[481] HW #4 x7 Qs

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§Ch.10 "POINT ESTIMATION"

[10.53] Given RAND sample size n from POISSION pop. Q: USE "METHOD of MOMENTS" to obtain an estimator for parameter λ.

- want to recover an "estimator for λ" — $\hat{\lambda}$ by solving the following EQ:

$$\{ \underbrace{\mu_1'}_{\text{1st pop moment}} \overset{\text{SET}}{=} \underbrace{m_1'}_{\text{1st sample moment}} \}$$

- Suppose $X \sim POISSON(\lambda)$ where

$$P(\underset{\sim}{X}=x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x = 0, 1, 2, \ldots$$

- $$\mu = \mu_1' = \sum x\, p(x) = \sum_{x=0}^{\infty} x \left(\frac{\lambda^x e^{-\lambda}}{x!} \right) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} (e^{\lambda}) = \lambda$$

- Thus $\boxed{\mu = \mu_1' = \lambda}$
- We know $m_1' = \frac{\sum_{i=1} x_i}{n} = \bar{X} \longrightarrow \boxed{m_1' = \bar{X}}$
- Since $\mu_1' = m_1'$ we know $\lambda = \bar{X}$. THEN the method of moments estimator of λ is $\boxed{\hat{\lambda} = \bar{X}}$ *

[10.56] $\{X_1, \dots X_n\}$ rand sample size n

from pop : $g(x; \theta, \delta) = \begin{cases} \frac{1}{\theta} \cdot e^{-\frac{(x-\delta)}{\theta}}, & x > \delta \\ 0 & \text{elsewhere.} \end{cases}$

- find estimators for δ & θ by method of moments
- This is the "two parameter exponential dist"

IF $\theta = 1$ then ~~[redacted]~~ the dist is

$f(x) = \begin{cases} e^{-x-\delta}, & x > \delta \\ 0 & \text{elsewhere.} \end{cases}$

(from ex 10.3)

$\mu = \mu_1' = \int_{-\infty}^{\infty} x\, g(x; \theta, \delta) = \frac{1}{\theta} \int_{x=\delta}^{\infty} x\, e^{-(x-\delta)/\theta}\, dx$

U-SUB!
$u = x - \delta$
$du = dx$

$= \frac{1}{\theta} \left[\int_{u=0}^{\infty} (u+\delta) e^{-u/\theta}\, du \right] = \frac{1}{\theta}[\theta^2]$

$= \theta$

$= \begin{bmatrix} a = u + \delta & db = e^{-u/\theta}\, du \\ da = du & b = -\theta e^{-u/\theta} \\ ab - \int b\, da & \end{bmatrix}$

"integration by parts technique"

$-\theta(u+\delta) e^{-u/\theta} \Big|_{u=0}^{\infty} + \theta \int_0^{\infty} e^{-u/\theta}\, du$

$= -\theta \left[(\infty e^{-\infty/\theta}) - \delta (e^0) \right] - \theta^2 \left[e^{-u/\theta} \right]_{u=0}^{u=\infty}$

$= -\theta^2 \left[e^{-\infty} - e^0 \right]$

$\infty e^{-\infty/\theta} \to 0$, $e^0 = 1$, $e^{-\infty} = 0$, $e^0 = 1$

$e^{-\infty} \to 0$ faster than polynomial growth $\to \infty$ ... so this term converges to 0.

0

$(+\theta\delta + \theta^2)$

$= \frac{1}{\theta}[\theta\delta + \theta^2]$

$\boxed{\delta + \theta = \mu_1'}$ ★

(... CONT NEXT PG.

$$\bullet\ \mu_2' = \int_{-\infty}^{\infty} x^2 g(x;\theta,\delta)\,dx = \frac{1}{\theta}\int_{\delta}^{\infty} x^2 e^{-(x-\delta)/\theta}\,dx$$

let $u = x-\delta \Rightarrow \begin{matrix} x = u+\delta \\ du = dx \end{matrix}$ . . . THEN,

$$= \frac{1}{\theta}\int_{u=0}^{\infty}(u+\delta)^2 e^{-(1/\theta)u}\,du$$

$$\boxed{\begin{matrix} w = (u+\delta)^2 & dv = e^{-1/\theta\, u}du \\ dw = 2(u+\delta) & v = -\theta e^{-u/\theta} \\ wv - \int v\,dw & \end{matrix}}$$

