MATH 350:H1, EXAM 1

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Do all 5 problems.

Show all your work and justify your answers

Good luck!

Problem number	Possible points	Points earned (out of 100):
1	20	19
2	20	20
3	20	20
4	20	19
5	20	20
Total points earned:		98

One to one $\longleftrightarrow$ $T(x)=T(y) \Longrightarrow x=y \quad \forall\, x,y \in V$

onto $\longleftrightarrow$ $\forall y \in W,\ \exists\, x \in V$ s.t. $T(x)=y$, $T: V \to W$

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(20) 1. Let V and W be vector spaces over a field $\mathbb{F}$ and let $T: V \to W$ be an invertible linear transformation. Prove that $T^{-1}: W \to V$ is a linear transformation.

- We want to show $\forall x,y \in W$ and $\forall c \in \mathbb{F}$ that $T^{-1}(cx+y) = cT^{-1}(x)+T^{-1}(y) \Rightarrow T^{-1}$ is Linear.
- By def of T being invertible, $\exists!\, T^{-1}$ (unique) Such that $T^{-1}T = I = TT^{-1}$; $T^{-1}: W \to V$
- We know $\forall\, x,y \in V$, $\exists!\, w, v \in W$ (since T is invertible it is One-to-one and onto) ~~so~~ such that $T(x)=v$, and $T(y)=w$.

  $\Rightarrow x = T^{-1}(v)$ and $y = T^{-1}(w)$

-1 Rather, should begin: Let $v, w \in W$ be arbitrary. Then $\exists!\, x, y \in V \ni T(x)=v,\ T(y)=w$.

- For any arbitrary $v, w \in W$ and $\forall c \in \mathbb{F}$
- We see that,

$$T^{-1}(cv+w) = T^{-1}[cT(x)+T(y)]$$
$$= \underbrace{T^{-1}[T}_{T^{-1}T=I}(cx+y)] \text{ (by linearity of } T)$$
$$= cx+y$$
$$= cT^{-1}(v)+T^{-1}(w), \text{ as desired.}$$

- Therefore by ~~thm~~ definition of linearity, we know since $\forall c \in \mathbb{F}$ and, any arbitrary $v, w \in W$ that $\{T^{-1}(cv+w) = cT^{-1}(v)+T^{-1}(w)\}$ then that T^{-1} is LINEAR transformation.

$L.C =$ linear combination

wrt | with respect to

2 20

(20) 2. Let V and W be finite-dimensional vector spaces over a field $\mathbb{F}$ having ordered bases β and γ, respectively. Let $T : V \to W$ be a linear transformation. Prove that for each $u \in V$, we have

$$\left\{ [T(u)]_\gamma = [T]^\gamma_\beta [u]_\beta . \right\}$$

Thm 2.14

$\forall u \in V$ since β is an ordered basis for V we have that u can be represented as a UNIQUE L.C. of vectors in $\beta = \{\mu_1, \dots, \mu_n\}$ so $u = \sum_{i=1}^n a_i \mu_i$ for unique scalars $a_1, \dots, a_n \in \mathbb{F}$. So coordinate vector representation of u wrt/ β $- [u]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Let $A = [T]^\gamma_\beta = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & & A_{mn} \end{bmatrix}_{(m \times n)}$

$T(u) = \sum_{j=1}^n a_j T(\mu_j)$ where $T(\mu_i) \in W$ so
(by Linearity of T) for $\gamma = \{\omega_1, \dots, \omega_m\}$, $T(u_i) = \sum_{j=1}^m A_{ij} \omega_i$

Then $T(u) = \sum_{j=1}^n a_j \left(\sum_{i=1}^m A_{ij} \omega_i \right) = \sum_{j=1}^n \left(\sum_{i=1}^m (a_j A_{ij}) \right) \omega_i = \sum_{i=1}^m \left(\sum_{j=1}^n a_j A_{ij} \right) \omega_i$

Thus $[T(u)]_\gamma = \begin{bmatrix} \sum_{j=1}^n a_j A_{1j} \\ \vdots \\ \sum_{j=1}^n a_j A_{mj} \end{bmatrix}$

Which is precisely $[T]^\gamma_\beta [u]_\beta = \underset{(m\times n)}{A} \underset{(n \times 1)}{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ & \ddots & \\ A_{m1} & & A_{mn} \end{bmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{bmatrix} \sum a_j A_{1j} \\ \vdots \\ \sum a_j A_{mj} \end{bmatrix}_{(m \times 1)}$ dot product

Therefore, $[T(u)]_\gamma = [T]^\gamma_\beta [u]_\beta$.

