INDIAN INSTITUTE OF TECHNOLOGY
GANDHINAGAR

GRÖBNER BASES OF RATIONAL NORMAL CURVES

AMOGH PARAB (14110089)

KSHITEEJ JITESH SHETH (14110068)

SUPERVISOR: DR. INDRANATH SENGUPTA

November 25, 2015

Abstract

Let $n \geq 3$ be a natural number. Let $R = k[x_0, \ldots, x_n]$ be the polynomial ring in the indeterminates $x_0, x_1, \ldots, x_n$ over a field k. Let $A = \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}$. Let $\mathcal{G}_n$ denote the set of all 2×2 minors of the matrix A, i.e., $\mathcal{G}_n = \{x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq n\}$. Let I denote the ideal generated by $\mathcal{G}_n$ in $k[x_0, x_1, ..., x_n]$. Suppose that the monomial ordering in $R = k[x_0, x_1, ..., x_n]$ is given by $x_{i_0} > x_{i_1} > \ldots > x_{i_n}$, with the lexicographic ordering of monomials in R, where $(i_0, i_1, \ldots, i_n)$ denotes a permutation of the set $\{0, 1, \ldots, n\}$. We have classified all possible permutations $(i_0, i_1, \ldots, i_n)$ of $\{0, 1, \ldots, n\}$ such that $\mathcal{G}_n$ is a Gröbner basis of the ideal I.

Definitions

Definition 1. Set $S_k \subset \mathbb{N}$:
If monomial ordering in $R = k[x_0, x_1, \ldots, x_n]$ is given by $x_{i_0} > x_{i_1} > \ldots > x_{i_k}, \ldots, x_{i_n}$, then the set S_k is defined as $S_k = \{i_k, i_{k+1}, i_{k+2}, \ldots, i_n\}$; $k = 0, 1, \ldots, n$

Remark: S_0 is full set $\{0, 1, \ldots, n\}$ and S_1 is singleton set $\{i_n\}$.

Definition 2. Property P_j for given monomial order:
If the monomial ordering in $R = k[x_0, x_1, \ldots, x_n]$ is given by $x_{i_0} > x_{i_1} > \ldots > x_{i_j} > \ldots > x_{i_n}$,where $0 \leq j \leq n$ then the given monomial order is said to satisfy property P_j if i_k is either $max(S_k)$ or $min(S_k)$ $\forall k \leq j$

Remark: If the given monomial order satisfies the property P_j, $0 \geq j \leq n$ then it satisfies property P_k $\forall k < j$.

Theorem:

Suppose that the monomial ordering in $k[x_0 > x_1 > \ldots > x_n]$ is given by $(n \geq 3)$. $x_{i_0} > x_{i_1} > \ldots > x_{i_n}$ with the lexicographic ordering. Let $\mathcal{G}_n$ denote the set of all 2×2 minors of the matrix A, i.e., $\mathcal{G}_n = \{x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq n\}$. Let I denote the ideal generated by $\mathcal{G}_n$ in $k[x_0, x_1, ..., x_n]$. The set $\mathcal{G}_n$ is a Gröebner basis with respect to the said monomial order if and only if
given monomial order satisfies the property P_{n-3}.
That is i_k is either $\min(S_k)$ or $\max(S_k)$

for $0 \leq k \leq n-3$

.

And for $n = 2$ G_n forms a Gröbner basis.

Remark: There is relaxation on properties P_{n-2}, P_{n-1} and P_n. The monomial order may or may not satisfy the Properties P_{n-2}, P_{n-1} and P_n.

Proof for Only If part:

The set $\mathcal{G}_n$; $n > 2$; is a Gröebner basis with respect to the said monomial order only if
given monomial order satisfies the property P_{n-3}.

Theorem 1. *Suppose that the monomial ordering in $k[x_0, x_1, \ldots, x_n]$ is given by $(n \geq 3)$. $x_{i_0} > x_{i_1} > \ldots > x_{i_n}$ with the lexicographic ordering. Let $\mathcal{G}_n$ denote the set of all 2×2 minors of the matrix A, i.e., $\mathcal{G}_n = \{x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq n\}$. Let I denote the ideal generated by $\mathcal{G}_n$ in $k[x_0, x_1, ..., x_n]$. The set $\mathcal{G}_n$ a Groebner basis with respect to the said monomial order only if i_0 is either 0 or n.*

Proof. By method of contradiction.