INTEGRATION by PARTS !!

$$= \frac{1}{\theta}\Bigg[\underbrace{-\theta\left(e^{-u/\theta}(u+\delta)^2\Big|_{u=0}^{\infty}\right)}_{=\ -\theta\left(\underbrace{e^{-\infty}\cdot\infty}_{=0}\right) - \theta(\delta^2)} + \underbrace{2\theta\left(\int_0^{\infty}(u+\delta)e^{-u/\theta}\,du\right)}_{=\ +2\theta(\delta\theta+\theta^2)}\Bigg]$$

(Since exponential decay is faster than polynomial growth)

(✱) computed EARLIER . .

$$= \frac{1}{\theta}\left(+\theta\delta^2 + 2\theta^2\delta + 2\theta^3\right)$$

$$\boxed{\mu_2' = \delta^2 + 2\theta\delta + 2\theta^2} \quad \text{and} \quad \boxed{\mu_1' = \delta+\theta}$$

→ we can express μ_2' in terms of μ_1' . . .

$$\mu_2' = (\delta^2+2\theta\delta+\theta^2)+\theta^2 = (\delta+\theta)^2+\theta^2 = (\mu_1')^2+\theta^2$$

By the method of moments let us set $\{m_1' = \mu_1'$ and $m_2' = \mu_2'\}$

THEN $\{m_1' = \delta+\theta$ and $m_2' = (\delta+\theta)^2+\theta^2 = (m_1')^2+\theta^2\}$

Therefore, $\left[\hat{\theta} = \sqrt{m_2' - (m_1')^2} \text{ and } \hat{\delta} = m_1' - \sqrt{m_2' + (m_1')^2}\right].$

[10.59] METHOD of MAX LIKELIHOOD to rework (10.53) $\longrightarrow$ rand sample size n from POISSION pop...

Q recover estimator $\hat{\lambda}$ for parameter λ.

where $\left\{ p(X=x;\lambda) = \frac{\lambda^x}{x!} e^{-\lambda} \right\}$ *

- $L(\lambda) = p(x_1, \cdots, x_n ; \lambda)$

$a^m a^n = a^{m+n}$ — "product rule"

$$= \prod_{i=1}^{n} p(x_i;\lambda) = \frac{(\prod \lambda^{x_i})}{\prod x_i!} e^{-n\lambda} = \frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod x_i!} (e^{-n\lambda})$$

$$\ln(L(\lambda)) = \left(\sum_{i=1}^{n} x_i\right) \ln(\lambda) - n\lambda(\underbrace{\ln(e)}_{1}) - \ln(\prod x_i!)$$

$$\frac{\partial}{\partial\lambda}(\ln(L(\lambda))) = \left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda} - n = 0$$

$$\Rightarrow \boxed{\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X}}$$

[10.66] USE Method of MAX Likelihood to rework (exercise 10.56)

- We know that $\{X_1, \ldots, X_n\}$ constitute RAND sample size n from pop s.t

$$g(x;\theta,\delta) = \begin{cases} \frac{1}{\theta} \cdot e^{-\frac{(x-\delta)}{\theta}}, & \text{for } x > \delta \\ 0, & \text{otherwise} \end{cases}$$

$\ln(uv) = \ln(u) + \ln(v)$

and

$\ln(u^n) = n\ln(u)$

RECAL . . .

