(HW #7) (4.3) 2*
(5.1) 3* 9*
(5.2) 4* 7* 11*

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2. Eigenvalue problem w/ ROBIN BC @ both ends:

$$\star \begin{cases} -X'' = \lambda X \\ X'(0) - a_0 X(0) = 0, \\ X'(l) + a_l X(l) = 0. \end{cases}$$

(a) SHOW that $(\lambda = 0)$ is an eigenvalue $\overset{\text{i.f.f.}}{\Longleftrightarrow}$

$$\{a_0 + a_l = -a_0 a_l l\}$$

- IF $(\lambda = 0)$, then ODE simplifies: $\{X'' = 0\}$
- Gen sol of $X \longrightarrow X = Ax + B \longrightarrow X' = A$

* apply x2 BC to determine A and B *

$$* \begin{cases} X'(0) - a_0 X(0) \Longleftrightarrow A - a_0 B = 0 \\ X'(l) + a_l X(l) \Longleftrightarrow A + a_l(A \cdot l + B) = 0 \end{cases}$$

THEN $(A = a_0 B)$, and substituting we get...

$$a_0 B + a_l a_0 B l + a_l B = 0$$

for $B \neq 0$ divide by B...

$$(a_0 + a_l a_0 l + a_l = 0) \Longleftrightarrow \left[a_0 + a_l = -a_0 a_l l\right]$$

$$(a_0 + a_l(a_0 l + 1) = 0)$$

IF $[(a_0+a_\ell) = -a_0 a_\ell \ell]$ THEN

let $\lambda = \beta^2 \geq 0$

$X(x) = A\cos(\beta x) + B\sin(\beta x)$

$X' = -A\beta\sin(\beta x) + B\beta\cos(\beta x)$

Since $X'(0) - a_0 X(0) = 0$, $B\beta - a_0 A = 0$

and $-A\beta\sin\beta\ell + B\beta\cos\beta\ell + a_\ell(A\cos\beta\ell + B\sin\beta\ell) = 0$

THEN

$-\beta\sin\beta\ell + a_0\cos\beta\ell + a_\ell\cos\beta\ell + \frac{a_0 a_\ell}{\beta}\sin\beta\ell =$

$(a_0 + a_\ell)\cos\beta\ell + \left(\frac{a_0 a_\ell}{\beta} - \beta\right)\sin\beta\ell = 0$

THEN $\Rightarrow$ $a_0 + a_\ell = 0$ && $\frac{a_0 a_\ell}{\beta} - \beta = 0$

$-a_0 a_\ell \ell = 0$

$\ell \neq 0$.

$a_0 a_\ell = 0 \to \frac{0}{\beta} - \beta = 0 \to \beta = 0$

$\boxed{\lambda = 0}$

(b) eigenfunctions ~ $\lambda \geq 0$?

(solve ODE for $X(x)$ ... not w/ sine/cosine)

...

found $(A = a_0 B)$

and

$a_0 + a l = -a_0 a l\, l$

$X(x) = Ax + B$

$= a_0 B x + B$

$= B(a_0 x + 1)$

$\boxed{X_n = B_n (a_0 x + 1)}$ eigenfunctions for some undetermined coefficents B_n.

[5.1.3] Consider f $\phi(x) \equiv x$ on $\underline{(0, l)}$

→ Sketch (on same graph)

(a) SUM 1st 3 terms fourier sine

NONZERO

(b) sum of 1st 3 terms fourier cosine

(a) Let $\phi(x) \equiv x$ have fourier sine expansion

s.t. $\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) = x$

$$\sum_{n=1}^{\infty} B_n \int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) = \int_0^l x \sin\left(\frac{m\pi x}{l}\right)$$

f integral of sum ≡ sum of integral even in this case of an infinite sum assuming sufficient conditions for convergence

By "orthogonality" we know:

$$\left(\int_0^l \sin\left(\frac{n\pi}{l}x\right) \sin\left(\frac{m\pi}{l}x\right) = 0\right) \text{ if } n \neq m$$

only term surviving is when $n = m$ ...

