HW #11 [423]

§ 6.3	1*, 3*
§ 7.1	2*, 3*
§ 7.2	2*

[6.3.1] SUPPOSE u is harmonic fn ~~in~~ in disk $D = \{r < 2\}$

& $u = 3\sin(2\theta) + 1$ for $r = 2$.

Without finding solution answer following questions:

(a) MAX Value of u in $\overline{D}$

- $\overline{D} = D \cup \text{bdy}(D) = \{r \leq 2\}$
- By the "max pr." the max value of u is attained on the bdy(D) and nowhere inside (unless $u \equiv$ constant)
- Therefore the maximum value of u in $\overline{D}$ occurs on the boundary, $r = 2$, on which we have

$$\{u = 3\sin(2\theta) + 1\} \rightsquigarrow \text{find max of this function.}$$

- We know $\sin(2\theta)$ achieves a maximum value of 1 when $\theta = \frac{\pi}{4} + 2\pi n$ $n = 0, 1, 2 \ldots$ THEN $u_* = 3(1) + 1$.
- The maximum value of u in $\overline{D}$ is 4.

(b) CALCULATE the value of u at the origin...

- By the (MVP) "Mean Value Property" the value of u at the center of D — the origin — is the average of u on its circumfrence.

(...)

$$u(0) = \frac{\int u(2,\theta)\,ds}{\int ds} = \frac{\int_0^{2\pi} u(2,\theta)(2\,d\theta)}{\int_0^{2\pi} 2\,d\theta} \longrightarrow \left(2\theta \Big|_{\theta=0}^{2\pi} \quad 4\pi \right)$$

Circumference

$s = r\theta$
$ds = r\,d\theta$

$$= \frac{\cancel{2}^{\,1}}{2\cancel{4\pi}} \int_0^{2\pi} (3\sin(2\theta)+7)\,d\theta$$

$$= \frac{1}{2\pi}\left[-\frac{3}{2}\cos(2\theta) + \theta \Big|_{\theta=0}^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\underbrace{\cancel{-\frac{3}{2}(1-1)}}_{0} + (2\pi - 0)\right]$$

$$= \frac{2\pi}{2\pi} = 1$$

Thus, value of u at origin $\Rightarrow u(0) = 1$.

[6.1.3] SOLVE $\{u_{xx} + u_{yy} = 0\}$ on disk $\{r < a\}$

w/ BC: $u = \sin^3\theta$ on $r = a$

* (identity) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow 4\sin^3\theta = 3\sin\theta - \sin(3\theta)$

$\Rightarrow \sin^3(\theta) = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$

Let us proceed by using the method of SoV in polar coordinates. So look for solutions of the form $u = R(r)\Theta(\theta)$.

$$0 = u_{xx} + u_{yy} = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$$
$$= R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta''$$

* (divide by $R\Theta$ & multiply by r^2)

$$= \frac{r^2R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} = 0$$

$$\Rightarrow \frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = +\lambda$$

$\frac{\partial\lambda}{\partial\theta} = 0$ and $\frac{\partial\lambda}{R} = 0$

So λ is constant in r and θ.

So, $-\Theta'' = +\lambda\Theta$ and $r^2R'' + rR' = \lambda R$

* $\{\Theta'' + \lambda\Theta = 0\}$ and $\{r^2R'' + rR' - \lambda R = 0\}$

PAIR of ODES!

with BC...

$$\Theta(\theta + 2\pi) = \Theta(\theta) \text{ for } -\infty < \theta < \infty$$

For eigenvalue λ let $\lambda = \beta^2 > 0$. THEN,

$\Theta(\theta) = A\cos(\beta\theta) + B\sin(\beta\theta)$ and imposing the periodic BC...

$\Theta(\theta + 2\pi) = A\cos(\beta(\theta + 2\pi)) + B\sin(\beta(\theta + 2\pi)) = A\cos(\beta(\theta)) + B\sin(\beta(\theta + 2\pi))$.