Q: FIND estimators $\hat{\delta}$ for δ and $\hat{\theta}$ for θ.

$$g(x;\theta,\delta) = \frac{1}{\theta} e^{-(x-\delta)/\theta} \quad \text{for } x > \delta$$

$$L(\theta,\delta) = g(x_1, \ldots, x_n; \theta, \delta) \quad \text{for } X_1, \ldots X_n \text{ INDEPENDENT}$$

$$= \prod_{i=1}^{n} g(x_i;\theta,\delta)$$

$$L(\theta,\delta) = \frac{1}{\theta^n} e^{-1/\theta \sum_{i=1}^{n}(x_i-\delta)}$$

$$\ln(L(\theta,\delta)) = \ln(\theta^{-n}) + \ln\left(e^{-1/\theta \sum_{i=1}^{n}(x_i-\delta)}\right)$$

$$= -n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n}(x_i-\delta) \underbrace{\cancel{\ln(e)}}_{1}$$

$$\frac{\partial}{\partial\theta} \ln(L(\theta,\delta)) = -\frac{n}{\theta} = \frac{1}{\theta^2}\left(\sum_{i=1}^{n}(x_i-\delta)\right) \overset{\text{SET}}{=} 0$$

$$\Rightarrow -n\frac{\cancel{\theta^2}}{\theta} = \cancel{\theta}\left(\sum_{i=1}^{n}(x_i-\delta)\right) \Rightarrow \boxed{\hat{\theta} = \frac{-\left(\sum_{i=1}^{n}(x_i-\delta)\right)}{n}}$$

AND to find MLE for $\delta \longrightarrow \frac{\partial}{\partial \delta}(\ln(L(\theta,\delta)) \stackrel{\text{SET}}{=} 0 \ldots$

$$\frac{\partial}{\partial \delta}(\ln(L(\theta,\delta)) = +\frac{1}{\theta}\left(\sum_{i=1}^{n} \frac{\partial}{\partial \delta}(-x_i+\delta)\right) = \frac{1}{\theta}\sum_{i=1}^{n} +1 = \frac{(+1)^n}{\theta} = 0$$

[10.3] Use formula for sampling dist of $\tilde{x}$ (on pg. 253) to show that for RAND samples of size $(n=3)$ ODD sample size! the median is an U.E. of parameter θ for a UNIFORM POP. w/ $\alpha = \theta - 1/2$ and $\beta = \theta + 1/2$

★RECALL:

(Y_r – rth order statistic) $Y_r(y_r) = \frac{n!}{(r-1)!\,(n-r)!} f(y_r)$

$$= \left(\int_{-\infty}^{y_r} f(x)\,dx \right)^{r-1} f(y_r) \left(\int_{y_r}^{\infty} f(x)\,dx \right)^{n-r}$$

- We know for a RAND SAMPLE of size $(n = 2m+1)$, the sample MEDIAN – $\tilde{X}$ is Y_{m+1} whose sampling dist...

$$h(\tilde{x}) = \frac{(2m+1)!}{m!\,m!} \left[\int_{-\infty}^{\tilde{x}} f(x)\,dx \right]^m f(\tilde{x}) \left[\int_{\tilde{x}}^{\infty} f(x)\,dx \right]^m$$

for $-\infty < \tilde{x} < \infty$

→ The sampling dist of MEDIAN given by:

$$h(\tilde{x}) = \begin{cases} \frac{(2m+1)!}{m!\,m!\,\theta} e^{-\tilde{x}(m+1)/\theta} \left(1 - e^{-\tilde{x}/\theta}\right)^m & \text{for } \tilde{x} > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

LET $\cdot\left[\alpha = \theta - 1/2 \;,\; \beta = \theta + 1/2\right]$

$\cdot (n=3) \equiv (n = 2m+1 \text{ for } m=1)$

UNIFORM POP:

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} = 1 & \text{for } \alpha<x<\beta \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu = E(x) = \frac{\beta+\alpha}{2} = \theta$$

$$\sigma^2 = \frac{(\beta-\alpha)^2}{12} = \frac{1}{12}$$

→ SHOW MEDIAN IS U.E.

$$h(\tilde{x}) = \frac{(2m+1)!}{m!\,m!}\left(\int_{\theta-1/2}^{\tilde{x}} dx\right)^m \cdot 1 \cdot \left(\int_{\tilde{x}}^{\theta+1/2} dx\right)^m$$

$$= \frac{(2m+1)!}{m!\,m!}\left(\tilde{x}-\theta+1/2\right)^m\left(\theta+1/2-\tilde{x}\right)^m$$