We know,

$\Rightarrow$

$$\int_0^l \sin^2\left(\frac{n\pi}{l}x\right)dx = \frac{1}{2}\int_0^l 1 + \cos\left(\frac{n\pi}{l}x\right)dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\frac{l}{n\pi}\sin\left(\frac{n\pi}{l}x\right)\right]\Bigg|_{x=0}^{x=l}$$

$$= \frac{1}{2}[l-0] - \frac{l}{2n\pi}\left[\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin(0)}_{0}\right]$$

$$= l/2.$$

- Thus, we simplify the equation to get... integration by parts

$$\Rightarrow \boxed{B_n \cancel{l/2} = \frac{2}{l}\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx}$$

$$u = x \quad dv = \sin\left(\frac{n\pi}{l}x\right)$$
$$du = 1\,dx \quad v = -\frac{l}{n\pi}\cos\left(\frac{n\pi}{l}x\right)$$
$$uv - \int v\,du$$

$$\Rightarrow B_n = \frac{2}{l}\left[\underbrace{-\frac{l}{n\pi}x\cos\left(\frac{n\pi}{l}x\right)\Big|_0^l}_{\parallel} + \frac{l}{n\pi}\underbrace{\int_0^l \cos\left(\frac{n\pi}{l}x\right)dx}_{\parallel}\right]$$

$$\underbrace{\frac{-l}{n\pi}\left(l\underbrace{\cos(n\pi)}_{(-1)^n} - 0\right)}_{\parallel}$$

$$-\frac{l^2}{n\pi}(-1)^n$$

$$\frac{l}{n\pi}\cdot\sin\left(\frac{n\pi}{l}x\right)\Big|_{x=0}^{x=l}$$

$$\underbrace{\underbrace{\sin(n\pi)}_{0} - \underbrace{\sin(0)}_{0}}_{=0.}$$

$$\Rightarrow B_n = +\frac{2l}{n\pi}(-1)^{n+1}$$

- THEREFORE, [Fourier Sine series expansion of $\phi(x) \equiv x$]

$$\therefore \sum_{n=1}^{\infty}\frac{2l}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi}{l}x\right) = x.$$

(1st 3 nonzero terms ??)

$$\underbrace{\left[\frac{2l}{\pi}\sin\left(\frac{\pi}{l}x\right) - \frac{l}{\pi}\sin\left(\frac{2\pi}{l}x\right) + \frac{2l}{3\pi}\sin\left(\frac{3\pi}{l}x\right)\right]}_{\parallel\ f(x)} \approx x.$$

PARTIAL SUM

(b) Fourier cosine series?! Let $\phi(x) \equiv x$ have Fourier cosine series

$$\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{\ell}x\right) = x \longrightarrow A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\ell}x\right) = x$$

• recover A_0:

$$\underbrace{\int_0^\ell A_0\,dx}_{A_0 \ell} + \underbrace{\sum_{n=1}^{\infty} A_n \int_0^\ell \cos\left(\frac{n\pi}{\ell}x\right)dx}_{} = \frac{x^2}{2}\Big|_0^\ell$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{\ell}{n\pi}\sin\left(\frac{n\pi}{\ell}x\right)\Big|_{x=0}^{x=\ell}}_{\underbrace{\cancel{\sin(n\pi) - \sin(0)}}_{0 \quad 0}} = \frac{\ell^2}{2}$$

$$= 0.$$

$$\Rightarrow \quad A_0 = \frac{\ell}{2}$$

• recover A_n:

$$\underbrace{A_0 \int_0^\ell \cos\left(\frac{m\pi}{\ell}x\right)dx}_{} + \sum_{n=1}^{\infty} A_n \int_0^\ell \cos\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{m\pi}{\ell}x\right)dx = \int_0^\ell x\cos\left(\frac{m\pi}{\ell}x\right)dx$$

$$\int_0^\ell x\cos\left(\frac{m\pi}{\ell}x\right)dx = \underbrace{A_0 \frac{\ell}{m\pi}\sin\left(\frac{m\pi x}{\ell}\right)\Big|_0^\ell}_{=0} + \underbrace{A_n \int_0^\ell \cos\left(\frac{m\pi}{\ell}x\right)^2 dx}_{}$$