Case I: n=3

$A = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{bmatrix}$.

$\mathcal{G}_3 = x_0 x_2 - x_1^2, x_0 x_3 - x_2 x_1, x_1 x_3 - x_2^2$

Assume i_0 is neither 0 or 3.

$\Rightarrow i_0 = 1$ or $i_0 = 2$

Subcase I] $i_0 = 1$ i.e. x_1 largest

consider the S polynomial

$S(x_0 x_3 - x_2 x_1, x_1 x_3 - x_2^2) = x_2^3 - x_0 x_3^2$

$x_2^3 - x_0 x_3^2$ does not tend to 0

as LT of each polynomial in $\mathcal{G}_3$ contains x_1 which is not present in $x_2^3 - x_0 x_3^2$

thus $\mathcal{G}_3$ does not form a Groebner Basis for $i_0 = 1$.

Subcase II] $i_0 = 2$ i.e. x_2 is largest.

Consider the S polynomial

$S(x_0 x_3 - x_2 x_1, x_0 x_2 - x_1^2) = x_1^3 - x_3 x_0^2$

$x_2^3 - x_0x_3^2$ does not tend to 0
as LT of each polynomial in $\mathcal{G}_3$ contains x_2 which is not present in $x_0^2x_3 - x_1^3$
thus $\mathcal{G}_3$ does not form a Groebner Basis for $i_0 = 2$.

Thus $\mathcal{G}_3$ does not form a Groebner Basis if i_0 is neither 0 nor 3 which is a contradiction for n = 3

Case II: n = 4
Assume i_0 is neither 0 nor 4
$\Rightarrow\ i_0 = 1$ or $i_0 = 2$ or $i_0 = 3$
$\mathcal{G}_3 = \{x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_0x_4 - x_1x_3, x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_2x_4 - x_3^2\}$
Subcase I] $i_0 = 1$ i.e. x_1 is largest.

Consider the S polynomial
$S(x_1x_3 - x_2^2, x_0x_4 - x_1x_3) = x_0x_4 - x_2^2$
which does not tend to 0 as except for $x_2x_4 - x_3^2$ all other polynomials LT contains x_1 which is not present in $x_0x_4 - x_2^2$ and $x_2x_4 - x_3^2$ does not divide the S polynomial. $\Rightarrow \mathcal{G}_3$ does not form a Groebner Basis for $i_0 = 1$

Subcase II] $i_0 = 2$ i.e. x_2 is largest.
Consider the S polynomial $S(x_0x_3 - x_1x_2, x_1x_4 - x_2x_3) = x_0x_3^2 - x_1^2x_4$ which does not tend to 0 as except for $x_0x_4 - x_1x_3$, all other polynomial's LT containx_2 which is not present in $x_0x_4 - x_1x_3$.
Thus $\mathcal{G}_3$ does not form a Gröebner Basis for $i_0 = 2$.

Subcase III] $i_0 = 3$ i.e. x_3 is largest.
Consider the S polynomial $S(x_3x_1 - x_4x_0, x_3x_1 - x_2^2) = x_2^2 - x_4x_0$ which do not tend to 0 as except for $x_0x_2 - x_1^2$, all other polynomial's LT contain x_3 which is not present in $x_2^2 - x_4x_0$.
Thus $\mathcal{G}_3$ does not form a Gröebner Basis for $i_0 = 3$.

Thus $\mathcal{G}_3$ does not form a Gröebner Basis if i_0 is neither 0 nor 4 which is a contradiction.

Lemma 1. *Let $x_{i_0} > x_{i_1} > \ldots > x_{i_n}$ be monomial ordering such that $S(x_ix_{i-2} - x_{i+1}x_{i-3}, x_ix_{i-2} - x_{i-1}^2) = x_{i+1}x_{i-3} - x_{i-1}^2$ where $4 \leq i \leq n-2$. And $LT(x_{i-1}^2 - x_ix_{i-2}) = x_ix_{i-2}$.*
Suppose we divide this polynomial with G_n if first 2×2 minor to divide is

$x_{i+1}x_{i-3} - x_{i+2}x_{i-4}$
(Remark: That means $LT(x_{i+1}x_{i-3} - x_{i+2}x_{i-4}) = x_{i+1}x_{i-3}$),
then remainder after division is non zero.