→ Substitute value $(m=1)$ * . . .

$$\boxed{h(\tilde{x}) = 6\left(\tilde{x}-\theta+1/2\right)\left(\theta+1/2-\tilde{x}\right) \text{ for } \theta-1/2<\tilde{x}<\theta+1/2}$$

OBJECTIVE:

(WANT to check if $\tilde{X}$ is an U.E. for parameter θ that is we want to verify that $E(\tilde{x}) = \theta$.)

$$E(\tilde{x}) = \int_{\theta-1/2}^{\theta+1/2} \tilde{x}\cdot h(\tilde{x}) = 6\int_{\theta-1/2}^{\theta+1/2} \tilde{x}\,\underbrace{(\tilde{x}-\theta+1/2)}_{\parallel}(\theta+1/2-\tilde{x})\,d\tilde{x}$$

LET $u = \tilde{x}-\theta+1/2 \rightarrow du = d\tilde{x}$

$$= 6\int_0^1 (u+\theta-1/2)\underbrace{(u)(1-u)}_{(u-u^2)}\,du$$

$$= 6\int_0^1 \Big[\underbrace{u^2-u^3+\theta u-\theta u^2-1/2\,u+1/2\,u^2}_{=}\Big]\,du$$

$$\left(u^2(3/2-\theta) + u(\theta-1/2) - u^3\right)$$

. . .

for $\alpha < x < \beta$

$$= 6\left[\left(\frac{1}{2}-\frac{\theta}{3}\right)\underset{=1-0}{u^3} + \left(\frac{\theta}{2}-\frac{1}{4}\right)\underset{=1-0}{u^2} - \frac{1}{4}\underset{=1-0}{u^4}\right]\Bigg|_{u=0}^{1}$$

$$= 6\left[\underset{0}{\cancel{\tfrac{1}{2}}\cancel{-\tfrac{1}{2}}} - \frac{2}{6}\theta + \frac{3}{6}\theta\right]$$

$$= 6\left(\frac{\theta}{6}\right) = \theta.$$

CONCLUDE:

- Since $E(\bar{x}) = \theta$ then $\bar{x}$ is indeed an U.E. for parameter θ.

[10.15] SHOW that mean of RAND sample of size n is a M.V.UE of parameter λ for POISSION(λ) pop...

*POISSION pop:

$$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\mu = \lambda, \quad \sigma^2 = \lambda, \quad var(\bar{x}) = \frac{\lambda}{n}$$

RECALL

CRLB alt. form?

$$\frac{-1}{n\,\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ln(f)(X)\right]}$$

$$\ln(f) = x\ln(\lambda) - \lambda\underset{1}{\ln(e)} - \ln(x!)$$

$$\frac{\partial}{\partial\lambda}(\ln(f)) = x\lambda^{-1} - 1$$

$$\frac{\partial^2}{\partial\lambda^2}(\ln(f)) = -x\lambda^{-2}$$

$$\mathbb{E}\left[\frac{\partial^2}{\partial\lambda^2}\ln(f)(X)\right] = \frac{-E[X]}{\lambda^2} = \frac{-\lambda}{\lambda^2} = \frac{-1}{\lambda} \ldots$$

CRLB $\Rightarrow \dfrac{-1}{n\,E} = \dfrac{-1}{n\left(\frac{-1}{\lambda}\right)} = \dfrac{\lambda}{n} = VAR(\bar{x})$

(Since $VAR(\bar{x}) = \frac{1}{n}\sigma_{pop}^2 = \frac{\lambda}{n}$)

- no U.E for λ can achieve a variance lower than CRLB $\equiv \frac{\lambda}{n}$ by Cramér-Rao inequality.
- It just so happens that $\bar{x}$ is an U.E. for λ as $var(\bar{x}) = \frac{1}{n}\sigma_{pop}^2 = \frac{\lambda}{n}$

$\therefore$ $\bar{x}$ is M.V.U.E. since the variance matches the CRLB.

