HW6: (3.4) 3*, 5, 6
(4.1) 2, 3, 4*
(4.2) 1, 2*, 3*, 4
[redacted]

SOURCE FN

$S(x-y, t-s) := \frac{1}{\sqrt{4\pi k(t-s)}} e^{\frac{-|x-y|^2}{4(t-s)}}$

[3.4.3*] SOLVE (*) $\begin{cases} u_{tt} = c^2 u_{xx} + \cos(x) \quad f(x,t) \\ u(x,0) = \sin(x) = \phi(x) \\ u_t(x,0) = 1 + x = \psi(x) \end{cases}$

- "wave EQ w/ source" on the whole line w/ two IC
- THM 1: UMQ SOL...

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x)\,dx$$

$$+ \frac{1}{2c}\iint_{\Delta} f \, dA \longrightarrow \text{"CHARACTERISTIC TRIANGLE"} = \text{"}\Delta\text{"}$$

where $\frac{1}{2c}\iint_\Delta f = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$

- $\frac{1}{2c}\int_{x-ct}^{x+ct} 1+y\,dy = \frac{1}{2c}\left[y + \frac{y^2}{2}\Big|_{x-ct}^{x+ct}\right] = \frac{1}{2c}\left[\left((x+ct) + \frac{(x+ct)^2}{2}\right) - (x-ct) + \frac{(x-ct)^2}{2}\right]$

$$= \frac{1}{2c}\left[(x+ct) + \frac{(x^2+2cxt+c^2t^2)}{2} - (x-ct) - \frac{(x^2-2cxt+c^2t^2)}{2}\right]$$

$$= \frac{1}{2c}\left[2ct + \frac{4cxt}{2}\right]$$

$$= t + 2cxt = t(1+x)$$

sum 3 terms

- $\frac{1}{2c}\iint_{\Delta} f = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos(y)\,dy\,ds = \frac{1}{2c}\int_0^t \left[\sin(y)\Big|_{x-c(t-s)}^{x+c(t-s)}\right] ds$

$$= \frac{1}{2c}\int_0^t \sin(x+c(t-s)) - \sin(x-c(t-s))\,ds$$

$-c$

$$= \frac{1}{2c}\left[+\frac{1}{c}\cos(x+c(t-s)) + \frac{1}{c}\cos(x-c(t-s)) \Big|_{s=0}^{s=t} \right]$$

$$= \frac{1}{2c^2}\Big[2\cos(x) - \underbrace{(\cos(x+ct) + \cos(x-ct))}_{\substack{= (\cos(x)\cos(ct) - \sin(x)\sin(ct) + \cos(x)\cos(ct) + \sin(x)\sin(ct)) \\ = (2\cos(x)\cos(ct))}}\Big]$$

3.

$$= \frac{1}{c^2}(\cos(x) - \cos(x)\cos(ct)) = \frac{1}{c^2}\cos(x)[1-\cos(ct)]$$

- $\frac{1}{2}(\phi(x+ct) + \phi(x-ct)) = \frac{1}{2}\Big[\underbrace{\sin(x+ct) + \sin(x-ct)}_{= (\sin(x)\cos(ct) + \sin(ct)\cos(x) + \sin(x)\cos(ct) - \sin(ct)\cos(x))}\Big]$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}\sin(x)\cos(ct)\right]$$

1.

- Taking the sum of the 3 terms, by DUHAMEL's principle we recover the sol.:

$$\boxed{u(x,t) = \sin(x)\cos(ct) + t(1+x) + \frac{1}{c^2}\cos(x)\left[1-\cos(ct)\right]}$$

* RECALL *

$$\sin(a\pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$$

$$\cos(a\pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

[4.1.4] • CONSIDER WAVES on a resistant medium that satisfy the following problem ($\star$) where r is a CONSTANT s.t

$$\left(\boxed{0 < r < \frac{2\pi c}{l}} \right)$$

Finite interval

$$(\star)\begin{cases} u_{tt} = c^2 u_{xx} - r u_t & \text{for } 0 < x < l \\ u = 0 & \text{at both ends — 2 (D) BC} \\ u(x,0) = \phi(x) & \\ u_t(x,0) = \psi(x) & \end{cases} \quad > 2 \text{ (IC)}$$

Q what is series ~~...~~ expansion of SOL — $u(x,t)$?