Proof. Let's divide $x_{i+1}x_{i-2} - x_{i-1}^2$ by $x_{i+1}x_{i-3} - x_{i+2}x_{i-4}$.
Thus quotient is 1 and remainder is $x_{i+2}x_{i-4} - x_{i-1}^2$.
Now there are two possibilities

1. $LT(x_{i+2}x_{i-4} - x_{i-1}^2) = x_{i-1}^2$;
this will give a nonzero remainder as $LT(x_{i-1}^2 - x_i x_{i-2}) = -x_i x_{i-2}$

2. $LT(x_{i+2}x_{i-4} - x_{i-1}^2) = x_{i+2}x_{i-4}$;
only possible divisor is $x_{i+2}x_{i-4} - x_{i+3}x_{i-5}$;
We observe that $x_{i+2}x_{i-4}$ is replaced by $x_{i+3}x_{i-5}$. If we carry out this process then difference between subscripts will go on increasing and finally will terminate at either one of them becomes x_0 or x_n.
Thus the term x_{i-1}^2 will not cancel leaving a nonzero remainder.
This proves our hypothesis. □

Case III: n $\geq$ 5
Assume i_0 is neither 0 nor n
Let $i_0 =$ i i.e. x_i is largest. such that $0 < i < n$
$\Rightarrow$ either $i - 3 \geq 0$ or $i + 3 \leq n$
for if $i - 3 < 0$
$\Rightarrow i + 3 < 6$
$\Rightarrow i + 3 \leq 5 \leq n \ldots$ as i is an integer.

Subcase I] $i - 3 \geq 0$
Consider the S polynomial
$S(x_i x_{i-2} - x_{i+1}x_{i-3}, x_i x_{i-2} - x_{i-1}^2) = x_{i-1}^2 - x_{i+1}x_{i-3}$
Only possible divisors are $x_{i-1}^2 - x_i x_{i-2}, x_{i+1}x_{i-3} - x_i x_{i-2}$ and $x_{i+1}x_{i-3} - x_{i+2}x_{i-4}$.
In this $x_{i-1}^2 - x_i x_{i-2}$ and $x_{i+1}x_{i-3} - x_i x_{i-2}$ will not divide the S-Polynomial as the leading term of S-Polynomial is not divisible by the leading terms of 2×2 minor.
Whereas $x_{i+1}x_{i-3} - x_{i+2}x_{i-4}$ gives nonzero remainder after division from lemma 1.
Thus S-Polynomial does not tend to 0 on division by $\mathcal{G}$. Thus $\mathcal{G}$ does not form a Gröebner Basis.

Subcase II] $i + 3 \leq n$

Consider the S polynomial
$S(x_i x_{i-2} - x_{i+1}x_{i-3}, x_i x_{i-2} - x_{i-1}^2) = x_{i-1}^2 - x_{i+1}x_{i-3}$
By same reasons as above S-Polynomial does not tend to 0 on division by $\mathcal{G}$.
Thus $\mathcal{G}$ does not form a Gröoebner Basis.
Thus $\mathcal{G}$ does not form a gröoebner basis for $n \geq 5$.
Thus $\mathcal{G}$ does not form a gröebner basis for any $n \geq 3$,
if i_0 is neither 0 nor n.
Hence the contradiction.
$\Rightarrow i_0 = 0$ or $i_0 = n$.

□

Lemma 2. *If monomial ordering is $x_{i_0} > \ldots > x_{i_n}$, with property P_{j-1}; $1 \leq j \leq n$ and if $min(S_j) \leq m \leq max_(S_j)$; then $m \in S_j$*
That is all the integers in between $min(S_j)$ and $max(S_j)$ are contained in S_j, i.e. S_j is of the form $\{i,\ i+1,\ \ldots i+k\}$.

Proof. Assume $m \notin S_j$
then $x_m > x_l \ \ \forall\ l \in S_j \ldots$ if $x_m < x_l$ for some $l \in S_j$ then $m \in S_j$ by definition.
$\Rightarrow \ m = i_p$ for some $p < j \ldots$ as $i_p \in S_j$ for $p \geq j$
$\Rightarrow \quad m = min(S_p)$ or $m = max(S_p)$
From definition of the set S_k we know that $S_j \subset S_p$ but $m \geq min(S_j) \in S_j \subset S_p$
thus m can't be $min(S_p)$
Similarly $m \leq max(S_j) \in S_j \subset S_p$
$\therefore m \neq max(S_p)$
Which is a contradiction.