$$= \frac{A_n}{2}\int_0^\ell 1 + \cos\left(\frac{2m\pi}{\ell}x\right)dx$$

$$= \frac{A_n}{2}\left[x + \frac{\ell}{2m\pi}\underbrace{\sin\left(\frac{2m\pi}{\ell}x\right)}_{}\right]\Big|_{x=0}^{x=\ell}$$

$$= \frac{A_n}{2}\left([\ell] + \underbrace{\sin(2\pi m) - \sin(0)}_{=0}\right)$$

$$= \frac{\ell}{2}A_n$$

$$\rightarrow \quad A_n = \frac{2}{\ell}\int_0^\ell x\cos\left(\frac{m\pi}{\ell}x\right)dx$$

* integration by parts *

$$A_n = \frac{2}{\ell} \cdot \int_0^{\ell} x \cos\left(\frac{n\pi}{\ell} x\right) dx$$

$u = x \quad dv = \cos\left(\frac{n\pi}{\ell} x\right)$

$du = dx \quad v = \frac{\ell}{n\pi} \sin\left(\frac{n\pi}{\ell} x\right)$

$uv - \int v\, du$

$$= \frac{2}{\ell}\left[\underbrace{\frac{\ell}{n\pi} x \sin\left(\frac{n\pi}{\ell} x\right)\Big|_{x=0}^{x=\ell}}_{=0.}\right] - \frac{2}{n\pi}\underbrace{\int_0^{\ell} \sin\left(\frac{n\pi}{\ell} x\right) dx}_{\Downarrow}$$

$$+ \frac{2}{n\pi} \cdot \frac{\ell}{n\pi}\left(\cos\left(\frac{n\pi}{\ell}\right) \Big|_{x=0}^{x=\ell}\right)$$

$\cos(n\pi) - \cos(0)$

$(-1)^n \quad - 1$

$$\boxed{A_n = \frac{2\ell}{n^2\pi^2}\left((-1)^n - 1\right)}$$

- Therefore the fourier cosine series expansion of $\phi(x) \equiv x$

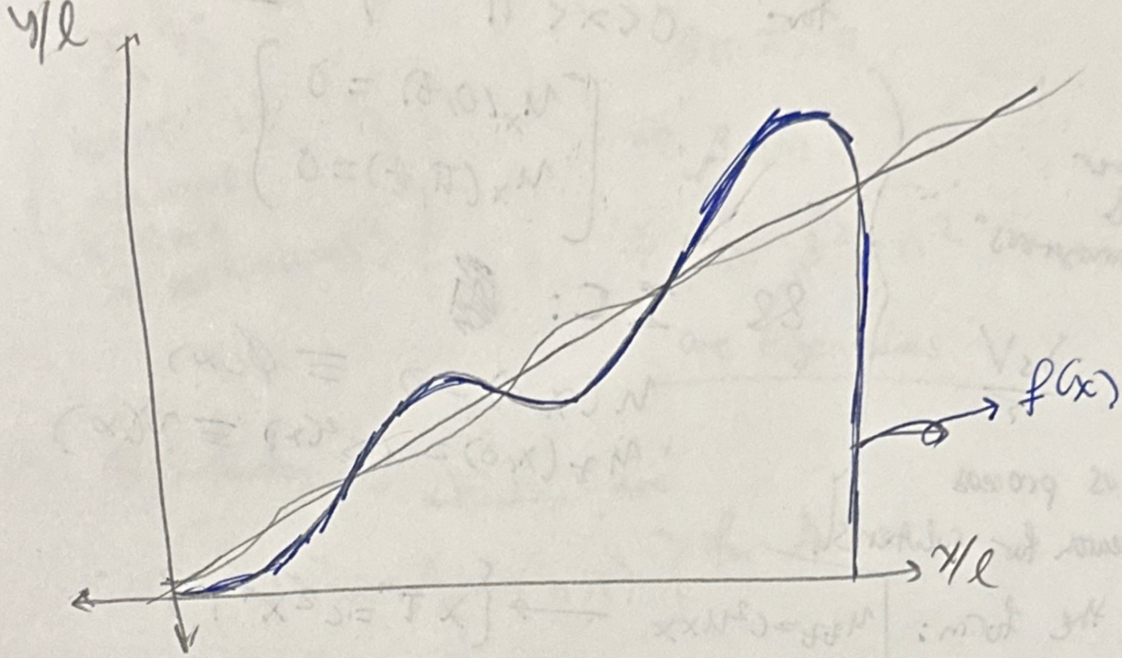
$$\frac{\ell}{2} + \sum_{n=1}^{\infty} \frac{2\ell}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi}{\ell} x\right) = x.$$