- Let us consider seperable solutions $u(x,t) = X(x)T(t)$ of this form..., THEN $\Rightarrow$ $X(x)T''(t) = c^2 X''(x)T(t) - rT'(t)X(x)$

$\Rightarrow$ $X(x)T''(t) + rX(x)T'(t) = c^2 X''(x)T(t)$

Suppose $X(x) \neq 0$ AND $T(t) \neq 0$ divide both sides by $X(x)T(t)$

$\stackrel{*}{\Rightarrow}$ $\star \left\{ \dfrac{T''(t) + rT'(t)}{c^2 T(t)} = \dfrac{\cancel{c^2} X''(x)}{X(x)} := -\lambda \right\}$ — $-\lambda$ is a

- constant in both space & time (x,t)
- why? $\left(\dfrac{\partial \lambda}{\partial x} = 0\right)$ and $\left(\dfrac{\partial \lambda}{\partial t} = 0\right)$.

- Since $u(0,t) = X(0)T(t) = 0$ AND $u(l,t) = X(l)T(t) = 0 \Rightarrow \underline{X(0) = X(l) = 0}$.

- ~~[illegible]~~

- VALUES of $-\lambda$ that satisfy the (BC) are <u>EIGENVALUES</u>; $X(x) \not\equiv 0$ (nontrivial) Solution to ODE in space variable: $\{X''(x) = -\lambda X(x)\}$ are the <u>EIGENFUNCTIONS</u>

- Let us proceed to consider different cases for the eigenvalues. Let $-\lambda = -\beta^2$

CASE: $\lambda > 0$, $\lambda < 0$, $\lambda = 0$, λ Complex.

- λ is a spectrum parameter.

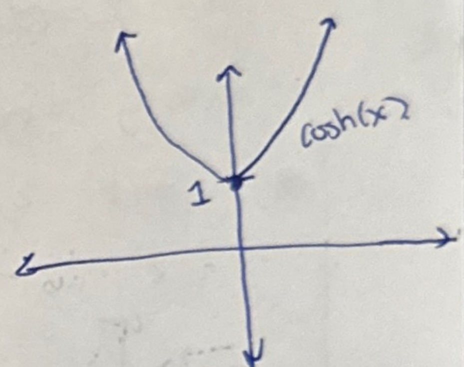
CASE $\boxed{\lambda = 0}$

- $X''(x) = 0$
  $X'(x) = A$
  $X(x) = Ax + B$ (for some constants A, B.)

  - Impose 1st Dirichlet (D) BC: $\xrightarrow{1} X(0) = B = 0$

  so $X(x) = Ax$.

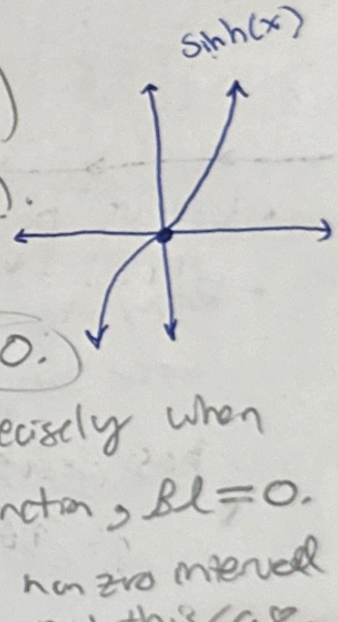
$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$



- impose 2nd (D) BC:
  $X(\ell) = A\underbrace{(\ell)}_{\neq 0} = 0 \Rightarrow A = 0.$
- THEN $X(x) = 0$ which is trivial!

$\therefore$ So $\lambda = 0$ is NOT an Eigenvalue.

$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$



CASE $\boxed{\lambda = -\beta^2 < 0}$

- THEN, $\left(X(x) = A\cosh(\beta x) + B\sinh(\beta x)\right)$
  so $X(0) = A = 0 \Rightarrow X(x) = B\sinh(\beta x)$.
- Imposing the 2nd (D) BC:
  $\longrightarrow X(\ell) = B\sinh(\beta \ell) = 0 \Rightarrow B = 0.$

REMARK: $\sinh(\beta \ell)$ achieves 0 precisely when the argument to the hyperbolic sine function, $\beta \ell = 0$. But $\ell \neq 0$ since we are considering a nonzero interval and $\beta = 0$ considered in $\lambda = 0$ case not this case. THEN $\beta \ell \neq 0$ so $\sinh(\beta \ell) \neq 0$ forcing $B = 0$.