$\blacksquare$

WTS: $(T(x) = T(y) \Rightarrow x = y)\ \forall x, y$

$\beta = \{v_1, \dots v_n\}$ basis for V

L. I = linearly independent

20 $\gamma = \{T(v_1), \dots, T(v_n)\} \subseteq W$, L. I. subset.

(20) 3. Let V and W be vector spaces over a field $\mathbb{F}$ and let $T : V \to W$ be a linear transformation. Let $\{v_1, \dots, v_n\}$ be a basis of V and suppose that the vectors $T(v_1), \dots, T(v_n)$ are linearly independent in W. Prove that T is one-to-one. (If you quote a theorem, you don't have to prove that theorem.)

- $\gamma := \{T(v_1), \dots, T(v_n)\} \subseteq W$ is a L. I. subset of W iff, $\dim(W) := m \geq n$ where n is the number of elements of γ. (by thm)

True, but not necessary to say this.

- For ARBITRARY $x, y \in V$, since $\beta = \{v_1, \dots, v_n\}$ is a basis for V by thm we can represent x and y as a UNIQUE L.C. of vectors in β.

So, for unique scalars $a_1, \dots, a_n \in \mathbb{F}$ and $b_1, \dots, b_n \in \mathbb{F}$ ∋ we have,

$\left\{ \begin{array}{l} x = \sum_{i=1}^{n} a_i v_i \\ y = \sum_{i=1}^{n} b_i v_i \end{array} \right\}$. IF $T(x) = T(y)$, THEN:

$\Rightarrow T(\sum_{i=1}^{n} a_i v_i) = T(\sum_{i=1}^{n} b_i v_i)$

$\Rightarrow \sum_{i=1}^{n} a_i T(v_i) = \sum_{i=1}^{n} b_i (T v_i)$ (by linearity of T).

$\Rightarrow \sum_{i=1}^{n} (a_i - b_i) T(v_i) = \underline{0}$ (by linearity of T).

- Since γ is L. I. we know the only representation of $\underline{0}$ as a L.C. of the vectors in γ is the trivial representation.

Thus, $\{(a_1 - b_1) = (a_2 - b_2) = \dots = (a_n - b_n) = 0\} \Longrightarrow \begin{pmatrix} a_1 = b_1 \\ a_2 = b_2 \\ \vdots \\ a_n = b_n \end{pmatrix}$.

- Thus since we said the "coefficients" of the L.C. of β vectors representation of x and y are UNIQUE this forces $x = y$ ∎

- Since for any arbitrary $x, y \in V$, $T(x) = T(y) \Rightarrow x = y$ by definition, T is "one-to-one".

Or, could invoke the thm. that $N(T) = \{\underline{0}_V\} \Longrightarrow T$ is one-to-one.

$\gamma = ((1,0,0), (0,1,0), (0,0,1))$ with e_1, e_2, e_3

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Let $f(x)$ be arbitrary $\xrightarrow{\text{s.t}}$ $f(x) = a + bx + cx^2 \in P_2(\mathbb{R})$

(20) 4. Define the linear transformation $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ by $f'(x) = b + 2cx \mid f'' = 2c$

$$T(f(x)) = \begin{pmatrix} f'(0) + f''(0) \\ f'(1) \\ f(0) \end{pmatrix} = \begin{pmatrix} b + 2c \\ b + 2c \\ a \end{pmatrix} \in \mathbb{R}^3 \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$$

($f'(x)$ and $f''(x)$ are the first two derivatives of $f(x)$.) You *don't* have to prove that T is linear.

(a) Determine the matrix $[T]_\beta^\gamma$ of T, where β is the standard ordered basis $\{1, x, x^2\}$ of $P_2(\mathbb{R})$ and where γ is the standard ordered basis of $\mathbb{R}^3$.