□

Theorem 2. *Suppose that the monomial ordering in $k[x_0 > x_1 > \ldots > x_n]$ is given by $(n \geq 3)$. $x_{i_0} > x_{i_1} > \ldots > x_{i_n}$ with the lexicographic ordering. Let $\mathcal{G}_n$ denote the set of all 2×2 minors of the matrix A, i.e., $\mathcal{G}_n = \{x_i x_{j+1} - x_{i+1}x_j \mid 0 \leq i < j \leq n\}$. Let I denote the ideal generated by $\mathcal{G}_n$ in $k[x_0, x_1, ..., x_n]$. The set $\mathcal{G}_n$ a Gröebner basis with respect to the said monomial order only if*
i_k is either $min(S_k)$ or $max(S_k)$
for $0 \leq k \leq n-3$
that is given monomial order satisfies the property P_{n-3}

Remark: Monomial ordering need not satisfy the property P_{n-2}, P_{n-1} or P_n

Proof.
Using the method of induction on subscript number of the property P_k
True for $k = 0$ from Theorem I
Consider true for $k = j - 1; 1 \leq j \leq n - 3$. i.e property P_{j-1} is satisfied and we have to show that property P_j is also satisfied by the monomial ordering.
Assume not true for $k = j \leq n - 3$
$\therefore\ min(S_j) < i_j < max(S_j)$
Now, $j \leq n - 3 \Rightarrow$ cardinality of S_j is at least 4

Case I: $|S_j| = 4$
From induction hypothesis, property P_{j-1} is satisfied and from lemma 2 we can say that S_j is of the form $\{i, i+1, i+2, i+3\}$.
Because $min(S_j) < i_j < max(S_j)$there are only two possibilities of S_j as follows.
$S_j = \{i_j - 1, i_j, i_j + 1, i_j + 2\}$ or $S_j = \{i_j - 2, i_{j-1}, i_j, i_j + 1\}$

Now, consider $S_j = \{i_j - 1, i_j, i_j + 1, i_j + 2\}$ then consider
$S(x_{i_j-1}x_{i_j+2} - x_{i_j}x_{i_j+1}, x_{i_j}x_{i_j+2} - x^2_{i_j+1}) = x_{i_j-1}x^2_{i_j+2} - x^3_{i_j+1}$

if $LT(x_{i_j-1}x^2_{i_j+2} - x^3_{i_j+1}) = x^3_{i_j+1}$
then $S \nrightarrow 0$ as only divisor to $x^3_{i_j+1}$ is $x^2_{i_j+1} - x_{i_j+2}x_{i_j}$ who's leading term is $x_{i_j+2}x_{i_j}$

if $LT(x_{i_j-1}x^2_{i_j+2} - x^3_{i_j+1}) = x_{i_j-1}x^2_{i_j+2}$
Then only possible divisors are $x^2_{i_j+2} - x_{i_j+1}x_{i_j+3}, x_{i_j-1}x_{i_j+2} - x_{i_j}x_{i_j+1}$ and $x_{i_j-1}x_{i_j+2} - x_{i_j-2}x_{i_j+3}$(if exists)
In the case of $x_{i_j-1}x_{i_j+2} - x_{i_j}x_{i_j+1}$, $LT(x_{i_j-1}x_{i_j+2} - x_{i_j}x_{i_j+1}) = -x_{i_j}x_{i_{j+1}}$ which does not divide $x_{i_j-1}x_{i_j+2}$.
In the case of $x_{i_j-1}x_{i_j+2} - x_{i_j-2}x_{i_j+3}$;
$LT(x_{i_j-1}x_{i_j+2} - x_{i_j-2}x_{i_j+3} = x_{i_j-2}x_{i_j+3}$ as $x_{i_j+3} > x_{i_j-1}, x_{i_j+2}$
and in the case of $x^2_{i_j+2} - x_{i_j+1}x_{i_j+3}$; $LT(x^2_{i_j+2} - x_{i_j+1}x_{i_j+3}) = x_{i_j+1}x_{i_j+3}$

