[481] (8.23, 6.7, 6.11, 8.45, 8.48) Khyaati Sigicherla.
[HW#3]

[8.23] Use THM 8.11 to show that for random sample of size n from a normal pop. w/ variance σ^2, the sampling dist of S^2 has mean σ^2 && variance $\left(\frac{2\sigma^4}{n-1}\right)$

(THM 8.11) IF $\bar{X}$, S^2 for RAND SAMPLE size n ~ Norm (μ, σ) pop.

$\bar{X}$ → mean, S^2 → variance

THEN →

1. $\bar{X}$, S^2 are INDEPENDENT.
2. r.v $\left(\frac{(n-1)S^2}{\sigma^2}\right)$ has ~ χ^2 dist $r = n-1$ w/ $(n-1)$ df.

- $E\left[\frac{(n-1)S^2}{\sigma^2}\right] = n-1$

and

- $Var\left[\frac{(n-1)S^2}{\sigma^2}\right] = 2(n-1)$

We know for a χ^2 distribution that

$\mu = r$

$\sigma^2 = 2r$

- Since $\left(\frac{(n-1)S^2}{\sigma^2}\right)$ follows a χ^2 dist $(r = n-1)$ df;

$$E(S^2) = \frac{\sigma^2}{(n-1)}(n-1) = \sigma^2$$

$$Var(S^2) = \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{(n-1)}$$

(6.7) Q: USE "integration by parts" to show that:

recursive definition of gamma fn.

$$\circledast \left\{ \Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1) \text{ for } \alpha > 1 \right\}$$

"GAMMA Fn":

$$\boxed{\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\,dx \quad \text{for } \alpha > 0.}$$

$uv - \int v\,du$

$$u = x^{\alpha-1} \qquad dv = e^{-x}$$
$$du = (\alpha-1)\,x^{\alpha-2}\,dx \qquad v = -e^{-x}$$

$$\Gamma(\alpha) = \underbrace{-x^{\alpha-1}e^{-x}\Big|_{0=x}^{\infty}}_{(-0 + \infty e^{-\infty}) = 0.} + (\alpha-1)\underbrace{\left(\int_0^{\infty} x^{\alpha-2} e^{-x}\,dx\right)}_{(\Gamma(\alpha-1))}$$

REMARK:
→ e decays faster to 0 than polynomial growth of $x^{\alpha-1}$.

Thus we find:

$$\Rightarrow \boxed{\Gamma(\alpha) = \alpha-1\;\Gamma(\alpha-1)} \circledast$$

as desired.

QED

(6.11) Q SHOW that a "Gamma dist" where $\alpha > 1$ has a relative maximum

at $x = \beta(\alpha - 1)$.

Q What happens when $0 < \alpha < 1$ and when $\alpha = 1$.

(recall)

- $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$ for $\alpha > 0$ | GAMMA FUNCTION.
- X has gamma dist, and it is referred to as a gamma rv. $\overset{\text{i.f.f}}{\longleftrightarrow}$ pdf is given by:

$$g(x; \alpha, \beta) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

WHERE $\alpha > 0$ $\beta > 0$.

- Let $X > 0$. Then, pdf is nonzero.
- Let us now find relative maxima for the gamma dist by taking $\left\{ \frac{\partial}{\partial x} g(x; \alpha, \beta) \overset{\text{SET}}{=} 0 \right\}$

(...) Cont.

for $x > 0$

$$\Rightarrow \frac{\partial}{\partial x} g(x;\alpha,\beta) = \frac{\partial}{\partial x}\left(\frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\right) = 0$$

$$k := \frac{1}{\beta^{\alpha}\Gamma(\alpha)}$$ which is a CONSTANT in x.