- 1st 3 non zero terms in series:

$$\Rightarrow \frac{\ell}{2} + \frac{2\ell}{\pi^2}\left((-1) - 1\right)\cos\left(\frac{\pi}{\ell}x\right) + \underbrace{\frac{\ell}{2\pi^2}(+1-1)}_{0} \cdots + \frac{2\ell}{9\pi^2}(-2)\cos\left(\frac{3\pi}{\ell}x\right) = 0$$

$$\Rightarrow \underbrace{\frac{\ell}{2} - \frac{4\ell}{\pi^2}\cos\left(\frac{\pi}{\ell}x\right) - \frac{4\ell}{9\pi^2}\cos\left(\frac{3\pi}{\ell}x\right)}_{g(x)} \approx x$$

Graphing f(x) & g(x) against one another



CONCLUDE:

[5.1.9*] SOLVE $u_{tt} = c^2 u_{xx}$

for $0 < x < \pi$ w/ BC:

$$\begin{bmatrix} u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \end{bmatrix}$$

"linear & homogeneous" ... USE S.o.V ?

&& IC:

$u(x,0) = 0 \equiv \phi(x)$

$u_t(x,0) = \cos^2(x) \equiv \psi(x)$

• Let us proceed to search for solutions, of the form:

$u(x,t) = X(x)\,T(t)$

$u_{tt} = c^2 u_{xx} \longrightarrow \{X T'' = c^2 X'' T\}$

$X'(0)\,T(t) = 0 \longrightarrow X'(0) = 0$

$X'(\pi)\,T(t) = 0 \longrightarrow X'(\pi) = 0$

THEN

$$\frac{T''}{c^2 T} = \frac{X''}{X} =: -\lambda$$

Some constant in space & time since $\left(\frac{\partial \lambda}{\partial x} = 0\right)$ and $\left(\frac{\partial \lambda}{\partial t} = 0\right)$

$\lambda \sim$ eigenvalues

$X(x) \sim$ eigenfunctions.

CASE 1: $(\lambda > 0)$ let $\lambda = \beta^2$

$$\begin{cases} X'' = -\beta^2 X \\ X'(0) = X'(\pi) = 0 \end{cases} \Rightarrow X(x) = A\cos(\beta x) + B\sin(\beta x)$$

$X'(x) = -A\beta\sin(\beta x) + B\beta\cos(\beta x)$

$X'(0) = B\underbrace{\beta}_{\neq 0} = 0 \longrightarrow B = 0 \longrightarrow X(x) = A\cos(\beta x)$

$X'(x) = -A\beta\sin(\beta x)$

$$X'(\pi) = -\underbrace{A}_{\neq 0}\underbrace{\beta}_{\neq 0}\sin(\beta\pi) = 0 \longrightarrow \sin(\beta\pi) = 0$$

$$\beta\pi = n\pi$$

(WANT non trivial Solutions SO NOT $A = 0 = \beta$!)

$$\Rightarrow \beta = n$$

$$\Rightarrow \lambda = \beta^2 = n^2$$

are eigenvalues.