- So $X(x) = 0$ (trivial) so $\lambda < 0$ not eigenvalues.

CASE $\boxed{\lambda = \beta^2 > 0}$: ★

pair of ODEs
$$\begin{cases} (1)\ X''(x) + \beta^2 X(x) = 0 \\ (2)\ T''(t) + rT'(t) + \beta^2 c^2 T(t) = 0 \end{cases}$$

★ (1) $\{X(x) = C\cos(\beta x) + D\sin(\beta x)\}$ Let us consider the two homogeneous (D) BCs:
$X(0) = X(\ell) = 0$

- $X(0) = C = 0$ and $X(\ell) = D\sin(\beta\ell) = 0$.

- Since we are searching for non trivial solutions ($X(x) \neq 0$) and $C = 0$, necessarily we found, let us try to satisfy the second (D) BC without forcing $D = 0$.
- THEN $\sin(\beta\ell) = 0$, which occurs when $(\beta\ell = n\pi)$ for any $n \in \mathbb{N}$. $\Rightarrow \beta = n\pi/\ell$.

$$\Longrightarrow \left[\lambda_n = \left(\frac{n\pi}{\ell}\right)^2\right]\left[X_n(x) = D_n \sin\left(\frac{n\pi}{\ell}x\right)\right]$$ root of sine function.

are distinct solutions and each sine function may be multiplied by arbitrary constant D_n, so there are an infinite number of seperated solutions, one for each choice of n $(n \in \mathbb{N})$.

- Let us proceed to solve the ODE in the time variable.
- From the ODE (2) let us extract the coefficients and write the characteristic EQ: The constant coefficient ODE has solutions of the form $T(t) = e^{qt}$, Substitute $\rightarrow$

$$q^2 e^{qt} + rqe^{qt} + \beta^2 c^2 e^{qt} = 0 \Rightarrow \{q^2 + rq + c^2\beta^2 = 0\}$$
where $(\beta = n\pi/\ell)$...

- THEN,

$$q = \frac{-r \pm \sqrt{r^2 - 4c^2\beta^2}}{2} = \frac{-r \pm \sqrt{r^2 - n^2\left(\frac{2c\pi}{\ell}\right)^2}}{2} \ldots$$

- We were given that $0 < r < \frac{2\pi c}{\ell} \Rightarrow r^2 < \left(\frac{2\pi c}{\ell}\right)^2$

for some $n \in \mathbb{N}$, $n^2 > 0$ so $\Rightarrow r^2 < \left(\frac{2\pi c}{\ell}\right)^2 < n^2\left(\frac{2\pi c}{\ell}\right)^2$.

- THEN the quantity under square root < 0.

So, extract $\sqrt{-1} = i$ from under radical..

$$\Rightarrow q = \frac{-r \pm i\sqrt{n^2\left(\frac{2\pi c}{\ell}\right)^2 - r^2}}{2} = -\frac{r}{2} \pm i\underbrace{\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2\ell^2}}{2\ell}\right)}_{\gamma_n}$$

Sol of form $e^{qt} = T(t)$

$$\Rightarrow T(t) = e^{-r/2\, t}\left[E e^{i\gamma_n t} + F e^{-i\gamma_n t}\right]$$ for unknown coeffients E, F.

- Since $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

AND $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$

our solution $T(t)$ as a linear we can re-express combination of $\sin(\gamma)$ and $\cos(\gamma)$

THEN $\Rightarrow T_n(t) = e^{-r/2\, t}\left[A_n \cos(\gamma_n t) + B_n \sin(\gamma_n t)\right]$.

- Since our PDE is linear, applying the superposition principle we can write $u(x,t)$ as a linear combination of $T_n(t) X_n(t)$.

$\longrightarrow\longrightarrow\longrightarrow$

- Therefore, the general solution $u(x,t) =$

$$\sum_{n=1}^{\infty} e^{-r/2\, t}\left[A_n \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2\ell^2}}{2\ell} t\right) + B_n \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2\ell^2}}{2\ell} t\right)\right] \cdot \sin\left(\frac{n\pi}{\ell} x\right)$$

...