$$\Rightarrow k\left[x^{\alpha-1}\left(-\frac{1}{\beta} e^{-x/\beta}\right) + e^{-x/\beta}\left((\alpha-1)x^{\alpha-2}\right)\right] = 0$$

$$\Rightarrow \cancel{k}\left[\underbrace{\left(x^{\alpha-2}\right)}_{=:f_1}\underbrace{\left(e^{-x/\beta}\right)}_{=:f_2}\underbrace{\left(-\frac{x}{\beta}+\alpha-1\right)}_{=:f_3}\right] = 0$$

- $\begin{pmatrix}\alpha > 1\\ \beta > 0\end{pmatrix}$; $k = \frac{1}{\beta^{\alpha}\Gamma(\alpha)}$ which is nonzero so we can divide both sides by k

$\Rightarrow$ we recovered that the product of 3 factors is zero.

$\Rightarrow$ • That is, as f_i defined $\prod_{i=1}^{3} f_i = (f_1)(f_2)(f_3) = 0$.

$\Rightarrow$ • This is true i.f.f. atleast one of the three factors is zero.

- for $x > 0$ and $\alpha > 1$ and $\beta > 0$,

$\rightarrow$ the exponential function $e^{-x/\beta} = f_2$ can never be 0.

$\rightarrow$ the polynomial fn $x^{\alpha-2} = f_1$ can't be 0.

- Setting $f_3 = 0 \Longrightarrow \frac{x}{\beta} = \alpha - 1 \Rightarrow \{x = \beta(\alpha-1)\}^{\star}$

$\therefore$ Therefore the Gamma dist. achieves a relative max when $x = \beta(\alpha-1)$. as desired.

$(x>0,\ \beta>0)$

- Next let us consider the CASE when $(0<\alpha<1)$.

$c_1>0$

$$\Rightarrow x^{-c_1} e^{-x/\beta}\left(-\frac{x}{\beta} - c_2\right)$$

$c_2>0$

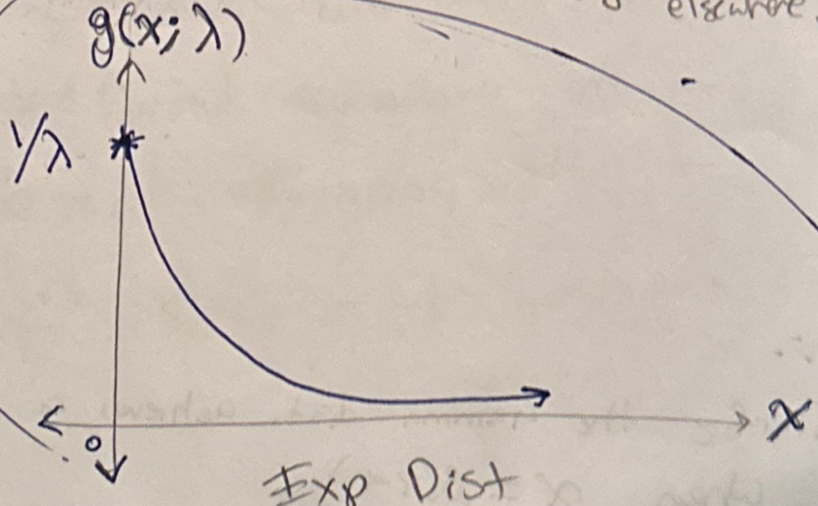
- Observe, as $x \to \infty$, the dominant term $e^{-x/\beta} \to 0$
  That is the behavior in the limit, of the expression derived previously.
- Primarily, notice that the gamma fn: $\frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$ (for $x>0$)
  tends $\to \infty$ as $x \to 0$ for the range of values $0<\alpha<1$.

limit analyisis. $\rightarrow$ $(e^{-0/\beta})$ factor becomes 1, and polynomial growth is unbounded. $\to \infty$

- Suppose instead we consider the CASE $\alpha = 1$.
  - Then the gamma distribution is given by the exponential distribution (which belongs to the gamma family).
  - That is, $\text{gamma}(\alpha=1, \beta) = \text{Exp}(\lambda=\beta)$

pdf of $X \sim$ exp dist. $g(x;\lambda) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda} & \text{for } x>0 \\ 0 & \text{elsewhere.} \end{cases}$

- Observe, gamma function when $\alpha=1$ achieves an absolute max* at $x=0$.