- Eigenfunctions ~ $\lambda = n^2$ are

$$\left[X_n(x) = A_n \cos(nx)\right] \text{ for } \lambda > 0$$

☆ $\boxed{\text{CASE 2: } (\lambda = 0)}$ $\longrightarrow \begin{cases} X'' = 0 \\ X'(0) = X'(\pi) = 0 \end{cases}$

ODE for span

$$X(x) = Ax + B \longrightarrow X'(x) = A$$

$$X'(0) = A = 0 \longrightarrow \boxed{X(x) = B}$$

$$X'(\pi) = A = 0$$

B remains arbitrary

* this is non-trivial eigenfunction so $\lambda = 0$ is indeed an eigenvalue. *

- The ODE $T(t) \longrightarrow \{T'' = 0\} \longrightarrow T(t) = Ct + D$

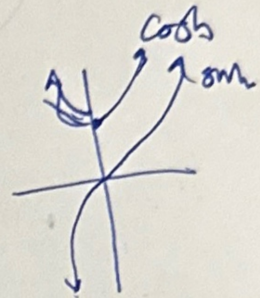
(...) HERE $X(x)T(t) = B(Ct + D)$

constants absorbed $\longrightarrow \quad = \tilde{C}t + \tilde{D}$

$\boxed{\text{CASE 3: } \lambda < 0}$ let $\lambda = \gamma^2$ THEN $\{X'' = \gamma^2 X\}$

$\longrightarrow$ $$X(x) = A\cosh(\gamma x) + B\sinh(\gamma x)$$

let us impose the BC: $X'(0) = 0$ and $X'(\pi) = 0$.



$\longrightarrow$ $X'(x) = A\gamma\sinh(\gamma x) + B\gamma\cosh(\gamma x)$

$X'(0) = \underbrace{B\gamma}_{\gamma \neq 0} = 0 \longrightarrow B = 0$

THEN $\Rightarrow \left(\begin{array}{l} X(x) = A\cosh(\gamma x) \\ X'(x) = A\gamma\sinh(\gamma x) \end{array} \right)$

$X'(\pi) = A\underbrace{\gamma}_{\neq 0}\underbrace{\sinh(\pi\gamma)}_{\neq 0} = 0 \Longrightarrow A = 0$

~~we know that sinh such~~

* we know that hyperbolic sine only achieves 0 when the input $= 0$

that is, ($\sinh(x) = 0 \overset{}{\underset{\text{iff.}}{\longleftrightarrow}} x = 0$.) *

- if $A \neq 0$ then $\sinh(\pi\gamma) = 0 \to \pi\gamma = 0 \to \gamma = 0$ ↯ this is a contradiction because we are considering $\lambda < 0$ CASE.

- Therefore $A = 0$ necessarily. The only sol is the trivial solution $X(x) = 0$.

So, $\lambda < 0$ not valid eigenvalues.

- return to $\left(\begin{matrix}\text{CASE 1} \\ \lambda > 0\end{matrix}\right)$ where we found
$X_n(x) = A_n \cos(nx)$

- ODE in time: $\{ T'' = -c^2 \beta^2 T \} \longrightarrow$

. $T(t) = C\cos(c\beta t) + D\sin(c\beta t) \longrightarrow T_n(t) = \tilde{A}\cos(cnt) + \tilde{B}\sin(cnt)$

By linear superposition, $u(x,t) = \sum_n T_n(t) X_n(t)$ over all eigenvalues.

Constants absorbed

$$u(x,t) = Ct + D + \sum_{n=1}^{\infty} \left(A_n \cos(cnt) + B_n \sin(cnt) \right) \cos(nx)$$

$$u_t(x,t) = C + \sum_{n=1}^{\infty} \left(-cnA_n \sin(cnt) + cnB_n \cos(cnt) \right) \cos(nx)$$

- Imposing the I.Cs $\longrightarrow u(x,0) = D + \sum_{n=1}^{\infty} A_n \cos(nx) = 0$

$$\longrightarrow u_t(x,0) = \left[C + \sum_{n=1}^{\infty} (cnB_n) \cos(nx) \right] = \left[\cos^2(x) \right]$$

*

$\Longrightarrow$ integrating over $(0,\pi)$ $\displaystyle\int_0^{\pi} C + \sum_{n=1}^{\infty} cnB_n \cos(nx)\, dx = \int_0^{\pi} \underbrace{\cos^2(x)}_{\frac{1}{2}(1+\cos(2x))} dx$

$$C\pi + \sum_{n=1}^{\infty} cnB_n \underbrace{\int_0^{\pi} \cos(nx)\,dx}_{\underbrace{\frac{1}{n}\sin(nx)\Big|_0^{\pi}}_{=0}}$$