- So $\boxed{\lambda = n^2}$ for $n = 1, 2, \dots$
- and $\boxed{\Theta(\theta) = A\cos(n\theta) + B\sin(n\theta)}$
- Also when $\lambda = 0$, $\Theta(\theta) = A$.
- We can solve the equation for R since it is Euler-type, with solutions of the form $R(r) = r^\alpha$, where $(\lambda = n^2)$.
- Substituting $\Rightarrow$ $r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^\alpha = 0$

$\Rightarrow \alpha(\alpha-1)\cancel{r^\alpha} + \alpha\cancel{r^\alpha} - n^2\cancel{r^\alpha} = 0$

$\Rightarrow \alpha(\alpha-1) + \alpha - n^2 = 0$

$\Rightarrow \alpha^2 \cancel{-\alpha} \cancel{+\alpha} - n^2 = 0$

$\Rightarrow \alpha = \pm n$ So $\boxed{R(r) = Cr^n + Dr^{-n}}$

$u = (Cr^n + Dr^{-n})(A\cos(n\theta) + B\sin(n\theta))$ by linear superposition.
for $n = 1, 2, 3, \dots$

WHEN $(n=0)$ a 2nd Linearly independent solution for $R(r)$ can be constructed as: $R(r) = \log(r)$
So the solution $\{u = C + D\log(r)\}$ is found.

- THESE solutions are "harmonic functions" on the disk D except at the origin $r = 0$ r^{-n} and $\log(r)$ are infinite! The requirement is that they are finite. So we reject and discard them. So we have,

$$\boxed{u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n\cos(n\theta) + B_n\sin(n\theta))}$$

• Now imposing the BC...

$$\frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)$$

• "matching terms" we can guess that $A_0 = 0$, $A_n = 0$ for all $n = 1, 2 \ldots$

and $aB_1 = \frac{3}{4}$, $a^3 B_3 = -\frac{1}{4}$ and $B_n = 0$ for all $n \in \mathbb{N} \setminus \{1, 3\}$.

THEREFORE our solution is...

$$\bigstar \left[u(r,\theta) = \frac{3r}{4a}\sin\theta - \frac{r^3}{4a^3}\sin(3\theta) \right].$$

SINCE ~~there~~ this is a valid solution to the problem it is necessarily the ONLY solution by uniqueness.

[7.1.2] PROVE UNIQUENESS UP TO CONSTANTS of the Neumann Problem using the Energy Method :

"The Neumann Problem in any domain D" :

$$\begin{cases} \Delta u = f(x) & \text{in } D \\ \frac{\partial u}{\partial n} = h(x) & \text{on } bdy(D) \end{cases}$$

where $\frac{\partial u}{\partial n} = \vec{n} \cdot \nabla \vec{u}$

s.t f & h satisfy the compatibility criterion: $\left\{ \iint_{bdy\, D} h\, dS = \iiint_D f\, d\vec{x} \right\}$

(...) Cont. next pg.

- Suppose we have two functions u_1, u_2 with the same boundary data that solve the Neuman problem.
- That is, u_1 s.t

$$*\begin{cases} \Delta u_1 = f & \text{in } D \\ \frac{\partial u_1}{\partial n} = h & \text{on bdy}(D) \end{cases}$$

and additionaly u_2 s.t

$$*\begin{cases} \Delta u_2 = f & \text{in } D \\ \frac{\partial u_2}{\partial n} = h & \text{on bdy}(D) \end{cases}$$

- THEN the difference of u_1 & u_2 — $u := u_1 - u_2$ is s.t

$$\begin{cases} \Delta u_1 - \Delta u_2 = f - f & \text{in } D \\ \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = h - h & \text{on bdy}D \end{cases} \Rightarrow \begin{cases} \Delta(u_1 - u_2) = 0 & \text{in } D \\ \frac{\partial}{\partial n}(u_1 - u_2) = 0 & \text{on bdy}D \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} \Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial n} = 0 & \text{on bdy}D \end{cases}}$$

Laplace's EQ.

- Take u and u as two arbitrary fn and substitute into (G1) Green's 1st identity.

