CMSC 441: Homework #1 Solutions

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Exercise 3.1–2

Show that for any real constants a and b, where $b > 0$,
$(n+a)^b = \Theta(n^b)$

Solution:

$$
\begin{aligned}
(n+a)^b &\leq (n+|a|)^b, \text{ where } n > 0 \\
&\leq (n+n)^b \text{ for } n \geq |a| \\
&= (2n)^b \\
&= c_1 \cdot n^b, \text{ where } \; c_1 = 2^b
\end{aligned}
$$

Thus

$$(n+a)^b = \Omega(n^b). \tag{1}$$

$$
\begin{aligned}
(n+a)^b &\geq (n-|a|)^b, \text{ where } n > 0 \\
&\geq (c_2' n)^b \text{ for } c_2' = 1/2 \text{ where } n \geq 2|a| \\
& \quad \text{as } n/2 \leq n - |a|, \text{ for } n \geq 2|a|
\end{aligned}
$$

Thus

$$(n+a)^b = O(n^b) \tag{2}$$

The result follows from 1 and 2 with $c_1 = 2^b, c_2 = 2^{-b}$, and $n_0 \geq 2|a|$.

Exercise 3.1–4

Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

Solution:

(a)

Is $2^{n+1} = O(2^n)$? Yes.

$2^{n+1} = 2 \; 2^n \leq c 2^n$ where $c \geq 2$.

(b)

Is $2^{2n} = O(2^n)$? No.

$2^{2n} = 2^n \cdot 2^n$. Suppose $2^{2n} = O(2^n)$. Then there is a constant $c > 0$ such that $c > 2^n$. Since 2^n is unbounded, no such c can exist.

Exercise 3.1–7

Prove that $o(g(n)) \cap \omega(g(n))$ is the empty set.

Solution:

Suppose not. Let $f(n) \in o(g(n)) \cap \omega(g(n))$ Now $f(n) = \omega(g(n))$ if and only if $g(n) = o(f(n))$ and $f(n) = o(g(n))$ by assumption. By transitivity property (page 49), $f(n) = o(f(n))$ i.e. for all constants $c > 0$, $f(n) < cf(n)$. Choose $c < 1$ and we have the desired contradiction from the asymptotic nonnegativity of $f(n)$.

Exercise 3.1–8

We can extend our notation to the case of two parameters n and m that can go to infinity independently at different rates. For a given function $g(n, m)$, we denote by $O(g(n, m))$ the set of functions

$O(g(n, m)) = \{ f(n, m) :$ there exist positive constants c, n_0, and m_0 such that $0 \leq f(n, m) \leq cg(n, m)$ for all $n \geq n_0$ and $m \geq m_0 \}$.
Give corresponding definitions for $\Omega(g(n, m))$ and $\Theta(g(n, m))$

Solution:
$\Omega(g(n, m)) = \{ f(n, m) :$ there exist positive constants c, n_0, and m_0 such that $0 \leq cg(n, m) \leq f(n, m)$ for all $n \geq n_0$ and $m \geq m_0 \}$.
$\Theta(g(n, m)) = \{ f(n, m) :$ there exist positive constants c_1, c_2, n_0, and m_0 such that $c_1 g(n, m) \leq f(n, m) \leq c_2 g(n, m)$ for all $n \geq n_0$ and $m \geq m_0 \}$.

Exercise 3.2–1

Show that if $f(n)$ and $g(n)$ are monotonically increasing functions, then so are $f(n) + g(n)$ and $f(g(n))$, and if $f(n)$ and $g(n)$ are in addition nonnegative, then $f(n) \cdot g(n)$ is monotonically increasing.

Solution:

(a)

We must show that if $f(n)$ and $g(n)$ are monotonically increasing functions, then so is $f(n) + g(n)$. Suppose not. Let $n_1 < n_2$ and $f(n_1) + g(n_1) > f(n_2) + g(n_2)$. Now, $f(n_1) \leq f(n_2)$ and $g(n_1) \leq g(n_2)$

$$\begin{aligned} f(n_1) &\leq f(n_2) \\ f(n_1) + g(n_1) &\leq f(n_2) + g(n_1) \\ f(n_1) + g(n_1) &\leq f(n_2) + g(n_2) \end{aligned}$$

This contradicts our assumption.

(b)

We must show that if $f(n)$ and $g(n)$ are monotonically increasing functions, then so is $f(g(n))$. Suppose not. Let $n_1 < n_2$ and $f(g(n_1)) > f(g(n_2))$. Let $m_1 = g(n_1), m_2 = g(n_2)$. $m_1 \leq m_2$. Clearly,

$f(m_1) \leq f(m_2)$, which contradicts our assumption.

(c)

We must show that if $f(n)$ and $g(n)$ are monotonically increasing functions, then so is $f(n)g(n))$ if f and g are nonnegative. Suppose not. Let $n_1 < n_2$ and $f(n_1) + g(n_1) > f(n_2) + g(n_2)$. Now, $f(n_1) \leq f(n_2)$ and $g(n_1) \leq g(n_2)$

$$\begin{aligned} f(n_1) &\leq f(n_2) && (3) \\ f(n_1)g(n_1) &\leq f(n_2)g(n_1) && (4) \\ f(n_1)g(n_1) &\leq f(n_2)g(n_2) && (5) \end{aligned}$$

This contradicts our assumption. Note that 4 and 5 hold since f and g nonnegative,