• To satisfy the (IC)s . . .

• $u(x,0)=\phi(x)$, so choose A_n s.t

→ $u(x,0)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)=\phi(x)$.

we know, for $m\neq n$ in $\mathbb{N}$

$$\int_0^l \sin(n\pi x/l)\sin(m\pi x/l)\,dx=0$$

$$\int_0^l \cos(n\pi x/l)\cos(m\pi x/l)\,dx=0$$

for $m\neq n$ in $\mathbb{Z}_{\geq 0}$

$$\int_0^l \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)dx=\int_0^l \phi(x)\sin\left(\frac{m\pi x}{l}\right)dx$$

$$=\sum_{n=1}^{\infty} A_n \underbrace{\int_0^l \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)dx}=$$

$(n,m\in\mathbb{Z})$ $$=\begin{cases}0 & n\neq m\\ ? & n=m\end{cases}$$

* every term in infinite series vanishes except when $n=m$. Analogous to orthogonal eigenvectors of real symmetric matrices whose dot product $=0$.

$$\Rightarrow \quad A_n \int_0^l \underbrace{\sin\left(\frac{m\pi}{l}x\right)^2}_{\frac{1}{2}\left(1-\cos\left(\frac{2m\pi}{l}x\right)\right)} dx$$

$$\Rightarrow \quad \frac{A_n}{2}\left[x-\frac{l}{2m\pi}\sin\left(\frac{2m\pi}{l}x\right)\right]\Bigg|_{x=0}^{x=l}$$

$$\Rightarrow \quad \frac{A_n}{2}\left[(l-0)-\frac{l}{2m}\underbrace{\cancel{\left(\sin(2m\pi)-\sin(0)\right)}}_{0}\right]$$

$$\Rightarrow \quad A_n\left(\frac{l}{2}\right)=\int_0^l \phi(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$\Rightarrow \quad \boxed{A_n=\frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi}{l}x\right)dx \text{ for } n\in\mathbb{N}}$$

☆

* To use the 2^{nd} IC $\quad u_t(x,0) = \psi(x)$ we must first differentiate.

* Differentiating our gen sol $u(x,t)$ with respect to t ...

Previously we let $\gamma_n := \dfrac{\sqrt{4n^2\pi^2c^2 - r^2\ell^2}}{2\ell}$ and found

$$u(x,t) = \sum_{n=1}^{\infty} e^{-r/2t}\left(A_n \cos(\gamma_n t) + B_n \sin(\gamma_n t)\right)\sin\left(\frac{n\pi}{\ell}x\right)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} e^{-r/2t}\left[-A_n\gamma_n \sin(\gamma_n t) + B_n\gamma_n \cos(\gamma_n t)\right]\sin\left(\frac{n\pi}{\ell}x\right)$$

$$+ \sum_{n=1}^{\infty} -\frac{r}{2} e^{-r/2t}\left[A_n\cos(\gamma_n t) + B_n \sin(\gamma_n t)\right]\sin\left(\frac{n\pi}{\ell}x\right)$$

* SET $t=0$ to recover IC ...

$$u_t(x,0) = \underbrace{\left(\sum_{n=1}^{\infty}\left(-\frac{r}{2}\right)\cancel{e}^{1} A_n \sin\left(\frac{n\pi}{\ell}x\right)\right)}_{=\left(-\frac{r}{2}\right)\underbrace{\left(\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right)\right)}_{=\phi(x)}} + \sum_{n=1}^{\infty} B_n\gamma_n \sin\left(\frac{n\pi}{\ell}x\right) = \psi(x).$$

$$\Rightarrow \int_0^{\ell} \sum_{n=1}^{\infty} B_n\gamma_n \sin\left(\frac{n\pi}{\ell}x\right)\cdot \sin\left(\frac{m\pi}{\ell}x\right)dx = \int_0^{\ell}\left(\psi(x) + \frac{r}{2}\phi(x)\right)\sin\left(\frac{m\pi}{\ell}x\right)dx$$