Similar arguments goes for $S_j = \{i_j - 2, i_j - 1, i_j, i_j + 1\}$
Thus for cardinality of $S_j = 4$, G_n does not form a Gröbner Basis. Hence contradiction.

case II: $n(S_j) = 5$
From assumption and lemma 2 only possible cases are,

$S(j) = i_j - 1, i_j, i_j + 1, i_j + 2, i_j + 3$
$S(j) = i_j - 2, i_j - 1, i_j, i_j + 1, i_j + 2$
$S(j) = i_j - 3, i_j - 2, i_j - 1, i_j, i_j + 1$

Consider following examples in each cases respectively.
$S(f_{i_j-1,i_j+2}, f_{i_j,i_j+1}) = x_{i_j-1}x_{i_j+3} - x_{i_j+1}^2$
$S(f_{i_j-2,i_j}, f_{i_j-1,i_j+1}) = x_{i_j-2}x_{i_j+1}^2 - x_{i_j-1}^2$
$S(f_{i_j-1,i_j+2}, f_{i_j,i_j+1}) = x_{i_j-1}x_{i_j+3} - x_{i_j+1}^2$
are counter examples to each case respectively. Arguments for example 1 and 3 are similar as in case I.
For case 2
If LT is $x_{i_j-1}^2$, then LT of only possible divisor i.e. $LT(x_{i_j-1}^2 - x_{i_j}x_{i_j-2}) = x_{i_j}x_{i_j-2}$ for the reason $x_{i_j} > x_{i_j-2}$. Thus does not divide.
And is LT is $x_{i_j-2}x_{i_j+1}^2$; then from lemma 1 we can say that G_n does not divide.
Thus for cardinality of $S_j = 5$, G_n does not form a Gröbner Basis. Hence contradiction.

Case III: $n(S_j) \geq 6$
From assumption and lemma 1.1 we can say that
either $\{i_j - 1, i_j, i_j + 1, i_j + 2, i_j + 3\} \in S_j$
or $\{i_j - 3, i_j - 2, i_j - 1, i_j, i_j + 1\} \in S_j$

Consider the case where,
$\{i_j - 3, i_j - 2, i_j - 1, i_j, i_j + 1\} \in S_j$

Consider,
$S(f_{i_j,i_j-3}, f_{i_j-2,i_j-1}) = x_{i_j-1}^2 - x_{i_j+1}x_{i_j-3}$
if $LT(S) = x_{i-1}^2$ then only possible divisor is $x_{i_j-1}^2 - x_{i_j}x_{i_j-2}$ who's leading term is $x_{i_j}x_{i_j-2}$

if $LT(S) = x_{i_j+1}x_{i_j-3}$ then possible divisors are $x_{i_j+1}x_{i_j-3} - x_{i_j}x_{i_j-2}$ and $x_{i_j+1}x_{i_j-3} - x_{i_j+2}x_{i_j-4}$
The first one is not possible as leading term is $x_{i_j}x_{i_j-2}$. For second possibility; from lemma 1 we can say that remainder after division by G_n is non zero.

Similar arguments will go for other possibility of the set S_j.

Thus for cardinality of $S_j \geq 6$, G_n does not form a Gröbner Basis. Hence contradiction.

So, the assumption we made was wrong
$\therefore$ the set "G_n" forms a Gröbner basis only if monomial order satisfies the property P_{n-3}. i.e.i_j is either $max(S_j)$ or $min(S_j)$; $\forall i_j \leq n-3$. $\square$

Definition 3. Mapping ϕ Suppose that the monomial ordering in $R_{n+1} = k[x_0, x_1, \ldots, x_{n+1}]$ is given by $x_{i_0} > x_{i_1} > \ldots > x_{i_{n+1}}$. Consider subset $A_{n+1} = \{x_0, x_1, \ldots, x_{n+1}\}$ of R_{n+1} and subset $A_n = \{x_0, x_1, \ldots, x_n\}$ of R_n.
Let $i_a \in A_{n+1}$ then mapping ϕ is defined as,
$\phi : A_{n+1}/\{i_a\} \rightarrow A_n$
$\phi(x_i) = x_i \qquad$ if $i < i_a$
$\phi(x_i) = x_{i-1} \quad$ if $i > i_a$
It is easy to show that this is a one-to-one and onto map.
Moreover, we extend the definition to all polynomials not containing i_a as,
$\phi(Ax^\alpha + Bx^\beta) = A\phi(x^\alpha) + B\phi(x^\beta)$ where $\alpha_{i_a} = \beta_{i_a} = 0$
$\phi(Af + Bg) = A\phi(f) + B\phi(g) \quad f, g \in k[x_0, \ldots, x_n]$
$\phi(x_0^{\alpha_0} x_1^{\alpha_1} \ldots x_{n+1}^{\alpha_{n+1}}) = \phi(x_0)^{\alpha_0}\phi(x_1)^{\alpha_1} \ldots \phi(x_{n+1})^{\alpha_{n+1}}$ where $\alpha_{i_a} = 0$
This is also a one to one and onto map.