$\begin{pmatrix} a=1 \\ b=0 \\ c=0 \end{pmatrix}$ — $T(1) = 0e_1 + 0e_2 + 1e_3$

$\begin{pmatrix} a=0 \\ b=1 \\ c=0 \end{pmatrix}$ — $T(x) = 1e_1 + 1e_2 + 0e_3$

$\begin{pmatrix} a=0 \\ b=0 \\ c=1 \end{pmatrix}$ — $T(x^2) = 2e_1 + 2e_2 + 0e_3$

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} =: A$$

(b) Find a basis for the range $R(T)$ of T, and justify that your set is in fact a basis. Use this to determine the rank of T. Series of elementary row operations to RREF $(A) \to (R)$.

$$r_1 \leftrightarrow r_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_3 + (-r_2) \to r_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} =: R$$

$x_1 \quad x_2 \quad x_3$

~~Let L_A~~ for A defined above. $L_A(\underline{x}) = T(\underline{x})$

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T doesn't have domain $\mathbb{R}^3$

• Basis for $R(T) = R(\cancel{L_A}) = \text{col}(A) \cancel{\neq} \cancel{\text{col}(R)}$.

The pivot columns of original matrix A form basis for $R(T)$.

$\beta = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $R(T)$. The rank$(T) = 2 = \#\dim(R(T))$.

• A BASIS $\boxed{(i)\ \beta \text{ is L.I}}$ since $a_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for $a_1, a_2 \in \mathbb{R}$ iff $a_1 = a_2 = 0 \Rightarrow \beta$ is L.I. $\boxed{(ii)\ \text{Span}(\beta) = R(T)}$ since $R(T) \subseteq \mathbb{R}^3$, $R(T)$ is a subspace of codomain $\mathbb{R}^3$ that contains β by thm $R(T) \supseteq \text{Span}(\beta)$. For arbitrary $\begin{pmatrix} b+2c \\ b+2c \\ a \end{pmatrix} \in R(T) \Rightarrow a \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (b+2c) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \text{Span}(\beta)$. So $R(T) \subseteq \text{span}(\beta)$.

# of elements in basis for $R(T)$.

(c) Use the **Dimension Theorem** to determine the nullity of T. $\text{rank}(T) = \text{rank}(L_A)$ $= \#$ pivot cols of $R = \text{Rank}(R) = \text{rank}(A) = 2$. The domain is $P_2(\mathbb{R})$ and $\dim(P_2(\mathbb{R})) = 3$. By dim thm, for linear transform $T: V \to W$ that $\{\text{nullity}(T) + \text{rank}(T) = \dim(V)\} \Rightarrow \{\text{nullity}(T) = 3 - 2\} = 1$

V, W are finite dimensional vector spaces.

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(20) 5. Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 - 2a_2 \\ a_1 - a_2 \end{pmatrix}.$$

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = 1e_1 + 1e_2$

$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix} = -2e_1 - 1e_2$

Let $\beta = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ be the standard ordered basis of $\mathbb{R}^2$, and also consider the ordered basis $\beta' = \left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ of $\mathbb{R}^2$.

(a) Find the matrix $[T]_\beta$ of T with respect to the ordered basis β. ($[T]_\beta$ can also be written as $[T]_\beta^\beta$.) Justify your answer.

$[T]_\beta = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

• Let $v \in V$ be arbitrary. Then since β is a basis for $\mathbb{R}^2 \Rightarrow v$ can be represented as a UNIQUE L.C. of β vectors. So $v = a_1\vec{e}_1 + a_2\vec{e}_2$ for unique scalars $a_1, a_2 \in \mathbb{F}$.

Then $[v]_\beta = \begin{pmatrix}a_1\\a_2\end{pmatrix}$. By THM $[T]_\beta[v]_\beta = \begin{bmatrix}1 & -2\\1 & -1\end{bmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}a_1 - 2a_2\\a_1 - a_2\end{pmatrix} = [T(v)]_\beta$.

alternate justification

(b) Find the matrix $[T]_\beta^{\beta'}$ of T with respect to the ordered bases β and β'. Justify your answer.