$$\Rightarrow \left(\sum_{n=1}^{\infty}\right) B_n\gamma_n \underbrace{\left(\int_0^{\ell}\sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{m\pi}{\ell}x\right)dx\right)}_{=\ell/2 \text{ for } n=m} = \int_0^{\ell}\left(\psi(x) + \frac{r}{2}\phi(x)\right)\sin\left(\frac{m\pi}{\ell}x\right)dx$$

found ~~for~~ previously & vanishes for $n \neq m$

THUS $\Rightarrow$

$$\boxed{B_n = \frac{2}{\ell}\gamma_n \int_0^{\ell}\left[\psi(x) + \frac{r}{2}\phi(x)\right]\sin\left(\frac{n\pi}{\ell}x\right)dx \text{ for } n\in\mathbb{N}}$$

REMARK: (Complex Eigenvalues for this question 4.1.4 disregard and 4.2.2 and 4.2.3)

What happens

Q: In the case when λ is any complex number. THEN $\beta = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$

THEN $\{X(x) = Ce^{\beta x} + De^{-\beta x}\}$

2 (D) BC $\begin{cases} X(0) = C + D \longrightarrow D = -C \\ X(\ell) = Ce^{\beta\ell} + De^{-\beta\ell} = 0 \Rightarrow De^{-\beta\ell} = -Ce^{\beta\ell} = De^{\beta\ell} \end{cases}$

$e^{z} = e^{n2\pi i} = 1$

THEN $RE(z) = 0$ AND $Im(z) = 2\pi n$ $\Rightarrow$ $(e^{2\beta\ell} = 1)$ $\rightarrow 2\beta\ell = 0$

- THEN $\cancel{2\ell}RE(\beta) = 0$ and $2\ell\, Im(\beta) = 2\pi n$ for some $n \in \mathbb{Z}$

THUS, $\beta = \frac{n\pi}{\ell} i \rightarrow \lambda = -\beta^2 = \left(\frac{n^2\pi^2}{\ell^2}\right) \in \mathbb{R}$ > 0

- Even upon considering the possibility of ~~imaginary~~ complex eigenvalues we find ourselves back in a situation with REAL and moreover positive eigenvalues in this case.

Consider the equation $\{u_{tt} = c^2 u_{xx}\}$ for $0<x<l$

[4.2.2] w/ (BC) $\underbrace{u_x(0,t)=0}_{(N)}$ AND $\underbrace{u(l,t)=0}_{(D)}$

(a) SHOW that eigenfunctions are $\cos[(n+\frac{1}{2})\pi x/l]$

- Let us look for seperable solutions of the form $u(x,t) = X(x)T(t)$... THEN $\{u_{tt}=c^2u_{xx}\} \Rightarrow \{XT''=c^2X''T\}$

$$\Rightarrow \left\{\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} =: -\lambda\right\}$$

(Suppose $X(x)\neq 0$ AND $T(t)\neq 0$ so divide by $X(x)T(t)$)

$-\lambda$ is a constant in (x,t) (Since $\frac{\partial\lambda}{\partial x}=0$ and $\frac{\partial\lambda}{\partial t}=0$.)

- We recover the (BC): $u_x(0,t) = X'(0)T(t) = 0 \Rightarrow X'(0)=0$

and $u(l,t) = X(l)T(t)=0 \Rightarrow X(l)=0$.

- $\lambda=\beta^2$ are the eigenvalues for which the BC: $\left(\begin{array}{l}X'(0)=0 \text{ and}\\ X(l)=0\end{array}\right)$ are satisfied and nontrivial functions $X(x)\neq 0$ are known as eigenfunctions.

CASE 1: $\boxed{\lambda=\beta^2>0}$

GEN SOL ...

$$\star\begin{cases} T''(t)+\beta^2c^2T(t)=0 \\ X''(x)+\beta^2X(x)=0 \\ \quad \text{s.t } X'(0)=0 \\ \quad \text{and } X(l)=0.\end{cases} \qquad \star\begin{cases} T(t) = A\cos(\beta c t) + B\sin(\beta c t) \\ X(x) = C\cos(\beta t) + D\sin(\beta t)\end{cases}$$

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- THEN differentiating $\Rightarrow X'(x) = -C\beta\sin(\beta t) + D\beta\cos(\beta t)$