$\Rightarrow \frac{1}{2}\left[x + \frac{1}{2}\sin(2x) \right]\Big|_{0=x}^{\pi=x}$

$\Rightarrow \frac{1}{2}\left([\pi] + \frac{1}{2} \underbrace{\sin(2\pi) - \sin(0)}_{=0} \right)$

$\Rightarrow \pi/2$

for $B \neq C$ divide

$\Rightarrow C\pi = \pi/2$

$\Rightarrow C = 1/2$

let us recall the value of B_n ...

mult by $\cos(mx)$

$$\int_0^{\pi} C\cos(mx) + \sum_{n\geq 1}^{\infty} c n B_n \cos(nx)\cos(mx)\,dx \qquad \int_0^{\pi} \cos^2(x)\cos(mx)\,dx$$

$$\Rightarrow C\int_0^{\pi}\cos(mx) + \sum_{n\geq1}^{\infty} C n B_n \underbrace{\int_0^{\pi}\cos(mx)\cos(nx)\,dx}_{\text{integrand}} = \frac{1}{2}\int_0^{\pi}(1+\cos(2x))\cos(mx)\,dx$$

$= 0$ if $n \neq m$

$= \cos^2(nx)$ if $n = m$

$$\underbrace{C\int_0^{\pi}\cos(mx)\,dx}_{=\ \frac{C}{n}\sin(mx)\Big|_0^{\pi}\ =\ 0} + C n B_n \cdot \frac{\pi}{2} = \frac{1}{2}\left[\underbrace{\int_0^{\pi}\cos(nx)\,dx}_{=0} + \int_0^{\pi}\cos(2x)\cos(mx)\,dx\right]$$

$$\Rightarrow B_n = \frac{1}{Cn\pi}\underbrace{\int_0^{\pi}\cos(2x)\cos(mx)\,dx}$$

$\Rightarrow$

$= 0$ if $m \neq 2$

$= \cos^2(2x)$ if $m = 2$

$= \frac{1}{2}(1+\cos(2x))$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right]\Big|_{x=0}^{x=\pi}$

$= \frac{\pi}{2}$

$$\rightarrow B_n = \frac{1}{Cn\,2} \text{ when } n=2$$

$$\boxed{B_n = \frac{1}{4C} \text{ when } n=2 \text{ and } 0 \text{ when } n\neq 2}$$

THEREFORE our general solution
to the PDE...

$$\cancel{\frac{t}{2} + B_2 \sin(2ct)\cos(2x)}$$

$$\boxed{u(x,t) = \frac{t}{2} + \frac{1}{4c}\sin(2ct)\cos(2x)}$$

[5.2.4] USE (5) $\left(\int_{-\ell}^{\ell} \text{ODD}\,dx = 0\right)$ && $\left(\int_{-\ell}^{\ell} \text{EVEN}\,dx = 2\int_{0}^{\ell} \text{EVEN}\,dx\right)$

(a) to prove that if $\phi(x)$ is an ODD function, its full F.S. on $(-\ell,\ell)$ has only sine terms...

By DEF ODD fn $\rightarrow$ $\phi(-x) = -\phi(x)$ && full F.S expansion on $(-\ell,\ell)$?

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\ell}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{\ell}x\right)$$

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \underbrace{\phi(x)\cos\left(\frac{n\pi x}{\ell}\right)}_{O\times E = ODD!} dx = 0 \quad (n=0,1,2,\ldots)$$

$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \phi(x)\sin\left(\frac{n\pi}{\ell}x\right)dx = \frac{2}{\ell}\int_{0}^{\ell} \underbrace{\phi(x)\sin\left(\frac{n\pi}{\ell}x\right)}_{O\times O = EVEN!} \quad n=1,2,\ldots$$

Thus $A_n = 0$ $(n=0,1,2,\ldots)$ and B_n is non zero!
Therefore only sine terms remain in Full F.S. on $(-\ell,\ell)$.