↳ β coordinates into β' coordinates

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = 0\begin{pmatrix}0\\1\end{pmatrix} + 1\begin{pmatrix}1\\1\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix} = a\begin{pmatrix}0\\1\end{pmatrix} + b\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}b\\a+b\end{pmatrix}$

$b = -2 \Rightarrow a + b = -1$

$a = 1$ $\quad a = -1 + 2 = 1$

$[T]_\beta^{\beta'} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$

By THM

$[T]_\beta^{\beta'}[v]_\beta = [\cancel{T(v)}]_\beta^{\beta'}$

$[T(v)]_{\beta'}$ You meant

So, $\underbrace{\begin{bmatrix}0 & 1\\1 & -2\end{bmatrix}}_{[T]_\beta^{\beta'}} \underbrace{\begin{pmatrix}a_1\\a_2\end{pmatrix}}_{[v]_\beta} = \underbrace{\begin{pmatrix}a_2\\a_1 - 2a_2\end{pmatrix}}_{[T(v)]_{\beta'}}$

alternate justification

RMK:

L.I — liner indepen

L.D. — liner dependn

L.C — lin combination

IH — inductive hyp.

Statement: Let V be an arbitrary vector space. Let G be a finite generating set for V with exactly n vectors.

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(20) 1. Formulate and then prove the Replacement Theorem (for a vector space V over a field $\mathbb{F}$).

• Let L be an arbitrary L.I subset of V containing exactly m vectors. THEN $\Rightarrow$ $m \leq n$ (i), AND: $\exists H \subseteq G$ with $n-m$ vectors s.t $L \cup H$ generates V. : (ii)

• Let $G := \{u_1, \ldots, u_n\}$ for arbitrary fixed value n. Let us proceed by induction on m. (BASE CASE):

IF $m=0$ then $L = \emptyset$. Want $H \subseteq G$ with $n-0=n$ vectors. Simply take $H=G$. Then $L \cup H = \emptyset \cup G = G$ generates V by hypothesis. Assume the statement of the replacement is true for arbitrary $m \geq 0$. Let us show it holds for $m+1$.

• Let $L = \{v_1, \ldots, v_{m+1}\}$ be arbitrary L.I subset of V. $L' = \{v_1, \ldots, v_m\} \subseteq L$ is L.I. as the subset of any linearly independent set is again L.I by thm. By I.H., $\exists H' = \underbrace{\{u_1, \ldots, u_{n-m}\}}_{\text{(in no particular order)}} \subseteq G$ with $n-m$ vectors s.t $L' \cup H'$ generates V, that is $\text{span}(L' \cup H') = V$.

• Since $v_{m+1} \in V$, v_{m+1} can be represented as a L.C. of vectors in $L' \cup H'$. For some scalars $a_1, \ldots, a_m, b_1, \ldots, b_{n-m} \in \mathbb{F}$; we have that: (1) $[v_{m+1} = (a_1 \underline{v_1} + \ldots + a_m \underline{v_m}) + (b_1 \underline{u_1} + \ldots + b_{n-m} \underline{u_{n-m}})]$

(i) WTS $(m+1 \leq n) \Leftrightarrow (m < n) \Leftrightarrow (n-m > 0)$. By I.H we know $(m \leq n) \Leftrightarrow (n-m \geq 0)$. Suppose for the sake of contradiction that $n-m=0$. Then v_{m+1} is a L.C. of elements in L' by EQ (1). That is, $v_{m+1} \in \text{Span}(L')$.

By thm $v_{m+1} \in \text{span}(L') \iff L' \cup \{v_{m+1}\} = L$ is L.D.
This contradicts our assumption that L is L.I. Hence,
$(n-m > 0) \iff \underline{m+1 \leq n}$, as desired.

(ii) Since $v_{m+1} \neq 0$ $\exists\, b_i \in \mathbb{F}$ $(1 \leq i \leq n-m)$ s.t. $b_i \neq 0$.
WLOG let $b_1 \neq 0$. Then, by field axiom $(b_1)^{-1} \in \mathbb{F}$ is the multiplicative inverse.