$\binom{1st}{BC} \Rightarrow X'(0) \overset{1st}{=} 0 + D\underbrace{\beta}_{\neq 0} \neq 0 \Rightarrow (D=0)$

$\binom{2nd}{BC} \Rightarrow X(l) = C\cos(\beta l) = 0 \Rightarrow \cos(\beta l)=0 \Rightarrow \ldots$

(looking for nontrivial sol so dont want, $C=0$ and $D=0$)

$\cos(\beta l) = 0$ when $\beta l = \frac{\pi}{2} + n\pi = \left(\frac{1}{2}+n\right)\pi$

$$\Rightarrow \left(\beta = \frac{(n+\frac{1}{2})\pi}{l} \right)$$

- Then, $X(x) = C\cos\left(\frac{(n+\frac{1}{2})\pi}{l}x\right)$. The eigenvalues are $\lambda = \beta^2$

(By linear superpos any scalar multiple of an eigenfunction (a sol) is another solution.)

$$\left[\lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2 . \right]$$

- The eigenfunctions are indeed

$$X_n(x) = \cos\left(\frac{(n+\frac{1}{2})\pi}{l}x\right) \text{ for } n = 0, 1, 2, \ldots$$

CASE2: $\boxed{\lambda = 0}$

- $X''(x) = 0 \Rightarrow X(x) = Ax + B \Rightarrow X'(x) = A$.

$\left(\begin{smallmatrix}1st\\BC\end{smallmatrix}\right) \Rightarrow X'(0) = A = 0$. THEN $X(x) = B$

$\left(\begin{smallmatrix}2nd\\BC\end{smallmatrix}\right) \Rightarrow X(l) = B = 0$. THEN $X(x) = 0$ (trivial sol.)

- Therefore, $\lambda = 0$ CANNOT be an eigenvalue.

CASE3: $\boxed{\lambda = -\beta^2 < 0}$

- $X''(x) - \beta^2 X(x) = 0 \Rightarrow X(x) = C\cosh(\beta x) + D\sinh(\beta x)$.

THEN $\Rightarrow X'(x) = C\beta\sinh(\beta x) + D\beta\cosh(\beta x)$

- $\left(\begin{smallmatrix}1st\\BC\end{smallmatrix}\right)$. $X'(0) = \underbrace{C\beta\sinh(0)}_{0} + D\beta\underbrace{\cosh(0)}_{1} = 0 \Rightarrow D\underbrace{(\beta)}_{\neq 0} = 0$.

- So $X(x) = C\cosh(\beta x)$. $\left(\begin{smallmatrix}2nd\\BC\end{smallmatrix}\right)$ $X(l) = C\underbrace{(\cosh(\beta l))}_{\neq 0} = 0$. $\Rightarrow D = 0$

$\Rightarrow C = 0$

- Therefore since $X(x) = 0$ (trivial) $\lambda < 0$ CAN NOT be an eigenvalue.

- So only for $\lambda > 0$ do we have eigenfunctions, which we found above.

CASE λ is complex then $\beta = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$

THEN $X(x) = Ce^{\beta x} + De^{-\beta x}$

$X'(x) = C\beta e^{\beta x} + D\beta e^{-\beta x}$

$\begin{pmatrix} X'(0)=0 \\ X(l)=0 \end{pmatrix}\begin{matrix}(N) \\ (D)\end{matrix}$
$\begin{cases} X'(0) = C\beta - D\beta = 0 \implies C = D \\ X(l) = Ce^{\beta l} + De^{-\beta l} = 0 \end{cases}$

$$\implies De^{-\beta l} = -Ce^{\beta l} = -De^{\beta l}$$

$$\implies -1 = e^{2\beta l}$$

$e^{(z)} = e^{n\pi i} = -1$

THEN $Re(z) = 0$ and $Im(z) = n\pi$

HERE ~~2l~~ $RE(\beta) = 0$ and $2l\, Im(\beta) = \pi n$ for som $n \in \mathbb{Z}$

So, $\beta = \frac{n\pi}{2l} i \rightarrow \lambda = -\beta^2 = \left(\frac{n^2\pi^2}{4l^2}\right) \in \mathbb{R}_{>0}$

* Even upon consdering the possibility of complex Eigenvalue we find ourselves back in a situation with REAL & moreover positive eigenvalues for this problem.