Definition 4. Mapping ϕ on the order Suppose that the monomial ordering $>_{n+1}$ in $R_{n+1} = k[x_0, x_1, \ldots, x_{n+1}]$ is given by $x_{i_0} > x_{i_1} > \ldots > x_{i_{n+1}}$. Then the monomial ordering $\phi(>_{n+1})$ in $R_n = k[x_0, x_1, \ldots, x_n]$ is defined by $\phi(x_{i_0}) > \phi(x_{i_1}) > \ldots > \phi(x_{i_{n+1}})$ where nothing maps at the pace of i_a.

Lemma 3. *Let $f = x_i x_{j+1} - x_{i+1} x_j$ be 2×2 minor of from G_{n+1} such that f does not contain x_{i_a}, then $\phi(f)$ is also a 2×2 minor from G_n.*

Proof. Consider $x_i x_{j+1} - x_j x_{i+1}$ which is a 2×2 minor and neither of i, i+1, j, j+1 is i_a. It is enough to show that if x_i maps to $x_{i'}$ then x_{i+1} maps to $x_{i'+1}$.

Case 1 $\quad i < i_a$
$\Rightarrow i + 1 < i_a$
Thus i maps to i and i+1 maps to i+1

Case 2 $\quad i > i_a$
$\Rightarrow i + 1 > i_a$

Thus i maps to i-1 and i+1 maps to i

Thus if x_i maps to $x_{i'}$ then x_{i+1} maps to $x_{i'}+1$, which proves that polynomials are nothing but 2×2 minor of $2\times n$ Matrix. $\square$

Lemma 4. *Suppose that* $<_1$ *and* $<_2$ *denote the monomial orders of* $k[x_0, x_1, \ldots, x_{n+1}]$ *and* $k[x_0, x_1, \ldots, x_n]$ *respectively, such that* $\phi(<_1) = <_2$. *If* $0 \neq f \in k[x_0, \ldots, x_{n+1}]$ *and* x_{i_a} *does not occur in* f *then,*
$\phi(LT_{<_1}(f)) = LT_{<_2}(\phi(f))$

Proof. Let $<_1 = (x_{i_0} > x_{i_1} > \ldots > x_{i_{n+1}})$.
Let $x = x_{i_0}x_{i_1}\ldots x_{i_{n+1}}$.
Let $\alpha = (\alpha_0\alpha_1\ldots\alpha_{n+1})$ such that $\alpha_a = 0$. Let $LT_{<_1}(f) = x^\alpha$.
Let x^{α_1} be any arbitrary term in f other than x^α
Let i^{th} entry of $\alpha - \alpha_1$ be non zero.
As $x^\alpha = LT_{<_1}(f)$, $\alpha(i) - \alpha_1(i) > 0$.
We have $<_2 = (\phi(x_{i_0}) > \phi(x_{i_1}) > \ldots > \phi(x_{i_{n+1}}))$
and $\phi(x) = \phi(x_{i_0})\phi(x_{i_1})\ldots\phi(x_{i_{n+1}})$.
Then from definition 3 we have, $\phi(x^\alpha) = \phi(x)^\alpha$ which is a term of $\phi(f)$. Similarly $\phi(x^{\alpha_1}) = \phi(x)^{\alpha_1}$ is also a term of $\phi(f)$.
Now, as first non-negative entry of $\alpha - \alpha_1$ is positive and as α hence α_1 is arbitrary $\phi(x^\alpha)$ is leading term of $\phi(f)$ with respect to monomial order $<_2$.
Hence, $\phi(LT_{<_1}(f)) = LT_{<_2}(\phi(f))$ is proved.
$\square$