[10.18] Q SHOW that for U.E. of ex 10.4, $\left(\frac{n+1}{n}\right) \cdot Y_n$ the Cramér-Rao inequality is NOT satisfied...

- RECALL: (from EX 10.4)
- $\left(\frac{n+1}{n}\right) Y_n$ is U.E. of parameter β...
- UNIF pop on (α, β) where $\alpha = 0 \longrightarrow f(x) = \begin{cases} \frac{1}{\beta} & \text{for } 0 \le x \le \beta \\ 0 & \text{elsewhere.} \end{cases}$
- found pdf of Y_n: $g_n(y_n) = \frac{n}{\beta^n} \cdot y_n^{\,n-1}$

  and $\star\ \boxed{\mathbb{E}(Y_n) = \frac{n}{n+1}\beta} \neq \beta$ so Y_n is biased estimator of parameter β.
- Suppose we LET $U := \left(\frac{n+1}{n}\right) Y_n$, THEN...

$$\mathbb{E}(U) = \beta \longrightarrow \text{unbiased estimator.}$$

- $$E(Y_n^2) = \frac{n}{\beta^n} \int_0^\beta y_n^{\,n+1}\, dy_n = \frac{n}{\beta^n}\left[\frac{1}{n+2}\, y_n^{\,n+2} \Big|_{0 = y_n}^{\beta}\right]$$
$$= \frac{n}{(n+2)\beta^n}\left(\beta^{n+2} - 0\right) = \left(\frac{n}{n+2}\right)\beta^2$$

$$\star\ \boxed{\mathbb{E}(Y_n^2) = \left(\frac{n}{n+2}\right)\beta^2}$$

THEREFORE, $VAR(Y_n) = \mathbb{E}(Y_n^2) - \left(\mathbb{E}(Y_n)\right)^2$

$$= \left(\frac{n}{n+2}\right)\beta^2 - \frac{n^2}{(n+1)^2}\beta^2$$

$$= \beta^2\left[\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}\right] = \beta^2\left(\frac{n}{(n+2)(n+1)^2}\right)$$

$$n^3 + 2n^2 + n$$
$$-n^3 - 2n^2$$
$$= n$$

SO,

$$\boxed{VAR(Y_n) = \frac{n}{(n+2)(n+1)^2}\beta^2}$$

(...)

- $VAR(\hat{U}) = VAR\left(\left(\frac{n+1}{n}\right)Y_n\right) = \frac{(n+1)^2}{n^2} VAR(Y_n)$

$$= \frac{\cancel{(n+1)^2}}{\cancel{n^2}\,n} \cdot \left(\frac{\cancel{(n)}}{(n+2)\,\cancel{(n+1)^2}}\right) \cdot \beta^2 = \frac{\beta^2}{n\,(n+2)}$$

- CRLB?
  - $f(x) = \frac{1}{\beta} \longrightarrow \ln(f) = -\ln(\beta)$
    $= \beta^{-1}$
  - THEN $\frac{\partial}{\partial \beta}(\ln(f)) = -\frac{1}{\beta} = -\beta^{-1}$

    are $\left(\frac{\partial}{\partial \beta}(\ln(f))\right)^2 = +\beta^{-2} = \frac{1}{\beta^2}$

$$CRLB = \frac{+1}{n\,E\left[\left(\frac{\partial}{\partial \beta}\ln(f)\right)^2\right]} = \frac{+1}{n\left(\frac{1}{\beta^2}\right)} = \frac{+\beta^2}{n}$$

RECALL
- {IF $\hat{\theta}$ is UE of θ it can be shown variance of θ must satisfy the Cramér-Rao inequality:

$$VAR(\hat{\theta}) \geq \frac{+1}{n\,E\left[\left(\frac{\partial \ln(f(x))}{\partial \theta}\right)^2\right]} = \text{"CRLB"}$$
}

SINCE $\underline{CRLB = \frac{\beta^2}{n} > \frac{\beta^2}{n(n+2)}} = VAR(U)$

CONCLUDE: $\longrightarrow$ $\therefore$ Therefore Cramér-Rao inequality NOT satisfied.