(b) We found previously that

$$* \begin{cases} X(x) = C\cos\left(\frac{(n+\frac{1}{2})\pi}{l}x\right) \text{ and} \\ T(t) = A\cos(\beta c t) + B\sin(\beta c t). \end{cases}$$

- Since the PDE: $\{u_{tt} = c^2 u_{xx}\}$ is linear we can apply the superposition principle to recover a general solution $u(x,t)$ which is a linear combination of all $X_n(x)T_n(t)$ for all eigenvalues

- $$u(x,t) = X(x)T(t) = \sum_{n=0}^{\infty}\left(A_n\cos(\beta c t) + B_n\sin(\beta c t)\right)C_n\cos\left(\frac{(n+\frac{1}{2})\pi}{l}x\right)$$

(we can absorb the constant C_n into A_n and B_n ...)

$$\left[u(x,t) = \sum_{n=0}^{\infty}\left[A_n\cos(\beta c t) + B_n\sin(\beta c t)\right]\cos\left(\frac{(n+\frac{1}{2})\pi}{l}x\right)\right]$$

(series expansion for sol $u(x,t)$)

where we leave A_n & B_n as undetermined coefficients since we are not given initial conditions.

[4.2.3] SOLVE Schrödinger EQ $\{\overset{\text{PDE}}{u_t = ik\,u_{xx}}\}$

for $k \in \mathbb{R}$ in finite interval $0 < x < l$

w/ B.C. $\underbrace{u_x(0,t) = 0}_{(N)}$ AND $\underbrace{u(l,t) = 0}_{(D)}$

- Let us search for "seperable solutions" of the form: $u(x,t) = X(x)\,T(t)$. Substituting and differentiating as appropriate in the PDE we find that,

$$\{X(x)\,T'(t) = ik\,X''(x)\,T(t)\}$$ For $T(t) \neq 0$ and $X(x) \neq 0$ let us divide by $X(x)\,T(t)$ ...

$$\Rightarrow \frac{T'(t)}{ik\,T(t)} = \frac{X''(x)}{X(x)} =: (-\lambda)$$

constant in space & time (x,t) as $\frac{\partial \lambda}{\partial x} = 0$ and $\frac{\partial \lambda}{\partial t} = 0$.

- We know that by the two I.C. ...

$u_x(0,t) = X'(0)\,T(t) = 0 \Rightarrow X'(0) = 0$ AND

$u(l,t) = X(l)\,T(t) = 0 \Rightarrow X(l) = 0$

THEN $\boxed{X'(0) = X(l) = 0}$

(...)

CONT next pg.

· CASE 1: $\boxed{\lambda = \beta^2 > 0}$

gen sol?

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$$\begin{cases} T'(t) + \beta^2 ik\, T(t) = 0 \\ X''(x) + \beta^2 X(x) = 0 \end{cases} \qquad \begin{cases} T(t) = A\, e^{-\beta^2 ik\, t} \\ X(x) = C\cos(\beta x) + D\sin(\beta x). \end{cases}$$

s.t. (BC) $X'(0) = 0$ AND $X(\ell) = 0$

$$\begin{cases} \dfrac{\partial T}{\partial t} = T'(t) = (-\beta^2 ik)T \\ \Rightarrow \displaystyle\int \frac{1}{T}\, \partial T = \int (-\beta^2 ik)\, \partial t \quad \text{separable} \\ \Rightarrow \ln(T) = (-\beta^2 ik)t \\ \Rightarrow T = e^{-\beta^2 ik\, t} \end{cases}$$

- $X'(x) = -C\beta\sin(\beta x) + D\beta\cos(\beta x)$

  $X'(0) = \underbrace{D\beta}_{\beta \neq 0} = 0 \Rightarrow D = 0$

- $X(x) = C\cos(\beta x)$

  $X(\ell) = \underbrace{C}_{\neq 0}\cos(\beta\ell) = 0 \Longrightarrow \cos(\beta\ell) = 0$

(Dont want $C = D = 0$ since we are searching for nontrivial solutions $X(x) \neq 0$).