Theorem 3. *Suppose that the monomial ordering in* $k[x_0, x_1, \ldots, x_n]$ *is given by* $(n \geq 3)$. $x_{i_0} > x_{i_1} > \ldots > x_{i_n}$ *with the lexicographic ordering. Let* $\mathcal{G}_n$ *denote the set of all* 2×2 *minors of the matrix* A, *i.e.,* $\mathcal{G}_n = \{x_ix_{j+1} - x_{i+1}x_j \mid 0 \leq i < j \leq n\}$. *Let* I *denote the ideal generated by* $\mathcal{G}_n$ *in* $k[x_0, x_1, ..., x_n]$. *The set* $\mathcal{G}_n$ *a Gröbner basis with respect to the said monomial order if given monomial order satisfy the property* P_{n-3}

Proof. The proof will follow the method of induction over the number of variables "N".
The statement is trivial in the case $N = 2$ but we will go one step ahead and show that using a computer program (appended below) that the statement is also true for $N = 1, 2, \ldots, 7$.
Lets assume the statement is True for $N = n$, then we have to show that the statement is also True for $N = n + 1$.

Now, consider the S-Polynomial of 2×2 Minors $f = x_i x_{j+1} - x_j x_{i+1}$ and $g = x_l x_{m+1} - x_m x_{l+1}$ as $S\,(f, g)$
As $n > 7$, there exists a x_{i_a} such that x_{i_a} does not occur in any one of the 4 monomials appearing in f and g, for the reason that there can be as the most 8 distinct variables that may occur in 4 monomials.

Now consider x_{i_a} is not present in given pair of S-polynomial where monomial ordering is $<_1 = x_{i_0} > \ldots > x_{i_{a-1}} > x_{i_a} > x_{i_{a+1}} > \ldots > x_{i_{n+1}}$.
Here we can apply mapping ϕ as defined in the definition 3.
Lemma 2 tells that $\phi(f)$ and $\phi(g)$ are both 2×2 minors from set G_n.
As $\phi(f)$ and $\phi(g)$ are both 2×2 minors from set G_n, using Induction Hypothesis we can say that the S-Polynomial of $\phi(f)$ and $\phi(g)$ is divisible by G_n. More precisely

$$S(\phi(f), \phi(g)) = \sum_i a_{i,j',k'} f_{j'} g_{k'} \qquad a_{i,j',k'} \in k[x_0, \ldots, x_n].$$

Division algorithm tells that
$multideg(a_{i'j'k'}) \leq multideg(S(\phi(f), \phi(g)))$

As ϕ being a one to one and onto map from $A_{n+1}/\{i_a\}$ to A_n. We can define ϕ^{-1}.
Applying ϕ^{-1} to above equation, we get
$\phi^{-1}(S(\phi(f), \phi(g))) = \phi^{-1}(\sum_i a_{i,j',k'} f_{j'} g_{k'})$

But $\phi^{-1}(S(\phi(f), \phi(g)))$ is nothing but $S(f, g)$
and $\phi^{-1}(\sum_i a_{i,j',k'} f_{j'} g_{k'})$ is nothing but $\sum_i a'_{i,j,k} f_j g_k$ where f_j and g_k are both 2×2 minors in R_{n+1}

Applying lemma 3 to above in-equation we will get
$multideg(a'_{i,j,k}) \leq multideg(S(f, g))$
This shows that $S(f, g)$ is divisible by G_{n+1}.
Hence G_{n+1} is also a Gröbner basis.
This completes the proof

□

Result

Using Theorem 2 and Theorem 3 we can state the following.

Suppose that the monomial ordering in $k[x_0 > x_1 > \ldots > x_n]$ is given by $(n \geq 3)$. $x_{i_0} > x_{i_1} > \ldots > x_{i_n}$ with the lexicographic ordering. Let $\mathcal{G}_n$ denote the set of all 2×2 minors of the matrix A, i.e., $\mathcal{G}_n = \{x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq n\}$. Let I denote the ideal generated by $\mathcal{G}_n$ in $k[x_0, x_1, ..., x_n]$. The set $\mathcal{G}_n$ a Gröebner basis with respect to the said monomial order if and only if
i_k is either $\min(S_k)$ or $\max(S_k)$
for $0 \geq k \geq n-3$
that is given monomial order satisfies the property P_{n-3}