leq |\alpha_k(x)f'(\xi_1)(x - x_k)| + |\alpha_{k+1}(x)f'(\xi_2)(x - x_{k+1})| + ||f'||_\infty \frac{8}{27}h_k$$

$$\leq |\alpha_k(x)f'(\xi_1)|h_k + |\alpha_{k+1}(x)f'(\xi_2)|h_k + ||f'||_\infty \frac{8}{27}h_k$$

由于 α_k, α_{k+1} 均是非负函数：

$$\leq \alpha_k(x)||f'||_\infty h_k + \alpha_{k+1}(x)||f'||_\infty h_k + ||f'||_\infty \frac{8}{27}h_k$$

$$= \frac{35}{27}||f'||_\infty h_k$$

证毕。

5. 利用 MALAB 编程完成以下数值实验：

(1) 给定 $f(x) = sinh(x), f(0) = 0, f(0.20) = 0.2013360, f(0.30) = 0.3045203, f(0.50) = 0.5210953$，试求 Newton 插值多项式 $N_3(x)$，并利用插值多项式计算 $f(0.23)$ 的近似值，并利用 MATLAB 自带函数所求结果计算截断误差

(2) 给定 $f(x) = e^{0.1x^2}, f(1) = 1.105170918, f\ (1) = 0.2210341836, f(1.5) = 1.252322716, f\ (1.5) = 0.3756968148$，试求 Hermite 插值多项式 $H_3(x)$，并利用插值多项式计算 $f(1.25)$ 的近似值，并利用 MATLAB 自带函数所求结果计算截断误差.

(1) 编写 MATLAB 的 Newton 插值函数：

```
%x,y为已知插值节点的信息，p为需要求解的点
function res = newton1(x, y, p)
lenx = length(x);
leny = length(y);
 if lenx ~= leny , error('len(x) != len(y)'); end;
c = zeros(lenx, lenx);
c(:,1) = y';
for i = 2:lenx
    for j = i: lenx
        c(j, i) = (c(j, i - 1) - c(j - 1, i - 1))./(x(j) - x(j - i + 1));
    end
```

```
        end
        res = 0;
        for k = 1 :lenx
            tmp = 1;
            for u = 1:k−1
                tmp = tmp .* (p − x(u));
            end
            res = res + tmp.* c(k, k);
        end
```

输入

$$x = [0, 0.2, 0.3, 0.5]$$

及

$$y = [0, 0.2013360, 0.3045203, 0.5210953]$$

调用插值函数得到结果 $f(0.23) = 0.232031850820000$，再利用 MATLAB 自带 $sinh(x)$，计算得到截断误差为 $1.352893071904226 \times 10^{-6}$，插值结果较为理想。

(2) 这是一个典型的两点三次 Hermite 插值，之前我们已经计算得到：

$$\begin{cases} \alpha_k(x) = \left(\frac{x_{k+1}-3x_k+2x}{x_{k+1}-x_k}\right)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2 \\ \alpha_{k+1}(x) = \left(\frac{x_k-3x_{k+1}+2x}{x_k-x_{k+1}}\right)\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2 \\ \beta_k(x) = (x-x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2 \\ \beta_{k+1}(x) = (x-x_{k+1})\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2 \end{cases}$$

则利用结果编写 MATLAB 程序：

```
function res = hermite(x, y, y1, p)
A = (1 + 2.*(p − x(1))./ (x(2) − x(1))).*(((p − x(2))./(x(1) − x(2))).^2);
B = (1 + 2.*(p − x(2))./ (x(1) − x(2))).*(((p − x(1))./(x(2) − x(1))).^2);
C = (p − x(1)).*(((p − x(2)) ./ (x(1) − x(2))).^2);
D = (p − x(2)).*(((p − x(1)) ./ (x(2) − x(1))).^2);
res = A.*y(1) + B .*y(2) + C.*y1(1) + D.*y1(2);
end
```

计算插值结果得到 $H_3(1.25) = 1.169080402550000$
利用 MATLAB 自带函数计算截断误差得到 $3.804361950443536 \times 10^{-5}$

5. 思考许久，无力解答。