* SINCE u is harmonic $\Rightarrow \Delta u = 0$ so the second term on RHS is 0.

(By G1) $\Rightarrow$

$$\underbrace{\iint_{\text{bdy}D} u \underbrace{\frac{\partial u}{\partial n}}_{=0} d\vec{S}}_{=0} = \iiint_D |\nabla u|^2 d\vec{x} + \underbrace{\iiint_D u \underbrace{\Delta u}_{=0} d\vec{x}}_{=0}$$

* SINCE $\frac{\partial u}{\partial n} = 0$ on bdy(D) LHS vanishes.

$\Rightarrow \left\{ 0 = \iiint_D |\nabla u|^2 d\vec{x} \right\}$ and by first vanishing thm we can conclude that $(|\nabla u|^2 \equiv 0)$ in D.

- A fn w/ a vanishing gradient is necessarily constant given D is connected. Therefore, $u(\vec{x}) \equiv C$ throughout D.

$\{|\nabla u|^2 \equiv 0\} \Rightarrow \{\nabla u = 0\} \Rightarrow u \equiv$ constant

THEREFORE, $\left[\begin{array}{l} u = C \\ u_1 - u_2 = C \\ (u_1 = u_2 + C) \end{array}\right]$. The Solution is UNIQUE up to a constant!

[7.1.3] PROVE the UNIQUENESS of the ROBIN problem $\left\{ \frac{\partial u}{\partial n} + a(\vec{x})\, u(\vec{x}) = h(\vec{x}) \right\}$

* GIVEN: $a(\vec{x}) > 0$ on bdy (D)

· Suppose we have two fn u_1, u_2 that have the same boundary data. THEN

u_1 s.t $\begin{cases} \Delta u_1 = f & \text{in } D \\ \frac{\partial u_1}{\partial n} + a u_1 = h & \text{on bdy } D \end{cases}$ and u_2 s.t $\begin{cases} \Delta u_2 = f & \text{in } D \\ \frac{\partial u_2}{\partial n} + a u_2 = h & \text{on bdy } D \end{cases}$

THEN $u = u_1 - u_2$ s.t

$$\left| \begin{array}{ll} \Delta u_1 - \Delta u_2 = \Delta(u_1 - u_2) = f - f = 0 & \text{in } D \\ \frac{\partial}{\partial n}(u_1 - u_2) + a(u_1 - u_2) = h - h = 0 & \text{on bdy } D \end{array} \right.$$

u s.t $\begin{cases} \Delta u = 0 & \text{in } D \\ \frac{\partial u}{\partial n} + a u = 0 & \text{on bdy } D \end{cases}$

↓ harmonic fn.

· Let us use (G1) and choose two fn to be u and u.

· THEN, $\left\{ \iint_{\text{bdy } D} u \underbrace{\frac{\partial u}{\partial n}}_{= -au} dS = \iiint_D |\nabla u|^2 d\vec{x} + \cancel{\iiint_D u \underbrace{\Delta u}_{=0} d\vec{x}} \right.$

$=0$

(...)

$$\Rightarrow -\iint_{\text{bdy } D} (a \cdot u^2)\, dS = \iiint_D |\nabla u|^2 d\vec{x} = 0$$

- SINCE $a > 0$ and u^2 is squared quantity integral on LHS is positive and due to $(-)$ sign LHS ≤ 0.
- INTEGRAND on RHS $|\nabla u|^2$ is a squared quantity so positive and the integral is then positive. Thus RHS ≥ 0.
- THEN both sides must equal 0!

$$\Rightarrow \begin{cases} \iint_{\text{bdy } D} a u^2\, dS = 0 \\ \iiint_D |\nabla u|^2 d\vec{x} = 0 \end{cases} \Rightarrow \text{By vanishing thm both integrands are } 0!$$

$$\Rightarrow \begin{cases} a u^2 = 0 \Rightarrow u^2 = 0 \Rightarrow u = 0 & \text{on bdy } D \\ |\nabla u|^2 = 0 \Rightarrow \nabla u = 0 \Rightarrow u = \underbrace{\text{constant}}_{=0} & \text{in } D \end{cases}$$

- By maximum pr for u to achieve its max and min somewhere on the bdy D the constant must be 0.