Exp Dist

<SAMPLING DIST>

(8.45) verify the results of EX 8.4

that is, the sampling distributions

of Y_1, Y_n, and $\tilde{X}$ shown there

for random samples from an

exp pop.

ex 8.4 rand samples — IID — size n

from exp pop w/ param θ

VERIFY!

- samply dist Y_1, Y_n are given by

$$\rightarrow g_1(y_1) = \begin{cases} \frac{n}{\theta} \cdot e^{-ny_1/\theta} & \text{for } y_1 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$\rightarrow g_n(y_n) = \begin{cases} \frac{n}{\theta} e^{-y_n/\theta} (1 - e^{-y_n/\theta})^{n-1} & \text{for } y_n > 0 \\ 0 & \text{elsewhere} \end{cases}$$

and

- for RAND SAMPLE size $n = 2m+1$ . . .

the sampling dist of the median given by,

$$\rightarrow h(\tilde{x}) = \begin{cases} \frac{(2m+1)!}{m!\,m!\,\theta} * e^{-\tilde{x}(m+1)/\theta} (1 - e^{-\tilde{x}/\theta})^m & \text{for } \tilde{x} > 0. \\ 0, & \text{elsewhere.} \end{cases}$$

(THM 8.16)

- For rand sample size n from infinite pop.
- PDF of rth ORDER STAT Y_r is given by:

$$g_r(y_r) = \left(\frac{n!}{(r-1)!(n-r)!}\right) \left[\int_{-\infty}^{y_r} f(x)\,dx\right]^{r-1} f(y_r) \left[\int_{y_r}^{\infty} f(x)\,dx\right]^{n-r}$$

for $-\infty < y_r < \infty$

PDF for EXP $f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{for } x>0 \\ 0 & \text{elsewhere.} \end{cases}$

exp pdf $f(x;\theta)$

★ CASE 1: Y_1

$$g_1(y_1) = \underbrace{\left(\frac{n!}{\cancel{0!}\,(n-1)!}\right)}_{n} \cdot \underbrace{\left(\int_0^{y_1} f(x)\,dx\right)^{0}}_{=1} \cdot \frac{1}{\theta} e^{-y_1/\theta} \left(\int_{y_1}^{\infty} \frac{1}{\theta} e^{-x/\theta}\,dx\right)^{n-1}$$

(for $0<y_1<\infty$)

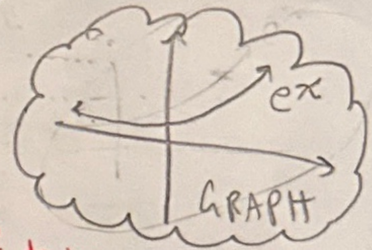
$$\Rightarrow \left(\frac{1}{\theta}\cdot\frac{-\theta}{1}\right) e^{-x/\theta}\Big|_{x=y_1}^{x=\infty}$$

$$\Rightarrow -1\left[\cancel{e^{-\infty}}_0 - e^{-y_1/\theta}\right]$$

$$\Rightarrow +e^{-y_1/\theta}$$

$$\Rightarrow g_1(y_1) = \frac{n}{\theta} e^{-y_1/\theta}\left(+e^{-y_1/\theta}\right)^{n-1}$$

$$* \; g_1(y_1) = \begin{cases} \frac{n}{\theta} e^{-n y_1/\theta} & \text{for } y_1>0 \quad 0<y_1<\infty \\ 0 & \text{elsewhere.} \end{cases}$$



★ CASE 2: Y_n

$0<y_n<\infty$, $y_n>0$ for pdf to be nonzero.