- THEN, $\beta\ell = (n+\frac{1}{2})\pi \Rightarrow \left(\beta = \dfrac{(n+\frac{1}{2})\pi}{\ell}\right)$
- SO, $X_n(x) = C_n \cos\left(\dfrac{(n+\frac{1}{2})\pi}{\ell} x\right)$ for $n = 0, 1, 2, \dots$
- We know that $T_n(t) = A_n\, e^{-\left(\frac{(n+\frac{1}{2})\pi}{\ell}\right)^2 ik\, t}$
- $u(x,t) = X(x)T(t) = \displaystyle\sum_{n=0}^{\infty} X_n(x)\, T_n(t)$

By linear superposition AND absorbing constants (i.e. C_n into A_n)

we find that

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} A_n\, e^{-\left(\frac{(n+\frac{1}{2})\pi}{\ell}\right)^2 ikt} \cos\left(\frac{(n+\frac{1}{2})\pi}{\ell} x\right)}$$

(An is <u>undetermined</u> coefficent since no I.C. given.) (...)

CONT...

CASE 2: $\boxed{\lambda = 0} \Rightarrow \{X''(x) = 0\} \Rightarrow X' = A \Rightarrow \boxed{X(x) = Cx + D}$

THEN $X'(x) = C$.

- Imposing the BC ...

$X'(0) = C = 0$. SO, $X(x) = D$.

AND $X(\ell) = D = 0$. THUS, $X(x) = 0$ (trivial sol).

- THEREFORE $\lambda = 0$ is NOT an eigenvalue.

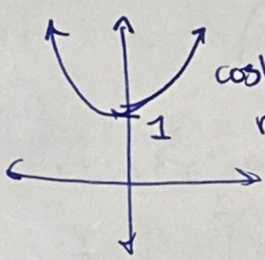
CASE 3: $\boxed{\lambda = -\beta^2 < 0} \Rightarrow \{X''(x) - \beta^2 X(x) = 0\} \Rightarrow$

$X(x) = C\cosh(\beta x) + D\sinh(\beta x)$,

$X'(x) = C\beta\sinh(\beta x) + D\beta\cosh(\beta x)$.

(1st BC) $\cdot$ $X'(0) = D\underset{\neq 0}{(\beta)} = 0 \Rightarrow D = 0$. THEN $X(x) = C\cosh(\beta x)$.

(2nd BC) $\cdot$ $X(\ell) = C\underbrace{(\cosh(\beta\ell))}_{\neq 0} = 0 \Rightarrow C = 0$



cosh(x) never achieves 0!!!

THUS, $X(x) = 0$.

So $\lambda < 0$ is NOT an eigenvalue!!

CASE 4: λ complex?

↳ By the exact same procedure as in our λ Complex CASE analysis in the previous question 4.2.2 which has same ODE in space with BC $\underbrace{X'(0) = 0}_{(N)}$ and $\underbrace{X(\ell) = 0}_{(D)}$ ... we FIND that by examining the Complex case we find ourselves back in the situation with REAL positive eigenvalues...

CASE λ is complex then $\beta = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$

THEN $X(x) = Ce^{\beta x} + De^{-\beta x}$

$X'(x) = C\beta e^{\beta x} - D\beta e^{-\beta x}$

$\begin{pmatrix} X'(0)=0 \\ X(l)=0 \end{pmatrix} \begin{matrix} (N) \\ (D) \end{matrix}$

$$\begin{cases} X'(0) = C\beta - D\beta = 0 \implies C = D \\ X(l) = Ce^{\beta l} + De^{-\beta l} = 0 \end{cases}$$

$$\implies De^{-\beta l} = -Ce^{\beta l} = -De^{\beta l}$$

$$\implies -1 = e^{2\beta l}$$

$e^{(z)} = e^{n\pi i} = -1$

THEN $Re(z) = 0$ and $Im(z) = n\pi$

HERE $\cancel{2l}$ $RE(\beta) = 0$ and $2l\, Im(\beta) = \pi n$ for some $n \in \cancel{\mathbb{Z}}$

$$\text{So, } \beta = \frac{n\pi}{2l} i \rightarrow \lambda = -\beta^2 = \left(\frac{n^2\pi^2}{4l^2}\right) \in \mathbb{R}_{>0}$$

* Even upon considering the possibility of complex Eigenvalue we find ourselves back in a situation ~~of~~ with REAL & moreover positive eigenvalues for this problem.