- THEREFORE $u \equiv 0$ everywhere in $\bar{D} = D \cup \text{bdy}(D)$.
- SO, $u_1 = u_2$; the ROBIN BC problem is thus UNIQUE!

[7.2.2] Let $\phi(x)$ be any C^2 function defined on all of the 3-D space that vanishes outside some sphere. Show that $\phi(\vec{0}) = -\iiint \frac{1}{|\vec{x}|} \Delta\phi(\vec{x}) \frac{d\vec{x}}{4\pi}$

integration is taken over the region where $\phi(\vec{x})$ is not zero.

- (G2) holds for any two fn u, v in D.

$$(G2)\quad \left[\iiint_D (u\Delta v - v\Delta u)\, d\vec{x} = \iint_{\text{bdy}D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \right.$$

- CHOOSE $u = \phi$ in D and $v = -\frac{1}{4\pi r}$ in D,

for $r = |\vec{x}| = \sqrt{x^2+y^2+z^2}$. LET $D :=$ all pts exterior to sphere w/ radius R centered at origin.

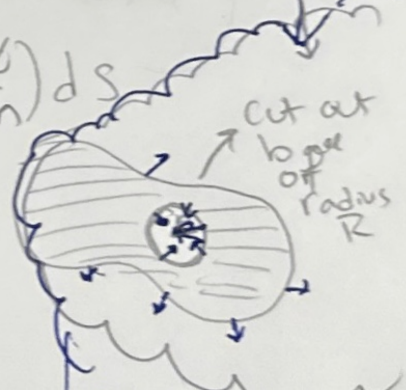
SINCE it vanishes outside of sphere we say $\Delta v = 0$ in D

$$(G2) \Rightarrow \iiint_D \phi \underbrace{\Delta v}_{=0} - \underbrace{v}_{\left(-\frac{1}{4\pi r}\right)} \Delta\phi \, d\vec{x} = \iint_{\text{bdy}(D)} \phi \frac{\partial v}{\partial n} - v\frac{\partial \phi}{\partial n}\, dS$$

$$\Rightarrow \iiint_D \frac{\Delta\phi}{4\pi r}\, d\vec{x} = \frac{1}{4\pi}\iint_{\text{bdy}(D)} -\phi(\vec{x})\frac{\partial}{\partial n}(r) + \left(\frac{1}{r}\right)\frac{\partial\phi}{\partial n}\Big)\, dS$$

$$= \frac{1}{4\pi}\iint_{r=R} \left[-\phi(\vec{x}) \underbrace{\left(\frac{1}{r^2}\right)}_{\frac{1}{R^2}} - \underbrace{\left(\frac{1}{r}\right)}_{\frac{1}{R}} \frac{\partial\phi}{\partial r}\right] dS$$

(...)



on sphere $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$

$$= -\frac{1}{4\pi R^2}\iint_{r=R} \phi(\vec{x})\,dS - \frac{R}{4\pi R^2}\iint_{r=R} \frac{\partial\phi}{\partial r}\,dS$$

$$= -\frac{\iint_{r=R} \phi(\vec{x})\,dS}{\iint_{r=R} dS} - R\,\frac{\iint_{r=R} \frac{\partial\phi}{\partial r}\,dS}{\iint_{r=R} dS}$$

Taking the limit of [redacted] as $R \to 0$ the 2^{nd} term $\to 0$, and first term which is the average of ϕ over sphere of radius R tends to $\phi(0)$ by mean value property.

SO,

$$\iiint_D \frac{1}{4\pi r}\,\Delta\phi(\vec{x})\,d\vec{x} = -\phi(\vec{0})$$

THEN $(r = |\vec{x}|)$

$$\phi(0) = -\iiint_D \frac{1}{|\vec{x}|}\,\Delta\phi(\vec{x})\,\frac{1}{4\pi}\,d\vec{x}.$$