$$g_n(y_n) = \underbrace{\left(\frac{n!}{(n-1)!\,\cancel{0!}}\right)}_{n} \cdot \underbrace{\left(\int_0^{y_n} \frac{1}{\theta} e^{-x/\theta}\,dx\right)^{n-1}}_{n} \cdot \left(\frac{1}{\theta} e^{-y_n/\theta}\right) \underbrace{\left(\int_{y_n}^{\infty} f(x)\,dx\right)^{0}}_{=1}$$

$$\left(-1\, e^{-x/\theta}\Big|_{x=0}^{x=y_n} \quad -1\left(e^{-y_n/\theta} - \cancel{e^{0}}_1\right)\right)$$

$$* \; g_n(y_n) = \begin{cases} \frac{n}{\theta} e^{-y_n/\theta}\left(1-e^{-y_n/\theta}\right)^{n-1} & \text{for } y_n>0 \quad 0<y_n<\infty \\ 0 & \text{elsewhere} \end{cases}$$

★ CASE 3: size $(n=2m+1)$ from exp dist pop.
the sampling dist of MEDIAN:

* $\tilde{x}>0$ for PDF to be non zero.

$$h(\tilde{x})=\left(\frac{(2m+1)!}{\underbrace{((m+1)-1)!}_{m!}\,\underbrace{((2m+1)-(m+1))!}_{m!}}\right)\left(\int_0^{\tilde{x}}\frac{1}{\theta}e^{-x/\theta}\right)^m\frac{1}{\theta}e^{-\tilde{x}/\theta}\left(\int_{\tilde{x}}^{\infty}\frac{1}{\theta}e^{-x/\theta}dx\right)^m$$

* evaluate integrals.

$$=\left(-1\left(e^{-x/\theta}\right)\Big|_0^{\tilde{x}} \quad -1\left(e^{-\tilde{x}/\theta}-\cancel{e^0}^{\,1}\right) \quad 1-e^{-\tilde{x}/\theta}\right)\left(-1\left(e^{-x/\theta}\right)\Big|_{\tilde{x}}^{\infty} \quad -1\left(\cancel{e^{-\infty}}^{\,0}-e^{-\tilde{x}/\theta}\right) \quad +e^{-\tilde{x}/\theta}\right)$$

$$=\left(\frac{(2m+1)!}{m!\,m!}\right)\left(1-e^{-\tilde{x}/\theta}\right)^m\frac{1}{\theta}e^{-\tilde{x}/\theta}\left(e^{-\tilde{x}/\theta}\right)^m$$

$$\Rightarrow h(\tilde{x})=\begin{cases}\left(\frac{(2m+1)!}{m!\,m!\,\theta}\right)\left(e^{-(m+1)\tilde{x}/\theta}\right)\left(1-e^{-\tilde{x}/\theta}\right)^m & \text{for } \tilde{x}>0 \quad 0<\tilde{x}<\infty\\ 0 & \text{elsewhere}\end{cases}$$

μ σ^2

[8.48] find mean & variance of the sampling distribution of Y_1 for the random samples of size n from pop of excerices 8.46.

sampling Dist for continuous UNIF pop. $\alpha = 0, \beta = 1,$

• Y_1 – rv of interest. * $0 < y_1 < 1$

$$g_1(y_1) = \underbrace{\left(\frac{n!}{0!(n-1)!}\right)}_{= n} \underbrace{\left(\int_0^{y_1} 1\,dx\right)^0}_{=1} \cdot 1 \cdot \left(\int_{y_1}^{1} dx\right)^{n-1}$$

$$g_1(y_1) = \begin{cases} \boxed{n(1-y_1)^{n-1}} & \text{for } 0 < y_1 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

REMARK: uniform dist:

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{for } \alpha < x < \beta \\ 0 & \text{elsewhere} \end{cases}$$

WHEN $\alpha = 0$, $\beta = 1$ ...