(2) $\mu_1 = (b_1)^{-1} \cdot [v_{m+1} - a_1 v_1 - \dots - a_m v_m - b_2\mu_2 - \dots - b_{n-m}\mu_{n-m}]$

- Since μ_1 is written in EQ (2) as L.C. of vectors in $L \cup H$ where $H := \{\mu_2, \dots, \mu_{n-m}\} \subseteq H'$, $\underline{\mu_1 \in \text{Span}(L \cup H)}$.
$L' \subseteq L \subseteq L \cup H$ and $H \subseteq L \cup H$ so $L' \subseteq \text{span}(L \cup H)$ and $\underline{H \subseteq \text{span}(L \cup H)}$. Since $H' = \{\mu_1\} \cup H \Rightarrow H' \subseteq \text{span}(L \cup H)$.
Thus, $\underline{L' \cup H' \subseteq \text{Span}(L \cup H)}$ (3). $\underline{\text{WTS}}$: $\text{Span}(L \cup H) = V$.
- Since $L \cup H \subseteq V$, $\text{Span}(L \cup H)$ is a subspace of V containing $L \cup H$ that is $\underline{L \cup H \subseteq \text{span}(L \cup H) \subseteq V}$ (4) (by thm).
- Since $\text{Span}(L \cup H)$ is a subspace of V containing $L' \cup H'$ by (3) it follows $\text{Span}(L \cup H) \supseteq \text{span}(L' \cup H')$ by thm. By I.H.: $V = \text{Span}(L' \cup H')$ so $\underline{\text{Span}(L \cup H) \supseteq V}$ (5). EQ (4) & (5) together, implies $V = \text{span}(L \cup H)$, and $H \subseteq H' \subseteq G$ with $(n-m) - 1 = n - (m+1)$ vectors. Thus completing the induction on m.

$\blacksquare$

Lemma
★ CCP: Let $A \sim B$ denote "row equivalence" which is an equivalence relation, for $A, B \in M_{m\times n}(\mathbb{F})$.

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Let $A := (\underline{a_1} \cdots \underline{a_n})$ and $B := (\underline{b_1} \cdots \underline{b_n})$

(20) 2. Formulate and then prove a theorem that asserts that the reduced row-echelon form for a given $m \times n$ matrix over a field $\mathbb{F}$ is unique. You don't have to write every detail of the proof as long as your reasoning is clear. In the course of your proof, state and then sketch a proof of the Column Correspondence Principle. (Do not spend time *defining* "RREF" or "row-equivalence of matrices" or "pivot columns." *Use* such concepts in your proof as appropriate.)

"CCP" ≈

and $c_1, \dots, c_n \in \mathbb{F}$. THEN, $\mathrm{Null}(A) = \mathrm{Null}(B)$.

That is $\left(\sum_{j=1}^{n} c_j \underline{a_j} = \underline{0}\right) \iff \left(\sum_{j=1}^{n} c_j \underline{b_j} = \underline{0}\right)$.

(pf.) Since $A \sim B$ we have that $\{A\underline{c} = \underline{0}\} \iff \{B\underline{c} = \underline{0}\}$ for $\underline{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{F}$. That is both homogeras matrix EQ have the same solution set as a consequence of row EQUIVALENCE and $\mathrm{Null}(A) = \{\underline{x} \in \mathbb{F}^n \mid A\underline{x} = \underline{0}\}$ similarly $\mathrm{Null}(B) = \{\underline{x} \in \mathbb{F}^n \mid B\underline{x} = \underline{0}\}$ So $\mathrm{Null}(A) = \mathrm{Null}(B)$. The column correspondence Property gives us that for the special case when $B = R =: \mathrm{rref}(A)$, $\left(a_j = \sum_{i \neq j} c_i \underline{a_i}\right) \iff \left(\underline{r_j} = \sum_{i \neq j} c_i \underline{r_i}\right)$. That is the linear dependence and independence relations among columns of A correspond exactly to that of the columns of R.

• **THM**: RREF of a matrix $A \in M_{m\times n}(\mathbb{F})$ is unique. Let R and S be in rref and both are row equivalent to A. THEN we have that $R = S$.

(pf.) By induction on j, the columns of A and R. $(1 \le j \le n)$ First observe $R \sim A$ and $S \sim A$. Row Equiv is an equivalence relation so by symmetry $A \sim S$. Then by transitivity since $R \sim A$ and $A \sim S \Rightarrow R \sim S$. By CCP, $\mathrm{Null}(R) = \mathrm{Null}(S)$.