(b) IF $\underline{\phi(x)}$ is even function... show its full F.S. on $(-l,l)$ has only cosine terms...

- By definition $\phi(-x)=\phi(x)$ and we know that the full F.S. expansion on $(-l,l)$ is as follows:

$$\phi(x)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{l}x\right)+\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{l}\right).$$

* consider the parity of the integrand in the integral formula for the coefficients.

$$A_n=\frac{1}{l}\int_{-l}^{l}\underbrace{\phi(x)\cos\left(\frac{n\pi}{l}x\right)}_{E\times E=E}=\frac{2}{l}\int_0^l\phi(x)\cos\left(\frac{n\pi}{l}x\right)\quad(n=0,1,2,\dots)$$

AND

$$B_n=\frac{1}{l}\int_{-l}^{l}\underbrace{\phi(x)\sin\left(\frac{n\pi}{l}x\right)}_{E\times O=ODD!}dx=0$$

→ So $B_n=0$! all sine terms vanish.

- Only $A_n\neq 0$ so only cosines terms persist.

[5.2.7] Show how full F.S. on $(-\ell,\ell)$ can be derived from full series on $(-\pi,\pi)$ by changing variables $\boxed{w=(\pi/\ell)x}$

CHANGE of SCALE/

(one unit x axis $\longleftrightarrow$ π/ℓ units w axis)

- The full F.S. on $(-\pi,\pi)$ for some function $\phi(w)$ is ...

$$\phi(w) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos(nw) + \sum_{n=1}^{\infty} B_n\sin(nw)$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(w)\cos(nw)\,dw \qquad n=0,1\ldots$$

AND

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(w)\sin(nw) \qquad n=1,2,\ldots$$

$$\left[\begin{array}{lll} w=\frac{\pi}{\ell}x & \longrightarrow & x=\frac{\ell}{\pi}w \\ dw=\frac{\pi}{\ell}dx & \longrightarrow & \frac{\ell}{\pi}dw=dx \end{array}\right] \quad \phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{\ell}x\right) + B_n\sin\left(\frac{n\pi}{\ell}x\right)$$

for

$$A_n = \frac{1}{\pi}\int_{-\ell}^{\ell}\phi(x)\cos\left(\frac{n\pi}{\ell}x\right)\left(\frac{\pi}{\ell}\right)dx$$

$$= \frac{1}{\ell}\int_{-\ell}^{\ell}\phi(x)\cos\left(\frac{n\pi}{\ell}x\right)dx \quad , n=0,1,2\ldots$$

AND

$$B_n = \frac{1}{\pi}\int_{-\ell}^{\ell}\phi(x)\sin\left(\frac{n\pi}{\ell}x\right)\left(\frac{\pi}{\ell}\right)dx \quad , n=1,2,\ldots$$

$$= \frac{1}{\ell}\int_{-\ell}^{\ell}\phi(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$$

Q: FIND full fourier series of e^x on $(-l, l)$ in its REAL & COMPLEX forms....

[5.2.11*]

(convenient to find complex form first)

★ COMPLEX F.S.

$$e^x = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$$

for $m \in \mathbb{Z}$ $(n \neq m)$

$$\Rightarrow \int_{-l}^{l} e^{x(1-im\pi/l)}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-l}^{l} e^{i(n-m)\pi x/l}\,dx$$

$$\Rightarrow \underbrace{\frac{1}{1-im\pi/l}\, e^{x(1-im\pi/l)}\Big|_{-l}^{l}}_{=\ \frac{l}{l-im\pi}\left(e^{l-im\pi} - e^{-l+im\pi}\right)} = \sum_{n=-\infty}^{\infty} c_n \cdot \frac{1}{i(n-m)\pi/l}\, e^{i(n-m)\pi x/l}\Big|_{-l}^{l}$$

$$\frac{l}{l-im\pi} = \frac{l(l+im\pi)}{(l-im\pi)(l+im\pi)}$$

$n \neq m$

$$\frac{1}{i(n-m)\pi/l}\left(e^{i(n-m)\pi} - e^{-i(n-m)\pi}\right)$$

$$= \cancel{\cos[(n-m)\pi]} + i\underbrace{\sin[(n-m)\pi]}_{=0} - \cancel{\cos[(n-m)\pi]} + i\underbrace{\sin[(n-m)\pi]}_{=0}$$

for $n \neq m$ RHS $= 0$...