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{elsewh} \end{cases}$$

$\mu = \frac{\alpha + \beta}{2} = \frac{1}{2}$

$\sigma^2 = \frac{1}{12}(\beta - \alpha)^2 = \frac{1}{12}$

$E(x) = \int_{-\infty}^{\infty} x f(x) dx$

$$E(y_1) = n \int_0^1 y_1 (1-y_1)^{n-1} dy_1$$

$u = 1 - y_1$
$du = -dy_1$

$$= +n \int_0^1 (1-u) u^{n-1} du$$

$u = u^{n-1}$
$du = (n-1)u^{n-2}$

$dv = 1 - u$
$v = u - \frac{u^2}{2}$

$$= +n \Big[\underbrace{u^n - \tfrac{1}{2} u^{n+1} \Big|_{u=0}^{u=1}}_{1} - (n-1) \int_0^1 u^{n-1} - \frac{u^n}{2}\, du \Big]$$

$$+n \left[\left(1 - \tfrac{1}{2}\right) - (n-1) \left[\tfrac{1}{n} u^n - \tfrac{1}{2(n+1)} u^{n+1} \Big|_{0}^{1} \right] \right]$$

$$+n\left[+\frac{1}{2} - (n-1)\left(\left(\frac{1}{n} - \frac{1}{2(n+1)}\right) - (0)\right)\right]$$

$$n\left(+\frac{1}{2} - (n-1)\frac{2(n+1) - n}{2(n+1)(n)}\right)$$

$$n\left(\frac{(+n)(n+1) - (n-1)(n+2)}{2(n+1)(n)}\right)$$

$$-n^2 + n + n^2 - n + 2$$

$$+2$$

$$n\left(\frac{2}{2(n+1)(n)}\right) = n\left(\frac{+1}{(n+1)(n)}\right)$$

$$\boxed{E(y_1) = \frac{+1}{(n+1)}}$$

$$E(y_1^2) = n\int_0^1 y_1^2(1-y_1)^{n-1}\,dy_1$$

$u = 1 - y_1$

$$= n\int_0^1 (1-u)^2 u^{n-1}\,du$$

$$= n\left[\underbrace{-\frac{1}{3}(1-u)^3 u^{n-1}\Big|_0^1}_{(+1/3)} + \frac{1}{3}\int_0^1 (1-u)^3 u^{n-1}\,du\right]$$

$u = u^{n-1}$	$dv = (1-u)^2\,du$
$du = n-1\, u^{n-2}\,du$	$v = -\frac{1}{3}(1-u)^3$

$a = u^{n-1}$

$da = (n-1)u^{n-2}\,du$

$db = (1-u)^3$

$b = -\frac{1}{4}(1-u)^4$

used mathematica

$$= n\left[\frac{u^{n+2}}{n+2} - \frac{2u^{n+1}}{(n+1)} + \frac{u^n}{n}\Big|_0^1\right]$$

...

$$\Rightarrow \quad \frac{2n}{n^3+3n^2+2n} = \frac{2}{(n^2+3n+2)} = \frac{2}{(n+2)(n+1)}$$

- Thus the second moment is,

$$\boxed{E(y_1^2) = \frac{2}{(n+1)(n+2)}}$$

- we found previously that the mean is,

$$\boxed{M = E(y_1) = \frac{1}{n+1}}$$

"mean"

- We know the variance is the second moment minus the square of the first moment...

that is, $\text{Var}(y_1) = E(y_1^2) - \left(E(y_1)\right)^2$

$$\Rightarrow \text{VAR}(y_1) = \frac{2}{(n+1)(n+2)} - \left(\frac{1}{(n+1)}\right)^2 = \frac{2(n+1)-(n+2)}{(n+1)^2(n+2)}$$

$$\boxed{\text{var}(y_1) = \frac{n}{(n+1)^2(n+2)}}$$

$2n+2-n-2$

$= n$