Then $\sum_{j=1}^{n} c_j \underline{r_j} = \underline{0} \iff \sum_{j=1}^{n} c_j \underline{s_j} = \underline{0}$.

(BASE CASE) WTS $\underline{r_1 = s_1}$. There is a dichotomy either r_1 (and by CCP s_1) is either L.D or L.I. If $\underline{r_1} = \{\underline{0}\}$, since $\underline{0}$ is trivially L.D, and if r_1 were any nonzero vector $\underline{r_1} \neq \underline{0}$ then $\{\underline{r_1}\}$ would be L.I. by thm. In the latter case, since R (and S) are in RREF $\underline{r_1}$ (and $\underline{s_1}$) would be a pivot column. Thus either both $\underline{r_1}$ and $\underline{s_1}$ are $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Assume $\underline{r_1} = \underline{s_1}, \underline{r_2} = \underline{s_2}, \ldots, \underline{r_{j-1}} = \underline{s_{j-1}}$.

(WTS: $r_j = s_j$). Either r_j is L.I or L.D on previous columns.

CASE 1: $\underline{r_j}$ is L.D on previous columns $r_1, \ldots, r_{j-1} \iff \underline{r_j}$ is L.D on previous pivot columns among $r_1, \ldots, r_{j-1}$. Call these $r_{i_1}, r_{i_2}, \ldots, r_{i_k}$ $(0 \le k \le j-1)$.

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k\text{th position}$$

Then $\left(\underline{r_j} = \sum_{l=1}^{k} c_{i_l} \underline{r_{i_l}}\right) \implies$ by CCP $\iff$ by CCP $\left(\underline{s_j} = \sum_{l=1}^{k} c_{i_l} \underline{s_{i_l}}\right)$ for same expansion coeffizents $c_{i_1}, \ldots, c_{i_k}$.

By I.H $r_{i_1} = s_{i_1}, \ldots, r_{i_k} = s_{i_k}$. Thus in this case

$$\underline{r_j} = \underline{s_j} = \begin{pmatrix} c_{i_1} \\ c_{i_2} \\ \vdots \\ c_{i_k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

CASE 2: Otherwise r_j is L.I and by CCP $\underline{s_j}$ is L.I. Since R and S are in RREF ~~Since~~ we have that $\underline{r_j} = \underline{s_j} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow (k+1)^{th}$ position is the $(k+1)^{th}$ pivot column given k pivot columns among $r_1, \ldots, r_{j-1}$.

In either case $\underline{s_j} = \underline{r_j}$ completing induction and proving $R = S$ and the uniqueness of RREF of A. ▨

Again, could have saved lots of time with a lot fewer details as in my example solution

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$W = \text{span}\{z, T(z), T^2(z), \ldots\}$

T cyclic subspace gen by z

smallest T invariant subspace containing z

(20) 4. Let $V = P_2(\mathbb{R})$. Define the linear operator $T : V \to V$ by $T(f(x)) = f'(x) - f(1)$ for $f(x) \in V$. Let W be the T-cyclic subspace of V generated by the polynomial $1 - x^2 \in V$.

z

(a) Find an ordered basis for W.

$z = (1-x^2)$

$T(z) = (-2x) - 0 = -2x$

$T^2(z) = T(-2x) = -2 + 2 = 0$ L.D

$T^2(z) = 0 = 0z + 0T(z)$

$W = \{z, T(z)\}$, $\dim(W) = k = 2$. Then by thm, ordered basis for W is, $\{z, T(z)\}$ or $\beta = \{(1-x^2), (-2x)\}$.

By thm, since $(\underset{a_0}{0}\, z + \underset{a_{k-1}}{\underset{\shortparallel}{\underset{a_1}{0}}}\, T(z) + \underbrace{T^2(z)}_{\shortparallel\, 0} = 0)$,

then the char poly of T_W call this $g(t)$ is,

$$\boxed{g(t) := 0 + 0t + t^2 = \boxed{t^2.}}$$

(b) Find the characteristic polynomial of the restriction T_W of T to W. (You may use any method.)

w/o using determinants ...

$$\left\{\begin{array}{l} a_0 z + a_1 T(z) + \cdots + a_{k-1} T^{k-1}(z) + T^k(z) = 0 \\ \Rightarrow \quad f(t) = a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k \end{array}\right\}$$