$$\left(\frac{l(l+im\pi)}{(l-im\pi)(l+im\pi)}\right)\left(e^{l-im\pi} - e^{-l+im\pi}\right) = c_n \int_{-l}^{l} e^0\,dx$$

$$= c_n\, 2l$$

$$\frac{l(l+im\pi)}{(l-im\pi)(l+im\pi)} = \frac{l(l+im\pi)}{l^2+m^2\pi^2}$$

(...)

$$c_n = \frac{\cancel{l}\,(l + in\pi)}{l^2+n^2\pi^2} \cdot \frac{1}{\cancel{l}} \cdot \frac{e^{i(-il-n\pi)} - e^{-i(-il-n\pi)}}{2i} \cdot i$$

$$= i\sin(-il-n\pi)$$

$$-i\,\underbrace{\sin(il+n\pi)}_{=\left[\sin(il)\underbrace{\cos(n\pi)}_{(-1)^n} + \cos(il)\underbrace{\sin(n\pi)}_{=0}\right]}$$

$$\frac{l+in\pi}{l^2+n^2\pi^2}\left(\underbrace{+i(-1)^{n+1}\sin(il)}_{=(-1)^n\sinh(l)}\right)$$

$\langle \sin(ix) = i\sinh(x) \rangle$ *

SO, complex F.S of e^x on $(-l, l)$ is

$$\boxed{e^x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{l+in\pi}{l^2+n^2\pi^2} \sinh(l)\, e^{in\pi x/l}}$$

REAL F.S.

$$e^x = \sum_{n=-\infty}^{-1} (-1)^n \frac{l + in\pi}{l^2+n^2\pi^2} (\sinh l) e^{in\pi x/l}$$

$$+ \frac{\sinh l}{l} + \sum_{n=1}^{\infty} (-1)^n \frac{l + in\pi}{l^2+n^2\pi^2} (\sinh(l)) e^{n\pi x/l}$$

(SUB $n=-k$ & $n=k$)

$(-1)^{-k} = (-1)^k$

$$= \sum_{k=1}^{\infty} (-1)^k \frac{l - ik\pi}{l^2+k^2\pi^2} \sinh(l)\, e^{-ik\pi x/l}$$

$$+ \frac{\sinh(l)}{l} + \sum_{k=1}^{\infty} (-1)^k \frac{l + ik\pi}{l^2+k^2\pi^2} \sinh(l)\, e^{ik\pi x/l}$$

THEN,

$$e^x = \frac{\sinh l}{l} + \sum_{k=1}^{\infty} (-1)^k \frac{\sinh(l)}{l^2+k^2\pi^2} \underbrace{\left((l - ik\pi) e^{-ik\pi x/l} + (l+ik\pi) e^{ik\pi x/l} \right)}$$

$$(l - ik\pi)\left(\cos\left(\frac{k\pi}{l}x\right) - i\sin\left(\frac{k\pi}{l}x\right)\right)$$
$$+ (l+ik\pi)\left(\cos\left(\frac{k\pi x}{l}\right) + i\sin\left(\frac{k\pi x}{l}\right)\right)$$

$$= \frac{\sinh l}{l} + \sum_{k=1}^{\infty} (-1)^k \frac{\sinh(l)}{l^2+k^2\pi^2} \left(2l\cos\frac{k\pi x}{l} - 2k\pi \sin\frac{k\pi x}{l} \right)$$

replace k with n...

TH EN

$$e^x = \frac{\sinh(l)}{l} + 2\sinh(l) \sum_{n\geq 1}^{\infty} \frac{(-1)^n}{l^2+n^2\pi^2} \left(l\cos\left(\frac{n\pi x}{l}\right) - n\pi \sin\left(\frac{n\pi x}{l}\right)\right)$$